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PONTRYAGIN OPTIMALITY CONDITIONS FOR GENERALIZED BILEVEL OPTIMAL CONTROL PROBLEMS WITH PURE STATE INEQUALITY CONSTRAINTS

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Abstract. In this paper, we study a generalized bilevel optimal control problem that has a variational inequality parametrized by the final state on the follower and pure state constraints on the leader. After reducing the problem with a gap function to an analogous single-level optimal control problem, we focus on the development of a necessary optimality condition of the Pontryagin type. We highlight some significant issues originating from the generalized bilevel structure and its pure state constraints on the leader, which give rise to a degenerated maximum principle in the absence of constraint qualifications. To ensure the nondegeneracy of the derived maximum principle, we employ a partial penalization strategy and a well-known regularity criterion for optimal control problems with pure state constraints.

Keywords. Gap function; Generalized bilevel programming; Maximum principle; Optimal control problem; Optimality conditions.

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1. INTRODUCTION

Since 1950, optimal control problems have been studied with two goals. The maximum principle, a set of necessary conditions for an optimal control function, was one of them. The second method was called "dynamic programming," which simplifies the work of determining the best control function for solving a Hamilton-Jacobi partial differential equation. In the same vein, bilevel optimal control problems, which are bilevel optimization problems with control at least one level, have been considered because of its numerous applications; see, e.g., [1, 2, 3, 4, 5, 6, 7] and the references therein. Many methods have been used to investigate such problems, including theoretical studies [8, 9, 10] and numerical studies [4, 11, 12].

Finding Pontryagin optimality conditions is under the spotlight for addressing optimal control problems recently. In 1995, Hartl, Sethi, and Vickson [13] reviewed a number of maximum principle approaches via two basic strategies: direct adjoining and indirect adjoining. Additionally, they also gave special attention to the connections between the various multipliers that arise in these approaches and providing examples to support their claims. In [14],

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Hoehener suggested new second-order necessary optimality conditions via a variational approach. The approach enables to produce direct proofs, in contrast to the conventional method of establishing second-order required conditions (by using an abstract infinite-dimensional optimization problem). The optimal controls need only be measurable, and there are no convexity assumptions made for the constraints. For other second-order optimality conditions, we refer to [15, 16, 17]. In terms of numerical resolution, Nikoobin and Moradi [18] examined two significant problems with indirect techniques from a practical standpoint: the convergence issue and constraint handling. More precisely, the homotopy continuation method was expanded to include ordinary differential equations, equations dealing with boundary conditions, and performance indexes in their study. Recently, Benita, Dempe, and Mehlitz [19] deduced Pontryagin optimality conditions for an original optimistic bilevel programming problem, which consists of a pure state-constrained optimal control problem in the leader and a parametric but convex finite-dimensional optimization problem in the follower, whose parameter is the final state of the leader's state function. To ensure that the improved maximum principle does not degenerate, the partial penalization principle and some enhanced assumptions were employed.

In this paper, we investigate a generalized bilevel optimal control problem (GBOCP) in which the leader controls the state function (of the dynamic system) x and the control u; and contains a pure state constraint. The follower, on the other hand, is a variational inequality constraint parametrized by the final state of x. Our approach requires employing a gap function to convert the parametric variational inequality into terminal constraints. The result is an optimal control problem with non-smooth equality constraints. The essential constraint containing the gap function was moved to the objective functional, leading to an exact penalization, using the partial calmness regularity to identify the necessary optimality conditions.

The following is how the remainder of this article is structured: Section 2 goes over some preliminaries and fundamental concepts. Section 3 is devoted to stating the problem under consideration and reformulating it. In addition, some results on the exact penalization and the principle uniform parametric error bound based on the R-regularity of the solution map solution are presented. Furthermore, we develop several Pontryagin optimality conditions in Section 4. After that, we provide an example to illustrate the results obtained. In Section 5, a discussion on the Pontryagin optimality conditions is presented. Finally, Section 6 ends this paper.

2. Preliminaries

2.1. **Basic tools.** In this paper, $\langle \cdot,\cdot\rangle_{\mathbb{R}^n}$ and $\|\cdot\|_\infty$ represent the scalar product and the maximum-norm of $\mathbb{R}^n$, respectively, and $\mathbb{R}^n_+$ is a collection of vectors $x\in\mathbb{R}^n$ with nonnegative components.

For a subset $\Omega\subset\mathbb{R}^n$, the convex hull of Ω is denoted by $\operatorname{co}(\Omega)$. In addition, we define the function χ_Ω as follows: $\chi_\Omega(\Lambda)=\Omega\cap\Lambda$ for any $\Lambda\subset\mathbb{R}^n$. Let $(E,\|\cdot\|_E)$ be a Banach space, $x\in E$, and $\varepsilon>0$. We denote the open ball around x with radius ε by $\mathbb{U}^\varepsilon_E(x)$. For a subset Δ of E and a vector x of E, $d_\Delta(x)=\inf\{\|x-y\|_E : y\in\Delta\}$ is the distance between Δ and x. Given scalars $t_i,t_f>0$, we consider the following set:

$$\mathscr{C}_0\left([t_i,t_f],\mathbb{R}^n\right):=\left\{\text{continuous functions } v \text{ with } n \text{ components over } [t_i,t_f]\right\}.$$

We denote by $\mathscr{C}_0^\star\left([t_i,t_f],\mathbb{R}^n\right)$ the dual space of $\mathscr{C}_0\left([t_i,t_f],\mathbb{R}^n\right)$. By $\mathscr{C}_0^p\left([t_i,t_f]\right)$, we refer to the set of all elements of $\mathscr{C}_0^\star\left([t_i,t_f],\mathbb{R}\right)$, which on nonnegative functions in $\mathscr{C}_0\left([t_i,t_f],\mathbb{R}\right)$ takes nonnegative values. For any measure $\lambda\in\mathscr{C}_0^p\left([t_i,t_f]\right)$, $\operatorname{supp}(\lambda)$ denotes the smallest closed set

$X \subset [t_i, t_f]$ with $\lambda(Y) = 0$ for any relatively open subset $Y \subset [t_i, t_f] \setminus X$. We denote by $\mathscr{L}$ the sigma-algebra generated by $[t_i, t_f]$ and by $\mathscr{B}^n$ the Borelean sigma-algebra on $\mathbb{R}^n$.

We indicate by $M\left([t_i, t_f], \mathbb{R}^n\right)$ the set of all functions $\varphi_1 : [t_i, t_f] \to \mathbb{R}^n$, which are measurable, by $L^1\left([t_i, t_f], \mathbb{R}^n\right)$ the space of all functions $\varphi_2 : [t_i, t_f] \to \mathbb{R}^n$, which are Lebesgue-integrable, and by $L^\infty\left([t_i, t_f], \mathbb{R}^n\right)$ the space of all measurable functions $\varphi_3 : [t_i, t_f] \to \mathbb{R}^n$, which are bounded almost everywhere on $[t_i, t_f]$, while the Sobolev space $\mathscr{W}[n]$ is described by

$$\mathscr{W}[n] = \left\{ v : [t_i, t_f] \to \mathbb{R}^n \ : \ v \in L^1\left([t_i, t_f], \mathbb{R}^n\right) \ \text{ and } \ \dot{v} \in L^1\left([t_i, t_f], \mathbb{R}^n\right) \right\},$$

where $\dot{v}$ denotes the (weak) derivative of v.

Lemma 2.1 ([20])**.** *Let $\mathscr{W}[n]$ be a Sobolev space. Then, we can locate a scalar $C_{emb} > 0$ such that, for any $v \in \mathscr{W}[n]$, and for any $t \in \left[t_i, t_f\right]$,*

$$\|v(t)\|_\infty \le \|v\|_{\mathscr{C}\left(\left[t_i, t_f\right], \mathbb{R}^n\right)} \le C_{emb} \|v\|_{\mathscr{W}[n]}.$$

Let $\mathscr{Q} \subset \mathbb{R}^n$ be a set, and $\overline{q} \in \mathscr{Q}$. The Fréchet normal cone to $\mathscr{Q}$ at $\overline{q}$ is defined by

$$\hat{N}(\mathscr{Q}, \overline{q}) := \left\{ \eta \in \mathbb{R}^n \ : \ \limsup_{q \to \overline{q}, q \in \mathscr{Q}} \frac{\eta^\top (q - \overline{q})}{\|q - \overline{q}\|_\infty} \le 0 \right\},$$

while the basic (or Mordukhovich) normal cone to $\mathscr{Q}$ at $\overline{q}$ is given by

$$N(\mathscr{Q}, \overline{q}) := \limsup_{q \to \overline{q}, q \in \mathscr{Q}} \hat{N}(\mathscr{Q}, q).$$

Consider $F : \mathbb{R}^n \to \mathbb{R}^d$ as a differentiable function. For $x \in \mathbb{R}^n$, $\nabla F(x) \in \mathbb{R}^{d \times n}$ refers to the Jacobian of F at x; when $d = 1$, we consider the gradient $\nabla F(x)$ as a vector of $\mathbb{R}^n$, and $\nabla^2 F(x) := \nabla\nabla F(x)$. Given a function $f : \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, $\mathrm{dom}(f) := \{x \in \mathbb{R}^n : f(x) < +\infty\}$ and $\mathrm{epi}(f) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : f(x) \le y\}$ denote the effective domain and the epigraph of f, respectively. The basic (or Mordukhovich) subdifferential of f at $\overline{a} \in \mathrm{dom}(f)$ is given by

$$\partial f(\overline{a}) := \{\alpha \in \mathbb{R}^n \ : \ (\alpha, -1) \in N(\mathrm{epi}(f), (\overline{a}, f(\overline{a})))\}.$$

If f is locally Lipschitz at $\overline{a}$, the Clarke subdifferential is defined by $\partial^c f(\overline{a}) := \mathrm{co}\, \partial f(\overline{a})$. Therefore, when f is strictly differentiable at $\overline{a}$, we have $\partial^c f(\overline{a}) = \{\nabla f(\overline{a})\}$. Besides, let $\varphi : \mathbb{R}^n \to \mathbb{R}$ and $\psi : \mathbb{R}^n \to \mathbb{R}^q$ be Lipschitz continuous functions. Consider the following nonlinear optimization problem:

$$\min_x \varphi(x) \quad \text{subject to} \quad \psi(x) \le 0. \tag{2.1}$$

Definition 2.2. [21] Let $\overline{a}$ be feasible to problem (2.1).

