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ON THE LOWER SEMICONTINUITY AND SUBDIFFERENTIABILITY OF THE VALUE FUNCTION FOR CONIC LINEAR PROGRAMMING PROBLEMS

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Abstract. The lemma 1 from the paper [N.E. Gretsky, J.M. Ostroy, W.R. Zame, Subdifferentiability and the duality gap, Positivity 6 (2002), 261-274] asserts that the value function v of an infinite dimensional linear programming problem in standard form is lower semicontinuous whenever v is proper and the involved spaces are normed vector spaces. In this paper, one shows that this statement is false even in finite-dimensional spaces, and one provides an example of linear programming problem in Hilbert spaces whose (proper) value function is not lower semicontinuous (hence it is not subdifferentiable) at any point in its domain. One shows that the restriction of the value function to its domain in Kretschmer's gap example is not bounded on any neighborhood of any point of the domain, and discusses other assertions done in the same paper.

Keywords. Assignment model; Counterexample; Duality gap; Lower semicontinuity; Subdifferentiability; Value function.

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1. INTRODUCTION

The following conical linear programming problem

(P) minimize $c^*(x)$ s.t. $x \in P,\ Ax - b \in Q,$

and its dual

(D) maximize $y^*(b)$ s.t. $y^* \in Q^+,\ c^* - A^*y^* \in P^+,$

were studied in [1], where X, Y are Hausdorff locally convex spaces, X^* and Y^* are their topological dual spaces, $A : X \to Y$ is a continuous linear operator, $A^* : Y^* \to X^*$ is the adjoint of A, $P \subset X$ and $Q \subset Y$ are convex cones, $P^+ \subset X^*$ and $Q^+ \subset Y^*$ are the positive dual cones of P and Q, $b \in Y$, and $c \in X^*$.

The main results from [1] are: Theorem 1 which states that the value function v associated to (P) is subdifferentiable at b if and only if (D) has optimal solutions and there is no duality gap, and its use for proving the Duffin–Karlovitz no-gap theorem; Lemma 1 which states that v is lower semicontinuous whenever it is proper and the involved spaces are normed vector spaces; the modification of Kretschmer's gap example in order to get a convex function which

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is subdifferentiable at a point but is not continuous there; Proposition 2 which provides sufficient conditions, adequate for the assignment model, to ensure that v is Lipschitz on Q.

Unfortunately, [1, Lem. 1] is not true even in finite dimensional spaces, which makes its use inadequate in the proof of the Duffin–Karlovitz no-gap theorem, while the proof of [1, Prop. 2] needs, in our opinion, serious clarifications. Moreover, there are also other inaccuracies in the paper. Having in view the remark that "This paper should be on the reading list of any advanced mathematical economics course which has a focus on extremal methods in infinite-dimensional spaces" from the review MR1932651 (2003i:90111) of [1] in Mathematical Reviews, we consider that there is a strong motivation for an attentive reading of this paper.

subadditive, in Section 2,

The paper is organized as follows. Having in view that the value function associated to problem (P) is positively homogeneous and subadditive, in Section 2, we underline some specific properties of such functions, pointing out the differences between the (lower-, upper-, local Lipschitz) continuity of such functions and their restrictions to the domain. In Section 3, we essentially discuss the proof of the Duffin–Karlovitz no-gap theorem, while in Section 4, we provide two counter-examples to [1, Lem. 1], the first in finite-dimensional spaces and the second in Hilbert spaces. Section 5 is dedicated to Kretschmer's gap example, while in the last section, Section 6, we comment the proof of [1, Prop. 2].

Below, we introduce the basic notations and some preliminary results used in the paper.

Throughout this paper, the considered spaces are real Hausdorff locally convex spaces (H.l.c.s. for short) if not mentioned explicitly otherwise. Having X an H.l.c.s., X^* is its topological dual endowed with its weakly-star topology $w^* := \sigma(X^*, X)$. The value $x^*(x)$ of $x^* \in X^*$ at $x \in X$ is denoted by $\langle x, x^*\rangle$. It is well known that $(X^*, w^*)^*$ can be identified with X, what we do in the sequel. Having $(\emptyset \neq)\ K \subseteq X$ a convex cone (that is, $x + x' \in K$ and $tx \in K$ for all $x, x' \in K$ and $t \in \mathbb{R}_+ := [0, \infty[$), we set $x \leq_K x'$ (equivalently $x' \geq_K x$) for $x, x' \in X$ with $x' - x \in K$; clearly $\leq_K$ is a *preorder* on X, that is, $\leq_K$ is reflexive and transitive. For $E \subseteq X$, by $\operatorname{span} E$, $\operatorname{aff} E$, $\operatorname{int} E$ and $\operatorname{cl} E$ one denotes the *linear hull*, the *affine hull*, the *interior* and the *closure* of E, respectively. Moreover, $\overline{\operatorname{span}} E := \operatorname{cl}(\operatorname{span} E)$, the *intrinsic core* (or *relative algebraic interior*) of $\emptyset \neq E \subseteq X$ is the set

$$\operatorname{icr} E := \{x \in X \mid \forall x' \in \operatorname{aff} E,\ \exists \delta \in \mathbb{P},\ \forall \lambda \in \mathbb{R} : |\lambda| \leq \delta \Rightarrow (1-\lambda)x + \lambda x' \in E\},$$

where $\mathbb{P} :=]0, \infty[$, while the *core* (or *algebraic interior*) of E, denoted $\operatorname{cor} E$, is $\operatorname{icr} E$ if $\operatorname{aff} E = X$ and the empty set otherwise. For $\emptyset \neq A \subseteq X$ (and similarly for $\emptyset \neq B \subseteq X^*$) we set $A^+ := \{x^* \in X^* \mid \forall x \in A : \langle x, x^*\rangle \geq 0\}$ for the *positive dual cone* of A. It is well known that A^+ $(\subseteq X^*)$ is a w^*-closed convex cone and $(K^+)^+ = \operatorname{cl} K$ whenever K is a convex cone.

Having a function $f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$, its *domain* is the set $\operatorname{dom} f := \{x \in X \mid f(x) < \infty\}$; f is *proper* if $\operatorname{dom} f \neq \emptyset$ and $f(x) \neq -\infty$ for all $x \in X$; f is *convex* if its *epigraph* $\operatorname{epi} f := \{(x,t) \in X \times \mathbb{R} \mid f(x) \leq t\}$ is convex; f is *positively homogeneous* if $f(tx) = tf(x)$ for all $t \in \mathbb{P}$ and $x \in X$; f is *subadditive* if $f(x + x') \leq f(x) + f(x')$ for all $x, x' \in \operatorname{dom} f$, where $-\infty + \infty := \infty - \infty := \infty$; f is *sublinear* if f is positively homogeneous, subadditive and $f(0) = 0$; f is *lower semicontinuous* (l.s.c. for short) *at* $x \in X$ if $\liminf_{x' \to x} f(x') \geq f(x)$, where $\overline{\mathbb{R}}$ is endowed with its usual topology (for example, the topology induced by the metric d defined by $d(t,t') := |\arctan t - \arctan t'|$ with $\arctan(\pm\infty) := \pm\pi/2$); f is *l.s.c.* if f is l.s.c. at any $x \in X$; the *l.s.c. envelope* of f is the function $\overline{f} : X \to \overline{\mathbb{R}}$ such that $\operatorname{epi} \overline{f} = \operatorname{cl}(\operatorname{epi} f)$, and so $\overline{f}$ is convex if f is so; f is *upper semicontinuous* (u.s.c. for short) *at* $x \in X$ (resp. on X) if $-f$ is l.s.c. at $x \in X$ (resp.

