J. Appl. Numer. Optim. 7 (2025), No. 2, pp. 199-213
Available online at http://jano.biemdas.com
https://doi.org/10.23952/jano.7.2025.2.04

CALCULUS RULES OF GENERALIZED PROPER ε-SUBDIFFERENTIAL FOR VECTOR VALUED MAPPINGS AND APPLICATIONS

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Abstract. This paper deals with a new concept of subdifferential defined in the Pareto sense and adapted to nonconvex vector mappings, called generalized proper ε-subdifferential. Some existence theorems and properties are discussed. We establish some formulas of the generalized proper ε-subdifferential for the sum and the difference of two vector valued mappings. As an application of the calculus rules, we establish necessary and sufficient optimality conditions for a constrained vector optimization problem with the difference of two vector valued mappings.

Keywords. Generalized ε-proper subdifferential; Vector optimization problem; Vector valued mappings; ε-proper minimizer; ε-blunt proper minimizer.
2020 Mathematics Subject Classification. 49J52, 49J53, 90C31.

1. INTRODUCTION

The notion of subdifferential defined by Rockafellar for convex functions plays a crucial role in optimization. This classical notion of subdifferentiability has been generalized to nonconvex functions. In particular, the concept of subdifferential has been subject to a more general extension to vector-valued mappings. In the last few decades, authors introduced the approximate subdifferential for non-differentiable vector-valued mappings due to their wide field of applications in economics, finance, game theory and optimal control. The approximate vector subdifferential, as a generalization of the vector subdifferential for extended vector-valued mappings, was extensively studied from the theoretical viewpoint; see, e.g., [6, 7, 8, 9, 10, 12, 13, 14, 15]. The motivation of this investigation is multiple since approximate solutions are always efficient and useful in all optimization problems especially when it is difficult to calculate the exact solutions. Therefore, techniques of approximation are very important in both vector and set-valued optimization, which allows to characterize the approximate efficient optimal solutions such as generalized strict weak ε-quasi ℓ-solutions and ℓ- weak minimum. We refer here to the recent works [2, 3, 4, 11] wherein the concept of approximate subdifferential plays a fundamental role and offers a large range of solutions. Recently, in [14], a concept of approximate subdifferential of a vector mapping was introduced, which was defined by a norm, called generalized ε-subdifferential. This subdifferential is a global notion, well-adapted to nonconvex mappings

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Received 12 November 2024; Accepted 20 March 2025; Published online 30 July 2025.

and weaker than the strong subdifferential. In fact, The authors of [14] studied the properties of their approximate subdifferential by providing some characterizations together with the related calculus rules.

In this paper, motivated by [14], we introduce and study a new concept of approximate subdifferential in the Pareto sense. Actually, our aim here is to introduce in the Pareto sense a new concept of approximate subdifferential, based on the concept of Henig proper efficient solution, called generalized proper ε-subdifferential. In particular, we focus our attention on studying its properties and stating the corresponding calculus rules. Moreover, we establish some new formulas for the generalized proper ε-subdifferential of the sum and the difference of two vector-valued mappings and give necessary and sufficient optimality conditions for proper efficient minimizers of vector optimization problems.

The outline of the paper is as follows. Section 2 presents some preliminaries which are needed in the sequel. In Section 3, we prove the existence theorem of generalized proper ε-subgradient by using the so called generalized properly lower ε-Lipschitz mappings. We also recall that the concepts discussed in [3, 4, 14]. In Section 4, we study some proprieties of generalized proper ε-subdifferential and the relationship between the generalized proper ε-subdifferential and the directional derivative of the considered vector-valued mappings. In Section 5, we give the calculus rules of generalized proper ε-subdifferential for the sum and the difference of two vector valued mappings. In Section 6, by virtue of these results, we are able to derive necessary and sufficient optimality conditions for the ε-blunt proper minimizer of constrained vector optimization problems in the framework of the so-called D.C. map (difference of two maps).

2. Preliminaries

In the following, we describe some notations, definitions and results which are used in the sequel. Let $(X,\|.\|_X)$ and $(Y,\|.\|_Y)$ be two normed spaces with topological dual spaces denoted by X^* and Y^*, respectively, and the dual pairing in X (respectively in Y) is denoted by $\langle x^*,x\rangle$, $x^*\in X^*$, $x\in X$ (respectively $\langle y^*,y\rangle$, $y^*\in Y^*$, $y\in Y$). Here we start by establishing the requirements needed for Y, the range space of the vector valued mappings involved in the sequel. In what follows, Y is endowed with a convex cone Y_+ with apex at the origin such that $Y_+\cap -Y_+=\{0_Y\}$.

One introduces on Y a partial ordering by, for any $y_1,y_2\in Y$, $y_1\leq_{Y_+} y_2 \iff y_2-y_1\in Y_+$. Furthermore, we define $y_1\lneq_{Y_+} y_2 \iff y_2-y_1\in Y_+\setminus\{0_Y\}$. We enlarge Y by attaching to Y a greatest elements $+\infty_Y$ with respect to $\leq_{Y_+}$. Hence, it holds $y\leq_{Y_+}+\infty_Y$ for any $y\in Y$. We suppose that $y-(+\infty_Y)\notin Y_+$ for any $y\in Y$. We also assume by convention that

$$\begin{aligned} y+(+\infty_Y) &= (+\infty_Y)+y=+\infty_Y, \quad \forall y\in Y\\ (+\infty_Y)-(+\infty_Y) &= +\infty_Y\\ \alpha.(+\infty_Y) &= +\infty_Y, \quad \forall \alpha\geq 0. \end{aligned}$$

Let $L(X,Y)$ be the set of linear continuous operators from X to Y. The positive polar cone Y_+^* of Y_+ is the set of nonnegative forms $y^*\in Y^*$ i.e. $y^*(Y_+)\subseteq\mathbb{R}_+$, while the strict polar cone $\left(Y_+^*\right)^\circ$ is the set of $y^*\in Y^*$ such that $y^*(Y_+\setminus\{0_Y\})\subset\mathbb{R}_+\setminus\{0\}$. Obviously $\left(Y_+^*\right)^\circ\subseteq Y_+^*\setminus\{0_Y\}$. Let $F:X\to Y\cup\{+\infty_Y\}$ be a vector mapping. F is said to be Y_+-convex if, for any $x_1,x_2\in X$ and for any $\alpha\in[0,1]$, $F(\alpha x_1+(1-\alpha)x_2)\leq_{Y_+}\alpha F(x_1)+(1-\alpha)F(x_2)$. F is said to be Y_+-concave

if $-F$ is Y_+-convex. F is said to be proper if its effective domain is nonempty, i.e.,

$$\text{dom}F := \{x \in X : \ F(x) \in Y\} \neq \emptyset.$$

Let us recall the well-known properties.

Lemma 2.1. [5] *If Y_+ is a convex pointed cone, then*

i) $\text{int}Y_+ \subset Y_+ \backslash \{0_Y\}$;
ii) $Y_+ + \text{int}Y_+ \subset Y_+ \backslash \{0_Y\}$;
iii) $Y_+ + Y_+ \backslash \{0_Y\} \subset Y_+ \backslash \{0_Y\}$;
iv) $Y_+ \backslash \{0_Y\} + \text{int}Y_+ \subset \text{int}Y_+$;

Definition 2.1. ([5, 14]) Let $F : X \to Y \cup \{+\infty_Y\}$ be a given mapping and $\bar{x} \in X$.

1) The strong subdifferential of F at $\bar{x}$ is defined by

$$\partial^s F(\bar{x}) := \left\{T \in L(X,Y) : T(x-\bar{x}) \leq_{Y_+} F(x) - F(\bar{x}), \forall x \in X\right\}.$$

2) The proper subdifferential of F at $\bar{x}$ is defined by

$$\partial^p F(\bar{x}) := \left\{ \begin{array}{l} T \in L(X,Y) : \exists \hat{Y}_+ \in \mathscr{D}(Y_+) \text{ and } \nexists x \in X \text{ such that} \\ F(x) - F(\bar{x}) \leq_{\hat{Y}_+} T(x-\bar{x}) \end{array} \right\}$$

where $\mathscr{D}(Y_+) := \left\{\hat{Y}_+ \subset Y \text{convex cone such that } Y_+ \backslash \{0_Y\} \subseteq \text{int}\hat{Y}_+ \right\}$.

3) Let $\varepsilon \in Y_+$. The generalized strong ε-subdifferential of F at $\bar{x}$ is defined by

$$\partial^s_\varepsilon F(\bar{x}) := \left\{T \in L(X,Y) : T(x-\bar{x}) - \|x-\bar{x}\|_X \varepsilon \leq_{Y_+} F(x) - F(\bar{x}), \forall x \in X\right\}.$$

Let us notice that $\partial^s_\varepsilon F(\bar{x}) = \partial^s \left(F + \|.-\bar{x}\|_X \varepsilon\right)(\bar{x})$.