($\mathscr{D}_1$)**:** The set of abnormal multipliers corresponding to $\overline{a}$ is given by

$$\mathscr{A}(\overline{a}) := \left\{ \eta \in \mathbb{R}^q \ : \ 0 \in \partial^c \psi(\overline{a})^\top \eta,\ \langle \eta, \psi(\overline{a}) \rangle_{\mathbb{R}^q} = 0,\ \eta \ge 0 \right\}.$$

($\mathscr{D}_2$)**:** The set of normal multipliers corresponding to $\overline{a}$ is given by

$$\mathscr{B}(\overline{a}) := \left\{ \eta \in \mathbb{R}^q \ : \ 0 \in \partial^c \varphi(\overline{a}) + \partial^c \psi(\overline{a})^\top \eta,\ \langle \eta, \psi(\overline{a}) \rangle_{\mathbb{R}^q} = 0,\ \eta \ge 0 \right\}.$$

Definition 2.3. Let $\overline{a}$ be a feasible point of problem (2.1), and let $I_\psi(\overline{a}) := \{k \ : \ \psi_k(\overline{a}) = 0\}$. Assume that ψ is a differentiable function at $\overline{a}$. The Mangasarian Fromovitz constraint qualification (MFCQ) is satisfied for $\overline{a}$ if there exists a vector $\eta \in \mathbb{R}^n$ that verifies $\langle \nabla \psi_k(\overline{a}), \eta \rangle_{\mathbb{R}^n} < 0$ for any $k \in I_\psi(\overline{a})$.

We point out that the Mangasarian Fromovitz constraint qualification is satisfied for $\overline{a}$ if and only if the abnormal cone $\mathscr{A}(\overline{a})$ equals $\{0\}$. Finally, let $\Psi : \mathbb{R}^n \rightrightarrows \mathbb{R}^d$ be a set-valued mapping with

$$\begin{array}{lcl} \mathrm{dom}(\Psi) & := & \{a \in \mathbb{R}^n : \Psi(a) \neq \varnothing\}, \\ \mathrm{gph}(\Psi) & := & \left\{(a,b) \in \mathbb{R}^n \times \mathbb{R}^d : a \in \mathrm{dom}(\Psi), b \in \Psi(a)\right\}. \end{array}$$

Definition 2.4. (R-regular) Let $g : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^l$ and $h : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^q$ be two continuous functions, and let $\Xi : \mathbb{R}^n \rightrightarrows \mathbb{R}^d$ defined by $\Xi(a) := \left\{b \in \mathbb{R}^d : g(a,b) \leq 0 \text{ and } h(a,b) = 0\right\}$. We say that Ξ is R-regular at a point $\left(\overline{a}, \overline{b}\right) \in \mathrm{gph}(\Xi)$ with respect to a subset Ω of $\mathbb{R}^n$ if there exist $\sigma, \varepsilon_1, \varepsilon_2 > 0$ such that $d_{\Xi(a)}(b) \leq \sigma\{0, \max\{g_s(a,b) : s = 1, \ldots, l\}, \max\{|h_s(a,b)| : s = 1, \ldots, q\}\}$, for any $(a,b) \in \left(\mathbb{U}^{\varepsilon_1}_{\mathbb{R}^n}(\overline{a}) \cap \Omega\right) \times \mathbb{U}^{\varepsilon_2}_{\mathbb{R}^d}\left(\overline{b}\right)$.

2.2. **Optimal control problems with pure state inequality constraints.** In this subsection, we briefly discuss the maximum principle context, following the approaches stated in [22, Theorem 9.3.1] and [23, Theorem 4.1].

To proceed, we consider a single-level optimal control problem with two state functions, $x : \left[t_i, t_f\right] \to \mathbb{R}^n$ and $y : \left[t_i, t_f\right] \to \mathbb{R}^d$. We suppose that the trajectory x has a fixed begin point $x(t_i) := x_0$ in $\mathbb{R}^n$; furthermore, we suppose that it is controlled by a control function $u \in M\left(\left[t_i, t_f\right], \mathbb{R}^m\right)$ and it is influenced by a pure state constraint $\psi : \left[t_i, t_f\right] \times \mathbb{R}^n \to \mathbb{R}$. For the second trajectory, we assume that it is controlled by a control function $v \in M\left(\left[t_i, t_f\right], \mathbb{R}^l\right)$. Mathematically, the body of the optimal control problem under consideration is as follows:

$$\begin{cases} \min\limits_{(x,u),(y,v)} J\left(x(t_f), y(t_f)\right) & \\ \dot{x}(t) = \phi_1(t, x(t), u(t)) & \text{a.e. } t \in \left[t_i, t_f\right] \\ \dot{y}(t) = \phi_2(t, y(t), v(t)) & \text{a.e. } t \in \left[t_i, t_f\right] \\ x(t_i) = x_0 & \\ \psi(t, x(t)) \leq 0 & \forall t \in \left[t_i, t_f\right] \\ u(t) \in \mathtt{U} & \text{a.e. } t \in \left[t_i, t_f\right] \\ v(t) \in \mathtt{V} & \text{a.e. } t \in \left[t_i, t_f\right] \\ \left(x(t_f), y(t_f)\right) \in C, & \end{cases} \tag{2.2}$$

where $J : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$ is the objective function, $\phi_1 : \left[t_i, t_f\right] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $\phi_2 : \left[t_i, t_f\right] \times \mathbb{R}^d \times \mathbb{R}^l \to \mathbb{R}^d$ represent the dynamic functions, $\mathtt{U}$ and $\mathtt{V}$ are nonemty and Borel measurable subsets of $\mathbb{R}^m$ and $\mathbb{R}^l$ respectively, and $C \subset \mathbb{R}^n \times \mathbb{R}^d$.

To continue, we consider the following definition.

Definition 2.5. A point $((x^*, u^*), (y^*, v^*))$ is said to be a $\mathscr{W}[n+d]$-local minimal solution to (2.2) if $((x^*, u^*), (y^*, v^*))$ is feasible to (2.2) and there exists $\varepsilon > 0$ such that $J\left(x^*(t_f), y^*(t_f)\right) \leq J\left(x(t_f), y(t_f)\right)$ for each feasible point $((x,u),(y,v))$ of (2.2) which verifies $(x,y) \in \mathbb{U}^{\varepsilon}_{\mathscr{W}[n+d]}(x^*, y^*)$.

Now, we need to make some hypotheses for its $\mathscr{W}[n+d]$-local optimal solutions. For this purpose, let $((x^*, u^*), (y^*, v^*))$ be a $\mathscr{W}[n+d]$-local minimal solution of (2.2).

($\mathscr{X}_1$)**:** J is Lipschitz continuous around $\left(x^*(t_f), y^*(t_f)\right)$.

($\mathscr{X}_2$): For each $(x,y) \in \mathbb{R}^n \times \mathbb{R}^d$, the mappings $(t,u) \mapsto \phi_1(t,x,u)$ and $(t,v) \mapsto \phi_2(t,y,v)$ are $\mathscr{L} \times \mathscr{B}^m$ and $\mathscr{L} \times \mathscr{B}^l$-measurable functions, respectively. For each $t \in [t_i,t_f]$, and $(u,v) \in \mathtt{U} \times \mathtt{V}$, the mappings $x \mapsto \phi_1(t,x,u)$ and $y \mapsto \phi_2(t,y,v)$ are continuously differentiable functions. There are two measurable functions $\gamma_x : [t_i,t_f] \times \mathbb{R}^m \to \mathbb{R}$, $\gamma_y : [t_i,t_f] \times \mathbb{R}^l \to \mathbb{R}$, and scalars $\varepsilon_1, \varepsilon_2 > 0$ such that $t \mapsto \gamma_x(t,u^*(t))$, $t \mapsto \gamma_y(t,v^*(t))$ are integrable, and for almost every $t \in [t_i,t_f]$

$$\|\phi_1(t,x_1,u) - \phi_1(t,x_2,u)\|_\infty \leq \gamma_x(t,u)\|x_1 - x_2\|_\infty \text{ and}$$
$$\|\phi_2(t,y_1,v) - \phi(t,y_2,v)\|_\infty \leq \gamma_y(t,v)\|y_1 - y_2\|_\infty,$$

for each $x_1,x_2 \in \mathbb{U}^{\varepsilon_1}_{\mathbb{R}^n}(x^*(t))$, $u \in \mathtt{U}$, $y_1,y_2 \in \mathbb{U}^{\varepsilon_2}_{\mathbb{R}^d}(y^*(t))$, $v \in \mathtt{V}$. ϕ_1 and ϕ_2 are continuous functions.

($\mathscr{X}_3$): For each $t \in [t_i,t_f]$, $x \mapsto \psi(t,x)$ is continuously differentiable, and $\nabla_x \psi(t,x^*(t)) \neq 0$. ψ is an upper semicontinuous function. There are scalars $\varepsilon_\psi, L_\psi > 0$ such that

$$|\psi(t,x_1) - \psi(t,x_2)| \leq L_\psi \|x_1 - x_2\|_\infty,$$

for each $t \in [t_i,t_f]$, $x_1,x_2 \in \mathbb{U}^{\varepsilon_\psi}_{\mathbb{R}^n}(x^*(t))$.

($\mathscr{X}_4$): C is non-empty and locally closed around $(x^*(t_f), y^*(t_f))$.

($\mathscr{X}_5$): There is $\kappa > 0$ such that $\{\phi_1(t,x,u) : u \in \mathtt{U}\}$ is a convex set for all $x \in \mathbb{U}^{\kappa}_{\mathbb{R}^n}(x_0)$, $t \in [t_i, \kappa[$.
If $\psi(t_i,x_0) = 0$, then there are scalars $\nu_1,\nu_2,\nu_3,\nu_4 > 0$ and $\widehat{u} \in L^1([t_i,t_f],\mathbb{R}^m)$ verifying $\widehat{u} \in \mathtt{U}$ a.e on $[t_i,t_f]$ such that

$$\|\phi_1(t,x_0,u^*(t))\|_\infty \leq \nu_3, \quad \|\phi_1(t,x_0,\widehat{u}(t))\|_\infty \leq \nu_3 \text{ and}$$
$$\int_{t_i}^{t} \nabla_x \psi(a,x)^\top [\phi_1(\tau,x_0,\widehat{u}(\tau)) - \phi_1(\tau,x_0,u^*(\tau))]\,d\tau \leq -\nu_4 t,$$

for all $a,t \in [t_i,\nu_1[$ and $x \in \mathbb{U}^{\nu_2}_{\mathbb{R}^n}(x_0)$.