on X); the *subdifferential* of f at $x \in X$ with $f(x) \in \mathbb{R}$ is the set

$$\partial f(x) := \{x^* \in X^* \mid \forall x' \in X : \langle x' - x, x^* \rangle \leq f(x') - f(x)\}$$

and $\partial f(x) := \emptyset$ if $f(x) \notin \mathbb{R}$; f is *subdifferentiable* at $x \in X$ if $\partial f(x) \neq \emptyset$. The *conjugate* of f is the function $f^* : X^* \to \overline{\mathbb{R}}$ defined by

$$f^*(x^*) := \sup\{\langle x, x^* \rangle - f(x) \mid x \in X\} = \sup\{\langle x, x^* \rangle - f(x) \mid x \in \operatorname{dom} f\} \quad (x^* \in X^*),$$

where $\sup \emptyset := -\infty$; the conjugate $g^* : X \to \overline{\mathbb{R}}$ of $g^* : X^* \to \overline{\mathbb{R}}$ is defined similarly. Clearly, f^* is a w^*-l.s.c. convex function, $f^* = (\overline{f})^*$ and $f^{**} := (f^*)^* \leq \overline{f} \leq f$; moreover, for $x \in X$ and $x^* \in X^*$ one has

$$x^* \in \partial f(x) \Leftrightarrow [f(x) \in \mathbb{R} \wedge f(x) + f^*(x^*) = \langle x, x^* \rangle] \Rightarrow \overline{f}(x) = f(x) \in \mathbb{R} \Rightarrow \partial \overline{f}(x) = \partial f(x). \tag{1.1}$$

As in [2], the class of proper convex functions defined on X is denoted by $\Lambda(X)$. Using [2, Ths. 2.2.6, 2.3.4] for the convex function $f : X \to \overline{\mathbb{R}}$, one has

$$\overline{f} \in \Lambda(X) \Leftrightarrow \left[\exists x \in X : \overline{f}(x) \in \mathbb{R}\right] \Leftrightarrow [\exists x^* \in X^* : f^*(x^*) \in \mathbb{R}] \Rightarrow f^{**} = \overline{f}, \tag{1.2}$$

$$\left[\exists x \in X : \overline{f}(x) = -\infty\right] \Leftrightarrow \left[\forall x \in \operatorname{cl}(\operatorname{dom} f) : \overline{f}(x) = -\infty\right] \Leftrightarrow f^* = \infty \Leftrightarrow f^{**} = -\infty. \tag{1.3}$$

The *indicator function* of $E \subseteq X$ is $\iota_E : X \to \overline{\mathbb{R}}$ defined by $\iota_E(x) := 0$ for $x \in E$ and $\iota_E(x) := \infty$ for $x \in X \setminus E$; notice that ι_E is l.s.c. iff E is closed, and ι_E is convex iff E is convex. In the sequel, the sequences are indexed by $\mathbb{N}^* := \{1, 2, \ldots, n, \ldots\}$.

2. Some Properties of Sublinear Functions

Because the value function of a linear programming problem is positively homogeneous and subadditive (hence sublinear when it vanishes at 0), it is useful to point out some specific properties of such functions.

Let $g : X \to \overline{\mathbb{R}}$ be positively homogeneous and subadditive. First, observe that

$$\forall x, x' \in \operatorname{dom} g, \ \forall \lambda \in \mathbb{P} \ : \ x + x' \in \operatorname{dom} g \ \text{ and } \ \lambda x \in \operatorname{dom} g.$$

Consequently, for $x, x' \in \operatorname{dom} g$ and $\lambda \in \left]0, 1\right[$, one has

$$g(\lambda x + (1 - \lambda)x') \leq g(\lambda x) + g((1 - \lambda)x') = \lambda g(x) + (1 - \lambda)g(x').$$

So, g is convex.

Because g is positively homogeneous, one has $g(0) = g(\lambda 0) = \lambda g(0)$ for $\lambda \in \mathbb{P}$, and so $g(0) \in \{-\infty, 0, \infty\}$. Moreover, if $g(x_0) = -\infty$ for some $x_0 \in X$, then $g(x + \lambda x_0) \leq g(x) + \lambda g(x_0) = -\infty$, whence $g(x + \lambda x_0) = -\infty$, for all $\lambda \in \mathbb{P}$ and $x \in \operatorname{dom} g$. Consequently, $g(x) = -\infty$ for all $x \in \operatorname{dom} g$ if $g(0) = -\infty$.

Assume that $g(0) = \infty$. Then $\widetilde{g} : X \to \overline{\mathbb{R}}$ defined by $\widetilde{g}(x) := g(x)$ for $x \neq 0$ and $\widetilde{g}(0) := 0$ is sublinear. Indeed, take $x', x'' \in \operatorname{dom} \widetilde{g} = \{0\} \cup \operatorname{dom} g$. If $x', x'' \in \operatorname{dom} g$, then $x' + x'' \in \operatorname{dom} g$ (hence $x', x'', x' + x'' \neq 0$), and so $\widetilde{g}(x' + x'') = g(x' + x'') \leq g(x') + g(x'') = \widetilde{g}(x') + \widetilde{g}(x'')$; if $x' = 0$ (and similarly for $x'' = 0$), then $\widetilde{g}(x' + x'') = \widetilde{g}(x'') = \widetilde{g}(x') + \widetilde{g}(x'')$. Because $\widetilde{g}$ is clearly positively homogeneous, $\widetilde{g}$ is sublinear.

In the rest of this section, $g : X \to \overline{\mathbb{R}}$ is a sublinear function. Hence $g(0) = 0$. Using [2, Th. 2.4.14] one obtains that

$$[\partial g(0) \neq \emptyset \Leftrightarrow g \text{ is l.s.c. at } 0], \quad g^* = \iota_{\partial g(0)}, \quad \operatorname{dom} g^* = \partial g(0), \tag{2.1}$$

$$\partial g(0) \neq \emptyset \Rightarrow [\forall x \in X : \overline{g}(x) = \sup\{\langle x, x^* \rangle \mid x^* \in \partial g(0)\} = g^{**}(x)], \tag{2.2}$$

$$\forall x \in X : \partial g(x) = \{x^* \in \partial g(0) \mid \langle x, x^* \rangle = g(x)\}. \tag{2.3}$$

Also note that [g is u.s.c. at 0] $\Leftrightarrow$ [$\operatorname{dom} g = X$ and g is continuous on X]; moreover, [g is l.s.c. at 0] $\Leftrightarrow$ [$g|_{\operatorname{dom} g}$ is l.s.c. at 0], where $\operatorname{dom} g$ is endowed with its trace (induced) topology.

Assume now that X is a normed vector space. One also has:

$$g \text{ is l.s.c. at } 0 \Leftrightarrow [\exists L > 0, \ \forall x \in X : g(x) \geq -L\|x\|]; \tag{2.4}$$

$$g \text{ is u.s.c. at } 0 \Leftrightarrow [\exists L > 0, \ \forall x \in X : g(x) \leq L\|x\|] \tag{2.5}$$

$$\Leftrightarrow \operatorname{dom} g = X \text{ and } g \text{ is } (L\text{-})\text{Lipschitz on } X. \tag{2.6}$$

Indeed, the implications $\Leftarrow$ from (2.4)–(2.6) are obvious because $g(0) = 0$. Assume that g is l.s.c. at 0. Then there exists $r > 0$ such that $g(x) \geq -1$ for $x \in X$ with $\|x\| \leq r$. Taking $x \in X \setminus \{0\}$ and $x' := \frac{r}{\|x\|}x$, one has $\|x'\| \leq r$, whence $-1 \leq g(x') = \frac{r}{\|x\|}g(x)$, and so $g(x) \geq -r^{-1}\|x\|$; hence the implication $\Rightarrow$ holds in (2.4). The proof of the implication $\Rightarrow$ from (2.5) is similar. Assume now that $g(x) \leq L\|x\|$ for $x \in X$; hence $\operatorname{dom} g = X$. Taking $x \in X$, one has $0 = g(0) = g(x + (-x)) \leq g(x) + g(-x)$, whence $g(x) \in \mathbb{R}$. Take now $x, x' \in X$; we (may) assume that $g(x) \geq g(x')$. Then

$$g(x) = g((x - x') + x') \leq g(x - x') + g(x') \leq L\|x - x'\| + g(x'),$$

whence $|g(x) - g(x')| = g(x) - g(x') \leq L\|x - x'\|$. Therefore, g is L-Lipschitz.