Remark 2.1. If F is a convex function ($Y = \mathbb{R}$, $Y_+ = \mathbb{R}_+$), then the above subdifferentials with $\varepsilon = 0$ reduces to the well known Fenchel subdifferential

$$\partial F(\bar{x}) := \{x^* \in X^* : \langle x^*, x-\bar{x}\rangle \leq F(x) - F(\bar{x}), \forall x \in X\}.$$

We also need the following result.

Proposition 2.1. [1] *Let X be a locally convex topological vector space and $F, G : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be two convex proper functions. If F is continuous at some point of* dom$G$*, then, for any $x \in X$, $\partial\left(F+G\right)(x) = \partial F\left(x\right) + \partial G\left(x\right)$.*

3. Generalized Proper ε-subdifferential

Motivated by [14] and [5], we introduce the new concept of generalized proper ε-subdifferential for vector valued mappings.

Definition 3.1. Let $F : X \to Y \cup \{+\infty_Y\}$ be a given mapping, $\varepsilon \in Y_+$, and $\bar{x} \in \text{dom}F$. The set

$$\partial^p_\varepsilon F(\bar{x}) := \left\{ \begin{array}{l} T \in L(X,Y) : \exists \hat{Y}_+ \in \mathscr{D}(Y_+) \text{ such that} \\ \nexists x \in X\,,\ F(x) - F(\bar{x}) \leq_{\hat{Y}_+} T(x-\bar{x}) - \|x-\bar{x}\|_X \varepsilon \end{array} \right\}$$

is called the generalized proper ε-subdifferential of F at $\bar{x}$. Every $T \in \partial^p_\varepsilon F(\bar{x})$ is called a generalized proper ε-subgradient of F at $\bar{x}$. If $\bar{x} \notin \text{dom}F$, we set $\partial^p_\varepsilon F(\bar{x}) = \emptyset$. In the case where $intY_+ \neq \emptyset$, the cone $\hat{Y}_+$ can be chosen as $\hat{Y}_+ := intY_+ \cup \{0_Y\}$

Notice that $\partial^p_0 F(\bar{x}) = \partial^p F(\bar{x})$ and $\partial^p_\varepsilon F(\bar{x}) = \partial^p(F + \|.-\bar{x}\|_X \varepsilon)(\bar{x})$.

Example 3.1. Consider the following mapping

$$\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R}^2 \\ & x & \longmapsto & F(x)=(-|x|,-|x|) \end{array}$$

The space $\mathbb{R}^2$ is equipped with its natural order $Y_+=\mathbb{R}^2_+$. For $\bar{x}=0$ and $\varepsilon=(\varepsilon_1,\varepsilon_2)\in\mathbb{R}^2_+$, also for some $x_0\in\mathbb{R}$ and $(a,b)\in\mathbb{R}^2$. We have

$$F(x_0)-F(0)-(a,b)(x_0-0)+|x_0-0|\varepsilon=(|x_0|(\varepsilon_1-1)-ax_0,|x_0|(\varepsilon_2-1)-bx_0)$$

and

$$|x_0|(\varepsilon_1-1)-ax_0+|x_0|(\varepsilon_2-1)-bx_0=|x_0|(\varepsilon_1+\varepsilon_1-2)-(a+b)x_0.$$

By taking the convex pointed cone $\hat{Y}_+:=\{(x,y)\in\mathbb{R}^2/x+y>0\}\sqcup\{(0,0)\}\in\mathscr{D}(Y_+)$, we see that if $b=-a$ *and* $\varepsilon=(\varepsilon_1,\varepsilon_2)\in([1,+\infty[)^2$, then there not exists any $x_0\in\mathbb{R}$ such that

$$F(x_0)-F(0)-(a,-a)(x_0-0)+|x_0-0|\varepsilon\in-\hat{Y}_+\backslash\{(0,0)\},$$

which yields $(a,-a)\in\partial^p_\varepsilon F(0)$. Notice that if $\varepsilon=(1,1)$, and $a=0$, then $(0,0)\in\partial^p_\varepsilon F(0)$. Taking into account that

$$F(x_0)-F(0)-(0,0)(x_0-0)+|x_0-0|\varepsilon=(-|x_0|+|x_0|,-|x_0|+|x_0)|=(0,0)\notin-\hat{Y}_+\backslash\{(0,0)\},$$

for any convex pointed cone $\hat{Y}_+\in\mathscr{D}(Y_+)$, we see that $(0,0)\in\partial^p_\varepsilon F(0)$ does not depend on the choice of $\hat{Y}_+$.

In order to establish existence of generalized proper ε-subgradient, we introduce the following class of vector valued mappings.

Definition 3.2. Let $\varepsilon\in Y_+$. A mapping $F:X\longrightarrow Y\cup\{+\infty_Y\}$ is called generalized properly lower locally ε-Lipschitz at $\bar{x}$ if there exists a neighborhood $V(\bar{x})$ of $\bar{x}$ such that

$$\begin{cases} \exists\hat{Y}_+\in\mathscr{D}(Y_+),\ \nexists x\in V(\bar{x}) \text{ such that} \\ F(x)-F(\bar{x})\leq_{\hat{Y}_+}-\|x-\bar{x}\|_X\varepsilon. \end{cases}$$

If the above definition does not holds for any $x\in X$, then F is called generalized properly lower ε-Lipschitz at $\bar{x}$.

Remark 3.1. The mapping presented in Example 3.1 is generalized properly lower ε-Lipschitz at $\bar{x}$.

Theorem 3.1. *Let $F:X\longrightarrow Y\cup\{+\infty_Y\}$ be a given mapping and $\varepsilon\in Y_+$. If F is generalized properly lower ε-Lipschitz at $\bar{x}$, then F is generalized properly ε-subdifferentiable at $\bar{x}$, i.e., $\partial^p_\varepsilon F(\bar{x})\neq\emptyset$.*

Proof. As F is generalized properly lower ε-Lipschitz at $\bar{x}$, i.e.,

$$\begin{cases} \exists\hat{Y}_+\in\mathscr{D}(Y_+),\ \nexists x\in X \text{ such that} \\ F(x)-F(\bar{x})\leq_{\hat{Y}_+}-\|x-\bar{x}\|_X\varepsilon. \end{cases}$$

one sees that

$$\begin{cases} \exists\hat{Y}_+\in\mathscr{D}(Y_+),\ \nexists x\in X \text{ such that} \\ F(x)-F(\bar{x})\leq_{\hat{Y}_+}0\,(x-\bar{x})-\|x-\bar{x}\|_X\varepsilon. \end{cases}$$

That is, $0\in\partial^p_\varepsilon F(\bar{x})$. □

Remark 3.2. Note that the class of mappings $F : X \longrightarrow Y \cup \{+\infty_Y\}$ satisfying $0 \in \partial_\varepsilon^p F(\bar{x})$ are generalized properly lower ε-Lipschitz at $\bar{x}$.

Theorem 3.2. *Let $F : X \longrightarrow Y \cup \{+\infty_Y\}$ be a Y_+-convex mapping and $\varepsilon \in Y_+$. If F is generalized properly lower locally ε-Lipschitz at $\bar{x}$, then F is generalized proper ε-subdifferentiable at $\bar{x}$.*

Proof. Following the above theorem, it suffices to prove that F is generalized properly lower ε-Lipschitz at $\bar{x}$. As F is generalized properly lower locally ε-Lipschitz at $\bar{x}$, one sees that there exists a neighborhood $V(\bar{x})$ of $\bar{x}$ such that

$$\begin{cases} \exists \hat{Y}_+ \in \mathscr{D}(Y_+),\ \nexists x \in V(\bar{x}) \text{ such that} \\ F(x) - F(\bar{x}) \lneq_{\hat{Y}_+} -\|x-\bar{x}\|_X \varepsilon. \end{cases}$$

We proceed by contradiction. If F is not generalized properly lower ε-Lipschitz at $\bar{x}$, then by taking the convex cone $\widetilde{Y}_+ := \text{int}\hat{Y}_+ \cup \{0_Y\}$, one sees that there exists some $x_0 \in X$ such that $F(x_0) - F(\bar{x}) \lneq_{\widetilde{Y}_+} -\|x_0 - \bar{x}\|_X \varepsilon$, i.e.,

$$F(x_0) - F(\bar{x}) + \|x_0 - \bar{x}\|_X \varepsilon \in -\widetilde{Y}_+ \setminus \{0_Y\} = -\text{int}\hat{Y}_+. \tag{3.1}$$