Remark 2.6. As it is known that the main objective of the optimality conditions is to reduce the cardinal of the feasible set. Sadly, sometimes we find that any feasible point verifies the derived necessary optimality conditions and is therefore useless. For that, and as in [23], we put the last hypothesis ($\mathscr{X}_5$) to make sure that our derived Pontryagin optimality conditions, in Theorem 2.7, of optimal control problem (2.2) do not degenerate.

An immediate consequence of [22, Theorem 9.3.1] and [23, Theorem 4.1] is given below.

Theorem 2.7. *Let $((x^*,u^*),(y^*,v^*))$ be a $\mathscr{W}[n+d]$-local minimal solution to (2.2) such that hypotheses $(\mathscr{X}_1)-(\mathscr{X}_5)$ hold. Then, there are two functions $(p_1,p_2) \in \mathscr{W}[n] \times \mathscr{W}[d]$, a measure $\lambda \in \mathscr{C}^p_0([t_i,t_f])$, and a constant $\tau \geq 0$ with the following optimality conditions:*

- ▷: *the enhanced nontriviality condition:* $\|R\|_{L^\infty([t_i,t_f],\mathbb{R}^n)} + \|p_2\|_{\mathscr{W}[d]} + \lambda(]t_i,t_f]) + \tau > 0$;
- ▷: *the adjoint equations: for almost every $t \in [t_i,t_f]$*

$$-\dot{p}_1(t)^\top = R(t)^\top \nabla_x \phi_1(t,x^*(t),u^*(t)),$$
$$-\dot{p}_2(t)^\top = p_2(t)^\top \nabla_y \phi_2(t,y^*(t),v^*(t));$$

▷: *the Weierstrass-Pontryagin conditions: for almost every* $t \in [t_i, t_f]$

$$R(t)^\top \phi_1(t, x^*(t), u^*(t)) = \max_{w \in \mathtt{U}} R(t)^\top \phi_1(t, x^*(t), w),$$
$$p_2(t)^\top \phi_2(t, y^*(t), v^*(t)) = \max_{w \in \mathtt{V}} p_2(t)^\top \phi_2(t, y^*(t), w);$$

▷: *the transversality condition:*

$$p_2(t_i) = 0,$$
$$\left(-R(t_f), -p_2(t_f)\right) \in \tau\, \partial^c J\left(x^*(t_f), y^*(t_f)\right) + N\left(C, \left(x^*(t_f), y^*(t_f)\right)\right);$$

▷: *the support condition:* $\operatorname{supp}(\lambda) \subseteq \left\{t \in [t_i, t_f] : \psi(t, x^*(t)) = 0\right\}$;

where the function $R : [t_i, t_f] \to \mathbb{R}^n$ *is given by*

$$R(t) := \begin{cases} p_1(t) + \displaystyle\int_{[t_i, t_f[} \nabla_x \psi(a, x^*(a))\, \lambda(\mathrm{d}a) & \text{if } t_i < t < t_f, \\ p_1(t_f) + \displaystyle\int_{[t_i, t_f]} \nabla_x \psi(a, x^*(a))\, \lambda(\mathrm{d}a) & \text{if } t = t_f. \end{cases}$$

Remark 2.8. Without imposing the validity of hypothesis $(\mathscr{X}_5)$, the enhanced nontriviality condition in Theorem 2.7 receives $(p_1, p_2, \lambda, \tau) \neq (0,0,0,0)$. In this case, there is nothing that guarantees the nondegeneracy of the derived maximum principle in Theorem 2.7.

3. The Problem and Its Reformulation

To keep things simple, this work is concentrated on a Mayer-type cost function. Examine the following bilevel optimal control problem with variational inequality constraints:

$$\begin{cases} \min\limits_{x,u,y} f\left(x(t_f), y\right) \\ \text{subject to } \dot{x}(t) = \phi(t, x(t), u(t)) & \text{a.e. } t \in [t_i, t_f] \\ \qquad x(t_i) = x_0 \\ \qquad g(t, x(t)) \leq 0 & \forall t \in [t_i, t_f] \\ \qquad u(t) \in \mathscr{U} & \text{a.e. } t \in [t_i, t_f] \\ \qquad y \in S\left(x(t_f)\right), \end{cases} \tag{GBOCP}$$

where $S(x(t_f))$ is the solution set of the following variational inequality parameterized by the final state:

$$\left\langle F(x(t_f), y), y - z \right\rangle_{\mathbb{R}^d} \leq 0, \quad \forall z \in K(x(t_f)). \tag{3.1}$$

The material provided for this problem includes an interval $[t_i, t_f]$, the objective function $f : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$, the dynamic function of system $\phi : [t_i, t_f] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, the initial state condition fixed $x_0 \in \mathbb{R}^n$, the scalar function $g : [t_i, t_f] \times \mathbb{R}^n \to \mathbb{R}$, a nonempty and Borel measurable set $\mathscr{U} \subset \mathbb{R}^m$, the set-valued map $K : \mathbb{R}^n \to \mathbb{R}^d$, and the vector function $F : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^d$. Here, x and u are respectively the state function and the measurable control of our problem, while parameter t describes the time, and $\dot{x}$ refers to us as the (weak) derivative of x w.r.t. time. One can observe that function S symbolizes the solution mapping of $\min_z \left\langle F(x(t_f), y), z\right\rangle_{\mathbb{R}^d}$ subject to $z \in K(x(t_f))$.

From now on, we work with a specific type of a local and global minimizer. For this purpose, we indicate the feasible leader set $\mathbb{O}_s^u$ and the feasible set $\mathbb{O}_s$ of (GBOCP) by

$$\mathbb{O}_s^u := \left\{ (x,u) \in \mathscr{W}[n] \times M\left([t_i,t_f],\mathbb{R}^m\right) \;\middle|\; \begin{array}{ll} \dot{x}(t) = \phi\left(t,x(t),u(t)\right) & \text{a.e. } t \in [t_i,t_f] \\ x(t_i) = x_0 & \\ g\left(t,x(t)\right) \leq 0 & \forall t \in [t_i,t_f] \\ u(t) \in \mathscr{U} & \text{a.e. } t \in [t_i,t_f] \end{array} \right\},$$

$$\mathbb{O}_s := \left\{ (x,u,y) \in \mathscr{W}[n] \times M\left([t_i,t_f],\mathbb{R}^m\right) \times \mathbb{R}^d : (x,u) \in \mathbb{O}_s^u \text{ and } y \in S\left(x(t_f)\right) \right\},$$

respectively. Hence, for $\Theta := \mathscr{W}[n] \times \mathbb{R}^d$, we make use of the definitions below.

Definition 3.1 (Θ-Local minimal solution)**.** A point $(x^*,u^*,y^*) \in \mathbb{O}_s$ is a Θ-local minimal solution to (GBOCP) if we can find $\varepsilon > 0$ such that, for all $(x,u,y) \in \mathbb{O}_s$ with $(x,y) \in \mathbb{U}_\Theta^\varepsilon(x^*,y^*)$, $f\left(x^*(t_f),y^*\right) \leq f\left(x(t_f),y\right)$.

Definition 3.2 (Global minimal solution)**.** A global minimal solution to problem (GBOCP) is a Θ-local minimal solution to problem (GBOCP) with $\varepsilon = +\infty$.

Next, we suppose that $K\left(x(t_f)\right) = \left\{y \in \mathbb{R}^d \,:\, G\left(x(t_f),y\right) \leq 0\right\}$, with $G : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^q$ being a vector function with components G_k, $k = 1,\dots,q$.

We assume for the rest of this study that the following requirements hold true for all Θ-local minimal solutions (x^*,u^*,y^*) of (GBOCP):

(C_1) f, F, and G are continuously differentiable functions w.r.t their variables.

(C_2) ϕ is a continuous function w.r.t its variables.
$x \mapsto \phi\left(t,x,u\right)$ is continuously differentiable for all $(t,u) \in [t_i,t_f] \times \mathscr{U}$.
For x to be fixed, $(t,u) \mapsto \phi\left(t,x,u\right)$ is $\mathscr{L} \times \mathscr{B}^m$-measurable.
There exist $\varsigma : [t_i,t_f] \times \mathbb{R}^m \to \mathbb{R}$ a measurable function and $\varepsilon > 0$ such that $t \mapsto \varsigma\left(t,u^*(t)\right)$ is in $L^1\left([t_i,t_f],\mathbb{R}\right)$ and

$$\left\|\phi\left(t,x_1,u\right) - \phi\left(t,x_2,u\right)\right\|_\infty \leq \varsigma(t,u)\left\|x_1 - x_2\right\|_\infty \quad \forall x_1,x_2 \in \mathbb{U}_{\mathbb{R}^n}^\varepsilon\left(x^*(t)\right), \forall u \in \mathscr{U}.$$

(C_3) For each $t \in [t_i,t_f]$, $x \mapsto g(t,x)$ is continuously differentiable, $\nabla_x g\left(t,x^*(t)\right) \neq 0$ for all $t \in [t_i,t_f]$, and g is an upper semicontinuous function. There are scalars $\varepsilon_g, L_g > 0$ such that $\left|g\left(t,x_1\right) - g\left(t,x_2\right)\right| \leq L_g\left\|x_1 - x_2\right\|_\infty$, for each $t \in \left[t_i,t_f\right]$ $x_1,x_2 \in \mathbb{B}_{\mathbb{R}^n}^{\varepsilon_g}\left(x^*(t)\right)$.

(C_4) The MFCQ holds at all $y \in K\left(x^*(t_f)\right)$.

With the aim to derive an equivalent single-level optimal control problem for generalized bileve optimal control problems, we define the gap function $\varphi_{GF} : \mathbb{R}^n \times \mathbb{R}^d \to \overline{\mathbb{R}}$ for problem (GBOCP) for all $(a,b) \in \mathbb{R}^n \times \mathbb{R}^d$, by $\varphi_{GF}\left(a,b\right) := \sup\left\{\langle F\left(a,b\right), b - z\rangle_{\mathbb{R}^d} : z \in K\left(a\right)\right\}$.

