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ITERATIVE SOLUTIONS OF SPLIT FIXED POINT AND MONOTONE INCLUSION PROBLEMS IN HILBERT SPACES

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Abstract. Our purpose in this paper is to propose an iterative method involving a step-size selected in such a way that its implementation does not require the computation or an estimate of the spectral radius. Using our algorithm, we state and prove a strong convergence theorem of a common solution to a monotone inclusion problem and a fixed point problem of multi-valued Lipschitz hemicontractive-type mappings, whose image under a bounded linear operator is a fixed point of a demicontractive mapping. Our result generalizes some important and recent results in the literature.

Keywords. Algorithm implementation; Demicontractive mapping; Equilibrium problem; Monotone inclusion; Minimization problem.

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1. Introduction

Let H be a real Hilbert space, and let C be a nonempty, convex, and closed subset of H. Let $T : C \to C$ be a mapping, and let $F(T) := \{x \in C : x = Tx\}$ denote the set of fixed points of T. Recall that T is said to be

(i) *nonexpansive* if $\|T(x) - T(y)\| \le \|x - y\|$ for all $x, y \in C$;
(ii) *quasi-nonexpansive* if $F(T)$ is not empty and $\|T(x) - q\| \le \|x - q\|$ fr all $x \in C$ and $q \in F(T)$;
(iii) μ*-demicontractive* if $F(T)$ is not empty and there exists a constant $\mu \in [0, 1)$ such that

$$\|T(x) - q\|^2 \le \|x - q\|^2 + k\|T(x) - q\|^2 \quad \forall\ x \in C, \quad q \in F(T),$$

which is equivalent to

$$\langle Tx - Tq, x - q\rangle \le \|x - q\|^2 - \frac{1-k}{2}\|Tx - x\|^2 \quad \forall\ x \in C,\ q \in F(T). \tag{1.1}$$

Let $CB(C)$ denote the family of nonempty, closed, and bounded subset of C. The Hausdorff metric on $CB(C)$ is defined by

$$\mathscr{H}(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}$$

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for all $A,B \in CB(C)$, where $d(x,B) = \inf\{\|x-b\| : b \in B\}$. A multi-valued mapping $S : C \to CB(C)$ is called Lipschitzian if there exists $L \geq 0$ such that $\mathscr{H}(Sx,Sy) \leq L\|x-y\|$ for all $x,y \in C$. If $L = 1$, then S is called a nonexpansive mapping. If $L \in (0,1)$, then S is called a contraction. An element $x \in C$ is called a fixed point of $S : C \to CB(C)$ provide $x \in Sx$. Let $T : H \to H$ be a mapping. We denote the fixed point set of T by $F(T)$, that is, $F(T) := \{x \in H : Tx = x\}$. Fixed point problems of single-valued or set-valued nonlinear operator have various applications; see, e.g., [1, 2, 3, 4, 5] and the references therein.

Recall that a mapping $S : C \to CB(C)$ is said to be a multivalued hemicontractive-type if $F(S) \neq \emptyset$ and, for all $p \in F(S)$, $x \in C$, $\mathscr{H}^2(Sx,Sp) \leq \|x-p\|^2 + \|x-u\|^2$ for all $u \in Sx$. For any point $u \in H$, there exists a unique point $P_C u \in C$ such that $\|u - P_C u\| \leq \|u-y\|$ for all $y \in C$, where P_C is called the metric projection of H onto C. We recall that P_C is nonexpansive from H onto C and satisfies $\langle x-y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2$ for all $x,y \in H$. $P_C x$ is also characterized by $\langle x - P_C x, P_C x - y\rangle \geq 0$ for all $y \in C$.

Recall that a mapping $T : H \to H$ is said to be

(i) α*-strongly monotone* if there exists a constant $\alpha > 0$ such that $\langle Tx - Ty, x-y\rangle \geq \alpha||x-y||^2$ for all $x,y \in H$;

(ii) β*-inverse strongly monotone* (β-ism) if there exists a constant $\beta > 0$ such that $\langle Tx - Ty, x-y\rangle \geq \beta||Tx-Ty||^2$ for all $x,y \in H$.

Recall that a set valued mapping $M : H \to 2^H$ is called *monotone* if, for all $x,y \in H$ with $u \in M(x)$ and $v \in M(y)$, $\langle x-y, u-v\rangle \geq 0$. A monotone mapping M is said to be *maximal* if the graph of M, denoted as $G(M)$, is not properly contained in the graph of any other monotone mapping, where for multi-valued mapping M, $G(M) = \{(x,y) : y \in M(x)\}$. It is known that M is maximal if and only if, for $(x,u) \in H \times H$, $\langle x-y, u-v\rangle \geq 0$ for all $(y,v) \in G(M)$ implies $u \in M(x)$. Its resolvent operator with λ, introduced by Moreau [6], is the mapping $J_\lambda^M : H \to H$ defined by $J_\lambda^M(x) = (I+\lambda M)^{-1}x$ for all $x \in H, \lambda > 0$. One knows that $J_\lambda^M(x)$ is single-valued, nonexpansive, and 1-inverse strongly monotone. The inverse-strongly monotone (also referred as co-coercive) operators were widely used to solve optimization problems; see, e.g., [7, 8, 9, 10, 11] and the references therien. It can be easily seen that (i) if T is nonexpansive, then $I - T$ is monotone; (ii) the projection mapping P_C is 1-ism.

A fundamental problem is to find a zero of a maximal monotone operator $M : H \to 2^H$ in real Hilbert space H. That is,

$$\text{find } \ x \in H : \quad 0 \in Mx. \tag{1.2}$$

It includes non-smooth convex optimization problems and convex-concave saddle-point problems as special cases and finds various applications in machine learning. It is known that the solution of (1.2) is a fixed point of J_λ^M and the set $M^{-1}(0) := \{x \in H : 0 \in Tx\}$ is closed and convex. The classical algorithm to solve (1.2) is the proximal point algorithm, which can be traced back to Minty [12] and Martinet [13]. The proximal point algorithm generates a sequence $\{x_n\}$ by $x_{n+1} = (I+\lambda_n M)^{-1}(x_n)$, where λ_n is a positive regularization parameter. Recall that Rockafellar [14] proved that the sequence $\{x_n\}$ generated by the proximal point algorithm converges weakly to a point x^* with $0 \in Mx^*$. To reduce the computational complexity, T can be written as the sum of two monotone operators, i.e., $T = M + B$, where $(I+\lambda A)^{-1}$ or $(I+\lambda B)^{-1}$ is easier to compute than $(I+\lambda T)^{-1}$. Let us recall two splitting algorithm: the Peaceman-Rachford splitting algorithm [15],

$$x_{n+1} = (I+\lambda B)^{-1}(I-\lambda A)(I+\lambda M)^{-1}(I-\lambda B)(x_n),$$

and the Douglas-Rachford splitting algorithm [16],

$$x_{n+1} = (I+\lambda B)^{-1}[(I+\lambda M)^{-1}(I-\lambda B)+\lambda B](x_n).$$