One can find $b \in V(\bar{x})$ and $\lambda \in]0,1]$ such that $b - \bar{x} = \lambda(x_0 - \bar{x})$, i.e., $b = \lambda x_0 + (1-\lambda)\bar{x}$. Since F is Y_+-convex, we obtain

$$F(b) = F(\lambda x_0 + (1-\lambda)\bar{x}) \leq_{Y_+} \lambda F(x_0) + (1-\lambda)F(\bar{x}),$$

that is,

$$F(b) - \lambda F(x_0) - (1-\lambda)F(\bar{x}) \in -Y_+, \tag{3.2}$$

Since $\lambda\left(\text{int}\hat{Y}_+\right) \subseteq int\hat{Y}_+$, it follows from (3.1) that

$$\lambda F(x_0) - \lambda F(\bar{x}) + \|\lambda(x_0 - \bar{x})\|_X \varepsilon \in -\text{int}\hat{Y}_+ \tag{3.3}$$

Adding (3.2) and (3.3) and taking into account that $-Y_+ - \text{int}\hat{Y}_+ \subseteq -\text{int}\hat{Y}_+$, we have

$$F(b) - F(\bar{x}) + \|b - \bar{x}\|_X \varepsilon \in -\text{int}\hat{Y}_+ \subseteq -\hat{Y}_+ \setminus \{0_Y\},$$

which contradicts the fact that F is generalized properly lower locally ε-Lipschitz at $\bar{x}$. The proof is complete. □

Theorem 3.3. *Let $F : X \longrightarrow Y \cup \{+\infty_Y\}$ be a given mapping. If F is properly subdifferentiable at $\bar{x} \in \text{dom}F$ (i.e. $\partial^p F(\bar{x}) \neq \emptyset$, then $\partial^p F(\bar{x}) \subseteq \partial_\varepsilon^p F(\bar{x})$ for all $\varepsilon \in Y_+$.*

Proof. The case that $\varepsilon = 0_Y$ is obvious. Since F is properly subdifferentiable at $\bar{x}$, i.e., $\partial^p F(\bar{x}) \neq \emptyset$. Let $T \in \partial^p F(\bar{x})$. Then there exists a convex cone $\hat{Y}_+ \subsetneq Y$ with $Y_+ \setminus \{0_Y\} \subset \text{int}\hat{Y}_+$ such that

$$\nexists x \in X,\ F(x) - F(\bar{x}) - T(x - \bar{x}) \in -\hat{Y}_+ \setminus \{0_Y\}. \tag{3.4}$$

If we suppose that there exists some $\varepsilon \in Y_+ \setminus \{0_Y\}$ such that $T \notin \partial_\varepsilon^p F(\bar{x})$, we obtain by taking the convex cone $\widetilde{Y}_+ := \text{int}\hat{Y}_+ \cup \{0_Y\}$ that there exists some $x_0 \in X$ such that

$$F(x_0) - F(\bar{x}) - T(x_0 - \bar{x}) + \|x_0 - \bar{x}\|_X \varepsilon \in -\widetilde{Y}_+ \setminus \{0_Y\}. \tag{3.5}$$

If $x_0 = \bar{x}$, then $0_Y \in -\hat{Y}_+ \setminus \{0_Y\}$, which is a contradiction. Now, if $x_0 \neq \bar{x}$, it follows that $\varepsilon\|x_0 - \bar{x}\|_X \in Y_+ \setminus \{0_Y\}$. Since $Y_+ \setminus \{0_Y\} \subseteq \text{int}\hat{Y}_+$ and $\text{int}\hat{Y}_+ = \widetilde{Y}_+ \setminus \{0\}$, we have

$$-\|x_0 - \bar{x}\|\varepsilon \in -\widetilde{Y}_+ \setminus \{0_Y\}. \tag{3.6}$$

By adding (3.5) and (3.6), we obtain $F(x_0)-F(\bar{x})-T(x_0-\bar{x}) \in -\widetilde{Y}_+\backslash\{0_Y\} = -\text{int}\hat{Y}_+$. In view of $-\text{int}\hat{Y}_+ \subseteq -\hat{Y}_+\backslash\{0_Y\}$, we have $F(x_0)-F(\bar{x})-T(x_0-\bar{x}) \in -\hat{Y}_+\backslash\{0_Y\}$, which contradicts (3.4). Hence, for any $\varepsilon \in Y_+$, we have $T \in \partial_\varepsilon^p F(\bar{x})$. The proof is complete. □

Remark 3.3. According to Theorem 3.3, we obtain $\partial^p F(\bar{x}) \subset \partial_\varepsilon^p F(\bar{x})$ for any $\varepsilon \in Y_+$, but the reverse inclusion does not holds. In fact, let us consider the space $\mathbb{R}^2$ equipped with its natural order $Y_+ = \mathbb{R}^2_+$ and the following mapping

$$\begin{array}{rccc} F: & \mathbb{R} & \longrightarrow & \mathbb{R}^2 \\ & x & \longmapsto & F(x)=(0,-x). \end{array}$$

Taking any convex cone $\hat{Y}_+$ such that $\mathbb{R}^2_+\backslash\{(0,0)\} \subseteq \text{int}\hat{Y}_+$, we have for $\bar{x}=0$, $x_0=1$ and $T=(1,0)$

$$F(1)-F(0)-(1-0)(1,0) = (-1,-1) \in -\text{int}\mathbb{R}^2_+ \subseteq -\hat{Y}_+\backslash\{(0,0)\},$$

which means that $(1,0) \notin \partial^p F(0)$. Now, let us prove that $(1,0) \in \partial_\varepsilon^p F(0)$ for $\varepsilon=(1,1) \in \mathbb{R}^2_+$. We suppose that $(1,0) \notin \partial_\varepsilon^p F(0)$. By taking any convex pointed cone $\hat{Y}_+ \in \mathscr{D}(Y_+)$, we have

$$\exists x_0 \in \mathbb{R} : F(x_0)-F(0)-(x_0-0)(1,0)+\|x_0-0\|_X\,\varepsilon \in -\hat{Y}_+\backslash\{(0,0)\},$$

that is,

$$\exists x_0 \in \mathbb{R} : (|x_0|-x_0, |x_0|-x_0) \in -\hat{Y}_+\backslash\{(0,0)\}. \tag{3.7}$$

Since $|x_0| \geq x_0$, it follows from (3.7) that $|x_0| > x_0$ and also $(|x_0|-x_0, |x_0|-x_0) \in \mathbb{R}^2_+\backslash\{(0,0)\} \subseteq \hat{Y}_+\backslash\{(0,0)\}$. Hence, we deduce from the fact that $\hat{Y}_+$ is pointed that $|x_0| = x_0$, contradiction. Thus $(1,0) \in \partial_\varepsilon^p F(0)$.

4. Properties of Generalized Proper ε-subdifferential

Proposition 4.1. *Let $F: X \to Y \cup \{+\infty_Y\}$ be a given mapping. For any $\bar{x} \in \text{dom}F$, $\partial^p F(\bar{x}) = \bigcap_{\varepsilon \in Y_+} \partial_\varepsilon^p F(\bar{x})$.*

Proof. If $\bar{x} \notin \text{dom}F$. Then the equality is obvious. Let $\bar{x} \in \text{dom}F$. Then $\bigcap\limits_{\varepsilon \in Y_+} \partial_\varepsilon^p F(\bar{x}) \subseteq \partial_0^p F(\bar{x}) = \partial^p F(\bar{x})$. For the reverse inclusion and according to Theorem 3.3, we have $\partial^p F(\bar{x}) \subseteq \partial_\varepsilon^p F(\bar{x})$ for all $\varepsilon \in Y_+$, which yields $\partial^p F(\bar{x}) \subseteq \bigcap\limits_{\varepsilon \in Y_+} \partial_\varepsilon^p F(\bar{x})$. □

Corollary 4.1. *Let $F: X \longrightarrow Y \cup \{+\infty_Y\}$ be a given mapping. For any $\varepsilon \in Y_+$, $\partial_\varepsilon^p F(\bar{x}) = \bigcap_{\eta \in Y_+} \partial_{\varepsilon+\eta}^p F(\bar{x})$.*

Proof. It follows from Proposition 4.1 that

$$\begin{aligned} \bigcap_{\eta \in Y_+} \partial_{\varepsilon+\eta}^p F(\bar{x}) &= \bigcap_{\eta \in Y_+} \partial^p\left(F+\|.-\bar{x}\|_X\,\varepsilon+\|.-\bar{x}\|_X\,\eta\right)(\bar{x}) \\ &= \bigcap_{\eta \in Y_+} \partial_\eta^p\left(F+\|.-\bar{x}\|_X\,\varepsilon\right)(\bar{x}) \\ &= \partial^p\left(F+\|.-\bar{x}\|_X\,\varepsilon\right)(\bar{x}) \\ &= \partial_\varepsilon^p F(\bar{x}). \end{aligned}$$

This completes the proof. □

When F is a Y_+-convex vector valued mapping, the following proposition shows that the generalized proper ε-subdifferential can be characterized by the directional derivative.