For any $a \in \mathbb{R}^n$, the two assertions that follow are clear:

$$\begin{cases} \varphi_{GF}\left(a,b\right) \geq 0 \ \text{ for all } \ b \in K\left(a\right), \\ \varphi_{GF}\left(a,b\right) = 0 \ \text{ if and only if } \ b \in S\left(a\right). \end{cases} \tag{3.2}$$

Using the gap function, we can then rewrite the solution set of (3.1) for a state function $x \in \mathscr{W}[n]$ as follows:

$$S\left(x(t_f)\right) := \left\{y \in \mathbb{R}^d : G\left(x(t_f),y\right) \leq 0 \ \text{ and } \ \varphi_{GF}\left(x(t_f),y\right) = 0\right\}. \tag{3.3}$$

Hence, problem (GBOCP) is identical to the next one-level optimal control problem:

$$\begin{cases} \min\limits_{x,u,y} f\left(x(t_f),y\right) \\ \text{subject to } \dot{x}(t) = \phi\left(t,x(t),u(t)\right) \quad \text{a.e. } t \in [t_i,t_f] \\ \qquad x(t_i) = x_0 \\ \qquad g\left(t,x(t)\right) \le 0 \qquad \forall t \in [t_i,t_f] \\ \qquad u(t) \in \mathscr{U} \qquad \text{a.e. } t \in [t_i,t_f] \\ \qquad \varphi_{GF}\left(x(t_f),y\right) = 0 \\ \qquad G\left(x(t_f),y\right) \le 0. \end{cases} \tag{3.4}$$

From now on, $\Lambda\left(x(t_f),y\right)$ is borrowed to stand for the collection of all vectors where $\varphi_{GF}\left(x(t_f),y\right)$ attains its maximum. It is worth mentioning that the standard constraint qualifications, like the MFCQ, never hold for problem (3.4). Consider the simple case that (x^*,u^*,y^*) is a Θ-local minimal solution to (GBOCP), $K(x(t_f)) = \mathbb{R}^d$, $\mathbb{O}_s^u = \mathscr{W}[n] \times M\left([t_i,t_f),\mathbb{R}^m\right)$, and φ_{GF} is a Lipschitz continuous function. Then, assertions (3.2) yield inclusion $0 \in \partial^c \varphi_{GF}\left(x^*(T),y^*\right)$; demonstrating that the abnormal multipliers (see Definition 2.2 ($\mathscr{D}_1$)) always exist for problem (3.4). As a result, the Mangasarian Fromovitz constraint qualification is not verified. We adopt Ye and Zhu's theories [24], in which the authors discovered some constraint qualifications by using an exact penalization. This principle, which is now widely known, is completely connected to the property of calmness. First, let us go over the concept of partial calmness.

Definition 3.3. Problem (3.4) is said to be partially calm at a $(x^*,u^*,y^*) \in \mathbb{O}_s$ if there exist $\varepsilon,\beta > 0$ such that $f\left(x(t_f),y\right) - f\left(x^*(t_f),y^*\right) + \beta\sigma \ge 0$ for all $(x,u,y,\sigma) \in \mathbb{O}_s^u \times K\left(x(t_f)\right) \times \mathbb{R}_+$ satisfying $(x,y,\sigma) \in \mathbb{U}^{\varepsilon}_{\Theta\times\mathbb{R}}(x^*,y^*,0)$ and $\varphi_{GF}\left(x(t_f),y\right) = \sigma$.

The following result describes the partial calmness property at a defined Θ-local minimum solution of (3.4) using the precise penalization concept when the gap function φ_{GF} is a locally Lipschitz continuous function.

Proposition 3.4. *Let (x^*,u^*,y^*) be a Θ-local minimal solution of (3.4). Assume that φ_{GF} is Lipschitz continuous function around $\left(x^*(t_f),y^*\right)$. Then, problem (3.4) is partially calm at (x^*,u^*,y^*) if and only if there exists some $\beta > 0$ such that (x^*,u^*,y^*) is a Θ-local minimal solution for the following optimal control problem:*

$$\begin{cases} \min\limits_{x,u,y} f\left(x(t_f),y\right) + \beta\varphi_{GF}\left(x(t_f),y\right) \\ \textit{subject to } \dot{x}(t) = \phi\left(t,x(t),u(t)\right) \quad \textit{a.e. } t \in [t_i,t_f] \\ \qquad x(t_i) = x_0 \\ \qquad g\left(t,x(t)\right) \le 0 \qquad \forall t \in [t_i,t_f] \\ \qquad u(t) \in \mathscr{U} \qquad \textit{a.e. } t \in [t_i,t_f] \\ \qquad G\left(x(t_f),y\right) \le 0. \end{cases} \tag{$P[\beta]$}$$

Proof. First, assume that problem (3.4) is partially calm at (x^*,u^*,y^*). Then, there exist $\varepsilon,\beta > 0$ such that for all $(x,u,y,\sigma) \in \mathbb{O}_s^u \times K\left(x(t_f)\right) \times \mathbb{R}_+$ satisfying $(x,y,\sigma) \in \mathbb{U}^{\varepsilon}_{\Theta\times\mathbb{R}}(x^*,y^*,0)$ and $\varphi_{GF}\left(x(t_f),y\right) = \sigma$, $f\left(x(t_f),y\right) - f\left(x^*(t_f),y^*\right) + \beta\sigma \ge 0$. Since φ_{GF} is Lipschitz continuous around $\left(x^*(t_f),y^*\right)$, then there exist $\varepsilon_1, L_{\varphi_{GF}} > 0$ such that

$$\|\varphi_{GF}(a_1,b_1)-\varphi_{GF}(a_2,b_2)\|_\infty \leq L_{\varphi_{GF}} \|(a_1,b_1)-(a_2,b_2)\|_\infty, \tag{3.5}$$

for all $(a_1,b_1),(a_2,b_2) \in \mathbb{U}^{\varepsilon_1}_{\mathbb{R}^n\times\mathbb{R}^d}\left(x^*(t_f),y^*\right)$.

Now, let $(x,u,y) \in \mathbb{O}^u_s \times K\left(x(t_f)\right)$ which satisfies $(x,y) \in \mathbb{U}^{\varepsilon^\star}_\Theta(x^*,y^*)$ with

$$\varepsilon^\star := \min\left\{\varepsilon, \frac{\varepsilon}{C_{emb}}, \frac{\varepsilon_1}{C_{emb}}, \frac{\varepsilon}{\max\left\{L_{\varphi_{GF}}, C_{emb}L_{\varphi_{GF}}\right\}}\right\},$$

where C_{emb} is the constant in Lemma 2.1 satisfying

$$\|x(t)\|_\infty \leq \|x\|_{\mathscr{C}\left([t_i,t_f],\mathbb{R}^n\right)} \leq C_{emb}\|x\|_{\mathscr{W}[n]}, \tag{3.6}$$

for all $t \in [t_i,t_f]$.

Next, fix a scalar σ such that $\varphi_{GF}\left(x(t_f),y\right) = \sigma$. It follows that

$$\begin{aligned}
\sigma &\overset{(1^*)}{=} \varphi_{GF}\left(x(t_f),y\right) - \varphi_{GF}\left(x^*(t_f),y^*\right)\\
&\overset{(2^*)}{\leq} L_{\varphi_{GF}}\left\|\left(x(t_f),y\right)-\left(x^*(t_f),y^*\right)\right\|_\infty\\
&\overset{(3^*)}{\leq} L_{\varphi_{GF}}\max\left\{\left\|x(t_f)-x^*(t_f)\right\|_\infty \ , \ \|y-y^*\|_\infty\right\}\\
&\overset{(4^*)}{\leq} L_{\varphi_{GF}}\max\left\{\|x-x^*\|_{\mathscr{C}\left([t_i,t_f],\mathbb{R}^n\right)} \ , \ \|y-y^*\|_\infty\right\}\\
&\overset{(5^*)}{\leq} \max\left\{L_{\varphi_{GF}} \ , \ L_{\varphi_{GF}}C_{emb}\right\}\max\left\{\|x-x^*\|_{\mathscr{W}[n]} \ , \ \|y-y^*\|_\infty\right\}\\
&\overset{(6^*)}{<} \max\left\{L_{\varphi_{GF}} \ , \ L_{\varphi_{GF}}C_{emb}\right\}\varepsilon^\star\\
&\overset{(7^*)}{\leq} \varepsilon,
\end{aligned}$$

where (1^*) results from $y^* \in S\left(x^*(t_f)\right)$, (2^*) from (3.5), (3^*) is the definition of $\|\cdot\|_\infty$, (4^*) and (5^*) corresponds to assertion (3.6), (6^*) is a consequence of $(x,y) \in \mathbb{U}^{\varepsilon^\star}_\Theta(x^*,y^*)$, and (7^*) relates to how $\varepsilon^\star$ is defined.

Consequently, $(x,y,\sigma) \in \mathbb{U}^{\varepsilon}_{\Theta\times\mathbb{R}}(x^*,y^*,0)$. Finally, from the fact that $\varphi_{GF}\left(x(t_f),y\right) = \sigma$, the partial calmness of (3.4) at (x^*,u^*,y^*), and the assumption $y^* \in S\left(x^*(t_f)\right)$, we have that

$$f\left(x(t_f),y\right) + \beta\varphi_{GF}\left(x(t_f),y\right) \geq f\left(x^*(t_f),y^*\right) + \beta\varphi_{GF}\left(x^*(t_f),y^*\right),$$

hold for all feasible points (x,u,y) of problem ($P[\beta]$) with $(x,y) \in \mathbb{U}^{\varepsilon^\star}_\Theta(x^*,y^*)$, which proves that (x^*,u^*,y^*) is a Θ-local minimal solution to problem ($P[\beta]$).

Conversely, assume that there is $\beta > 0$ such that (x^*,u^*,y^*) is a Θ-local minimal solution to the problem of ($P[\beta]$). Then, there exists some $\varepsilon > 0$ such that $f\left(x^*(t_f),y^*\right) \leq f\left(x(t_f),y\right) + \beta\varphi_{GF}\left(x(t_f),y\right)$ for all $(x,u,y) \in \mathbb{O}^u_s \times K\left(x(t_f)\right)$ with $(x,y) \in \mathbb{U}^{\varepsilon}_\Theta(x^*,y^*)$. Let us fix (x,u,y,σ) in $\mathbb{O}^u_s \times K\left(x(t_f)\right) \times \mathbb{R}_+$ with $\varphi_{GF}\left(x(t_f),y\right) = \sigma$ and $(x,y,\sigma) \in \mathbb{U}^{\varepsilon}_{\Theta\times\mathbb{R}}(x^*,y^*,0)$. Hence, we can write $f\left(x(t_f),y\right) - f\left(x^*(t_f),y^*\right) + \beta\sigma \geq 0$. As a consequence, problem (3.4) is partially calm at (x^*,u^*,y^*). □

Subsequently, we offer sufficient conditions to ensure that problem (3.4) remains partially calm. We start with the uniform parametric error bound, which uses the distance function.