In what concern the continuity and the upper semicontinuity of $g|_{\operatorname{dom} g}$, in a similar way as for (2.5), one obtains:

$$g|_{\operatorname{dom} g} \text{ is u.s.c. at } 0 \Leftrightarrow [\exists L > 0, \ \forall x \in \operatorname{dom} g : g(x) \leq L\|x\|],$$

$$g|_{\operatorname{dom} g} \text{ is continuous at } 0 \Leftrightarrow [\exists L > 0, \ \forall x \in \operatorname{dom} g : |g(x)| \leq L\|x\|].$$

The next example shows the big differences among the continuity properties of the functions g and $g|_{\operatorname{dom} g}$.

Example 2.1. Let X be an infinite-dimensional normed vector space and let $\varphi : X \to \mathbb{R}$ be a linear, not continuous functional. Consider the proper sublinear functions $g_1, g_2, g_3 : X \to \overline{\mathbb{R}}$ defined by

$$g_1 := \max\{0, \varphi\}, \quad g_2(x) := \begin{cases} 0 & \text{if } x \in [\varphi \leq 0], \\ \infty & \text{if } x \in [\varphi > 0], \end{cases} \quad g_3(x) := \begin{cases} \varphi(x) & \text{if } x \in [\varphi \leq 0], \\ \infty & \text{if } x \in [\varphi > 0], \end{cases}$$

where $[\varphi \leq 0] := \{x \in X \mid \varphi(x) \leq 0\}$ and similarly for $[\varphi > 0]$ (and the like). The following assertions hold:

(i) $\operatorname{dom} g_1 = X$, $\partial g_1(x) = \{0\}$ for $x \in [\varphi \leq 0]$, $\partial g_1(x) = \emptyset$ for $x \in [\varphi > 0]$, $(g_1)^* = \iota_{\{0\}}$, $\overline{g_1} = 0$, and so g_1 is l.s.c. at x iff $x \in [\varphi \leq 0]$; moreover, g_1 is not u.s.c. at each $x \in X$.

(ii) $\operatorname{dom} g_2 = [\varphi \leq 0]$, $\partial g_2(x) = \{0\}$ for $x \in [\varphi \leq 0]$, $\partial g_2(x) = \emptyset$ for $x \in [\varphi > 0]$, $(g_2)^* = \iota_{\{0\}}$, $\overline{g_2} = 0$, and so g_2 is l.s.c. at x iff $x \in [\varphi \leq 0]$; moreover, g_2 is u.s.c. at x iff $x \in [\varphi > 0]$ and $g_2|_{\operatorname{dom} g_2}$ is Lipschitz.

(iii) $\operatorname{dom} g_3 = [\varphi \le 0]$, $\partial g_3(x) = \emptyset$ for $x \in X$, $(g_3)^* = \infty$, $\overline{g_3} = -\infty$, and so g_3 is not l.s.c. at each $x \in X$; moreover, g_3 is u.s.c. at x iff $x \in [\varphi > 0]$, $g_3|_{\operatorname{dom} g_3}$ is u.s.c. at x iff $x \in [\varphi = 0]$, and $g_3|_{\operatorname{dom} g_3}$ is not l.s.c. at each $x \in \operatorname{dom} g_3$.

Proof. (i) It is clear that $\operatorname{dom} g_1 = X$ and $0 \in \partial g_1(0)$. Consider $x^* \in \partial g_1(0)$; then, obviously, $\langle x, x^* \rangle \le g_1(x) = 0$ for $x \in [\varphi = 0]$, and so $\langle x, x^* \rangle = 0$ for $x \in [\varphi = 0]$. Using [3, Lem. 3.9], there exists $\lambda \in \mathbb{R}$ such that $x^* = \lambda \varphi$, and so $\lambda = 0$ because φ is not continuous. Hence $\partial g_1(0) = \{0\}$, whence, $\partial g_1(x) = \{0\} \Leftrightarrow g_1(x) = 0 \Leftrightarrow \varphi(x) \le 0$, and $\partial g_1(x) = \emptyset$ when $\varphi(x) > 0$ by (2.3). Using (2.1) one gets $(g_1)^* = \iota_{\{0\}}$, while using (2.2) one gets $\overline{g_1} = 0$, and so g_1 is l.s.c. at x if and only if $\varphi(x) \le 0$. Assume that g_1 is u.s.c. at $x \in X$; then g_1 is Lipschitz on $\operatorname{dom} g_1 = X$ by (2.6), and so g_1 is l.s.c. on X; this contradiction proves that g_1 is not u.s.c. at each $x \in X$.

(ii) It is obvious that $\operatorname{dom} g_2 = [\varphi \le 0]$ and $0 \in \partial g_2(0)$. Similar to the proof of (i) one gets that $\partial g_2(x) = \{0\}$ if $x \in [\varphi \le 0]$ and $\partial g_2(x) = \emptyset$ otherwise, that $\overline{g_2} = 0$, and so g_2 is l.s.c. at x if and only if $x \in [\varphi \le 0]$. Because $g_2(x) = \infty \ge g_2(x')$ for all $x \in [\varphi > 0]$ and $x' \in X$, g_2 is obviously u.s.c. at $x \in [\varphi > 0]$. Assuming that g_2 is u.s.c. at some $x \in \operatorname{dom} g_2$, one obtains that $x \in \operatorname{int}[\varphi \le 0]$, and so one gets the contradiction that φ is continuous; hence g_2 is u.s.c. at x iff $x \in [\varphi > 0]$. Because $g_2|_{\operatorname{dom} g_2} = 0$, $g_2|_{\operatorname{dom} g_2}$ is Lipschitz.

(iii) Clearly, $\operatorname{dom} g_3 = [\varphi \le 0]$. One has that $x^* \in \partial g_3(0)$ iff $\langle x, x^* \rangle \le \varphi(x)$ for all $x \in [\varphi \le 0]$; using similar arguments to those in the proof of (i), one gets $\partial g_3(0) = \emptyset$, and so $\partial g_3(x) = \emptyset$ for every $x \in X$. Because $\partial g_3(0) = \emptyset$, one has that $(g_3)^* = \iota_\emptyset = \infty$, and that g_3 is not l.s.c. at 0, whence $\overline{g_3}(0) = -\infty$, and so $\overline{g_3}(x) = -\infty$ for all $x \in \operatorname{cl}(\operatorname{dom} g_3) = X$ because $[\varphi \le 0]$ is dense in X as φ is not continuous. Hence $\overline{g_3} = -\infty$, and so g_3 is not l.s.c. at any $x \in X$. As in the proof of (ii), g_3 is u.s.c. at x iff $x \in [\varphi > 0]$. Because $g_3|_{\operatorname{dom} g_3}(x) \ge g_3|_{\operatorname{dom} g_3}(x')$ for all $x \in [\varphi = 0]$ and $x' \in \operatorname{dom} g_3$, $g_3|_{\operatorname{dom} g_3}$ is u.s.c. at each $x \in [\varphi = 0]$.

Because φ is not continuous, $\sup\{\varphi(x) \mid x \in S_X\} = \infty$, where $S_X := \{x \in X \mid \|x\| = 1\}$, and so there exists a sequence $(x_n) \subseteq S_X$ such that $\varphi(x_n) \ge n$ for $n \in \mathbb{N}^*$. Consider $x \in [\varphi < 0]$ and take $x'_n := x - [\varphi(x)/\varphi(x_n)]x_n$ for $n \ge 1$. Then $\varphi(x'_n) = 0$ (whence $x'_n \in \operatorname{dom} g_3$) for $n \in \mathbb{N}^*$, $x'_n \to x$, and $\limsup_{x' \to x} \varphi(x') \ge \limsup_{n\to\infty} \varphi(x'_n) = 0 > \varphi(x) = g_3(x)$. Hence $g_3|_{\operatorname{dom} g_3}$ is not u.s.c. at x. Therefore, $g_3|_{\operatorname{dom} g_3}$ is not u.s.c. at each $x \in [\varphi < 0]$, proving so that $g_3|_{\operatorname{dom} g_3}$ is u.s.c. at $x \in \operatorname{dom} g_3$ iff $x \in [\varphi = 0]$. Consider now $x \in [\varphi \le 0]$ and take $x''_n := x - [1/\varphi(x_n)]x_n$ for $n \ge 1$. Then $\varphi(x''_n) = \varphi(x) - 1 \le 0$ (whence $x''_n \in \operatorname{dom} g_3$) for $n \in \mathbb{N}^*$, $x''_n \to x$, and $\liminf_{x'' \to x} \varphi(x'') \le \liminf_{n\to\infty} \varphi(x''_n) = \varphi(x) - 1 < \varphi(x) = g_3(x)$. Hence $g_3|_{\operatorname{dom} g_3}$ is not l.s.c. at each $x \in \operatorname{dom} g_3$. □

The next result seems to be quite relevant in the context of [1].