Observe that the two splitting algorithm were originally proposed in the context of linear operators and systems. In [17], Lion and Mercier analysed and developed the splitting algorithms. Their idea was to perform a change of variables $x_n = (I+\lambda B)^{-1}(v_n)$ such that the Peaceman-Rachford and Douglas-Rachford splitting algorithms are efficient for A and B being multi-valued operators. Regarding convergence of the algorithms, the Peaceman-Rachford algorithm still needs to assume that B is single-valued but the Douglas-Rachford algorithm converges even in the general setting, where $M+B$ is just maximally monotone. Another important line of splitting methods was given by the so-called forward-backward splitting technique [17, 18]. In contrast to the more complicated splitting technique discussed above, the forward-backward scheme is only based on the recursive application of an explicit forward step with respect to B, followed by an implicit backward step with respect to M. The forward-backward algorithm is written as: $x_{n+1} = (I+\lambda_n M)^{-1}(I-\lambda_n B)(x_n)$. In the most general setting, the convergence result is rather weak [19] if both M and B are general monotone operators. Basically, λ_n has to fulfil the same step-size restrictions as unconstrained subgradient descend schemes. In addition, if B is single-valued and Lipschtz, that is, B is the gradient of a smooth convex function, the situation becomes much more beneficial. In fact, if B is L- Lipschitz, and λ_n is chosen such that $\lambda_n < \frac{2}{L}$, the forward-backward algorithm converges to zero of $T = M+B$ [20, 21]. Recently, the forward-backward algorithm is under the spotlight of research. It has been proposed and further improved in the context of sparse signal recovery, image processing, and machine learning. One refers to [22, 23, 24, 25] for various modifications of the modifications of forward-backward algorithm.

Let C be a nonempty, convex, and closed set in a Hilbert space H_1, and let Q a nonempty, convex, and closed set in Hilbert space H_2. Let $A : H_1 \to H_2$ be an operator, which is assumed to be both bounded and linear. A Split Feasibility Problem (SFP) is to find a point x in C with Ax in Q. The SFP was first introduced by Censor and Elfving [26] for the problems arising from medical image reconstruction. Moreover, it has been found that the SFP can also be used in image restoration, computer tomograph, and radiation therapy treatment planning [27, 28]. In the past decade, various efficient solution methods were devised and investigated for solving the SFP and its related optimization problems; see, e.g, [29, 30, 31, 32, 33] and the references therein.

Let $B : H_1 \to 2^{H_1}$ be a multivalued mapping, and let $T : H_2 \to H_2$ be a single-valued mapping. Let $A : H_1 \to H_2$ be a bounded linear operator. Recently, Takahashi et al. [33] studied the following problems: find $x \in H_1$ such that $0 \in B(x)$ and $Ax \in F(T)$. We denote its solution set by Ω, that is, $\Omega := \{x \in H_1 : 0 \in B(x) \text{ and } Ax \in F(T)\}$. Takahashi et al. [33] stated and proved the following two weak convergence results.

Theorem 1.1. *([33]) Let H_1 and H_2 be Hilbert spaces. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping and let $J_\lambda^B = (I+\lambda B)^{-1}$ be the resolvent for B for $\lambda > 0$. Let $T : H_2 \to H_2$ be a nonexpansive mapping and $A : H_1 \to H_2$ a bounded linear operator. Suppose $B^{-1}(0) \cap A^{-1}F(T) \neq \emptyset$. For any $x_1 = x \in H_1$, define $x_{n+1} = J_{\lambda_n}^B(I-\gamma_n A^*(I-T)A)x_n$ for all $n \in \mathbb{N}$, where the sequences $\{\lambda_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:*

(i) $0 < \liminf_{n\to\infty} \lambda_n \limsup_{n\to\infty} \lambda_n < \infty$,
(ii) $0 < \liminf_{n\to\infty} \gamma_n \limsup_{n\to\infty} \gamma_n < \frac{1}{\|A\|^2}$.

Then $\{x_n\}$ converges weakly to a point $z_0 \in B^{-1}(0) \cap A^{-1}F(T)$, which is a strong limit of the projections of $\{x_n\}$ onto $B^{-1}(0) \cap A^{-1}F(T)$, that is $z_0 = \lim_{n\to\infty} P_{B^{-1}(0)\cap A^{-1}F(T)} x_n$.

Theorem 1.2. [33] *Let H_1 and H_2 be Hilbert spaces. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping and let $J_\lambda^B = (I + \lambda B)^{-1}$ be the resolvent for B for $\lambda > 0$. Let $T : H_2 \to H_2$ be a nonexpansive mapping and $A : H_1 \to H_2$ a bounded linear operator. Suppose $B^{-1}(0) \cap A^{-1}F(T) \neq \emptyset$. For any $x_1 = x \in H_1$, define $x_{n+1} = \beta_n x_n + (1-\beta_n) J_{\lambda_n}^B (I - \gamma_n A^*(I-T)A)x_n$ for all $n \in \mathbb{N}$, where the sequences $\{\beta_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0,\infty)$ satisfy the conditions:*

(a1) $\sum_{n=1}^\infty \beta_n(1-\beta_n) < \infty$,

(a2) $0 < a \leq \gamma_n \leq \frac{1}{\|A\|^2}$ *and* $\sum_{n=1}^\infty |\lambda_n - \lambda_{n+1}| < \infty$.

Then $x_n \rightharpoonup z_0 \in B^{-1}(0) \cap A^{-1}F(T)$, where $z_0 = \lim_{n\to\infty} P_{B^{-1}(0)\cap A^{-1}F(T)} x_n$.

In this paper, we introduce an iterative algorithm and proved a strong convergence theorem for finding a fixed point of a multi-valued Lipschitz hemicontractive-type mapping, which is also a solution to monotone variational inclusion problem (1.2), where $T = M + B$ with M being a maximal monotone operator and B an α-inverse strongly monotone mapping and whose image under a bounded linear operator is a fixed point of a demicontractive mapping. In our result, the step-size is selected in such a way that its implementation does not involve the computation or an estimate of the operator norm. Hence Our result improve and extend many known results in this direction.

2. Preliminaries

In this section, we give some definitions, lemmas, and results that are needed in the main results.

Let H be a real Hilbert space. For all $x_i \in H$ and $\alpha_i \in [0,1]$ for $i = 1,2,...,n$ such that $\alpha_1 + \alpha_2 + ... + \alpha_n = 1$, the following equality holds:

$$\|\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n\|^2 = \sum_{i=1}^n \alpha_i \|x_i\|^2 - \sum_{1\leq i,j\leq n} \alpha_i \alpha_j \|x_i - x_j\|^2.$$

One the other hand, one also has the following celebrated identities

(i) $\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x,y\rangle$;

(ii) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$;

(iii) $\|\lambda x + (1-\lambda)y - z\|^2 = \lambda\|x-z\|^2 + (1-\lambda)\|y-z\|^2 - \lambda(1-\lambda)\|x-y\|^2$, for any $\lambda \in (0,1)$, $x,y,z \in H$.