Proposition 4.2. *Let $F : X \longrightarrow Y \cup \{+\infty_Y\}$ be a Y_+-convex vector mapping, $\varepsilon \in Y_+$, and $\bar{x} \in$ dom$F$. Suppose that Y_+ is closed and the directional derivative $F'(\bar{x},.)$ of F at $\bar{x}$ exists, that is, for every $d \in X$,*

$$\lim_{t\to 0^+} \frac{F(\bar{x}+td)-F(\bar{x})}{t} = F'(\bar{x},d)$$

exists in sense of norm convergence in Y. Then

$$T \in \partial_\varepsilon^p F(\bar{x}) \Longleftrightarrow \begin{cases} T \in L(X,Y),\ \exists \hat{Y}_+ \in \mathscr{D}(Y_+),\ \nexists d \in X \text{ such that} \\ F'(\bar{x},d) \precneqq_{\hat{Y}_+} T(d) - \|d\|_X \varepsilon. \end{cases}$$

Proof. Let $T \in \partial_\varepsilon^p F(\bar{x})$. Then there exists a convex cone $\hat{Y}_+ \subsetneq Y$ such that $Y_+\backslash\{0_Y\} \subset \text{int}\hat{Y}_+$ and for any $x \in X$

$$F(x) - F(\bar{x}) - T(x-\bar{x}) + \|x-\bar{x}\|_X \varepsilon \in (-\hat{Y}_+\backslash\{0_Y\})^c.$$

Letting $d \in X$ and taking $x - \bar{x} = td$ with $t > 0$ we obtain

$$F(\bar{x}+td) - F(\bar{x}) - T(td) + \|td\|_X \varepsilon \in (-\hat{Y}_+\backslash\{0_Y\})^c, \forall d \in X.$$

As $\frac{1}{t}(-\hat{Y}_+\backslash\{0_Y\})^c = (-\hat{Y}_+\backslash\{0_Y\})^c$ and int $\hat{Y}_+ \subset \hat{Y}_+\backslash\{0_Y\}$, we find

$$\frac{F(\bar{x}+td)-F(\bar{x})}{t} - T(d) + \|d\|_X \varepsilon \in (-\text{int}\hat{Y}_+)^c, \forall d \in X$$

The fact that $(-\text{int}\hat{Y}_+)^c$ is closed, it follows when $t \longrightarrow 0^+$ that

$$F'(\bar{x},d) - T(d) + \|d\|_X \varepsilon \in (-\text{int}\hat{Y}_+)^c,\ \forall d \in X.$$

By taking $\widetilde{Y}_+ := \text{int}\hat{Y}_+ \cup \{0_Y\}$, one can check that $\widetilde{Y}_+ \in \mathscr{D}(Y_+)$. Thus

$$\begin{cases} T \in L(X,Y), \nexists d \in X \text{ such that} \\ F'(\bar{x},d) \precneqq_{\widetilde{Y}_+} T(d) - \|d\|_X \varepsilon. \end{cases}$$

Conversely, let $T \in L(X,Y)$ satisfy

$$\begin{cases} \exists \hat{Y}_+ \in \mathscr{D}(Y_+), \nexists d \in X \text{ such that} \\ F'(\bar{x},d) \precneqq_{\hat{Y}_+} T(d) - \|d\|_X \varepsilon. \end{cases} \tag{4.1}$$

We prove that $T \in \partial_\varepsilon^p F(\bar{x})$. Since F is Y_+-convex, then, for any $t \in]0,1]$ and $d \in X$,

$$F(t(\bar{x}+d) + (1-t)\bar{x}) \leq_{Y_+} tF(\bar{x}+d) + (1-t)F(\bar{x}),$$

which yields that $F(\bar{x}+td) \leq_{Y_+} tF(\bar{x}+d) - tF(\bar{x}) + F(\bar{x})$ for all $d \in X$, so

$$\frac{F(\bar{x}+td)-F(\bar{x})}{t} \leq_{Y_+} F(\bar{x}+d) - F(\bar{x}).$$

Observe that Y_+ is closed. As $t \longrightarrow 0^+$, we have $F'(\bar{x},d) \leq_{Y_+} F(\bar{x}+d) - F(\bar{x})$ for all $d \in X$, which means that

$$F'(\bar{x},d) - F(\bar{x}+d) + F(\bar{x}) \in -Y_+, \forall d \in X. \tag{4.2}$$

If $T \notin \partial_\varepsilon^p F(\bar{x})$, then by using the cone $\widetilde{Y}_+ := \text{int}\hat{Y}_+ \cup \{0_Y\}$, one sees that there exists some $x_0 \in X$ such that

$$F(x_0) - F(\bar{x}) - T(x_0-\bar{x}) + \|x_0-\bar{x}\|_X \varepsilon \in -\widetilde{Y}_+\backslash\{0_Y\}. \tag{4.3}$$

By taking $d_0 = x_0 - \bar{x}$, oee sees from (4.3) that

$$F(d_0+\bar{x}) - F(\bar{x}) - T(d_0) + \|d_0\|_X \varepsilon \in -\widetilde{Y}_+ \backslash \{0_Y\}. \tag{4.4}$$

By taking $d = d_0$ in (4.2) we have $F'(\bar{x}, d_0) - F(\bar{x}+d_0) + F(\bar{x}) \in -Y_+$, which together with (4.4) yields $F'(\bar{x}, d_0) - T(d_0) + \|d_0\|_X \varepsilon \in -Y_+ - \widetilde{Y}_+ \backslash \{0_Y\}$. Since $Y_+ \backslash \{0_Y\} \subset \text{int}\hat{Y}_+$, it follows that $-Y_+ \subseteq -\widetilde{Y}_+$, which yields $-Y_+ - \widetilde{Y}_+ \backslash \{0_Y\} \subseteq -\hat{Y}_+ \backslash \{0_Y\}$. Hence, $F'(\bar{x}, d_0) - T(d_0) + \varepsilon \|d_0\|_X \in -\hat{Y}_+ \backslash \{0_Y\}$, which contradicts (4.1). Then $T \in \partial_\varepsilon^p F(\bar{x})$ and the proof is complete. □

5. Calculus Rules of Generalized Proper ε-subdifferential

In this section, we show the positive homogeneity of generalized proper ε-subdifferential and we establish the calculus rules of generalized proper ε-subdifferential for the sum and the difference of two vector valued mappings.

Proposition 5.1. *Let $F : X \to Y \cup \{+\infty_Y\}$ be a given mapping, $\varepsilon \in Y_+$, and $\bar{x} \in \text{dom}F$. Then, for all $\alpha > 0$, $\alpha \partial_\varepsilon^p F(\bar{x}) = \partial_{\alpha\varepsilon}^p (\alpha F)(\bar{x})$.*

Proof. Letting $\alpha > 0$, we have $T \in \alpha \partial_\varepsilon^p F(\bar{x}) \Longleftrightarrow \frac{T}{\alpha} \in \partial_\varepsilon^p F(\bar{x})$

$$\Longleftrightarrow \begin{cases} \exists \hat{Y}_+ \in \mathscr{D}(Y_+) \text{ such that} \\ F(x) - F(\bar{x}) - \frac{T}{\alpha}(x-\bar{x}) + \|x-\bar{x}\|_X \varepsilon \in \left(-\hat{Y}_+ \backslash \{0_Y\}\right)^c, \forall x \in X \end{cases}$$

$$\Longleftrightarrow \begin{cases} \exists \hat{Y}_+ \in \mathscr{D}(Y_+) \text{ such that} \\ \alpha F(x) - \alpha F(\bar{x}) - T(x-\bar{x}) - \alpha \|x-\bar{x}\|_Y \varepsilon \in \alpha \left(-\hat{Y}_+ \backslash \{0_Y\}\right)^c, \forall x \in X \end{cases}$$

Since $\alpha \left(-\hat{Y}_+ \backslash \{0_Y\}\right)^c = \left(-\hat{Y}_+ \backslash \{0_Y\}\right)^c$, it follows that

$$T \in \alpha \partial_\varepsilon^p F(\bar{x}) \Longleftrightarrow T \in \partial_{\alpha\varepsilon}^p (\alpha F)(\bar{x}).$$

□

Next, we recall the notion of star difference of two sets and also the concept of approximately pseudo-dissipative of a set-valued mapping [14] in order to characterize the generalized proper $\varepsilon-$subdifferential for the sum and the difference of two vector valued mappings.

Definition 5.1. [14] Let C and D be two subsets of X. The operation defined by $C \boxminus D := \{x \in X : x + D \subseteq C\}$ is called the star-difference between C and D.