Proposition 2.1. *Let X be a normed vector space and $g : X \to \overline{\mathbb{R}}$ be a proper sublinear function. Assume that $x \in \operatorname{dom} g$ and $\delta, L > 0$ are such that $|g(x') - g(x'')| \le L\|x' - x''\|$ for all $x', x'' \in B(x,\delta) \cap \operatorname{dom} g$, where $B(x,\delta) := \{x' \in X \mid \|x' - x\| < \delta\}$. Then*

$$\forall \gamma \in \mathbb{P},\ \forall x', x'' \in B(\gamma x, \gamma\delta) \cap \operatorname{dom} g : \left|g(x') - g(x'')\right| \le L\left\|x' - x''\right\|.$$

In particular, $g|_{\operatorname{dom} g}$ is L-Lipschitz if $x = 0$ (or, more generally, $\|x\| < \delta$).

Proof. Take $x', x'' \in B(\gamma x, \gamma\delta) \cap \operatorname{dom} g$. Then $\gamma^{-1}x', \gamma^{-1}x'' \in B(x,\delta) \cap \operatorname{dom} g$, and so

$$\gamma^{-1}\left|g(x') - g(x'')\right| = \left|g(\gamma^{-1}x') - g(\gamma^{-1}x'')\right| \le L\left\|\gamma^{-1}x' - \gamma^{-1}x''\right\| = L\gamma^{-1}\left\|x' - x''\right\|,$$

whence $|g(x') - g(x'')| \le L\|x' - x''\|$. □

Notice that $g|_{\operatorname{dom} g}$ is locally Lipschitz on $\operatorname{icr}(\operatorname{dom} g)$ whenever g is a proper sublinear function and $\dim X < \infty$ [or, more generally, $\dim X_g < \infty$, where $X_g := \operatorname{span}(\operatorname{dom} g) = \operatorname{dom} g - \operatorname{dom} g$; this is so because X_g is a linear space, $\widetilde{g} := g|_{X_g} \in \Gamma(X_g)$, $\operatorname{dom}\widetilde{g} = \operatorname{dom} g$ and $\operatorname{icr}(\operatorname{dom} g) = \operatorname{cor}(\operatorname{dom}\widetilde{g}) = \operatorname{int}(\operatorname{dom}\widetilde{g})$ and so $\widetilde{g}$ is continuous (and so locally Lipschitz) on $\operatorname{int}(\operatorname{dom}\widetilde{g})$].

3. An Alternative Proof for Theorem 1 From [1]

If not mentioned explicitly otherwise, the problems (P) and (D), as well as the corresponding data, are as in the Introduction. Moreover, the preorders defined by P, Q, P^+ and Q^+ are simply denoted by $\leq$.

The value function associated to problem (P) is

$$v : Y \to \overline{\mathbb{R}}, \quad v(y) := \inf\{\langle x, c^*\rangle \mid Ax \geq y,\ x \geq 0\} = \inf\{\langle x, c^*\rangle \mid x \in P,\ y \in Ax - Q\},$$

where $\inf \emptyset := \infty$. It is clear that $\operatorname{dom} v = A(P) - Q$, v is positively homogeneous and convex, $v(0) \leq 0$, and $v(y_1) \leq v(y_2)$ whenever $y_1 \leq y_2$. Hence $v(0) \in \{0, -\infty\}$, and so $v(y) = -\infty$ for $y \in \operatorname{dom} v$ if $v(0) = -\infty$. Moreover, by the definition of v, we have

$$\begin{aligned} v^*(y^*) &= \sup\big\{\, \langle y, y^*\rangle + \sup\{\langle x, -c^*\rangle \mid x \in P,\ Ax - y = q \in Q\,\} \mid y \in Y\big\} \\ &= \sup\{\langle Ax - q, y^*\rangle + \langle x, -c^*\rangle \mid x \in P,\ q \in Q\} \\ &= \sup\{\langle x, A^*y^* - c^*\rangle + \langle q, -y^*\rangle \mid x \in P,\ q \in Q\} \\ &= \iota_{P^+}(c^* - A^*y^*) + \iota_{Q^+}(y^*) \end{aligned}$$

for every $y^* \in Y^*$. Hence

$$\operatorname{dom} v^* = \{y^* \in Q^+ \mid c^* - A^*y^* \in P^+\} = Q^+ \cap (A^*)^{-1}(c^* - P^+), \quad v^* = \iota_{\operatorname{dom} v^*}. \tag{3.1}$$

So, $\operatorname{dom} v^*$ is the feasible set of problem (D). It follows that

$$v^{**}(y) = \sup\big\{\langle y, y^*\rangle - v^*(y^*) \mid y^* \in Y^*\big\} = \sup\big\{\langle y, y^*\rangle \mid y^* \in Q^+,\ c^* - A^*y^* \in P^+\big\}$$

for all $y \in Y$. Hence $\operatorname{val(D)} = v^{**}(b) \leq v(b) = \operatorname{val(P)}$.

Proposition 3.1. *For the problems* (P) *and* (D) *above, one has:* $\partial v(b) \neq \emptyset \Leftrightarrow [\operatorname{val(P)} = \operatorname{val(D)} \in \mathbb{R}$ *and* (D) *has optimal solutions*]*; moreover, if* $\partial v(b) \neq \emptyset$*, then* $\partial v(b)$ *is the set* Sol(D) *of the optimal solutions of dual problem* (D).

Proof. Assume that $\partial v(b) \neq \emptyset$ and take $y^* \in \partial v(b)$. Then $v(b) \in \mathbb{R}$ and $v(b) + v^*(y^*) = \langle b, y^*\rangle$ by (1.1). Consequently, $\operatorname{val(P)} = v(b) = \langle b, y^*\rangle - v^*(y^*) \leq \operatorname{val(D)}$, and so $\operatorname{val(P)} = \operatorname{val(D)}$. Hence $y^* \in \operatorname{Sol(D)}$. Conversely, assume that $[v(b) =]$ $\operatorname{val(P)} = \operatorname{val(D)} \in \mathbb{R}$ and (D) has an optimal solution y^*. Then $(\mathbb{R} \ni)\ v(b) = \langle b, y^*\rangle - v^*(y^*)$, and so $y^* \in \partial v(b)$; hence $\operatorname{Sol(D)} \subseteq \partial v(b)$. Therefore, $\operatorname{Sol(D)} = \partial v(b)$ whenever $\partial v(b) \neq \emptyset$. □

Proposition 3.1 is essentially [4, Prop. 2.5]. Its first part was established in [1, Th. 1]. In this context, it is worth recalling [1, Rem. 1, p. 274]:

Q1 – "REMARK 1. The Lipschitz property of the value function v in the assignment model ensures that the set of dual solutions coincides with the subdifferential of the value function. *This*, of course, *need not be true in more general economic models*." (Our emphasis.)

The second part of Proposition 3.1 shows that, for the general conic linear programming problem (P), the set of solutions of dual problem (D) coincides with the subdifferential of the value function whenever the latter is nonempty, not only for "the value function v in the assignment model".

In [1, p. 266], one says "we will consider only LP problems for which the value function is proper". Hence, in [1], v is a proper sublinear function with $\operatorname{dom} v = A(P) - Q$. Of course, in this case (that is, v is proper), if $\dim Y < \infty$, then v is subdifferentiable on $\operatorname{icr}(\operatorname{dom} v)$ $(\neq \emptyset)$, and so $\emptyset \neq \partial v(y)$ [$\subseteq \partial v(0)$ by (2.3)] for $y \in \operatorname{icr}(\operatorname{dom} v)$ whence $c^* \in P^+ + A^*(Q^+)$.