Definition 3.5. We say that variational inequality (3.1) possesses a uniformly parametric error bound (UPEB) at $\left(x^*(t_f),y^*\right) \in \mathrm{gph}(S)$ if there exist scalars $\varepsilon,\sigma>0$ such that

$$\forall\left(x(t_f),y\right) \in \mathbb{U}^{\varepsilon}_{\mathbb{R}^n\times\mathbb{R}^d}\left(x^*(t_f),y^*\right):\ \ y\in K\left(x(t_f)\right) \Rightarrow d_{S(x(t_f))}(y) \leq \sigma\varphi_{GF}\left(x(t_f),y\right).$$

Proposition 3.6. *Let (x^*,u^*,y^*) be a Θ-local minimal solution to (3.4). Assume that variational inequality (3.1) possesses a (UPEB) at $\left(x^*(t_f),y^*\right)$ with modulus $\sigma>0$. Then, there exists a scalar $\beta^*>0$ such that (x^*,u^*,y^*) is a Θ-local minimal solution to ($P[\beta]$) for all $\beta\geq\sigma\beta^*$.*

Proof. Firstly, observe that problem (GBOCP) is identical to the next optimal control problem:

$$\begin{cases} \min\limits_{x,u,y} f\left(x(t_f),y\right) \\ \text{subject to } \dot{x}(t)=\phi\left(t,x(t),u(t)\right) & \text{a.e. } t\in[t_i,t_f] \\ \qquad\qquad x(0)=x_0 \\ \qquad\qquad g\left(t,x(t)\right)\leq 0 & \forall t\in[t_i,t_f] \\ \qquad\qquad u(t)\in\mathscr{U} & \text{a.e. } t\in[t_i,t_f] \\ \qquad\qquad d_{S\left(x(t_f)\right)}(y)=0. \end{cases} \tag{3.7}$$

Furthermore, problem (3.7) is partially calm at (x^*,u^*,y^*). Indeed, let $\varepsilon>0$ be the constant that satisfy Definition 3.1 and $\varepsilon_1=\dfrac{\varepsilon}{2}$. Let $\varepsilon_2\in[0,\varepsilon_1[$, $(x,u)\in\mathbb{O}^u_s$ and $y\in\mathbb{R}^d$ such that $d_{S\left(x(t_f)\right)}(y)=\varepsilon_2$ and $(x,y)\in\mathbb{U}^{\varepsilon_1}_{\Theta}(x^*,y^*)$. Condition (C_1) implies that $S\left(x(t_f)\right)$ is closed. Then, there exists $\bar{y}\in S\left(x(t_f)\right)$ such that $\|\bar{y}-y\|_\infty=\varepsilon_2$. Thus $(x,u,\bar{y})$ is a feasible point of (3.7). Moreover,

$$\|(x^*,y^*)-(x,\bar{y})\|_\Theta \leq \|(x^*,y^*)-(x,y)\|_\Theta+\|(x,y)-(x,\bar{y})\|_\Theta<\varepsilon_1+\varepsilon_2<\varepsilon.$$

Consequently, $(x,\bar{y})\in\mathbb{U}^{\varepsilon}_{\Theta}(x^*,y^*)$. Since (x^*,u^*,y^*) is a Θ-local minimal solution to (3.4), then

$$f\left(x^*(t_f),y^*\right)\leq f\left(x(t_f),\bar{y}\right). \tag{3.8}$$

By multiplying inequality (3.8) by (-1) and adding $f\left(x(t_f),y\right)$ on both sides, we obtain

$$f\left(x(t_f),y\right)-f\left(x^*(t_f),y^*\right)\geq f\left(x(t_f),y\right)-f\left(x(t_f),\bar{y}\right).$$

Since (C_1) holds, f is locally Lipschitz continuous at $\left(x^*(t_f),y^*\right)$ with modulus $L_f>0$ on $\mathbb{U}^{\varepsilon}_{\mathbb{R}^n\times\mathbb{R}^d}\left(x^*(t_f),y^*\right)$, then $f\left(x(t_f),y\right)-f\left(x^*(t_f),y^*\right)+L_f\varepsilon_2\geq 0$, which proves that (3.7) is partially calm at (x^*,u^*,y^*).

Secondly, as in Proposition 3.4, there exists $\beta^*>0$ such that, for all $(x,u,y)\in\mathbb{O}_s$ with $(x,y)\in\mathbb{U}^{\varepsilon}_{\Theta}(x^*,y^*)$,

$$f\left(x^*(t_f),y^*\right)+\beta^* d_{S\left(x^*(t_f)\right)}\left(y^*\right)\leq f\left(x(t_f),y\right)+\beta^* d_{S\left(x(t_f)\right)}(y). \tag{3.9}$$

Now, let $(x,u,y)\in\mathbb{O}_s$ with $(x,y)\in\mathbb{U}^{\varepsilon}_{\Theta}(x^*,y^*)$. We have from $y^*\in S\left(x^*(t_f)\right)$, inequality (3.9), and the (UPEB) assumption that

$$f\left(x^*(t_f),y^*\right)+\sigma\beta^*\varphi_{GF}\left(x^*(t_f),y^*\right)\leq f\left(x(t_f),y\right)+\sigma\beta^*\varphi_{GF}\left(x(t_f),y\right).$$

Consequently, (x^*,u^*,y^*) is a Θ-local minimal solution to ($P[\beta]$) for all $\beta\geq\sigma\beta^*$. $\square$

Finally, keeping in mind that S is defined as in (3.3), we provide a sufficient condition for the (UPEB) based on the R-regularity of the set-valued mapping S.

Proposition 3.7. *Let (x^*,u^*,y^*) be a Θ-local minimal solution to problem (GBOCP). Assume that there exists some neighborhood $\mathscr{N}\subset\mathbb{R}^n$ of $x^*(t_f)$ verified* $\operatorname{dom}(K)\cap\mathscr{N}=\operatorname{dom}(S)\cap\mathscr{N}$. *Moreover, let S be R-regular at $\left(x^*(t_f),y^*\right)$ w.r.t* $\operatorname{dom}(S)$. *Then, variational inequality (3.1) possesses a (UPEB) at $\left(x^*(t_f),y^*\right)\in\operatorname{gph}(S)\cap\mathscr{N}\times\mathbb{R}^d$.*

Proof. Since S is R-regular at $\left(x^*(t_f),y^*\right)$ w.r.t $\operatorname{dom}(S)$, then there exist scalars $\sigma,\delta>0$ such that, for all $(x,u,y)\in\mathbb{O}_s$ with $\left(x(t_f),y\right)\in\mathbb{U}^{\delta}_{\mathbb{R}^n\times\mathbb{R}^d}\left(x^*(t_f),y^*\right)\cap\operatorname{dom}(S)\times\mathbb{R}^d$,

$$d_{S\left(x(t_f)\right)}(y)\leq\sigma\max\left\{0,G_1\left(x(t_f),y\right),\ldots,G_q\left(x(t_f),y\right),\varphi_{GF}\left(x(t_f),y\right)\right\}.$$

Hence, one can write $y\in K(x(t_f)\Rightarrow d_{S\left(x(t_f)\right)}(y)\leq\sigma\varphi_{GF}\left(x(t_f),y\right)$ for all $(x,u,y)\in\mathbb{O}_s$ verified $\left(x(t_f),y\right)\in\mathbb{U}^{\delta}_{\mathbb{R}^n\times\mathbb{R}^d}\left(x^*(t_f),y^*\right)\cap\operatorname{dom}(S)\times\mathbb{R}^d$. In view of $\operatorname{dom}(K)\cap\mathscr{N}=\operatorname{dom}(S)\cap\mathscr{N}$, for all $(x,u,y)\in\mathbb{O}_s$ satisfying $\left(x(t_f),y\right)\in\mathbb{U}^{\delta}_{\mathbb{R}^n\times\mathbb{R}^d}\left(x^*(t_f),y^*\right)\cap\mathscr{N}\times\mathbb{R}^d$, we obtain

$$y\in K(x(t_f)\Rightarrow d_{S\left(x(t_f)\right)}(y)\leq\sigma\varphi_{GF}\left(x(t_f),y\right),$$

which completes the proof. $\square$

4. Necessary Optimality Conditions

In this section, we use the maximal concept described in Section 2.2 to deal with Pontryagin optimality conditions for problem (GBOCP). Following Remark 2.6, we add the two conditions below to avoid the nondegeneracy of the necessary optimality conditions. For a Θ-local minimal solution (x^*,u^*,y^*) of (GBOCP), we assume that:

(C_5) There exists $\varepsilon>0$ such that, for all $x\in\mathbb{U}^{\varepsilon}_{\mathbb{R}^n}(x_0)$ and $t\in[t_i,\varepsilon[$, $\{\phi(t,x,u):u\in\mathscr{U}\}$ is convex.

(C_6) If $g(t_i,x_0)=0$, then there exist scalars $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4>0$ and a control function $\widehat{u}\in L^1\left([t_i,t_f],\mathbb{R}^m\right)$ verifying $\widehat{u}\in\mathscr{U}$ a.e on $[t_i,t_f]$ such that

$$\begin{cases}\|\phi\left(t,x_0,u^*(t)\right)\|_\infty\leq\varepsilon_3,\\ \|\phi\left(t,x_0,\widehat{u}(t)\right)\|_\infty\leq\varepsilon_3,\\ \int_{t_i}^{t}\nabla_x g(s,x)^\top\left[\phi\left(\tau,x_0,\widehat{u}(\tau)\right)-\phi\left(\tau,x_0,u^*(\tau)\right)\right]d\tau\leq-\varepsilon_4 t,\end{cases}$$

for all $s,t\in[t_i,\varepsilon_1[$ and $x\in\mathbb{U}^{\varepsilon_2}_{\mathbb{R}^n}(x_0)$.