Lemma 2.1. [34] *Let C be a nonempty, convex, and closed set in a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping. Then $I - T$ is demiclosed at 0, (i.e., if $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$, then $x = Tx$).*

Lemma 2.2. [35] *Let H be a real Hilbert space. Let $M : H \to 2^H$ be a maximal monotone operator, and let $B : H \to H$ be an α-inverse strongly monotone mapping. Then*

(i) for $r > 0$, $F(T_r) = (M+B)^{-1}(0) := \{x \in H : 0 \in (M+B)x\}$,

(ii) for $0 < s \leq r$ and $x \in E$, $||x - T_s x|| \leq 2||x - T_r x||$, where $T_r := (I + rM)^{-1}(I - rB) = J_r^M(I - rB)$.

Lemma 2.3. [36] *Let H be a Hilbert space. Let $A, B \in CB(H)$ and $a \in A$. Then, for $\varepsilon > 0$, there exists a point $b \in B$ such that $\|a - b\| \leq \mathscr{H}(A,B) + \varepsilon$. If $\varepsilon = \mathscr{H}(A,B)$, then $\|a-b\| \leq 2\mathscr{H}(A,B)$.*

Lemma 2.4. [37] *Let $\{a_n\}$ be a sequence of non-negative real numbers such that $a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \delta_n$ for all $n \geq 0$, where $\limsup \sigma_n \leq 0$, $\{\alpha_n\} \subset [0,1]$, $\sum_{n=0}^{\infty}\alpha_n = \infty$, $\delta_n \geq 0$, and $\sum_{n=0}^{\infty}\delta_n < \infty$. Then $a_n \to 0$ as $n \to \infty$.*

Lemma 2.5. [38] *Let $\{a_n\}$ be a real sequence with its subsequence $\{n_j\}$ of $\{n\}$ such that $a_{n_j} < a_{n_j+1}$ for all $j \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) number $k \in \mathbb{N}$: $a_{m_k} \leq a_{m_k+1}$ and $a_k \leq a_{m_k+1}$. In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.*

3. MAIN RESULTS

We now state and prove the following theorem.

Theorem 3.1. *Let H_1 and H_2 be two real Hilbert spaces, and let C be a nonempty, convex and closed subset of H_1. Let $A : H_1 \to H_2$ be a bounded and linear operator, and let A^* be the adjoint of A. Let $M : H_1 \to 2^{H_1}$ be a maximal monotone operator, and let $B : C \to H_1$ be a τ-inverse strongly monotone mapping. Let $S : H_1 \to CB(H_1)$ be a L-Lipschitz hemicontractive-type mapping, and let $T : H_2 \to H_2$ be an μ-demicontractive mapping such that $\Upsilon := F(S) \cap (M+B)^{-1}(0) \cap A^{-1}F(T) \neq \emptyset$. Let the step size γ_n be chosen such that for some $\varepsilon > 0$, $\gamma_n \in \left(\varepsilon, \frac{(1-\mu)\|TAx_n - Ax_n\|^2}{\|A^*(T-I)Ax_n\|^2} - \varepsilon\right)$, if $TAx_n \neq Ax_n$; otherwise $\gamma_n = \gamma$ (γ being any nonnegative real number). Suppose that $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\delta_n\}$ are sequences in $(0,1)$, and the following conditions are satisfied:*

(i) $\alpha_n + \beta_n + \delta_n = 1$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty}, \alpha_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\tau$;
(iv) $\alpha_n + \beta_n \leq \lambda_n \leq \lambda \leq \frac{1}{\sqrt{1+4L^2}+1}$,
(v) $Sp = \{p\}$ $\forall p \in \Upsilon$, $(I-T)$ *and* $(I-S)$ *are demiclosed at* 0.

Then the sequence $\{x_n\}$ generated below for any $x_1, u \in H_1$ by

$$\begin{cases} w_n = P_C(x_n + \gamma_n A^*(T-I)Ax_n), \\ z_n = (I + r_nM)^{-1}(w_n - r_nBw_n), \\ y_n = (1-\lambda_n)z_n + \lambda_n u_n, \\ x_{n+1} = \alpha_n u + \beta_n v_n + \delta_n z_n, \quad n \geq 1, \end{cases} \tag{3.1}$$

where $u_n \in Sz_n$, and $v_n \in Sy_n$ converges strongly to $q \in \Upsilon$ where $q = P_\Upsilon u$.

Proof. We first demonstrate that $\{x_n\}$ is a bounded sequence. Let $q = P_\Upsilon u$, and define $T_n := J_{r_n}^M(I - r_nB)$ for all $n \geq 1$. Then, T_n is nonexpansive for all $n \geq 1$ and

$$\|z_n - q\| = \|T_nw_n - T_nq\| \leq \|w_n - q\|.$$

From (1.1), (3.1), and $q \in \Upsilon$, we have

$$\begin{aligned}
\|w_n - q\|^2 &= \|P_C(x_n + \gamma_n A^*(T-I)Ax_n) - q\|^2 \\
&= \|x_n - q\|^2 + \gamma_n^2 \|A^*(T-I)Ax_n\|^2 + 2\gamma_n \langle x_n - q, A^*(T-I)Ax_n \rangle \\
&= \|x_n - q\|^2 + \gamma_n^2 \|A^*(T-I)Ax_n\|^2 \\
&\quad + 2\gamma_n [\langle Ax_n - Ap, TAx_n - Aq \rangle + \langle Ax_n - Aq, Aq - Ax_n \rangle] \\
&= \|x_n - q\|^2 + \gamma_n^2 \|A^*(T-I)Ax_n\|^2 \\
&\quad + 2\gamma_n [\langle Ax_n - Aq, TAx_n - Aq \rangle - \|Ax_n - Aq\|^2] \\
&\leq \|x_n - q\|^2 + \gamma_n^2 \|A^*(T-I)Ax_n\|^2 \\
&\quad + 2\gamma_n [\|Ax_n - Aq\|^2 - \frac{(1-\mu)}{2} \|TAx_n - Ax_n\|^2 - \|Ax_n - Aq\|^2] \\
&= \|x_n - q\|^2 + \gamma_n [\gamma_n \|A^*(T-I)Ax_n\|^2 + (\mu - 1)\|TAx_n - Ax_n\|^2]. \qquad (3.2)
\end{aligned}$$

From the choice of γ_n and (3.2), we see that $\|w_n - q\|^2 \leq \|x_n - q\|^2$. Since S is a hemicontractive-type mapping and $u_n \in Sz_n$, we obtain from (3.1) that

$$\begin{aligned}
\|y_n - q\|^2 &= (1-\lambda_n)\|z_n - q\|^2 + \lambda_n \|u_n - q\|^2 - \lambda_n(1-\lambda_n)\|z_n - u_n\|^2 \\
&\leq (1-\lambda_n)\|z_n - q\|^2 + \lambda_n \mathscr{H}^2(Sz_n, Sq) - \lambda_n(1-\lambda_n)\|z_n - u_n\|^2 \\
&\leq (1-\lambda)\|z_n - q\|^2 + \lambda_n \left(\|z_n - q\|^2 + \|z_n - u_n\|^2\right) - \lambda_n(1-\lambda_n)\|z_n - u_n\|^2 \\
&= \|z_n - q\|^2 + \lambda_n \|z_n - u_n\|^2 - \lambda_n(1-\lambda_n)\|z_n - u_n\|^2 \\
&\leq \|x_n - q\|^2 + \lambda_n^2 \|z_n - u_n\|^2.
\end{aligned}$$