The following statement seems to be an important result from [1] even if it is not mentioned explicitly throughout that article.

Q2 – "LEMMA 1. *If the value function v* for a linear programming problem in standard form on ordered normed linear spaces *is proper*, *then v is a lower semicontinuous* extended real-valued (convex and homogeneous) *function.*" (Our emphasis.)

We shall see in the next section that [1, Lem. 1] is false even in finite dimensional spaces.

Immediately after the proof of [1, Lem. 1], one finds the following text:

Q3 – "As promised in the Introduction, we can quickly derive the Duffin–Karlovitz [3][1] and the Charnes–Cooper–Kortanek [1, 2] no-gap theorems from Theorem 1. *Let X and Y be ordered normed linear spaces and consider a linear programming problem in standard form with data A, b, c^*.* The Duffin–Karlovitz theorem asserts that *if the positive cone Y_+ has non-empty interior, if there is a feasible solution $\widehat{x}$ for the primal problem such that $\widehat{x} \geq 0$ and $A\widehat{x} - b$ is in the interior of Y_+, and if the value of the primal is finite, then the dual problem has a solution and there is no gap.*[2] Since $\widehat{x}$ is feasible for b, b is in the domain of the value function v and, hence, $v(b) < +\infty$. By hypothesis there is an open ball U around $A\widehat{x} - b$ within F_+. Hence, $\widehat{x}$ is feasible for $b + u$ for all $u \in U$; viz. b is an interior point of the domain of v. The hypothesis that the value of the primal is finite is just the statement that $v(b) > -\infty$. Consequently, v is subdifferentiable at b and we conclude by Theorem 1 that there exists a dual solution and that there is no gap, as asserted." (Our emphasis. Of course, it must be Y_+ instead of F_+.)

We have to understand that the statement of the Duffin–Karlovitz theorem is given by the emphasized text from Q3. Let us analyze the given proof. We agree with the facts that $b + U \subseteq \operatorname{dom} v$ [and so $b \in \operatorname{int}(\operatorname{dom} v)$] and $v(b) \in \mathbb{R}$. From this, without providing a motivation, one concludes that $\partial v(b) \neq \emptyset$. Even if not mentioned, it is true that $v(y) \in \mathbb{R}$ for every $y \in \operatorname{dom} v$, and so v is proper. Using Lemma 1,[3] it follows that v is lower semicontinuous. Why is $\partial v(b)$ nonempty? Without being mentioned explicitly, probably, one uses the following statement from [1, p. 266]:

Q4 – "Suppose that $f : Y \to \mathbb{R} \cup \{+\infty\}$ is a proper convex function defined on the normed linear space Y and that $b \in \operatorname{dom} f$. Each of the following conditions implies the next and the last is equivalent to the subdifferentiability of f at b.

1. f is lower semicontinuous and b is an interior point of $\operatorname{dom} f$;

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[1] This is our reference [5].

[2] We did not find an assertion equivalent to the mentioned "Duffin–Karlovitz theorem" in [5]; in fact, in [5, p. 123], one says: "This theory makes very little use of topology so it is more like the theory of finite linear programming than like the theories given in [2] and [5]. The desirability of omitting topological considerations is emphasized by the paper of Charnes, Cooper and Kortanek. (However, in another paper [4] a topological approach to this and similar problems will be treated.)".

[3] The string "Lemma 1" appears only once in [1], more precisely in the text from Q2.

2. f is locally Lipschitz at the point b, i.e., there exists $\delta > 0$ such that f is Lipschitz on $\operatorname{dom} f \cap B(b;\delta)$;

3. f has bounded steepness at the point b, i.e., the quotients $(f(b) - f(y))/\|y-b\|$ are bounded above."

So, using [1, Lem. 1] and the above implications 1. $\Rightarrow$ 2. $\Rightarrow$ 3., one has $\partial v(b) \neq \emptyset$. Is the implication 1. $\Rightarrow$ 2. true?

On page 267 of [1], one mentions:

Q5 – "It is well-known (see Phelps [9][4]) that an extended real-valued proper lower semicontinuous convex function is locally Lipschitz and locally bounded on the interior of its domain."

We did not succeed to find this assertion in (our reference) [6], but we found the following two related results:

Q6 – "Proposition 1.6. If the convex function f is continuous at $x_0 \in D$, then it is locally Lipschitzian at x_0, that is, there exist $M > 0$ and $\delta > 0$ such that $B(x_0;\delta) \subset D$ and $|f(x) - f(y)| \leq M\|x-y\|$ whenever $x, y \in B(x_0;\delta)$.

Proposition 3.3. Suppose that f is a proper lower semicontinuous convex function on a Banach space E and that $D = \operatorname{int}\operatorname{dom}(f)$ is nonempty; then f is continuous on D."

In fact, this version of "the Duffin–Karlovitz theorem" is contained in [7, Th. 3] because $P := X_+$ and $Q := Y_+$ are tacitly assumed to be closed in [1]. Note that "the Duffin–Karlovitz theorem" is true even for P and Q not necessarily closed; for this one could apply [2, Th. 2.7.1] under its condition (iii) for $\Phi : X \times Y \to \overline{\mathbb{R}}$ defined by $\Phi(x,y) := \langle x, c^* \rangle + \iota_P(x) + \iota_Q(Ax - y - b)$.

4. Two Examples

In the sequel, the topological duals of Hilbert spaces (including $\mathbb{R}^m$ with $m \in \mathbb{N}^*$) are identified with themselves using Riesz' theorem.

The next example shows that [1, Lem. 1] is not true even in finite dimensional spaces.

Example 4.1. Consider $X := \mathbb{R}^2$, $Y := \mathbb{R}^2 \times \mathbb{R}$, $A : X \to Y$ with $A(x_1,x_2) := (x_1,x_2,0)$, $c^* := (0,1) \in \mathbb{R}^2$,

$$P := \mathbb{R} \times \mathbb{R}_+, \quad Q := \left\{(y_1,y_2,y_3) \in Y \mid y_1, y_3 \in \mathbb{R}_+,\ (y_2)^2 \leq 2y_1y_3\right\},$$

the conic linear programming problem

(P) minimize $\langle x, c^* \rangle$ s.t. $x \in P$, $Ax - b \in Q$,

and $v : Y \to \overline{\mathbb{R}}$ defined by $v(y) := \inf\{\langle x, c^* \rangle \mid x \in P,\ Ax - b \in Q\}$.

Then $v(y) = y_2$ if $y := (y_1,y_2,y_3) \in \mathbb{R} \times \mathbb{R}_+ \times \{0\}$, $v(y) = 0$ for $y \in \mathbb{R} \times \mathbb{R} \times (-\mathbb{P})$, and $v(y) = \infty$ elsewhere. Consequently, v is not lower semicontinuous at any $y \in \mathbb{R} \times \mathbb{P} \times \{0\}$.

Proof. Observe that $\langle x, c^* \rangle = x_2 \geq 0$ for $x \in P$, and so $v(y) \geq 0$ for every $y \in \operatorname{dom} v = A(P) - Q$. Take $y \in Y$ and $x \in P$. If $Ax - y = (x_1 - y_1, x_2 - y_2, -y_3) \in Q$, then $x_2 \geq 0$, $x_1 \geq y_1$, $y_3 \leq 0$, and $(x_2 - y_2)^2 \leq -2y_3(x_1 - y_1)$. Hence $y \notin \operatorname{dom} v$ for $y_3 > 0$, and so $v(y) = \infty$. Take $y_3 = 0$; then necessarily $x_2 = y_2$. Hence $y \notin \operatorname{dom} v$ for $y_2 < 0$, and so $v(y) = \infty$. If $y_2 \geq 0$, then $x_2 = y_2$; taking $x_1 = y_1$, $x = (x_1,x_2)$ is feasible for (P), and so one obtains $v(y) = y_2$ in this case. Take now $y_3 < 0$. Then $x = (x_1, 0)$ with $x_1 = y_1 - \frac{1}{2}(y_2)^2/y_3$ is feasible for (P), and so $v(y) = 0$ in this case.