Definition 5.2. [14] A set-valued mapping $M : X \rightrightarrows L(X,Y)$ is said to be approximately pseudo-dissipative at $\bar{x}$ if, for every $\varepsilon \in Y_+$, one can find a neighborhood $V(\bar{x})$ of $\bar{x}$ such that

$$\forall x \in V(\bar{x}), \exists T \in M(x), \exists T^* \in M(\bar{x}) : (T - T^*)(x-\bar{x}) \leq_{Y_+} \|x-\bar{x}\|_X \varepsilon.$$

Proposition 5.2. *Let $F, G : X \to Y \cup \{+\infty_Y\}$ be a given vector mappings and $\bar{x} \in \text{dom}F \cap \text{dom}G$. Then, for any $\varepsilon, \eta \in Y_+$, $\partial_\varepsilon^p (F-G)(\bar{x}) \subseteq \partial_{\varepsilon+\eta}^p F(\bar{x}) \boxminus \partial_\eta^s G(\bar{x})$.*

Proof. Let $T \in \partial_\varepsilon^p (F-G)(\bar{x})$. For any $S \in \partial_\eta^s G(\bar{x})$, we have to prove that $T + S \in \partial_{\varepsilon+\eta}^p F(\bar{x})$. Since $T \in \partial_\varepsilon^p (F-G)(\bar{x})$, there exists a convex cone $\hat{Y}_+ \subsetneq Y$ with $Y_+ \backslash \{0_Y\} \subset \text{int}(\hat{Y}_+)$ such that

$$\nexists x \in X, (F-G)(x) - (F-G)(\bar{x}) - T(x-\bar{x}) + \|x-\bar{x}\|_X \varepsilon \in -\hat{Y}_+ \backslash \{0_Y\}. \tag{5.1}$$

If $T + S \notin \partial_{\varepsilon+\eta}^p F(\bar{x})$, by using the cone $\widetilde{Y}_+ = \text{int}\hat{Y}_+ \cup \{0_Y\}$, we can find some $x_0 \in X$ such that

$$F(x_0) - F(\bar{x}) - (T+S)(x_0 - \bar{x}) + \|x_0 - \bar{x}\|_X (\varepsilon + \eta) \in -\widetilde{Y}_+ \backslash \{0_Y\}. \tag{5.2}$$

As $S \in \partial_\eta^s G(\bar{x})$, one sees that $G(x) - G(\bar{x}) - S(x-\bar{x}) + \|x-\bar{x}\|_X \eta \in Y_+$ for all $x \in X$. By taking $x = x_0$, one obtains $-G(x_0) + G(\bar{x}) + S(x_0-\bar{x}) - \|x_0-\bar{x}\|_X \eta \in -Y_+$, which together with (5.2) yields

$$(F-G)(x_0) - (F-G)(\bar{x}) - T(x_0-\bar{x}) + \|x_0-\bar{x}\|_X \varepsilon \in -Y_+ - \widetilde{Y}_+ \backslash \{0_Y\}.$$

Then $(F-G)(x_0) - (F-G)(\bar{x}) - T(x_0-\bar{x}) + \|x_0-\bar{x}\|_X \varepsilon \in -\hat{Y}_+ \backslash \{0_Y\}$, which is a contradiction with (5.1). Then $T + S \in \partial_{\varepsilon+\eta}^p F(\bar{x})$ for any $S \in \partial_\eta^s G(\bar{x})$. Hence $T + \partial_\eta^s G(\bar{x}) \subseteq \partial_{\varepsilon+\eta}^p F(\bar{x})$. Thus $\partial_\varepsilon^p (F-G)(\bar{x}) \subseteq \partial_{\varepsilon+\eta}^p F(\bar{x}) \boxminus \partial_\eta^s G(\bar{x})$. The proof is complete. □

By taking $\eta = 0$ in the above proposition, we have the following result immediately.

Corollary 5.1. *Let $F, G : X \longrightarrow Y \cup \{+\infty_Y\}$ be a given vector mappings, $\varepsilon \in Y_+$, and $\bar{x} \in \mathrm{dom}F \cap \mathrm{dom}G$. Then $\partial_\varepsilon^p (F-G)(\bar{x}) \subseteq \partial_\varepsilon^p F(\bar{x}) \boxminus \partial^s G(\bar{x})$.*

Lemma 5.1. *Let $G : X \to Y \cup \{+\infty_Y\}$ be a given vector mapping and $\bar{x} \in \mathrm{dom}G$. If G is Y_+-concave and $\partial^s G$ is approximately pseudo-dissipative at $\bar{x}$, then, for all $x \in X$, there exists $L_x \in \partial^s G(\bar{x})$ such that $G(x) - G(\bar{x}) - L_x(x-\bar{x}) \in -Y_+$.*

Proof. The set-valued mapping $\partial^s G$ is approximately pseudo-dissipative at $\bar{x}$. For $\varepsilon = 0$, there exist a neighborhood $V(\bar{x})$ of $\bar{x}$ such that, for any $z \in V(\bar{x})$, there exist $A_z \in \partial^s G(z)$ and $B_z \in \partial^s G(\bar{x})$ satisfying $(A_z - B_z)(z-\bar{x}) \in -Y_+$. Let $x \in X$. One can find some $y(x) \in V(\bar{x})$ and $\lambda \in]0,1]$ satisfying $y(x) - \bar{x} = \lambda(x-\bar{x})$. Therefore, there exist $L_{y(x)} \in \partial^s G(y(x))$ and $S_{y(x)} \in \partial^s G(\bar{x})$ satisfying

$$\left(L_{y(x)} - S_{y(x)}\right)(y(x)-\bar{x}) \in -Y_+ \Longleftrightarrow \lambda \left(L_{y(x)} - S_{y(x)}\right)(x-\bar{x}) \in -Y_+.$$

Since $\lambda > 0$ and Y_+ is a cone, we have

$$\left(L_{y(x)} - S_{y(x)}\right)(x-\bar{x}) \in -Y_+. \tag{5.3}$$

From $L_{y(x)} \in \partial^s G(y(x))$, we obtain $L_{y(x)}(\bar{x} - y(x)) - G(\bar{x}) + G(y(x)) \in -Y_+$. As $\bar{x} - y(x) = -\lambda(x-\bar{x})$ and $y(x) = \lambda x + (1-\lambda)\bar{x}$, we have

$$-\lambda L_{y(x)}(x-\bar{x}) - G(\bar{x}) + G(\lambda x + (1-\lambda)\bar{x}) \in -Y_+. \tag{5.4}$$

Since G is Y_+-concave, we obtain

$$\lambda G(x) + (1-\lambda) G(\bar{x}) - G(\lambda x + (1-\lambda)\bar{x}) \in -Y_+. \tag{5.5}$$

(5.4) and (5.5) imply $-\lambda L_{y(x)}(x-\bar{x}) + \lambda G(x) - \lambda G(\bar{x}) \in -Y_+$, that is,

$$-L_{y(x)}(x-\bar{x}) + G(x) - G(\bar{x}) \in -Y_+. \tag{5.6}$$

By adding (5.3) and (5.6) we have $-S_{y(x)}(x-\bar{x}) + G(x) - G(\bar{x}) \in -Y_+$. Setting $S_{y(x)} := T_x$ yields $G(x) - G(\bar{x}) - T_x(x-\bar{x}) \in -Y_+$. The proof is complete. □

Theorem 5.1. *Let $F, G : X \longrightarrow Y \cup \{+\infty_Y\}$ be a given mappings, $\varepsilon \in Y_+$, and $\bar{x} \in \mathrm{dom}F \cap \mathrm{dom}G$. If G is Y_+-concave and the set-valued mapping $\partial^s G$ is approximately pseudo-dissipative at $\bar{x}$, then $\partial_\varepsilon^p (F-G)(\bar{x}) = \partial_\varepsilon^p F(\bar{x}) \boxminus \partial^s G(\bar{x})$.*

Proof. In view of Corollary 5.1, we only need to prove that $\partial_\varepsilon^p F(\bar{x}) \boxminus \partial^s G(\bar{x}) \subseteq \partial_\varepsilon^p (F-G)(\bar{x})$. Let $A \in \partial_\varepsilon^p F(\bar{x}) \boxminus \partial^s G(\bar{x})$. Then, for any $T \in \partial^s G(\bar{x})$, $A + T \in \partial_\varepsilon^p F(\bar{x})$, which means that there exists a convex cone $\hat{Y}_+ \subsetneq Y$ such that $Y_+ \backslash \{0_Y\} \subset \mathrm{int}\hat{Y}_+$ and

$$\nexists x \in X : F(x) - F(\bar{x}) - (A+T)(x-\bar{x}) + \|x-\bar{x}\|_X \varepsilon \in -\hat{Y}_+ \backslash \{0_Y\}.$$

If $A \notin \partial_\varepsilon^p (F-G)(\bar{x})$, by using the cone $\widetilde{Y}_+ := \text{int}\hat{Y}_+ \cup \{0_Y\}$, one obtains that there exists $x_0 \in X$ such that

$$(F-G)(x_0) - (F-G)(\bar{x}) - A(x_0-\bar{x}) + \|x_0-\bar{x}\|_X \varepsilon \in -\widetilde{Y}_+ \backslash \{0_Y\}. \tag{5.7}$$