Since S is hemicontractive-type and $v_n \in Sy_n$, we conclude from (3.1) that

$$\begin{aligned}
\|v_n - q\|^2 &= (d(v_n, Sq))^2 \leq \mathscr{H}^2(Sy_n, Sq) \\
&\leq \|y_n - q\|^2 + \|y_n - v_n\|^2 \\
&\leq \|x_n - q\|^2 + \lambda_n^2 \|z_n - u_n\|^2 + \|y_n - v_n\|^2. \qquad (3.3)
\end{aligned}$$

In view of (3.1), we have

$$\|z_n - y_n\|^2 = \|z_n - ((1-\lambda_n)z_n + \lambda_n u_n)\|^2 = \lambda_n^2 \|z_n - u_n\|^2. \qquad (3.4)$$

Since S is L-Lipschitzian mapping and $\|u_n - v_n\| \leq 2\mathscr{H}(Sz_n, Sy_n)$, we obtain from (3.4) that

$$\begin{aligned}
\|y_n - v_n\|^2 &= (1-\lambda_n)\|z_n - v_n\|^2 + \lambda_n \|u_n - v_n\|^2 - \lambda_n(1-\lambda_n)\|z_n - u_n\|^2 \\
&\leq (1-\lambda_n)\|z_n - v_n\|^2 + \lambda_n \|u_n - v_n\|^2 - \lambda_n(1-\lambda_n)\|z_n - u_n\|^2 \\
&\leq (1-\lambda_n)\|z_n - v_n\|^2 + 4\lambda_n \mathscr{H}^2(Tz_n, Ty_n) - \lambda_n(1-\lambda_n)\|z_n - u_n\|^2 \\
&\leq (1-\lambda_n)\|z_n - v_n\|^2 + 4\lambda_n^3 L^2 \|z_n - u_n\|^2 - \lambda_n(1-\lambda_n)\|z_n - u_n\|^2. \qquad (3.5)
\end{aligned}$$

Hence, substituting (3.5) into (3.3), we obtain that

$$\|v_n - q\|^2 = \|x_n - q\|^2 + (1-\lambda_n)\|z_n - v_n\|^2 + \lambda_n(4L^2\lambda_n^2 + 2\lambda_n - 1)\|z_n - u_n\|^2. \qquad (3.6)$$

Thus, from (3.1) and (3.6), we have that

$$\begin{aligned}
& \|x_{n+1}-q\|^2 \\
= \;& \alpha_n\|u-q\|^2+\beta_n\|v_n-q\|^2+\delta_n\|z_n-q\|^2-\beta_n\delta_n\|z_n-v_n\|^2 \\
\leq \;& \|u-q\|^2+\beta_n\left[\|x_n-q\|^2+(1-\lambda_n)\|z_n-v_n\|^2+\lambda_n(4L^2\lambda_n^2+2\lambda_n-1)\|z_n-u_n\|^2\right] \\
& +\delta_n\|z_n-q\|^2-\beta_n\delta_n\|z_n-v_n\|^2 \\
\leq \;& \alpha_n\|u-q\|^2+(1-\alpha_n)\|x_n-q\|^2+\beta_n(1-\delta_n-\lambda_n)\|z_n-v_n\|^2 \\
& -\beta_n\lambda_n(1-4L^2\lambda_n^2-2\lambda_n)\|z_n-u_n\|^2. \qquad (3.7)
\end{aligned}$$

From assumption (iv), we have

$$1-4L^2\lambda_n^2-2\lambda_n \geq 1-4L^2\lambda^2-2\lambda > 0 \text{ and } (\alpha_n+\beta_n)-\lambda_n \leq 0, \quad \forall n \geq 1. \qquad (3.8)$$

Therefore, from (3.7) and (3.8), we have

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\leq \alpha_n\|u-q\|^2+(1-\alpha_n)\|x_n-q\|^2 \\
&\leq \max\{\|u-q\|^2, \|x_n-q\|^2\} \\
&\;\;\vdots \\
&\leq \max\{\|u-q\|^2, \|x_1-q\|^2\}.
\end{aligned}$$

Hence, $\{x_n\}$ is bounded. It follows from (3.1) that

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \|\alpha_n u+\beta_n v_n+\delta_n z_n-q\|^2 \\
&\leq \|\beta_n(v_n-q)+\delta_n(z_n-q)\|^2+2\alpha_n\langle u-q, x_{n+1}-q\rangle \\
&\leq \beta_n\|v_n-q\|^2+\delta_n\|z_n-q\|^2-\beta_n\delta_n\|z_n-u_n\|^2+2\alpha_n\langle u-q, x_{n+1}-q\rangle \\
&\leq \beta_n\left[\|x_n-q\|^2+\lambda_n(4L^2\lambda_n^2+2\lambda_n-1)\|z_n-u_n\|^2+(1-\lambda_n)\|z_n-v_n\|^2\rangle\right] \\
&\quad +\delta_n\|x_n-q\|^2-\beta_n\delta_n\|z_n-u_n\|^2+2\alpha_n\langle u-q, x_{n+1}-q\rangle \\
&= (1-\alpha_n)\|x_n-q\|^2-\beta_n\lambda_n(1-4\lambda_n^2L^2-2\lambda_n)\|z_n-u_n\|^2 \\
&\quad +\beta_n(\alpha_n+\beta_n-\lambda_n)\|z_n-v_n\|^2+2\alpha_n\langle u-q, x_{n+1}-q\rangle, \qquad (3.9)
\end{aligned}$$

which implies that

$$\|x_{n+1}-q\|^2 \leq (1-\alpha_n)\|x_n-q\|^2+2\alpha_n\langle u-q, x_{n+1}-q\rangle. \qquad (3.10)$$

Now, to obtain the strong convergence, we divide the proof into two cases

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n-q\|\}$ is decreasing for all $n \geq n_0$. Thus $\{\|x_n-q\|\}$ is convergent, and

$$\|x_{n+1}-q\|^2-\|x_n-q\|^2 \to 0, \quad n \to \infty. \qquad (3.11)$$

In view of (3.8) and (3.9), we have that

$$\beta_n\lambda_n(1-4L^2\lambda_n^2-2\lambda_n)\|z_n-u_n\|^2 \leq (1-\alpha_n)\|x_n-q\|^2-\|x_{n+1}-q\|^2+2\alpha_n\langle u-q, x_{n+1}-q\rangle.$$