[4] This is our reference [6].

Consequently $v(y) = y_2$ if $y \in \mathbb{R} \times \mathbb{R}_+ \times \{0\}$, $v(y) = 0$ for $y \in \mathbb{R} \times \mathbb{R} \times (-\mathbb{P})$, and $v(y) = \infty$ elsewhere. Hence,

$$\operatorname{dom} v = (\mathbb{R} \times \mathbb{R}_+ \times \{0\}) \cup (\mathbb{R} \times \mathbb{R} \times (-\mathbb{P})).$$

Clearly, v is convex (in fact sublinear), but v is not l.s.c. at any $y \in \mathbb{R}^3$ with $y_2 > 0$ and $y_3 = 0$. Indeed, in this case, $\operatorname{dom} v \ni \zeta_n := (y_1, y_2, -1/n) \to y := (y_1, y_2, 0)$ and $v(\zeta_n) = 0 \to 0 < y_2 = v(y)$. □

The next example is an adaptation of [8, Examp. 2.3] to the present context. It shows that the value function v can be proper and not lower semicontinuous at each $y \in \operatorname{dom} v$.

Example 4.2. Let X be a separable infinite-dimensional real Hilbert space with the orthonormal basis $(e_n)_{n\geq 1}$ and

$$P := \left\{ \sum\nolimits_{n\geq 1} \lambda_n z_n \mid (\lambda_n) \in (\ell_2)_+ \right\} \subseteq X, \quad c^* := \sum\nolimits_{n\geq 1} \eta_n e_{2n},$$

with $z_n := \eta_n e_{2n-1} - \mu_n e_{2n}$, where $\eta_n, \mu_n \in]0,1[$ are such that $\eta_n^2 + \mu_n^2 = 1$ for every $n \geq 1$ and $(\eta_n) \in \ell_2 := \{(\lambda_n) \in \mathbb{R}^{\mathbb{N}^*} \mid \sum_{n\geq 1} |\lambda_n|^2 < \infty\}$, and $(\ell_2)_+ := \{(\lambda_n) \in \ell_2 \mid \forall n \in \mathbb{N}^* : \lambda_n \geq 0\}$. Consider $\mathrm{Pr}_L : X \to X$ the orthogonal projection on

$$L := \overline{\operatorname{span}} \{e_{2n-1} \mid n \geq 1\} = \left\{ \sum\nolimits_{n\geq 1} \lambda_n e_{2n-1} \mid (\lambda_n) \in \ell_2 \right\},$$

$A : X \to Y := X$ defined by $Ax := \mathrm{Pr}_L x$, the conic linear programming problems

(P) minimize $\langle x, c^* \rangle$ s.t. $x \in P$, $Ax = b$,

and $v : Y \to \overline{\mathbb{R}}$ defined by $v(y) := \inf\{\langle x, c^* \rangle \mid x \in P, Ax = y\}$ for $y \in Y$.

Then

$$\operatorname{dom} v = \left\{ y := \sum\nolimits_{n\geq 1} \gamma_n e_{2n-1} \mid \left(\gamma_n/\eta_n\right) \in (\ell_2)_+ \right\} \subseteq L, \tag{4.1}$$

$$v(y) = -\sum\nolimits_{n\geq 1} \mu_n \gamma_n \leq 0 = v(0), \quad \forall y := \sum\nolimits_{n\geq 1} \gamma_n e_{2n-1} \in \operatorname{dom} v. \tag{4.2}$$

More precisely, (P) has a unique feasible solution (hence a unique optimal solution) for every $b \in \operatorname{dom} v$, and so v is a proper sublinear function. Moreover, the dual problem (D) of (P) has not feasible solutions for every $b \in Y$, proving so that v is not l.s.c. at any $b \in \operatorname{dom} v$; in particular, $\partial v(b) = \emptyset$ for every $b \in Y$.

Proof. Note that $\langle z_n, z_m \rangle = \delta_{nm}$ for $n, m \geq 1$ (δ_{nm} being the Kronecker's symbols). Clearly, P, Q ($Q = \{0\}$ in the present problem) are closed convex cones and $Q^+ = Y$. Because $\mathrm{Pr}_L = (\mathrm{Pr}_L)^*$, one obtains that

$$\operatorname{dom} v = A(P) - Q = \mathrm{Pr}_L(P), \quad \operatorname{dom} v^* = Q^+ \cap (A^*)^{-1}(c^* - P^+) = \mathrm{Pr}_L^{-1}(c^* - P^+).$$

Consider now $y \in \operatorname{dom} v$ ($\subseteq L$). Hence $y = \sum_{n\geq 1} \gamma_n e_{2n-1}$ with $(\gamma_n) \in \ell_2$, and there exists $x \in P$ such that $y = Ax$; hence $x = \sum_{n\geq 1} \lambda_n z_n = \sum_{n\geq 1} \lambda_n (\eta_n e_{2n-1} - \mu_n e_{2n})$ for some $(\lambda_n) \in (\ell_2)_+$. Therefore, $\gamma_n = \lambda_n \eta_n \geq 0$, whence $\lambda_n = \gamma_n/\eta_n$ for $n \geq 1$. This shows that the set $\{x \in P \mid y = Ax\}$ is a singleton $\{x_y\}$ for (every) $y \in \operatorname{dom} v$ and so

$$v(y) = \langle c^*, x_y \rangle = \sum\nolimits_{n\geq 1} \eta_n(-\lambda_n \mu_n) = \sum\nolimits_{n\geq 1} \eta_n(-\mu_n \cdot \gamma_n/\eta_n) = -\sum\nolimits_{n\geq 1} \mu_n \gamma_n.$$

Consequently, (4.1) and (4.2) hold. Assume that $x \in \operatorname{dom} v^*$. Then there exists $(\lambda_n) \in \ell_2$ such that $u := \mathrm{Pr}_L x = \sum_{n\geq 1} \lambda_n e_{2n-1} \in c^* - P^+$, that is, $c^* - u \in P^+$. Hence

$$0 \leq \langle c^* - u, z_k \rangle = -\lambda_k \eta_k - \eta_k \mu_k = -\eta_k(\lambda_k + \mu_k) \quad \forall k \geq 1.$$

It follows that $\lambda_k + \mu_k \le 0$ for all $k \ge 1$, contradicting the fact that $\lambda_n \to 0$ and $\mu_n \to 1$. Hence $\operatorname{dom} v^* = \emptyset$, and so (D_y) has not feasible solutions for every $y \in Y$. Consequently, $v^* = \infty$, and so $\overline{v}(y) = -\infty$ for every $y \in \operatorname{cl}(\operatorname{dom} v)$ by (1.3). Hence v is not l.s.c. at any $y \in \operatorname{dom} v$. □

5. On Kretschmer's Gap Example in Linear Programming

In [7, pp. 230, 231], Kretschmer considered $Y := L^2 := L^2[0,1]$ (with respect to the Lebesgue measure μ on $[0,1]$) endowed with the usual inner product and ordered by $Q := L^2_+$, as well as $X := L^2 \times \mathbb{R}$ endowed with the inner product defined by $\langle (x,r),(x',r')\rangle := \langle x,x'\rangle + rr'$ and ordered by $P := L^2_+ \times \mathbb{R}_+$. Obviously, $P^+ = P$ and $Q^+ = Q$. Moreover, one takes $A : X \to Y$ with $A(x,r) := y + re_0$ with $y(t) := \int_t^1 x(s)ds$ for $t \in [0,1]$ and $e_0 \in L^2$ with $e_0(t) := 1$ for $t \in [0,1]$. Furthermore, A is a continuous linear operator and A^* $(: Y \to X)$ is given by $A^*y = (x,r) \in X$ with $x(t) := \int_0^t y(s)ds$ for $t \in [0,1]$ and $r = \int_0^1 y(s)ds$.