Using Lemma 5.1 yields that there exists $L_{x_0} \in \partial^s G(\bar{x})$ such that $G(x_0) - G(\bar{x}) - L_{x_0}(x_0-\bar{x}) \in -Y_+$, which together with (5.7) yields

$$F(x_0) - F(\bar{x}) - (A+L_{x_0})(x_0-\bar{x}) + \|x_0-\bar{x}\|_X \varepsilon \in -Y_+ - \widetilde{Y}_+ \backslash \{0_Y\}$$

Since $Y_+ \backslash \{0_Y\} \subset \text{int}\hat{Y}_+$, it follows that $-Y_+ \subset -\widetilde{Y}_+$ and $-Y_+ - \widetilde{Y}_+ \backslash \{0_Y\} \subseteq -\hat{Y}_+ \backslash \{0\}$. Finally, we find $x_0 \in X$ and $L_{x_0} \in \partial^s G(\bar{x})$ such that

$$F(x_0) - F(\bar{x}) - (A+L_{x_0})(x_0-\bar{x}) + \|x_0-\bar{x}\|_X \varepsilon \in -\hat{Y}_+ \backslash \{0_Y\},$$

which means that $A + L_{x_0} \notin \partial_\varepsilon^p F(\bar{x})$, which contradicts the fact that $A + L_{x_0} \in \partial_\varepsilon^p F(\bar{x})$, with $L_{x_0} \in \partial^s G(\bar{x})$. Thus we have $\partial_\varepsilon^p (F-G)(\bar{x}) = \partial_\varepsilon^p F(\bar{x}) \boxminus \partial^s G(\bar{x})$. □

By taking $\varepsilon = 0$ in Theorem 5.1, we have the following result.

Corollary 5.2. *Let $F, G : X \longrightarrow Y \cup \{+\infty_Y\}$ be a given mappings and $\bar{x} \in \text{dom}F \cap \text{dom}G$. If G is Y_+-concave and the set-valued mapping $\partial^s G$ is approximately pseudo-dissipative at $\bar{x}$, then*

$$\partial^p (F-G)(\bar{x}) = \partial^p F(\bar{x}) \boxminus \partial^s G(\bar{x}).$$

Next, we deal with the calculus rule of generalized proper ε-subdifferential for the sum of two vector valued mappings.

Proposition 5.3. *Let $F, G : X \longrightarrow Y \cup \{+\infty_Y\}$ be a given mappings with $\bar{x} \in \text{dom}F \cap \text{dom}G$. For all $\varepsilon, \eta \in Y_+$, $\partial_\varepsilon^p F(\bar{x}) + \partial_\eta^s G(\bar{x}) \subseteq \partial_{\varepsilon+\eta}^p (F+G)(\bar{x})$.*

Proof. Let $A \in \partial_\varepsilon^p F(\bar{x})$. Then there exists a convex cone $\hat{Y}_+ \subsetneq Y$ such that $Y_+ \backslash \{0_Y\} \subset \text{int}\hat{Y}_+$ and

$$\nexists x \in X : F(x) - F(\bar{x}) - A(x-\bar{x}) + \|x-\bar{x}\|_X \varepsilon \in -\hat{Y}_+ \backslash \{0_Y\}. \tag{5.8}$$

Let $B \in \partial_\eta^s G(\bar{x})$. We prove that $A+B \in \partial_{\varepsilon+\eta}^p (F+G)(\bar{x})$. Suppose, in the contrary, that $A+B \notin \partial_{\varepsilon+\eta}^p (F+G)(\bar{x})$. By using the cone $\widetilde{Y}_+ := \text{int}\hat{Y}_+ \cup \{0_Y\}$ we have for some $x_0 \in X$ that

$$(F+G)(x_0) - (F+G)(\bar{x}) - (A+B)(x_0-\bar{x}) + \|x_0-\bar{x}\|_X (\varepsilon+\eta) \in -\widetilde{Y}_+ \backslash \{0_Y\}. \tag{5.9}$$

Since $B \in \partial_\eta^s G(\bar{x})$, then

$$-G(x_0) + G(\bar{x}) + B(x_0-\bar{x}) - \|x_0-\bar{x}\|_X \eta \in -Y_+. \tag{5.10}$$

Adding (5.9) and (5.10) and taking into account that $-Y_+ - \widetilde{Y}_+ \backslash \{0_Y\}) \subseteq -\hat{Y}_+ \backslash \{0_Y\}$, it follows that

$$F(x_0) - F(\bar{x}) - A(x_0-\bar{x}) + \|x_0-\bar{x}\|_X \varepsilon \in -\hat{Y}_+ \backslash \{0_Y\},$$

which contradicts (5.8). Then $A+B \in \partial_{\varepsilon+\eta}^p (F+G)(\bar{x})$ and the desired result is obtained. □

If we take $\eta = 0$ in Proposition 5.3, we have the following result.

Corollary 5.3. *Let $F, G : X \longrightarrow Y \cup \{+\infty_Y\}$ be a given vector mappings with $\bar{x} \in \text{dom}F \cap \text{dom}G$. For any $\varepsilon \in Y_+$, $\partial_\varepsilon^p F(\bar{x}) + \partial^s G(\bar{x}) \subseteq \partial_\varepsilon^p (F+G)(\bar{x})$.*

In order to establish the equality of the sum for the generalized proper ε-subdifferential, let us recall the concept of regular subdifferentiability of a vector valued mapping and also a scalarization theorem.

Definition 5.3. [5] Let $F : X \longrightarrow Y \cup \{+\infty_Y\}$ be a Y_+-convex mapping and $\bar{x} \in \mathrm{dom}F$. F is said to be proper-regular subdifferentiable at $\bar{x}$ if $\partial (y^* \circ F)(\bar{x}) = y^* \circ \partial^s F(\bar{x}), \forall y^* \in \left(Y_+^*\right)^\circ$.

Theorem 5.2. [5] *Let $F : X \longrightarrow Y \cup \{+\infty_Y\}$ be a Y_+-convex mapping and $\bar{x} \in \mathrm{dom}F$. Then*

$$\partial^p F(\bar{x}) = \bigcup_{y^* \in \left(Y_+^*\right)^\circ} \{A \in L(X,Y) : y^* \circ A \in \partial (y^* \circ F)(\bar{x})\}.$$

Theorem 5.3. *Let $F, G : X \longrightarrow Y \cup \{+\infty_Y\}$ be two proper Y_+-convex vector mappings and $\bar{x} \in \mathrm{dom}F \cap \mathrm{dom}G$. If G is properly regular subdifferentiable at $\bar{x}$ and continuous at some point of $\mathrm{dom}F \cap \mathrm{dom}G$, then, for any $\varepsilon \in Y_+$, $\partial_\varepsilon^p (F+G)(\bar{x}) = \partial_\varepsilon^p F(\bar{x}) + \partial^s G(\bar{x})$.*

Proof. According to Corollary 5.3, we only need to prove that $\partial_\varepsilon^p (F+G)(\bar{x}) \subseteq \partial_\varepsilon^p F(\bar{x}) + \partial^s G(\bar{x})$. To this end, we take an arbitrary $T \in \partial_\varepsilon^p (F+G)(\bar{x})$. By setting $F_\varepsilon := F + \varepsilon \|. - \bar{x}\|_X$, we have

$$T \in \partial_\varepsilon^p (F+G)(\bar{x}) = \partial^p (F + G + \|. - \bar{x}\|_X \varepsilon)(\bar{x}) = \partial^p (F_\varepsilon + G)(\bar{x}).$$

Since $\mathrm{dom}F_\varepsilon = \mathrm{dom}F$, then G is continuous at some point of $\mathrm{dom}F_\varepsilon \cap \mathrm{dom}G$. In view of Theorem 5.2, one sees that there exists some $y^* \in \left(Y_+^*\right)^\circ$ such that $y^* \circ T \in \partial (y^* \circ F_\varepsilon + y^* \circ G)(\bar{x})$. The scalar functions $y^* \circ F_\varepsilon$ and $y^* \circ G$ satisfy the assumptions of Proposition 2.1. Thus

$$y^* \circ T \in \partial (y^* \circ F_\varepsilon)(\bar{x}) + \partial (y^* \circ G)(\bar{x}). \tag{5.11}$$

By using the regular subdifferentiability of G at $\bar{x}$, we have $\partial (y^* \circ G)(\bar{x}) = y^* \circ \partial^s G(\bar{x})$. It follows that (5.11) becomes $y^* \circ T \in \partial (y^* \circ F_\varepsilon)(\bar{x}) + y^* \circ \partial^s G(\bar{x})$. Then, there exists $K \in \partial^s G(\bar{x})$ such that $y^* \circ (T - K) \in \partial (y^* \circ F_\varepsilon)(\bar{x})$. By virtue of Theorem 5.2, we have $T - K \in \partial^p F_\varepsilon(\bar{x})$. which means that $T \in \partial_\varepsilon^p F(\bar{x}) + \partial^s G(\bar{x})$. Thus $\partial_\varepsilon^p (F+G)(\bar{x}) \subseteq \partial_\varepsilon^p F(\bar{x}) + \partial^s G(\bar{x})$. The proof is complete. □

6. Application to Vector DC-optimization Problems

In this section, as an application of the general results of the previous sections, we consider a classical DC-vector optimization problem

$$(P) \begin{cases} \min(F(x) - G(x)) \\ x \in C \end{cases}$$

where $F, G : X \longrightarrow Y \cup \{+\infty_Y\}$ are two given mappings and C is nonempty subset of X. We introduce the notion of proper ε-blunt minimizer of a vector valued mapping.