Form (3.8), (3.11), and the fact that $\alpha_n \to 0$ as $n \to \infty$, we have that $\|z_n-u_n\| \to 0$ as n tends to ∞, which implies that

$$d(z_n, Sz_n) \leq \|z_n-u_n\| \to 0 \;\; as \;\; n \to \infty. \qquad (3.12)$$

Since S is Lipschitz, we obtain from (3.4) that

$$\begin{aligned} \|z_n - v_n\| &\leq \|z_n - u_n\| + \|u_n - v_n\| \\ &\leq \|z_n - u_n\| + 2L\|z_n - y_n\| = \|z_n - u_n\| + 2L\lambda_n\|z_n - u_n\| \to 0 \end{aligned} \tag{3.13}$$

as $n \to \infty$. If $TAx_n \neq Ax_n$, then $\gamma_n \in \left(\varepsilon, \frac{(1-\mu)\|TAx_n - Ax_n\|^2}{\|A^*(T-I)Ax_n\|^2} - \varepsilon\right)$. From (3.2), we have

$$\begin{aligned} \|w_n - q\|^2 &\leq \|x_n - q\|^2 + \gamma_n[\gamma_n\|A^*(T-I)Ax_n\|^2 + (\mu - 1)\|TAx_n - Ax_n\|^2] \\ &\leq \|x_n - q\|^2 - \gamma_n\varepsilon\|A^*(T-I)Ax_n\|^2. \end{aligned} \tag{3.14}$$

By using (3.1), (3.6), and (3.14), we have

$$\begin{aligned} &\|x_{n+1} - q\|^2 \\ \leq\;& \alpha_n\|u - q\|^2 + \beta_n\|v_n - q\|^2 + \delta_n\|z_n - q\|^2 - \beta_n\delta_n\|z_n - v_n\|^2 \\ \leq\;& \alpha_n\|u - q\|^2 + \delta_n\|w_n - q\|^2 - \delta_n\beta_n\|z_n - v_n\|^2 \\ &+\beta_n\left[\|x_n - q\|^2 + (1-\lambda_n)\|z_n - v_n\|^2 + \lambda_n(4L^2\lambda_n^2 + 2\lambda_n - 1)\|z_n - u_n\|^2\right] \\ \leq\;& \alpha_n\|u - q\|^2 + \beta_n\|x_n - q\|^2 + \beta_n(1-\lambda_n)\|z_n - v_n\|^2 - \delta_n\beta_n\|z_n - v_n\|^2 \\ &+\beta_n\lambda_n(4L^2\lambda^2 + 2\lambda_n - 1)\|z_n - u_n\|^2 + \delta_n\|x_n - q\|^2 \\ &-\delta_n\gamma_n\varepsilon\|A^*(T-I)Ax_n\|^2, \end{aligned} \tag{3.15}$$

which implies that

$$\begin{aligned} &\delta_n\gamma_n\varepsilon\|A^*(T-I)Ax_n\|^2 \\ \leq\;& \alpha_n\|u - q\|^2 - (1-\alpha_n)\|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\ &+\beta_n(1-\delta_n-\lambda_n)\|z_n - v_n\|^2 + \beta_n\lambda_n(4L^2\lambda_n^2 + 2\lambda_n - 1)\|z_n - u_n\|^2. \end{aligned} \tag{3.16}$$

From (3.11), (3.13), and the fact that $\alpha_n \to 0$ as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \|A^*(T-I)Ax_n\| = 0. \tag{3.17}$$

It also from (3.14) and (3.15) that

$$\begin{aligned} &\delta_n\gamma_n\|TAx_n - Ax_n\|^2 \\ \leq\;& \alpha_n\|u - q\|^2 + (1-\alpha_n)\|x_n - q\|^2 - \|x_{n+1} - q\|^2 + \beta_n(1-\delta_n-\lambda_n)\|z_n - v_n\|^2 \\ &+\beta_n\lambda_n(4L^2\lambda_n^2 + 2\lambda_n - 1)\|z_n - u_n\|^2 + \delta_n\gamma_n\|A^*(T-I)Ax_n\|^2. \end{aligned} \tag{3.18}$$

From (3.11), (3.13), (3.17), and the fact that $\alpha_n \to 0$ as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \|TAx_n - Ax_n\| = 0. \tag{3.19}$$

Now, from (3.1), we have

$$\begin{aligned} \|w_n - q\|^2 &\leq \langle w_n - q, x_n + \gamma_n A^*(T-I)Ax_n - q\rangle \\ &= \frac{1}{2}[\|w_n - q\|^2 + \|x_n - q\|^2 + \gamma_n[\gamma_n\|A^*(T-I)Ax_n\|^2 \\ &\quad +(\mu-1)\|TAx_n - Ax_n\|^2] - \|w_n - x_n - \gamma_n A^*(T-I)Ax_n - q\|^2] \\ &\leq \frac{1}{2}[\|w_n - q\|^2 + \|x_n - q\|^2 - \|w_n - x_n\|^2 + \gamma_n\|Aw_n - Ax_n\|\|(T-I)Ax_n\|]. \end{aligned}$$

That is,

$$\|w_n - q\|^2 \leq \|x_n - q\|^2 - \|w_n - x_n\|^2 + 2\gamma_n \|Aw_n - Ax_n\| \|(T-I)Ax_n\|. \tag{3.20}$$

From (3.16) and (3.20), we obtain

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
\leq\ & \alpha_n \|u-q\|^2 + \beta_n \left[\|x_n - q\|^2 + (1-\lambda_n)\|z_n - v_n\|^2 + \lambda_n(4L^2\lambda_n^2 + 2\lambda_n - 1)\|z_n - u_n\|^2\right] \\
& + \delta_n \|w_n - q\|^2 - \delta_n \beta_n \|z_n - v_n\|^2 \\
\leq\ & \alpha_n \|u-q\|^2 + \beta_n \|x_n - q\|^2 + \beta_n(1-\delta_n - \lambda_n)\|z_n - v_n\|^2 \\
& + \beta_n \lambda_n (4L^2\lambda_n^2 + 2\lambda_n - 1)\|z_n - u_n\|^2 + \delta_n \|x_n - q\|^2 - \delta_n \|w_n - x_n\|^2 \\
& + \delta_n \gamma_n \|Aw_n - Ax_n\| \|(T-I)Ax_n\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
\delta_n \|w_n - x_n\|^2 \leq\ & \alpha_n \|u-q\|^2 + (1-\alpha_n)\|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\
& + \beta_n (1-\delta_n - \lambda_n)\|z_n - v_n\|^2 + \beta_n \lambda_n (4L^2\lambda_n^2 + 2\lambda_n - 1)\|z_n - u_n\|^2 \\
& + \delta_n \gamma_n \|Aw_n - Ax_n\| \|(T-I)Ax_n\|.
\end{aligned}$$

From (3.11), (3.13), and the fact that $\alpha_n \to 0$ as $n \to \infty$, we obtain $\lim_{n\to\infty} \|w_n - x_n\| = 0$. Observe that