On the other hand, we need the following upper estimate for our gap function, which is a straightforward application of [25, Corollary 1 of Theorem 6.5.2]. To proceed, let $(x^*,u^*,y^*)\in\mathbb{O}_s$ and let $z\in K\left(x^*(t_f)\right)$. The abnormal and normal cones in smooth cases take, respectively, the following forms:

$$\begin{aligned}\mathscr{A}_{(x^*,y^*)}(z)&=\left\{\eta\in\mathbb{R}^q_+:\nabla_y G\left(x^*(t_f),z\right)^\top\eta=0,\eta^\top G\left(x^*(t_f),z\right)=0\right\},\\ \mathscr{B}_{(x^*,y^*)}(z)&=\left\{\eta\in\mathbb{R}^q_+:F\left(x^*(t_f),y^*\right)+\nabla_y G\left(x^*(t_f),z\right)^\top\eta=0,\eta^\top G\left(x^*(t_f),z\right)=0\right\}.\end{aligned}$$

Proposition 4.1. *Let $(x^*,u^*,y^*) \in \mathbb{O}_s$. Suppose that $\mathscr{A}_{(x^*,y^*)}\left(\Lambda\left(x^*(t_f),y^*\right)\right) = \{0_{\mathbb{R}^q}\}$. Then φ_{GF} is Lipschitz continuous near $\left(x^*(t_f),y^*\right)$ and*

$$\begin{aligned}\partial^c \varphi_{GF}\left(x^*(t_f),y^*\right) \subset \ & \mathrm{co}\Bigg\{\Bigg(\nabla_{x(t_f)} F\left(x^*(t_f),y^*\right)^\top (y^*-z) - \nabla_{x(t_f)} G\left(x^*(t_f),z\right)^\top \eta,\\ & \nabla_y F\left(x^*(t_f),y^*\right)^\top (y^*-z) + F\left(x^*(t_f),y^*\right)\Bigg) \text{ such that}\\ & z \in \Lambda\left(x^*(t_f),y^*\right), \eta \in \mathscr{B}_{(x^*,y^*)}(z)\Bigg\}.\end{aligned}$$

Now, we are ready to derive a maximum principle for the generalized bilevel optimal control problem under consideration.

Theorem 4.2. *Let (x^*,u^*,y^*) be a Θ-local minimal solution to problem (GBOCP). Suppose that $\mathscr{A}_{(x^*,y^*)}\left(\Lambda\left(x^*(t_f),y^*\right)\right) = \{0_{\mathbb{R}^q}\}$, conditions $(C_1)-(C_6)$ hold, and problem (3.4) is partially calm at (x^*,u^*,y^*). Then, there exist a function $p \in \mathscr{W}[n]$, an integer $l \in \mathbb{N}^*$, a finite family of scalars $(\alpha)_{2\le i\le l} \subseteq \mathbb{R}_+$ with $\sum_{i=2}^{l} \alpha_i = 1$, a finite family of vectors $(\eta_i)_{1\le i\le l} \subseteq \mathbb{R}^q_+$, a finite family of vectors $(z_i)_{2\le i\le l} \subset \Lambda\left(x^*(t_f),y^*\right)$ with for all $i \in \{2,\ldots,l\}$: $\eta_i \in \mathscr{B}_{(x^*,y^*)}(z_i)$, a measure $\lambda \in \mathscr{C}_0^p\left([t_i,t_f]\right)$, and $\tau \ge 0$, $\beta > 0$ such that*

- $(\mathscr{T}_1)$**:** *the enhanced nontriviality condition $\|R\|_{L^\infty_{\left([t_i,t_f],\mathbb{R}^n\right)}} + \lambda\left(]t_i,t_f]\right) + \tau > 0$;*
- $(\mathscr{T}_2)$**:** *the adjoint equation, for almost every $t \in [t_i,t_f]$, $-\dot{p}(t)^\top = R(t)^\top \nabla_x \phi\left(t,x^*(t),u^*(t)\right)$;*
- $(\mathscr{T}_3)$**:** *the Weierstrass–Pontryagin condition: for almost every $t \in [t_i,t_f]$*

$$R(t)^\top \phi\left(t,x^*(t),u^*(t)\right) = \max_{w\in\mathscr{U}} R(t)^\top \phi\left(t,x^*(t),w\right);$$

- $(\mathscr{T}_4)$**:** *the transversality condition:*

$$\begin{aligned}-R(t_f) =& \tau\Bigg(\nabla_{x(t_f)} f\left(x^*(t_f),y^*\right) + \beta\Bigg(\sum_{i=2}^{l}\alpha_i\Big[\nabla_{x(t_f)} F\left(x^*(t_f),y^*\right)^\top (y^*-z_i)\\ & -\nabla_{x(t_f)} G\left(x^*(t_f),z_i\right)^\top \eta_i\Big]\Bigg)\Bigg) + \nabla_{x(t_f)} G\left(x^*(t_f),y^*\right)^\top \eta_1;\end{aligned}$$

- $(\mathscr{T}_5)$**:** *the support condition:* $\mathrm{supp}(\lambda) \subseteq \left\{t \in [t_i,t_f] : g\left(t,x^*(t)\right) = 0\right\}$;
- $(\mathscr{T}_6)$**:** *the variational inequality condition: for all $i \in \{2,\ldots,l\}$*

$$F\left(x^*(t_f),y^*\right) + \nabla_y G\left(x^*(t_f),z_i\right)^\top \eta_i = 0, \quad \eta_i^\top G\left(x^*(t_f),z_i\right) = 0;$$

- $(\mathscr{T}_7)$**:** *the complementarity condition: $\eta_1^\top G\left(x^*(t_f),y^*\right) = 0$;*
- $(\mathscr{T}_8)$**:** *the multiplier condition:*

$$\begin{aligned}0 =& \tau\Bigg(\nabla_y f\left(x^*(t_f),y^*\right) + \beta\Bigg(\sum_{i=2}^{l}\alpha_i\Big[\nabla_y F\left(x^*(t_f),y^*\right)^\top (y^*-z_i) + F\left(x^*(t_f),y^*\right)\Big]\Bigg)\Bigg)\\ & + \nabla_y G\left(x^*(t_f),y^*\right)^\top \eta_1;\end{aligned}$$

where the function $R : [t_i, t_f] \to \mathbb{R}^n$ is given by:

$$R(t) := \begin{cases} p(t) + \displaystyle\int_{[t_i,t[} \nabla_x g(s, x^*(s))\,\lambda(ds) & \text{if } t_i \le t < t_f, \\ p(t_f) + \displaystyle\int_{[t_i,t_f]} \nabla_x g(s, x^*(s))\,\lambda(ds) & \text{if } t = t_f. \end{cases}$$

Proof. Since (x^*, u^*, y^*) is a Θ-local minimal solution to problem (GBOCP), then (x^*, u^*, y^*) is clearly a Θ-local minimal solution to problem (3.4). As (3.4) is partially calm at (x^*, u^*, y^*), according to Proposition 3.4, one has that (x^*, u^*, y^*) is a Θ-local minimal solution to ($P[\beta]$) for some $\beta > 0$. Set $\mathscr{Z} := \{(a,b) \in \mathbb{R}^n \times \mathbb{R}^d \,:\, G(a,b) \le 0\}$. Considering y^* as a constant state function, we see that (x^*, u^*, y^*) is a Θ-local minimal solution to the following optimal control problem:

$$\begin{cases} \min\limits_{(x,y),u} f\left(x(t_f), y(t_f)\right) + \beta \varphi_{GF}(x(t_f), y(t_f)) & \\ \text{subject to } \dot{x}(t) = \phi\left(t, x(t), u(t)\right) & \text{a.e. } t \in [t_i, t_f] \\ \qquad\qquad \dot{y}(t) = 0 & \text{a.e. } t \in [t_i, t_f] \\ \qquad\qquad x(t_i) = x_0 & \\ \qquad\qquad g\left(t, x(t)\right) \le 0 & \forall t \in [t_i, t_f] \\ \qquad\qquad u(t) \in \mathscr{U} & \text{a.e. } t \in [t_i, t_f] \\ \qquad\qquad \left(x(t_f), y(t_f)\right) \in \mathscr{Z}. & \end{cases}$$

It follows from Theorem 2.7 that there exist two functions $p_1 \in \mathscr{W}[n]$, $p_2 \in \mathscr{W}[d]$, a nonnegative Borel measure $\lambda \in \mathscr{C}_0^p\left([t_i, t_f]\right)$ and a constant $\tau \ge 0$ satisfying for almost every $t \in [t_i, t_f]$:

$$\|R\|_{L^\infty\left([t_i,t_f],\mathbb{R}^n\right)} + \|p_2\|_{\mathscr{W}[d]} + \lambda(]t_i, t_f]) + \tau > 0; \tag{4.1}$$

$$-\dot{p}_1(t)^\top = R(t)^\top \nabla_x \phi\left(t, x^*(t), u^*(t)\right); \tag{4.2}$$

$$-\dot{p}_2(t) = 0; \tag{4.3}$$

$$R(t)^\top \phi\left(t, x^*(t), u^*(t)\right) = \max_{w \in \mathscr{U}} R(t)^\top \phi\left(t, x^*(t), w\right); \tag{4.4}$$

$$p_2(0) = 0; \tag{4.5}$$

$$\begin{aligned}\left(-R(t_f), -p_2(t_f)\right) \in \Big\{ \Big(\tau \nabla_{x(t_f)} f\left(x^*(t_f), y^*(t_f)\right), \tau \nabla_{y(t_f)} f\left(x^*(t_f), y^*(t_f)\right) \Big) \Big\} \\ + \tau\beta\, \partial^c \varphi_{GF}\left(x^*(t_f), y^*(t_f)\right) + N\left(\mathscr{Z}, \left(x^*(t_f), y^*(t_f)\right)\right);\end{aligned} \tag{4.6}$$

$$\operatorname{supp}(([t_i, t_f], \mathbb{R}^n)) \subset \left\{ t \in [t_i, t_f] : g\left(t, x^*(t)\right) = 0 \right\}; \tag{4.7}$$

where $R : [t_i, t_f] \to \mathbb{R}^n$ is given by:

$$R(t) := \begin{cases} p_1(t) + \displaystyle\int_{[t_i,t[} \nabla_x g(s, x^*(s))\,\lambda(ds) & \text{if } t_i \le t < t_f, \\ p_1(t_f) + \displaystyle\int_{[t_i,t_f]} \nabla_x g(s, x^*(s))\,\lambda(ds) & \text{if } t = t_f. \end{cases}$$

Combining (4.3) and (4.5), we obtain $p_2 \equiv 0$. Then, the enhanced nontriviality condition ($\mathscr{T}_1$) follows from (4.1). Take $p \equiv p_1$. Then, the adjoint equation ($\mathscr{T}_2$) follows from (4.2) and the Weierstrass-Pontryagin condition ($\mathscr{T}_3$) follows from (4.4). Observe that condition (4.7) is the support condition $(\mathscr{T}_5)$. From (C_4), we have

$$N\left(\mathscr{Z},\left(x^*(t_f),y^*(t_f)\right)\right) \subseteq \left\{ \left(\nabla_{x(t_f)}G\left(x^*(t_f),y^*(t_f)\right)^\top\eta,\nabla_{y(t_f)}G\left(x^*(t_f),y^*(t_f)\right)^\top\eta\right) \right.$$
$$\left. \eta\geq 0,\ \eta^\top G\left(x^*(t_f),y^*(t_f)\right)=0\right\}.$$