Definition 6.1. Let $F : X \longrightarrow Y \cup \{+\infty_Y\}$ be a given vector mapping, C be a nonempty subset of X, $\bar{x} \in \mathrm{dom}F \cap C$, and $\varepsilon \in Y_+$. $\bar{x}$ is said to be a proper ε-blunt minimizer of F on C if there exists $\hat{Y}_+ \in \mathscr{D}(Y_+)$ such that $\nexists x \in C$, $F(x) - F(\bar{x}) + \|x - \bar{x}\|_X \varepsilon \in -\hat{Y}_+ \setminus \{0_Y\}$. The set of all proper ε-blunt minimizer of F on C is denoted by $S_\varepsilon^p(F,C)$.

Note that if $\varepsilon = 0$, proper ε-blunt minimizer becomes a proper (or Henig) minimizer, that is,

$$\exists \hat{Y}_+ \in \mathscr{D}(Y_+),\ \nexists x \in X \text{ such that } F(x) - F(\bar{x}) \in -\hat{Y}_+ \setminus \{0_Y\}.$$

From the definition of proper ε-blunt minimizers, we easily find the following proposition.

Proposition 6.1. *Let $F : X \longrightarrow Y \cup \{+\infty_Y\}$ be a given vector mapping, $\bar{x} \in \mathrm{dom}F$, and $\varepsilon \in Y_+$. Then $\bar{x}$ is a proper ε-blunt minimizer of F on X if and only if $0 \in \partial_\varepsilon^p F(\bar{x})$.*

Proof. We see that $\bar{x}$ is a proper ε-blunt minimizer of F at $\bar{x}$ if and only if

$$\begin{cases} \exists \hat{Y}_+ \in \mathscr{D}(Y_+), \nexists x \in X \text{ such that} \\ F(x) - F(\bar{x}) + \|x-\bar{x}\|_X \varepsilon \in -\hat{Y}_+ \backslash \{0_Y\}. \end{cases}$$

that is,

$$\begin{cases} \exists \hat{Y}_+ \in \mathscr{D}(Y_+), \nexists x \in X \text{ such that} \\ F(x) - F(\bar{x}) \lneq 0\,(x-\bar{x}) - \|x-\bar{x}\|_X \varepsilon. \end{cases}$$

which means that $0 \in \partial_\varepsilon^p F(\bar{x})$. □

Proposition 6.2. *Let $F : X \longrightarrow Y \cup \{+\infty_Y\}$ be a given proper mapping, C be a nonempty subset of X, $\bar{x} \in \mathrm{dom}F \cap C$, and $\varepsilon \in Y_+ \backslash \{0_Y\}$. If $\bar{x}$ is a proper minimizer of F on C, then $\bar{x}$ is a proper ε-blunt minimizer of F on C.*

Proof. Let $\bar{x}$ be a proper minimizer of F on C. That is,

$$\exists \hat{Y}_+ \in \mathscr{D}(Y_+),\ \nexists x \in X \text{ such that } F(x) - F(\bar{x}) \in -\hat{Y}_+ \backslash \{0_Y\}, \tag{6.1}$$

and $\bar{x}$ is not a proper ε-blunt minimizer of F on C. By taking the convex cone $\widetilde{Y}_+ := \mathrm{int}\hat{Y}_+ \cup \{0_Y\}$, one sees that there exists some $x_0 \in X$ such that

$$F(x_0) - F(\bar{x}) + \|x_0 - \bar{x}\|_X \varepsilon \in -\widetilde{Y}_+ \backslash \{0_Y\}. \tag{6.2}$$

Note that $\varepsilon\|x_0 - \bar{x}\|_X \in Y_+ \backslash \{0_Y\}$. Since $Y_+ \backslash \{0_Y\} \subseteq \mathrm{int}\hat{Y}_+$ and $\mathrm{int}\hat{Y}_+ = \widetilde{Y}_+ \backslash \{0\}$, we have $-\|x_0 - \bar{x}\|\varepsilon \in -\widetilde{Y}_+ \backslash \{0_Y\}$, which together with (6.2) yields $F(x_0) - F(\bar{x}) \in -\widetilde{Y}_+ \backslash \{0_Y\} = -\mathrm{int}\hat{Y}_+$. In view of $-\mathrm{int}\hat{Y}_+ \subseteq -\hat{Y}_+ \backslash \{0_Y\}$, we have $F(x_0) - F(\bar{x}) \in -\hat{Y}_+ \backslash \{0_Y\}$, which contradicts (6.1). Hence, $\bar{x}$ is a proper ε-blunt minimizer of F on C. The proof is complete. □

The following example demonstrates that the notion of the proper ε-blunt minimizer of F on C is weaker than the proper minimiser. Let $X = \mathbb{R}^2$, $Y = \mathbb{R}^2$, $Y_+ = \mathbb{R}^2_+$, $\bar{x} = (0,0)$, and $F(x_1,x_2) = (x_1,0)$. We have that F is $\mathbb{R}^2_+$-convex. By using the norm $\|(x_1,x_2)\| = \sqrt{\|x_1\|^2 + \|x_2\|^2}$ for $(x_1,x_2) \in \mathbb{R}^2$, it is easy to see that $(0,0) \notin \partial^p F(0,0)$ but, for $\varepsilon = (1,1)$, $(0,0) \in \partial_\varepsilon^p F(0,0)$, which means that $(0,0)$ is a proper $(1,1)$-blunt minimizer of the vector mapping F, but it is not a proper efficient solution to vector mapping F. The vector indicator mapping $\delta_C^v : X \longrightarrow Y \cup \{+\infty_Y\}$ is defined by

$$\delta_C^v(x) = \begin{cases} 0,\ x \in C \\ +\infty_Y,\ x \notin C \end{cases}$$

The normal cone at $\bar{x} \in C$ is defined by

$$N^v(C,\bar{x}) := \{T \in L(X,Y), A(x-\bar{x}) \leq_{Y_+} 0_Y, \forall x \in C\}.$$

When $Y = \mathbb{R}$ and $Y_+ = \mathbb{R}_+$, $N^v(C,\bar{x})$ is the normal convex cone of convex analysis. It is easy to see that $N^v(C,\bar{x}) = \partial^s \delta_C^v(\bar{x})$.

Lemma 6.1. *Let $F : X \longrightarrow Y \cup \{+\infty_Y\}$ be a mapping and C be a nonempty subset of X. Then $S_\varepsilon^p(F,C) = S_\varepsilon^p\left(F + \delta_C^v, X\right)$.*

Proof. Let $\bar{x} \in S_\varepsilon^p\left(F + \delta_C^v, X\right)$. Then $\bar{x} \in \mathrm{dom}F \cap C$ and

$$\begin{cases} \exists \hat{Y}_+ \in \mathscr{D}(Y_+), \nexists x \in X \text{ such that} \\ F(x) + \delta_C^v(x) - F(\bar{x}) - \delta_C^v(\bar{x}) + \|x-\bar{x}\|_X \varepsilon \in -\hat{Y}_+ \backslash \{0_Y\}. \end{cases}$$

Suppose that $\bar{x} \notin S_\varepsilon^p(F,C)$. By using the cone $\widetilde{Y}_+ := \text{int}\hat{Y}_+ \cup \{0_Y\}$, we can find some $x_0 \in C$ such that $F(x_0) - F(\bar{x}) + \|x_0 - \bar{x}\|_X \varepsilon \in -\widetilde{Y}_+ \backslash \{0_Y\}$. Since $x_0, \bar{x} \in C$ and $-\widetilde{Y}_+ \backslash \{0_Y\} = -\text{int}\hat{Y}_+ \subset -\hat{Y}_+ \backslash \{0_Y\}$, we find

$$F(x_0) + \delta_C^v(x_0) - F(\bar{x}) - \delta_C^v(\bar{x}) + \|x_0 - \bar{x}\|_X \varepsilon \in -\hat{Y}_+ \backslash \{0_Y\},$$

which contradicts the fact that $\bar{x} \in S_\varepsilon^p\left(F + \delta_C^v, X\right)$. Conversely, let $\bar{x} \in S_\varepsilon^p(F,C)$. Then $\bar{x} \in \text{dom}F \cap C$ and

$$\begin{cases} \exists \hat{D}_+ \in \mathscr{D}(Y_+), \nexists x \in C \text{ such that} \\ F(x) - F(\bar{x}) + \|x - \bar{x}\|_X \varepsilon \in -\hat{D}_+ \backslash \{0_Y\}. \end{cases} \tag{6.3}$$