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
\leq\ & \alpha_n \|u-q\|^2 + \beta_n \|v_n - q\|^2 + \delta_n \|z_n - q\|^2 - \delta_n \beta_n \|z_n - v_n\|^2 \\
\leq\ & \alpha_n \|u-q\|^2 + \beta_n \left[\|x_n - q\|^2 + (1-\lambda_n)\|z_n - v_n\|^2 + \lambda_n(4L^2\lambda_n^2 + 2\lambda_n - 1)\|z_n - u_n\|^2\right] \\
& + \delta_n \left[\|(w_n - r_n Bw_n) - (q - r_n Bq)\|^2\right] - \beta_n \delta_n \|z_n - v_n\|^2 \\
=\ & \alpha_n \|u-q\|^2 + \beta_n (1-\delta_n - \lambda_n)\|z_n - v_n\|^2 + \beta_n \lambda_n (4L^2\lambda_n^2 + 2\lambda_n - 1)\|z_n - u_n\|^2 \\
& + \beta_n \|x_n - q\|^2 + \delta_n \left[\|w_n - q\|^2 - 2r_n \langle w_n - q, Bw_n - Bq\rangle + r_n^2 \|Bw_n - Bq\|^2\right] \\
\leq\ & \alpha_n \|u-q\|^2 + (1-\alpha_n)\|x_n - q\|^2 + \delta_n r_n (r_n - 2\tau)\|Bw_n - Bq\|^2 \\
& + \beta_n (1-\delta_n - \lambda_n)\|z_n - v_n\|^2 + \beta_n \lambda_n (4L^2\lambda_n^2 + 2\lambda_n - 1)\|z_n - u_n\|^2,
\end{aligned}$$

and then

$$\begin{aligned}
&\delta_n r_n (2\tau - r_n)\|Bw_n - Bq\|^2 \\
\leq\ & \alpha_n \|u-q\|^2 + (1-\alpha_n)\|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\
& + \beta_n (1-\delta_n - \lambda_n)\|z_n - v_n\|^2 + \beta_n \lambda_n (4L^2\lambda_n^2 + 2\lambda_n - 1)\|z_n - u_n\|^2.
\end{aligned}$$

Condition (iii) yields that $\lim_{n\to\infty}\|Bw_n - Bq\| = 0$. Observe that $(I - r_nB)$ is nonexpansive and $J^M_{r_n}$ is firmly nonexpansive mapping. Thus

$$\begin{aligned}
\|z_n - q\|^2 &= \|J^M_{r_n}(q - r_nBq) - J^M_{r_n}(w_n - r_nBw_n)\|^2 \\
&\leq \langle (w_n - r_nBw_n) - (q - r_nBq, z_n - q\rangle \\
&= \frac{1}{2}\Big[\|w_n - r_nBw_n - (q - r_nBq)\|^2 + \|z_n - q\|^2 \\
&\quad - \|(w_n - r_nBw_n) - (q - r_nBq) - (z_n - q)\|^2\Big] \\
&\leq \frac{1}{2}\Big[\|w_n - q\|^2 + \|z_n - q\|^2 - \|(w_n - z_n) - r_n(Bw_n - Bq)\|^2\Big] \\
&= \frac{1}{2}\Big[\|w_n - q\|^2 + \|z_n - q\|^2 - \|w_n - z_n\|^2 + 2r_n\langle w_n - z_n, Bw_n - Bq\rangle \\
&\quad - r_n^2\|Bw_n - Bq\|^2\Big].
\end{aligned}$$

Therefore

$$\|z_n - q\|^2 \leq \|w_n - q\|^2 - \|w_n - z_n\|^2 + 2r_n\langle w_n - z_n, Bw_n - Bq\rangle - r_n^2\|Bw_n - Bq\|^2,$$

which together with (3.1) yields

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\leq \alpha_n\|u_n - q\|^2 + (1-\alpha_n)\|x_n - q\|^2 + \beta_n(1 - \delta_n - \lambda_n)\|z_n - v_n\|^2 \\
&\quad + \beta_n\lambda_n(4L^2\lambda_n^2 + 2\lambda_n - 1)\|z_n - u_n\|^2 - \delta_n\|w_n - z_n\|^2 \\
&\quad + 2\delta_n r_n\|w_n - z_n\|\|Bw_n - Bq\| - r_n^2\|Bw_n - Bq\|^2.
\end{aligned}$$

Thus

$$\begin{aligned}
\delta_n\|w_n - z_n\|^2 &\leq \alpha_n\|u - q\|^2 + (1-\alpha_n)\|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\
&\quad + \beta_n(1 - \delta_n - \lambda_n)\|z_n - v_n\|^2 + \beta_n\lambda_n(4L^2\lambda_n^2 + 2\lambda_n - 1)\|z_n - u_n\|^2 \\
&\quad + 2\delta_n r_n(\|w_n\| + \|z_n\|)\|Bw_n - Bq\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$ and both $\{z_n\}$ and $\{w_n\}$ are bounded, we have

$$\lim_{n\to\infty}\|T_nw_n - w_n\| = \lim_{n\to\infty}\|w_n - z_n\| = 0. \tag{3.21}$$

It follows from (3.21) that

$$\|z_n - x_n\| \leq \|z_n - w_n\| + \|w_n - x_n\| \to 0, \quad n \to \infty. \tag{3.22}$$

The fact that $\alpha_n \to 0$ as $n \to \infty$, (3.1), and (3.22) demonstrate that

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \|x_{n+1} - z_n\| + \|z_n - x_n\| \\
&= \alpha_n\|u - z_n\| + \beta_n\|v_n - z_n\| + \|z_n - x_n\| \to 0, \text{ as } n \to \infty.
\end{aligned} \tag{3.23}$$

Now, let $z = P_\Upsilon u$. We claim that $\lim_{n\to\infty}\langle u - z, x_{n+1} - z\rangle \leq 0$. Since $\{x_{n+1}\}$ is a bounded vector sequence in a Hilbert space H, which is a reflexive space, there exists a subsequence $\{x_{n_j+1}\}$ of $\{x_{n+1}\}$ and an element in H, say p, such that

$$x_{n_j+1} \rightharpoonup p \ \text{ and } \ \limsup_{n\to\infty}\langle u - z, x_{n+1} - z\rangle = \lim_{j\to\infty}\langle u - z, x_{n_j+1} - z\rangle.$$

Since C is weakly closed, we have $p \in C$ and from (3.23), which demonstrates that $x_{n_j} \rightharpoonup p$ as $j \to \infty$. From (3.22), we obtain that $z_{n_j} \rightharpoonup p$ as $j \to \infty$. Using the fact that $(I - S)$ is demiclosed at zero and (3.12), we conclude that $p \in F(S)$. Since $\liminf_{n\to\infty} r_n > 0$, there exists $\varepsilon > 0$ such that $r_n \geq \varepsilon$ for all $n \geq 1$. By Lemma 2.2, we have $\lim_{n\to\infty} ||T_\varepsilon w_n - w_n|| \leq 2\lim_{n\to\infty} ||T_n w_n - w_n|| = 0$. In view of Lemma 2.1, we conclude that $p \in F(T_\varepsilon) = (M+B)^{-1}(0)$. Moreover, since $\|w_n - x_n\| \to 0$, as $n \to \infty$, we have that Aw_{n_j} converges weakly to Ax^*. From (3.19) and the fact that $I-T$ is demiclosed at 0, we arrive at $Ap \in F(T)$. Hence $p \in \Upsilon$. Since $z = P_C u$ and $x_{n_j} \rightharpoonup p$, we conclude that

$$\limsup_{n\to\infty}\langle u-z, x_{n+1}-z\rangle = \lim_{j\to\infty}\langle u-z, x_{n+1}-z\rangle = \langle u-z, p-z\rangle \leq 0.$$

From $z \in \Upsilon$, (3.10), condition (ii), and Lemma 2.4, we see that $\|x_n - z\| \to 0$ as $n \to \infty$. Hence, $x_n \to z = P_\Upsilon u$.