Recall that $y^*(t_f)=y^*$. According to (4.6), there is $\xi=(\xi_1,\xi_2)^\top\in\partial^c\varphi_{GF}\left(x^*(t_f),y^*\right)$ satisfying complementarity condition $(\mathscr{T}_7)$ and $\eta_1\in\mathbb{R}^q_+$ such that

$$\begin{aligned} -R(t_f) =&\tau\left(\nabla_{x(t_f)}f\left(x^*(t_f),y^*\right)+\beta\xi_1\right)+\nabla_{x(t_f)}G\left(x^*(t_f),y^*\right)^\top\eta_1,\\ 0 =&\tau\left(\nabla_y f\left(x^*(t_f),y^*\right)+\beta\xi_2\right)+\nabla_y G\left(x^*(t_f),y^*\right)^\top\eta_1. \end{aligned} \tag{4.8}$$

Proposition 4.1 implies that there exists $l\in\mathbb{N}^*$, $(\alpha_i)_{2\leq i\leq l}\subset\mathbb{R}^+$ with $\sum\limits_{i=2}^{l}\alpha_i=1$, and for all $i\in\{2,\dots,l\}$, there exist $z_i\in\Lambda\left(x^*(t_f),y^*\right)$ and $\eta_i\in\mathscr{B}_{(x^*,y^*)}(z_i)$ satisfying variational inequality condition $(\mathscr{T}_6)$, and

$$\xi=\begin{pmatrix}\sum\limits_{i=2}^{l}\alpha_i\left[\nabla_{x(t_f)}F\left(x^*(t_f),y^*\right)^\top(y^*-z_i)-\nabla_{x(t_f)}G\left(x^*(t_f),z_i\right)^\top\eta_i\right]\\ \sum\limits_{i=2}^{l}\alpha_i\left[\nabla_y F\left(x^*(t_f),y^*\right)^\top(y^*-z_i)+F\left(x^*(t_f),y^*\right)\right]\end{pmatrix},$$

so we obtain transversality condition $(\mathscr{T}_4)$ and multiplier condition $(\mathscr{T}_8)$ from (4.8). The proof is complete. □

Example 4.3. To emphasize our findings, we take into consideration the subsequent problem:

$$\left\{\begin{array}{lll} \min\limits_{x,u,y}\ (x_1(1)+5)^2-2x_2(1)+y & & \\ \text{subject to } \dot{x}_1(t)=-2x_2(t)+u_1(t) & & \text{a.e. } t\in[0,1]\\ \qquad\qquad\quad \dot{x}_2(t)=2t^2-2t-4+u_2(t) & & \text{a.e. } t\in[0,1]\\ \qquad\qquad\quad x(0)=\left(\dfrac{1}{3},3\right)^\top & & \\ \qquad\qquad\quad -x_2(t)\leq 0 & & \forall t\in[0,1]\\ \qquad\qquad\quad u(t)\in[-1,3]\times[3,5] & & \text{a.e. } t\in[0,1]\\ \qquad\qquad\quad y\in S(x(1)), & & \end{array}\right. \tag{4.9}$$

where $S(x(1))$ is the solution set of the variational inequality: $y-z\leq 0$ for all $z\in K(x(1))$, with $K(x(1)):=\{z\in\mathbb{R}\ :\ 2x_2(1)-z\leq 0\ \text{ and }\ -z\leq 0\}$.

A comparison between problem (GBOCP) and problem (4.9) gives us:

$$
\begin{aligned}
t_i &:= 0, \\
t_f &:= 1, \\
x &:= (x_1, x_2)^\top, \\
u &:= (u_1, u_2)^\top, \\
f(x(1), y) &:= (x_1(1)+5)^2 - 2x_2(1) + y, \\
\phi(t, x(t), u(t)) &:= \left(-2x_2(t) + u_1(t), 2t^2 - 2t - 4 + u_2(t)\right)^\top, \\
x_0 &:= \left(\frac{1}{3}, 3\right)^\top, \\
g(t, x(t)) &:= -x_2(t), \\
\mathscr{U} &:= [-1,3] \times [3,5], \\
F(x(1), y) &:= 1, \\
G(x(1), y) &:= (2x_2(1) - y, -y)^\top, \\
n = m = q &:= 2 \\
d &:= 1.
\end{aligned}
$$

As we have already mentioned, $S(x(1))$ represents the solution set of the following parametric problem:

$$\min_z \langle F(x(1), y), z\rangle_{\mathbb{R}} \text{ subject to } z \in K(x(1)). \tag{4.10}$$

Then, $S(x(1))$ is given by

$$S(x(1)) := \begin{cases} \{0\} & \text{if } x_2(1) \leq 0, \\ \{2x_2(1)\} & \text{if } x_2(1) \geq 0. \end{cases}$$

Now, the point (x^*, u^*, y^*) which is defined by

$$\begin{cases} x_1^*(t) := \frac{2}{3}t^3 + t^2 - 7t + \frac{1}{3} & \forall t \in [0,1], \\ x_2^*(t) := -t^2 + t + 3 & \forall t \in [0,1], \\ u_1^*(t) := 4t - 1 & \forall t \in [0,1], \\ u_2^*(t) := -2t^2 + 5 & \forall t \in [0,1], \\ y^* := 6; \end{cases} \tag{4.11}$$

is a global minimal solution to (4.9). Thus(x^*, u^*, y^*) is a Θ-local minimal solution to (4.9). By definition of the gap function, we have

$$z^* \in S_0(x^*(1), y^*) \Leftrightarrow 0 = \langle F(x^*(1), y^*), y^* - z^*\rangle_{\mathbb{R}} \Leftrightarrow y^* = z^*.$$

Hence, $\Lambda(x^*(1), y^*) = \{6\}$.

By a simple calculation, we find that $\mathscr{A}_{(x^*,y^*)}\left(\Lambda\left(x^*(t_f), y^*\right)\right) = \{0\}$. By Proposition 4.1, we see that φ_{GF} is Lipschitz continuous near $(x^*(1), y^*)$. Problem (4.10) is fully linear. [24, Proposition 4.1] gives that problem (4.9) is partially calm at (x^*, u^*, y^*). By construction, all conditions (C_1)-(C_6) are satisfied.

Consequently, according to Theorem 4.2, there exist $p \in \mathscr{W}[2]$, an integer $l \in \mathbb{N}^*$, a finite family of scalars $(\alpha)_{2\leq i\leq l} \subset \mathbb{R}_+$ with $\sum_{i=2}^{l}\alpha_i = 1$, a finite family of vectors $(\eta_i)_{1\leq i\leq l} \subset \mathbb{R}^2_+$, a finite family of vectors $(z_i)_{1\leq i\leq l} \subset \Lambda(x^*(1),y^*)$ verified for all $i \in \{2,\dots,l\}$: $\eta_i \in \mathscr{B}_{(x^*,y^*)}(z_i)$, a measure $\lambda \in \mathscr{C}_0^p\left([0,1],\mathbb{R}^2\right)$ and some $\tau \geq 0$, $\beta > 0$ such that:

(1) the enhanced nontriviality condition: $\|R\|_{L^\infty\left([0,1],\mathbb{R}^2\right)} + \lambda\left(]0,1]\right) + \tau > 0$,

(2) the adjoint equation: for almost every $t \in [0,1]$ $\begin{pmatrix}\dot{p}_1(t)\\ \dot{p}_2(t)\end{pmatrix} = \begin{pmatrix}0\\ 2R_1(t)\end{pmatrix}$;

(3) the Weierstrass–Pontryagin condition: for almost every $t \in [0,1]$

$$\begin{aligned}R_1(t)\left(-2x_2^*(t)+u_1^*(t)\right)&+R_2(t)\left(2t^2-2t-4+u_2^*(t)\right)\\ &=\max_{w\in\mathscr{U}}\left[R_1(t)\left(-2x_2^*(t)+w_1\right)+R_2(t)\left(2t^2-2t-4+w_2\right)\right];\end{aligned}$$

(4) the transversality condition:

$$\begin{pmatrix}R_1(1)\\ R_2(1)\end{pmatrix} = 2\begin{pmatrix}0\\ \tau-\eta_{1,1}\end{pmatrix} + 2\tau\beta\sum_{i=2}^{l}\begin{pmatrix}0\\ \alpha_i\eta_{i,1}\end{pmatrix};$$

(5) the support condition: $\operatorname{supp}(\lambda) \subset \emptyset$;

(6) the variational inequality condition, for all $i \in \{2,\dots,l\}$, $1-\eta_{i,1}-\eta_{i,2} = 0$, $\eta_{i,1}(6-z_i) - \eta_{i,2}z_i = 0$;

(7) the complementarity condition: $-6\eta_{1,2} = 0$;

(8) the multiplier condition: $\tau(1+\beta) - \eta_{1,1} - \eta_{1,2} = 0$,

where the function R is defined by:

$$R(t) := \left(R_1(t),R_2(t)\right)^\top = \left(p_1(t), \begin{cases} p_2(t) - \displaystyle\int_{[0,t[}\lambda(ds) & \text{if } 0 \leq t < 1\\ p_2(1) - \displaystyle\int_{[0,1]}\lambda(ds) & \text{if } t = 1\end{cases}\right)^\top.$$

According to the complementarity condition and the multiplier condition, we find that $\eta_1 = \left(\tau(1+\beta),0\right)^\top$. Since $\Lambda(x^*(1),y^*) = \{6\}$, we have $z_i = 6$ for all $i \in \{2,\dots,l\}$. Therefore, under the variational inequality condition, we have $\eta_i = (1,0)^\top$ for all $i \in \{2,\dots,l\}$.

The function R's construction indicates $R_1 \equiv p_1$. R_1 and p_1 are both constant functions, according to the adjoint equation. The transversality condition gives us that $R(1) = 0$, therefore $R_1 \equiv p_1 \equiv 0$. Hence, p_2 is also a constant function. Let $\mu := \mu_{Leb} \circ \chi_F$ with μ_{Leb} be the Lebesgue measure and $F :=]-\infty,0[\cup]1,+\infty[$. It is easy to see that the support condition is verified by μ. Hence, we have $R_2(t) = p_2(t)$ for all $t \in [0,1]$. Consequently, from the transversality condition, we obtain $R_2(t) = p_2(t) = -\tau - 2\tau\beta$ for all $t \in [0,1]$. Finally, in our case, the choice of τ is unproblematic as long as $\tau > 0$.