Let $c^* := c^*_\alpha : X \to \mathbb{R}$ be defined by $c^*(x,r) := \int_0^1 tx(t)dt + \alpha r = \langle (x,r),(e_1,\alpha)\rangle$, where $\alpha \in \mathbb{R}_+$ and $e_1(t) := t$ for $t \in [0,1]$. Clearly, $c^* \in P^+$.

Consider the problem $(\mathrm{P}) := (\mathrm{P}_\alpha)$ and its dual $(\mathrm{D}) := (\mathrm{D}_\alpha)$ defined by

(P) minimize $\langle (x,r),c^*\rangle$ s.t. $(x,r) \ge 0,\ A(x,r) - b \ge 0,$

(D) maximize $\langle z,b\rangle$ s.t. $z \ge 0,\ A^*z \le c^*,$

as well as the value function

$$v := v_\alpha : Y \to \overline{\mathbb{R}}, \quad v_\alpha(y) := \inf\{\langle (x,r),c^*_\alpha\rangle \mid (x,r) \in F(y)\},$$

where

$$F(y) := \{(x,r) \in X \mid (x,r) \ge 0,\ A(x,r) \ge y\}$$

is the feasible set of problem (P). Notice that $F(y)$ is the same for all $\alpha \in \mathbb{P}$. Clearly, $v(y) \ge 0 = v(0)$ for $y \in Y$ because $c^* \in P^+$. Hence $0 \in \partial v(0)$. In fact, by (3.1), (2.1), (2.2), and (1.2), one has

$$\partial v(0) = Q^+ \cap (A^*)^{-1}(c^* - P^+) = Q \cap (A^*)^{-1}(c^* - P) \ \text{ and } \ v^{**} = \overline{v}.$$

Let us denote by $\mathscr{A}$ the class of measurable subsets of $[0,1]$. Without loss of generality, we assume that $y(t) \in \mathbb{R}$ for all $y \in L^2$ and $t \in [0,1]$. For $y \in L^2$ and $\gamma \in \mathbb{R}$, we set $[y \ge \gamma] := \{t \in [0,1] \mid y(t) \ge \gamma\}$ and $y_+ := \max\{y,0\}$. Clearly, $[y \ge \gamma] \in \mathscr{A}$ and $y_+ \in L^2_+$. Moreover, the *characteristic function* of $E \subseteq [0,1]$ is the function $\chi_E : [0,1] \to \mathbb{R}$ defined by $\chi_E(t) := 1$ for $t \in E$ and $\chi_E(t) := 0$ for $t \in [0,1] \setminus E$.

Lemma 5.1. *The following assertions hold:*

(i) *One has*

$$\operatorname{dom} v = \{y \in Y \mid \operatorname{ess\,sup} y < \infty\} = \{y \in L^2 \mid y_+ \in L^\infty\}.$$

In particular, $L^\infty \subseteq \operatorname{dom} v$, *and so* $\operatorname{cl}(\operatorname{dom} v) = Y$.

(ii) *Let* $A \in \mathscr{A}$ *be such that* $\beta := \mu(A) > 0$. *Then there exists a sequence* $(A_n)_{n\ge1} \subseteq \mathscr{A}$ *such that* $A = \cup_{n\ge1} A_n$, $A_n \cap A_m = \emptyset$ *for* $n \ne m$ *and* $\mu(A_n) = 2^{-n}\beta$ *for* $n \ge 1$.

(iii) *Let* A *and* $(A_n)_{n\ge1}$ *be as in* (ii) *and consider*

$$\widetilde{y}_n := \textstyle\sum_{k=1}^n 2^{k/4}\chi_{A_k} \ge 0 \ \ (n \ge 1), \quad \widetilde{y} := \sup_{n\ge1} \widetilde{y}_n \ge 0.$$

Then $\operatorname{ess\,sup} \widetilde{y}_n = 2^{n/4} \to \infty$, $\widetilde{y} \in L^2$, $\|\widetilde{y}_n\| < \|\widetilde{y}\| = \left[\beta(\sqrt{2}+1)\right]^{1/2}$ *and* $\|\widetilde{y}_n - \widetilde{y}\| \to 0$. *Consequently,* $\widetilde{y}_n \in L^\infty_+ \subseteq L^2_+$ *for* $n \ge 1$ *and* $\widetilde{y} \in L^2_+ \setminus L^\infty$.

Proof. (i) Let $y \in \operatorname{dom} v$, Then there exists $(x,r) \in P$ such that $y \le A(x,r)$, and so $y(t) \le \int_t^1 x(s)ds + r \le \int_0^1 x(s)ds + r \le \|x\| + r$ for $t \in [0,1]$ whence $\operatorname{ess\,sup} y < \infty$. Conversely, if $y \in L^2$ is such that $r := \operatorname{ess\,sup} y < \infty$, then $y \le_Q re_0 \le_Q r_+ e_0 = A(0,r_+)$, where $r_+ := \max\{0,r\}$, and so $v(y) \le \langle (0,r_+), c^* \rangle = \alpha r_+$. Hence $y \in \operatorname{dom} v$.

(ii) Because μ has not atoms, there exists $A_1 \subseteq A$ such that $A_1 \in \mathscr{A}$ and $\mu(A_1) = 2^{-1}\beta$ $(\in]0,\mu(A)[)$. Hence $A_1' := A \setminus A_1 \in \mathscr{A}$ and $\mu(A_1') = \mu(A) - \mu(A_1) > 2^{-2}\beta$, and so there exists $A_2 \subseteq A_1'$ such that $A_2 \in \mathscr{A}$ and $\mu(A_2) = 2^{-2}\beta$ $(\in]0,\mu(A_1')[)$; clearly, $A_1 \cap A_2 = \emptyset$. Continuing in the same way, we have the sequence $(A_n)_{n\ge1} \subseteq \mathscr{A}$ with $A_n \subseteq A$, $\mu(A_n) = 2^{-n}\beta$ and $A_n \cap A_m = \emptyset$ for $n,m \in \mathbb{N}^*$ with $n \ne m$. Setting $A' := \cup_{n\ge1} A_n$ one has that $A' \subseteq A$ and $\mu(A') = \sum_{n\ge1} \mu(A_n) = \sum_{n\ge1} 2^{-n}\beta = \beta = \mu(A)$, and so $\mu(A \setminus A') = 0$. Replacing A_1 with $A_1 \cup (A \setminus A')$, it follows that the sequence $(A_n)_{n\ge1}$ has the desired properties.

(iii) Because $A_k \in \mathscr{A}$ for $k \ge 1$, one has that $\widetilde{y}_n$ is measurable for $n \ge 1$. Moreover, because $A_n \cap A_m = \emptyset$ for $n \ne m$ and $\mu(A_n) = 2^{-n}\beta > 0$ for $n \ge 1$, it is clear that $\widetilde{y}_n(t) = 2^{n/4} \ge \widetilde{y}_n(s)$ for all $t \in A_n$ and $s \in [0,1]$, and so $\operatorname{ess\,sup} \widetilde{y}_n = \sup \widetilde{y}_n = 2^{n/4}$ for $n \ge 1$. Hence $(\widetilde{y}_n)_{n\ge1} \subseteq L_+^\infty \subseteq L_+^2$.