__Case 2.__ Assume that$\{\|x_n - q\|\}$ is not a monotonically decreasing sequence. Set $\Gamma_n = \|x_n - q\|^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) defined by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}.$$

Clearly, τ is non decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $0 \leq \Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ for all $n \geq n_0$, which implies that $\|x_{\tau(n)} - q\| \leq \|x_{\tau(n)+1} - q\|$ for all $n \geq n_0$. Thus $\lim_{n\to\infty}\|x_{\tau(n)} - q\|$ exists. Again, from (3.16), one has

$$\begin{aligned}
& \delta_{\tau(n)}\gamma_{\tau(n)}\varepsilon\|A^*(T-I)Ax_{\tau(n)}\|^2 \\
\leq\ & \alpha_{\tau(n)}\|u-q\|^2 - (1-\alpha_{\tau(n)})\|x_{\tau(n)} - q\|^2 - \|x_{\tau(n)+1} - q\|^2 \\
& +\beta_{\tau(n)}(1-\delta_{\tau(n)}-\lambda_{\tau(n)})\|z_{\tau(n)} - v_{\tau(n)}\|^2 + \beta_{\tau(n)}\lambda_{\tau(n)}(4L^2\lambda_{\tau(n)}^2 + 2\lambda_{\tau(n)} - 1)\|z_{\tau(n)} - u_{\tau(n)}\|^2.
\end{aligned}$$

Thus $\lim_{n\to\infty}\|A^*(T-I)Ax_{\tau(n)}\| = 0$. From (3.18), we have

$$\begin{aligned}
& \delta_{\tau(n)}\gamma_{\tau(n)}\|TAx_{\tau(n)} - Ax_{\tau(n)}\|^2 \\
\leq\ & \alpha_{\tau(n)}\|u-q\|^2 + (1-\alpha_{\tau(n)})\|x_{\tau(n)} - q\|^2 - \|x_{\tau(n)+1} - q\|^2 \\
& +\beta_{\tau(n)}(1-\delta_{\tau(n)}-\lambda_{\tau(n)})\|z_{\tau(n)} - v_{\tau(n)}\|^2 + \beta_{\tau(n)}\lambda_{\tau(n)}(4L^2\lambda_{\tau(n)}^2 \\
& +2\lambda_{\tau(n)} - 1)\|z_{\tau(n)} - u_{\tau(n)}\|^2 + \delta_{\tau(n)}\gamma_{\tau(n)}\|A^*(T-I)Ax_{\tau(n)}\|^2.
\end{aligned}$$

Hence $\lim_{n\to\infty}\|TAx_{\tau(n)} - Ax_{\tau(n)}\| = 0$. By using the same argument as in Case 1, we obtain that there exists a subsequence $\{x_{\tau(n_j)}\}$ of $\{x_{\tau(n)}\}$, which converges weakly to $x^* \in \Upsilon$ as $\tau(n_j) \to \infty$. From (3.10), we see that, for all $n \geq n_0$,

$$\begin{aligned}
0 &\leq \|x_{\tau(n)+1} - x^*\|^2 - \|x_{\tau(n)} - x^*\|^2 \\
&\leq \alpha_{\tau(n)}\left[2\langle u-x^*, x_{\tau(n)+1} - x^*\rangle - \|x_{\tau(n)} - x^*\|^2\right],
\end{aligned}$$

which implies that (due to $\alpha_{\tau(n)} > 0$) $\|x_{\tau(n)} - x^*\|^2 \leq 2\langle u - x^*, x_{\tau(n)+1} - x^*\rangle$. Thus

$$\limsup_{n\to\infty}\|x_{\tau(n)} - x^*\|^2 \leq 2\limsup_{n\to\infty}\langle u-x^*, x_{\tau(n)+1} - x^*\rangle \leq 0.$$

which demonstrates that $\lim_{n\to\infty}\|x_{\tau(n)} - x^*\| = 0$. It follows from (3.23) that

$$\|x_{\tau(n)+1} - x_{\tau(n)}\| \leq \alpha_{\tau(n)}\|u - z_{\tau(n)}\| + \beta_{\tau(n)}\|v_{\tau(n)} - z_{\tau(n)}\| + \|z_{\tau(n)} - x_{\tau(n)}\| \to 0$$

as $n \to \infty$. Hence, we deduce that

$$\|x_{\tau(n)+1} - x^*\| \leq \|x_{\tau(n)+1} - x_{\tau(n)}\| + \|x_{\tau(n)} - x^*\| \to 0$$

as $n \to \infty$, so $\lim_{n\to\infty}\Gamma_{\tau(n)} = \lim_{n\to\infty}\Gamma_{\tau(n)+1} = 0$. Furthermore, for $n \geq n_0$, it is easy to see that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is, $\tau(n) < n$) because $\Gamma_j \geq \Gamma_{j+1}$ for $\tau(n)+1 \leq j \leq n$. As a consequence, we obtain, for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

Hence $\lim_{n\to\infty}\Gamma_n = 0$, which demonstrates that $\{x_n\}$ converges strongly to x^*. This completes the proof. □

Corollary 3.1. *Let H_1 and H_2 be two real Hilbert spaces, and let C be a nonempty, convex, and closed subset of H_1. Let $A : H_1 \to H_2$ be a bounded and linear operator, and let A^* be the adjoint of A. Let $M : H_1 \to 2^{H_1}$ be a maximal monotone operator, and let $B : C \to H_1$ be a τ-inverse strongly monotone mapping. Let $S : H_1 \to CB(H_1)$ be a L-Lipschitz hemicontractive-type mapping, and let $T : H_2 \to H_2$ be a nonexpansive mapping such that $\Upsilon := F(S) \cap (M+B)^{-1}(0) \cap A^{-1}F(T) \neq \emptyset$. Let the step size γ_n be chosen such that for some $\varepsilon > 0$, $\gamma_n \in \left(\varepsilon, \frac{\|TAx_n - Ax_n\|^2}{\|A^*(T-I)Ax_n\|^2} - \varepsilon\right)$, if $TAx_n \neq Ax_n$; otherwise $\gamma_n = \gamma$ (γ being any nonnegative real number). Suppose $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\delta_n\}$ are sequences in $(0,1)$ and the following conditions are satisfied:*

(i) $\alpha_n + \beta_n + \delta_n = 1$;
(ii) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}, \alpha_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\tau$;
(iv) $\alpha_n + \beta_n \leq \lambda_n \leq \lambda \leq \frac{1}{\sqrt{1+4L^2}+1}$,
(v) $Sp = \{p\}$ $\forall p \in \Upsilon$, $(I-T)$ *and* $(I-S)$ *are demiclosed at* 0.