5. Discussion

In this section, we show that the Pontryagin optimality conditions for problem (GBOCP) gave more information than various optimal control problems. Here, we focus on bilevel optimal control problems. More precisely, we compare our maximum principle in Theorem 4.2 to that stated for bilevel optimal control problems in [19, Theorem 4.3], which both lead to two different Pontryagin optimality conditions.

Consider the bilevel optimal control problem shown below

$$\begin{cases} \min\limits_{x,u,y} f\left(x(t_f),y\right) \\ \text{subject to } \dot{x}(t) = \phi\left(t,x(t),u(t)\right) & \text{a.e. } t \in [t_i,t_f] \\ \qquad x(0) = x_0 \\ \qquad g\left(t,x(t)\right) \leq 0 & \forall t \in [t_i,t_f] \\ \qquad u(t) \in \mathscr{U} & \text{a.e. } t \in [t_i,t_f] \\ \qquad y \in \Pi\left(x(t_f)\right), \end{cases} \tag{BOCP}$$

where $\Pi\left(x(t_f)\right)$ is the solution set of the following fully convex parametric optimization problem:

$$\min_y \ h\left(x(t_f),y\right) \quad \text{subject to} \quad y \in K\left(x(t_f)\right), \tag{$P[x(t_f)]$}$$

with $h : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$. Using the value function $V : \mathbb{R}^n \to \overline{\mathbb{R}}$, which is defined by, for all $x(t_f) \in \mathbb{R}^n$, $V\left(x(t_f)\right) := \inf\limits_y \left\{h\left(x(t_f),y\right) : y \in K\left(x(t_f)\right)\right\}$, one has that problem (BOCP) is equivalent to:

$$\begin{cases} \min\limits_{x,u,y} f\left(x(t_f),y\right) \\ \text{subject to } \dot{x}(t) = \phi\left(t,x(t),u(t)\right) & \text{a.e. } t \in [t_i,t_f] \\ \qquad x(0) = x_0 \\ \qquad g\left(t,x(t)\right) \leq 0 & \forall t \in [t_i,t_f] \\ \qquad u(t) \in \mathscr{U} & \text{a.e. } t \in [t_i,t_f] \\ \qquad h\left(x(t_f),y\right) - V\left(x(t_f)\right) \leq 0 \\ \qquad G\left(x(t_f),y\right) \leq 0. \end{cases} \tag{5.1}$$

From the KKT optimality condition, one sees that $y^* \in \Pi\left(x(t_f)\right)$ is equivalent to

$$\left\langle \nabla_y h\left(x(t_f),y^*\right), y^* - z \right\rangle \leq 0, \quad \forall z \in K\left(x(t_f)\right).$$

Consequently, bilevel optimal control problem (BOCP) can now be considered an optimal control problem with variational inequality constraints (GBOCP).

Under the hypotheses of Theorem 4.2, we have the following Pontryagin optimality conditions for problem (GBOCP):

($\mathscr{T}_1$): the enhanced nontriviality condition: $\|R\|_{L^\infty_{\left([t_i,t_f],\mathbb{R}^n\right)}} + \lambda\left(]t_i,t_f]\right) + \tau > 0$;

($\mathscr{T}_2$): the adjoint equation, for almost every $t \in [t_i,t_f]$, $-\dot{p}(t)^\top = R(t)^\top \nabla_x \phi\left(t,x^*(t),u^*(t)\right)$;

($\mathscr{T}_3$): the Weierstrass–Pontryagin condition, for almost every $t \in [t_i,t_f]$,

$$R(t)^\top \phi\left(t,x^*(t),u^*(t)\right) = \max_{w \in \mathscr{U}} R(t)^\top \phi\left(t,x^*(t),w\right);$$

($\mathscr{T}_4$): the transversality condition:

$$\begin{aligned} -R(t_f) = &\tau \Bigg(\nabla_{x(t_f)} f\left(x^*(t_f),y^*\right) + \beta \Bigg(\sum_{i=2}^{l} \alpha_i \Big[\nabla_{x(t_f)} \nabla_y h\left(x^*(t_f),y^*\right)^\top (y^* - z_i) \\ &- \nabla_{x(t_f)} G\left(x^*(t_f),z_i\right)^\top \eta_i \Big] \Bigg) \Bigg) + \nabla_{x(t_f)} G\left(x^*(t_f),y^*\right)^\top \eta_1; \end{aligned}$$

($\mathscr{T}_5$): the support condition: $\operatorname{supp}(\lambda) \subseteq \left\{t \in [t_i,t_f] : g\left(t,x^*(t)\right) = 0\right\}$;
($\mathscr{T}_6$): the variational inequality condition: for all $i \in \{2,\ldots,l\}$

$$\nabla_y h\left(x^*(t_f),y^*\right) + \nabla_y G\left(x^*(t_f),z_i\right)^\top \eta_i = 0, \quad \eta_i^\top G\left(x^*(t_f),z_i\right) = 0;$$

($\mathscr{T}_7$): the complementarity condition: $\eta_1^\top G\left(x^*(t_f),y^*\right) = 0$;
($\mathscr{T}_8$): the multiplier condition:

$$0 = \tau\left(\nabla_y f\left(x^*(t_f),y^*\right) + \beta\left(\sum_{i=2}^{l}\alpha_i\left[\nabla_y^2 h\left(x^*(t_f),y^*\right)^\top (y^* - z_i) + \nabla_y h\left(x^*(t_f),y^*\right)\right]\right)\right) + \nabla_y G\left(x^*(t_f),y^*\right)^\top \eta_1;$$

where the function $R : [t_i,t_f] \to \mathbb{R}^n$ is given by:

$$R(t) := \begin{cases} p(t) + \int_{[t_i,t[} \nabla_x g\left(s,x^*(s)\right)\lambda(ds) & \text{if } t_i \le t < t_f, \\ p(t_f) + \int_{[t_i,t_f]} \nabla_x g\left(s,x^*(s)\right)\lambda(ds) & \text{if } t = t_f. \end{cases}$$

According to our knowledge, the first finding on optimality conditions for bilevel optimal control problems was identified in [19]. These conditions were detected via the notion of Clarke subdifferential by using the same reformulation stated in (5.1). The Pontryagin optimality conditions in the latter paper can easily be recovered from Theorem 4.2 while rewriting (BOCP) as an optimal control problem with variational inequality constraints (GBOCP). Indeed,

(1) The enhanced nontriviality condition, the adjoint equation, the Weierstrass-Pontryagin condition, and the support condition are the same in Theorem 4.2 and [19, Theorem 4.3] because all these conditions are associated with only leader data.
(2) Furthermore, since both problems (GBOCP) and (BOCP) have the same lower inequality constraints, their complementarity conditions are identical.
(3) In contrast to Theorem 4.2, the transversality condition, the follower's optimality condition, and the multiplier condition of [19, Theorem 4.3] have the following forms, respectively.
   (iv) $-R\left(t_f\right) = \nabla_{x(t_f)} G\left(x^*(t_f),y^*\right)^\top (\eta_1 - \tau\beta\eta_2) + \tau\nabla_{x(t_f)} f\left(x^*(t_f),y^*\right)$;
   (vi) $\nabla_y h\left(x^*(t_f),y^*\right) + \nabla_y G\left(x^*(t_f),y^*\right)^\top \eta_2 = 0, \quad \eta_2^\top G\left(x^*(t_f),y^*\right) = 0$;
   (viii) $\tau\left(\nabla_y f\left(x^*(t_f),y^*\right) + \beta\nabla_y h\left(x^*(t_f),y^*\right)\right) + \nabla_y G\left(x^*(t_f),y^*\right)^\top \eta_1 = 0$.

It is noticeable that conditions ($\mathscr{T}_4$), ($\mathscr{T}_6$), and ($\mathscr{T}_8$) significantly differ from (iv), (vi), and ($viii$), and the fundamental reason for this is the differing nature of the gap function applied. The absence of second-order terms in Theorem [19, Theorem 4.3] is the most visible difference between the results. This discovery implies that the subdifferential estimate of the optimal value function in [19, Theorem 4.3] plays a very restricted role. Imagine a situation in which the follower of (GBOCP) has no constraints, which results in $K\left(x(t_f)\right) = \mathbb{R}^d$. Since the inequality constraints $G\left(x(t_f),y\right) \le 0$ are absent, the conditions of (iv), (vi), and ($viii$) are reduced to the following form, respectively.

(iv) $-R\left(t_f\right) = \tau\nabla_{x(t_f)} f\left(x^*(t_f),y^*\right)$;
(vi) $\nabla_y h\left(x^*(t_f),y^*\right) = 0$;
(viii) $\tau\nabla_y f\left(x^*(t_f),y^*\right) = 0$.

While Theorem 4.2 continues to provide significantly greater information regarding the existence of V such that

($\mathscr{T}_4$): the transversality condition:

$$-R(t_f) = \tau\left(\nabla_{x(t_f)} f\left(x^*(t_f), y^*\right) + \beta \sum_{i=2}^{l} \alpha_i \left[\nabla_{x(t_f)} \nabla_y h\left(x^*(t_f), y^*\right)^\top (y^* - z_i)\right]\right);$$

($\mathscr{T}_6$): the variational inequality condition: $\nabla_y h\left(x^*(t_f), y^*\right) = 0$;

($\mathscr{T}_8$): the multiplier condition:

$$0 = \tau\left(\nabla_y f\left(x^*(t_f), y^*\right) + \beta \sum_{i=2}^{l} \alpha_i \left[\nabla_y^2 h\left(x^*(t_f), y^*\right)^\top (y^* - z_i)\right]\right).$$

6. Conclusions

In this paper, we developed Pontryagin optimality conditions for a generalized bilevel optimal control problem with pure state constraints in the leader. Using the gap function, we converted the problem under consideration into a single level optimal control problem. In order to accomplish our goal, we used a variety of tools, including partial penalization. Since the gap function is not differentiable, we used a few results of nonsmooth analysis. We specifically provided an estimation of the gap function's Clarke subdifferential. We imposed other hypotheses on our problem to make sure that the derived optimality conditions were not degenerate. Finally, we discussed the case that our results indicate that the maximum principle for generalized bilevel optimal control problems using the gap function approach is more efficient than the maximum principle for bilevel optimal control problems using the optimal value function approach when (BOCP) is expressed as a (GBOCP).

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