Suppose that $\bar{x} \notin S_\varepsilon^p\left(F + \delta_C^v, X\right)$. By using the cone $\widetilde{D}_+ := \text{int}\hat{D}_+ \cup \{0_Y\}$, we cane find some $x_0 \in X$ such that

$$F(x_0) + \delta_C^v(x_0) - F(\bar{x}) - \delta_C^v(\bar{x}) + \|x_0 - \bar{x}\|_X \varepsilon \in -\hat{D}_+ \backslash \{0_Y\}. \tag{6.4}$$

It follows that $x_0 \in C$, and hence (6.4) becomes $F(x_0) - F(\bar{x}) + \|x_0 - \bar{x}\|_X \varepsilon \in -\hat{D}_+ \backslash \{0_Y\}$, which contradicts (6.3). Then $\bar{x} \in S_\varepsilon^p\left(F + \delta_C^v, X\right)$ and the proof is complete. □

Now, we establish optimality conditions for problem (P). For this, let us recall that the vector indicator mapping δ_C^v is proper-regular subdifferentiable; see [5].

Theorem 6.1. *Let $F, G : X \longrightarrow Y \cup \{+\infty_Y\}$ be a given vector mappings, C is a nonempty convex subset of X, $\bar{x} \in \text{dom}F \cap \text{dom}G \cap C$, and $\varepsilon \in Y_+$. If F is Y_+-convex, finite and continuous at some point of C, G is Y_+-concave, and $\partial^s G$ is approximately pseudo-dissipative at $\bar{x}$, then $\bar{x}$ is a proper ε-blunt minimizer of (P) if and only if $\partial^s G(\bar{x}) \subseteq \partial_\varepsilon^p F(\bar{x}) + N^v(C,\bar{x})$.*

Proof. In view of Lemma 6.1, we have

$$\min_{x \in C}(F(x) - G(x)) = \min_{x \in X}(F(x) + \delta_C^v(x) - G(x)).$$

From Proposition 6.1, we have that $\bar{x}$ is a proper ε-blunt minimizer of (P) if and only if

$$0 \in \partial_\varepsilon^p(F + \delta_C^v - G)(\bar{x}). \tag{6.5}$$

According to Theorem 5.1, we have $\partial_\varepsilon^p\left(F + \delta_C^v - G\right)(\bar{x}) = \partial_\varepsilon^p\left(F + \delta_C^v\right)(\bar{x}) \boxminus \partial^s G(\bar{x})$. Then, (6.5) becomes equivalent to $\partial^s G(\bar{x}) \subset \partial_\varepsilon^p\left(F + \delta_C^v\right)(\bar{x})$. Since C is a nonempty and convex subset, we have that δ_C^v is proper and Y_+-convex. Let us recall that δ_C^v is proper-regular subdifferentiable. As F is Y_+-convex finite and continuous at some point of $\text{dom}\delta_C^v = C$, it follows from Theorem 5.3 that

$$\partial_\varepsilon^p(F + \delta_C^v)(\bar{x}) \quad = \quad \partial_\varepsilon^p F(\bar{x}) + \partial^s \delta_C^v(\bar{x}) = \partial_\varepsilon^p F(\bar{x}) + N^v(C,\bar{x}),$$

which yields $\partial^s G(\bar{x}) \subset \partial_\varepsilon^p F(\bar{x}) + N^v(C,\bar{x})$. The proof is complete. □

Next, we consider the cone-constrained DC vector optimization problem

$$(Q) \begin{cases} \min F(x) - G(x) \\ S(x) \in -Z_+ \end{cases}$$

where $F, G : X \longrightarrow Y \cup \{+\infty_Y\}$ and $S : X \longrightarrow Z \cup \{+\infty_Z\}$ are vector-valued mappings. We suppose that Z is ordered by the pointed convex cone $Z_+ \subseteq Z$. We denote by $L_+(Z,Y)$ the set of linear operator $T : Z \longrightarrow Y$ such that $T(Z_+) \subset Y_+$.

Theorem 6.2. *Let F be a Y_+-convex mapping, S be a Z_+-convex mapping, G be a Y_+-concave mapping, and $\partial^s G$ be approximately pseudo-dissipative at $\bar{x}$. Assume that F is proper-regular subdifferentiable at $\bar{x} \in \mathrm{dom}F$ and continuous at some point of* $\mathrm{dom}S$ *and there exists a linear mapping $T \in L_+(Z,Y)$ such that*

$$\begin{cases} \partial^s G(\bar{x}) \subseteq \partial_\varepsilon^p (T \circ S)(\bar{x}) + \partial^s F(\bar{x}) \\ (T \circ S)(\bar{x}) = 0 \end{cases}$$

Then $\bar{x}$ is a proper ε-blunt minimizer to problem (Q).

Proof. Observe that

$$\partial^s G(\bar{x}) \subseteq \partial_\varepsilon^p (T \circ S)(\bar{x}) + \partial^s F(\bar{x}) \Leftrightarrow 0 \in \left[\partial_\varepsilon^p (T \circ S)(\bar{x}) + \partial^s F(\bar{x})\right] \boxminus \partial^s G(\bar{x}). \tag{6.6}$$

Since S is Z_+-convex and $T \in L_+(Z,Y)$, one check easily that $T \circ S$ is Y_+-convex. As F is properly regular subdifferentiable at $\bar{x}$ and continuous at some point of $\mathrm{dom}S = \mathrm{dom}\,(T \circ S)$, one concludes by Theorem 5.3 that

$$\partial_\varepsilon^p (T \circ S + F)(\bar{x}) = \partial_\varepsilon^p (T \circ S)(\bar{x}) + \partial^s F(\bar{x}).$$

Thus (6.6) is reduced to $0 \in \partial_\varepsilon^p (T \circ S + F)(\bar{x}) \boxminus \partial G^s(\bar{x})$. The mappings $T \circ S + F$ and G satisfy the assumptions of Theorem 5.1. Thus $0 \in \partial_\varepsilon^p (T \circ S + F - G)(\bar{x})$ and there exists a convex cone $\hat{Y}_+ \in \mathscr{D}(Y_+)$ such that

$$\nexists x \in X : F(x) + (T \circ S)(x) - G(x) - F(\bar{x}) - (T \circ S)(\bar{x}) + G(\bar{x}) + \|x - \bar{x}\|_X \varepsilon \in -\hat{Y}_+ \backslash \{0_Y\}.$$

As $(T \circ S)(\bar{x}) = 0$, we see that

$$\nexists x \in X : F(x) + (T \circ S)(x) - G(x) - F(\bar{x}) + G(\bar{x}) + \|x - \bar{x}\|_X \varepsilon \in -\hat{Y}_+ \backslash \{0_Y\}.$$

By setting $C := \{x \in X : S(x) \in -Z_+\}$. Hence C is a nonempty convex subset of X. We next prove that $0 \in \partial_\varepsilon^p (F + \delta_C^v - G)(\bar{x})$, which yields according to Proposition 6.1 and Lemma 6.1 that $\bar{x}$ is a proper ε-blunt minimizer to problem (Q). Suppose, in the contrary, that $0 \notin \partial_\varepsilon^p (F + \delta_C^v - G)(\bar{x})$. By taking the cone $\widetilde{Y}_+ := \mathrm{int}\hat{Y}_+ \cup \{0_Y\}$, we see that there exists $x_0 \in X$ such that

$$F(x_0) + \delta_C^v(x_0) - G(x_0) - F(\bar{x}) - \delta_C^v(\bar{x}) + G(\bar{x}) + \|x_0 - \bar{x}\|_X \varepsilon \in -\widetilde{Y}_+ \backslash \{0_Y\}, \tag{6.7}$$

which yields $x_0 \in C$, i.e., $S(x_0) \in -Z_+$. Since $T \in L_+(Z,Y)$, then $(T \circ S)(x_0) \in -Y_+$, which together with (6.7) yields

$$(T \circ S)(x_0) + F(x_0) - G(x_0) - F(\bar{x}) + G(\bar{x}) + \|x_0 - \bar{x}\|_X \varepsilon \in -\hat{Y}_+ \backslash \{0_Y\}.$$

In view of $(T \circ S)(\bar{x}) = 0$, one obtains $0 \notin \partial_\varepsilon^p (T \circ S + F - G)(\bar{x})$, which contradicts the fact that $0 \in \partial_\varepsilon^p (T \circ S + F - G)(\bar{x})$. This obtains the desired result. □

Acknowledgments
The author would like to express his deep gratitude to the reviewer for the very nice comments on the primary version of this work, which helped him to improve the paper.

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