Then the sequence $\{x_n\}$ generated below for any $x_1, u \in H_1$ by

$$\begin{cases} w_n = P_C(x_n + \gamma_n A^*(T-I)Ax_n), \\ z_n = (I + r_n M)^{-1}(w_n - r_n B w_n), \\ y_n = (1-\lambda_n)z_n + \lambda_n u_n, \\ x_{n+1} = \alpha_n u + \beta_n v_n + \delta_n z_n, \quad n \geq 1, \end{cases}$$

where $u_n \in Sz_n$, and $v_n \in Sy_n$ converges strongly to $q \in \Upsilon$ where $q = P_\Upsilon u$.

Next, we give some theoretical applications of our main results. In particular, we apply our main results to the solution of minimization problems and equilibrium problems.

Let f and g two lower semi-continuous and convex functions from H to $\mathbb{R} \cup \{+\infty\}$ such that f is differentiable with L-Lipschitz continuous gradient, and g is "simple", that is, its "proximal map" can be directly computed

$$x \to \arg\min_{y\in H}\left\{g(y) + \frac{||x-y||^2}{2\tau}\right\}.$$

Let us consider the following minimization problem

$$\min_{x\in H} F(x) := \min_{x\in H}\{f(x) + g(x)\} \tag{3.24}$$

and assume that this problem has at least a solution.

Recall that the subdifferential of a function $g : H \to \mathbb{R}$ at x is the set-valued operator on a Hilbert space H defined by

$$\partial f(x) = \{z \in H : f(y) \geq f(x) + \langle z, y-x\rangle\};\ \forall y \in H,$$

and $\operatorname{prox}_{\gamma g}(x) = (I+\gamma\partial g)^{-1}(x),\ \gamma > 0$. It is known that a point $x^* \in H$ is a solution to problem (3.24), that is, x^* is a minimizer of $f(x)+g(x)$, if and only if $0 \in \nabla f(x^*)+\partial g(x^*)$, where ∇f is the gradient of f. For any $\gamma > 0$, this optimality condition holds if and only if the following equivalent statements hold:

$$\begin{array}{rcl} 0 & \in & \gamma\nabla f(x^*)+\gamma\partial g(x^*) \\ 0 & \in & \gamma\nabla f(x^*)-x^*+x^*+\gamma\partial g(x^*) \\ (I+\gamma\partial g)(x^*) & \in & (I-\gamma\nabla f)(x^*) \\ x^* & = & (I+\gamma\partial g)^{-1}(I-\gamma\nabla f)(x^*) \\ x^* & = & \operatorname{prox}_{\gamma g}(x^*-\gamma\nabla f(x^*)). \end{array}$$

The last two expressions hold with equality because the proximal operator is single-valued. The final statement says that x^* minimizes $f+g$ if and only if it is a fixed point of $\operatorname{prox}_{\gamma g}(I-\gamma\nabla f)$.

Recall that a mapping $T : H \to H$ is said to be averaged if it can be written as $T = (1-\alpha)I+\alpha S$, where $\alpha \in (0,1)$ and $S : H \to H$ is a nonexpansive mapping. The condition $\gamma \in (0, \frac{2}{L}]$, where L is the Lipschitz constant of ∇f, guarantees that $\operatorname{prox}_{\gamma g}(I-\gamma\nabla f)$ is averaged, which indicates that it is nonexpansive.

In Corollary 3.1, If $T := \operatorname{prox}_{\gamma_n g}(I-\gamma_n\nabla f)$ with $\gamma_n \in (0, \frac{2}{L}]$, then we obtain a strong convergence result of the fixed points of multivalued Lipschitz Hemicontractive mappings and the solution of monotone variational inclusion problems and the image under a bounded linear operator is a minimizer of the sum of two functions in real Hilbert spaces.

Let C be a nonempty, convex and closed subset of a real Hilbert space H, and let $F : C\times C \to \mathbb{R}$ be a bifunction. Consider the equilibrium problem: find x^* satisfying

$$F(x^*,y) \geq 0, \quad \forall\, y \in C. \tag{3.25}$$

Denote the solution set of problem (3.25) by $EP(F)$. For solving equilibrium problem (3.25), one assumes that F satisfies the following properties:

(A1) $F(x,x) = 0,\ \ \forall\, x \in C$;
(A2) F is monotone, i.e $F(x,y)+F(y,x) \leq 0, \forall\, x \in C$;
(A3) for each $x,y,z \in C$, $\limsup_{t\to 0^+} F(tz+(1-t)x,y) \leq F(x,y)$;
(A4) for each $x \in C$, $y \longmapsto F(x,y)$ is convex and lower semicontinuous;

Lemma 3.1. [39] *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $F : C\times C \to \mathbb{R}$ be a bifunction satisfying $(A1)-(A4)$ let $r > 0$ be a positive real number and $x \in H$. Then there exists $z \in C$, such that $F(y,x)+\frac{1}{r}\langle y-x, x-z\rangle < 0$ for all $y \in C$.*

Lemma 3.2. [40] *Assume that the bifunction $F : C\times C \to \mathbb{R}$ satisfy $(A1)-(A4)$. For $r > 0$ and for all $x \in H$, define a mapping $J_r^F : H \to C$ as follows:*

$$T_r^F x = \{z \in C : F(z,y)+\frac{1}{r}\langle y-z, z-x\rangle \geq 0,\ \ \forall y \in C\}.$$

Then the following hold:

(i) T_r^F is a single-valued firmly nonexpansive mapping;
(ii) $F(T_r^F) = EP(F)$ $EP(F)$ is closed and convex.

In Corollary 3.1, taking $T := T_r^F$, we obtain a strong convergence result for a common solution of a fixed point of multivalued Lipschitz Hemicontractive mappings and monotone variational inclusion problems whose image under a bounded linear operator is a solution of some equilibrium problem.

4. Concluding Remark

We studied the monotone inclusion and fixed point problems of multi-valued Lipschitz hemicontractive-type mappings in real Hilbert spaces. We proposed an iterative method with a stepsize selected in such a way that its implementation does not require the computation or an estimate of the spectral radius. Moreover, under some mild conditions on the control sequences, we proved that the sequence generated by our proposed method converges strongly to the common solution of the two problems. Finally, we applied our result to the minimization and equilibrium problems. Our result generalizes some important results in the literature.

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