Clearly, $\widetilde{y}$ is measurable and $\widetilde{y}(t) = \lim_{n\to\infty} \widetilde{y}_n(t) = \sum_{k=1}^\infty 2^{k/4} \chi_{A_k}(t)$ for $t \in [0,1]$. Because $\widetilde{y}_n \le \widetilde{y}$, one has $2^{n/4} = \operatorname{ess\,sup} \widetilde{y}_n \le \operatorname{ess\,sup} \widetilde{y}$ for $n \ge 1$, and so $\operatorname{ess\,sup} \widetilde{y} = \infty$; hence $\widetilde{y} \notin L^\infty$. On the other hand, for $t \in [0,1]$, one has

$$(\widetilde{y}_n)^2(t) = \sum\nolimits_{k=1}^n 2^{k/2} \chi_{A_k}(t) \ \ (n \ge 1), \quad \widetilde{y}^2(t) = \sum\nolimits_{k=1}^\infty 2^{k/2} \chi_{A_k}(t),$$

and so

$$\|\widetilde{y}_n\|^2 = \sum\nolimits_{k=1}^n 2^{k/2} \mu(A_k) = \beta \sum\nolimits_{k=1}^n 2^{-k/2} = (\sqrt{2}+1)(1-2^{-n/2})\beta,$$
$$\|\widetilde{y}\|^2 = \beta \sum\nolimits_{k=1}^\infty 2^{-k/2} = (\sqrt{2}+1)\beta, \quad \|\widetilde{y}_n - \widetilde{y}\|^2 = 2^{-n/2}(\sqrt{2}+1)\beta.$$

Therefore, $\widetilde{y} \in L_+^2 \setminus L^\infty$, $\|\widetilde{y}_n\| < \|\widetilde{y}\| = \left[\beta(\sqrt{2}+1)\right]^{1/2}$ and $\|\widetilde{y}_n - \widetilde{y}\| \to 0$. □

Proposition 5.1. *Assume that $\alpha > 0$. Then, for every $y \in \operatorname{dom} v$ and every $\rho > 0$, v is not bounded on $B(y,\rho) \cap \operatorname{dom} v$. In particular, $v|_{\operatorname{dom} v}$ is not continuous at each $y \in \operatorname{dom} v$.*

Proof. Consider $y \in \operatorname{dom} v$ and $\rho > 0$, as well as $0 < \eta_0 < \eta_1 < \min\{1,\alpha\}$ and $B := [0,\eta_0] \cup [\eta_1,1]$. Hence $\operatorname{ess\,sup} y < \infty$ by Lemma 5.1(i). For each $k \in \mathbb{N}^*$, set $E_k := [y \ge -k]$. Because $[0,1] = \cup_{k\ge1} E_k$ and $E_k \subseteq E_{k+1}$ for $k \ge 1$, it follows that $\mu(E_k) \to \mu([0,1]) = 1$. Consider $k_0 \ge 1$ such that $\mu(E_{k_0}) > \eta_0 + 1 - \eta_1 = \mu(B)$, and set $\gamma := -k_0$, $A := E_{k_0} \setminus B = E_{k_0} \cap]\eta_0,\eta_1[\subseteq]\eta_0,\eta_1[$ and $\beta := \mu(A)$. Clearly, $E_{k_0} \subseteq A \cup B$, and so $\mu(B) < \mu(E_{k_0}) \le \mu(A) + \mu(B) = \beta + \mu(B)$, whence $\beta > 0$; set also $\delta := \left[\beta(\sqrt{2}+1)\right]^{1/2} > 0$.

Consider now the sets A_n and the functions $\widetilde{y}_n$ for $n \ge 1$ provided by assertions (ii) and (iii) of Lemma 5.1, as well as $\widetilde{y} := \sup_{n\ge1} \widetilde{y}_n$. Hence

$$L_+^\infty \ni \widetilde{y}_n \to^{\|\cdot\|} \widetilde{y} \in L_+^2 \setminus L^\infty \ \text{ and } \ [\forall n \ge 1 : \|\widetilde{y}_n\| < \|\widetilde{y}\| = \delta].$$

Consider also $0 < \varepsilon < \rho/\delta$ and set $y_n := y + \varepsilon \widetilde{y}_n$ for $n \in \mathbb{N}^*$; clearly $y_n \in L^2$ and $\operatorname{ess\,sup} y_n \le \operatorname{ess\,sup} y + \varepsilon \operatorname{ess\,sup} \widetilde{y}_n < \infty$ whence $y_n \in \operatorname{dom} v$ and $\|y_n - y\| = \|\varepsilon \widetilde{y}_n\| < \varepsilon\delta < \rho$ for $n \ge 1$. Moreover, $y_n \to^{\|\cdot\|} y + \varepsilon \widetilde{y} \notin \operatorname{dom} v$. Therefore,

$$(y_n)_{n\ge1} \subseteq B(y,\rho) \cap \operatorname{dom} v \ \text{ and } \ B(y,\rho) \cap (Y \setminus \operatorname{dom} v) \ne \emptyset. \tag{5.1}$$

Let $n \ge 1$ be fixed and consider $(x,r) \in F(y_n)$. Hence

$$x \ge 0, \ \ r \ge 0, \ \text{ and } \ \int_t^1 x(s)ds + r \ge y(t) + \varepsilon \widetilde{y}_n(t) \ \text{ a.e. } t \in [0,1].$$

Because $A_n \subseteq A = E_{n_0} \cap]\eta_0, \eta_1[\subseteq [y \geq \gamma])$, one has

$$\int_t^1 x(s)ds + r \geq y(t) + \varepsilon \widetilde{y}_n(t) \geq \gamma + 2^{n/4}\varepsilon \quad \text{for a.e. } t \in A_n,$$

and so, for a.e. $t \in A_n$, one has

$$\begin{aligned}\langle (x,r), c^* \rangle &= \int_0^1 sx(s)ds + \alpha r \geq \int_t^1 sx(s)ds + \alpha r \geq t\int_t^1 x(s)ds + \alpha r \\ &\geq \eta_0(\gamma + 2^{n/4}\varepsilon - r) + \alpha r = \eta_0(\gamma + 2^{n/4}\varepsilon) + r(\alpha - \eta_0) \\ &\geq \eta_0(\gamma + 2^{n/4}\varepsilon) + r(\eta_1 - \eta_0) \geq \eta_0(\gamma + 2^{n/4}\varepsilon).\end{aligned}$$

Hence, $\langle (x,r), c^* \rangle \geq \eta_0(\gamma + 2^{n/4}\varepsilon)$. Because $(x,r) \in F(y_n)$ is arbitrary, it follows that $v(y_n) \geq \eta_0(\gamma + 2^{n/4}\varepsilon)$. Therefore, $v(y_n) \geq \eta_0(\gamma + 2^{n/4}\rho/\delta)$ for every $n \geq 1$, and so $v(y_n) \to \infty$. Taking into account (5.1), it follows that v is not bounded on $B(y,\rho) \cap \operatorname{dom} v$; moreover, $y \notin \operatorname{int}(\operatorname{dom} v)$ because $B(y,\rho) \cap (Y \setminus \operatorname{dom} v) \neq \emptyset$ for every $\rho > 0$, proving that $\operatorname{int}(\operatorname{dom} v) = \emptyset$. □

Observe that the case $\alpha = 0$ is very special. Indeed, as seen in the proof of Lemma 5.1(i), for $y \in \operatorname{dom} v$, $(0, r_+) \in F(y)$, where $r := \operatorname{ess\,sup} y$, and so $0 \leq v(y) \leq \langle (0, r_+), (e_1, 0) \rangle = 0$. Hence $v(y) = 0$ and the value $v(y)$ is attained. Therefore, $v = \iota_{\operatorname{dom} v}$. On the other hand, for $y \in Y$, z is feasible for dual problem (D_y) if and only if $z \geq 0$, $\int_0^t z(s)ds \leq e_1(t)$ a.e. $t \in [0,1]$ and $\int_0^1 z(s)ds \leq \alpha = 0$, and so $z = 0$ is the only feasible (hence optimal) solution of (D_y). Hence $v^{**}(y) = 0 = \overline{v}(y)$ for every $y \in Y$. Because $v = \iota_{\operatorname{dom} v}$, one has $\overline{v} = \iota_{\operatorname{cl}(\operatorname{dom} v)}$, confirming so that $\operatorname{cl}(\operatorname{dom} v) = Y$.

Taking $\alpha := 2$ and $b := e_0$, one obtains [7, Examp. 5.1]. This is also considered in [1, Examp. 1], as well as the one in which $b := b_0 := \chi_{[0,1/2]}$. The next two results are slight extensions of those related to the "modification" of [7, Examp. 5.1] used in [1, p. 270], the proofs using similar arguments to those in [1].

Proposition 5.2. *Consider $\alpha \in \mathbb{P}$ and $b := \chi_{I\cup J}$ with $I := [0,\delta]$, $J := [\gamma, 1]$, where $0 \leq \delta \leq \gamma < 1$. Then* val(P) $= \alpha$, val(D) $= \min\{1,\alpha\}$, *and* (P), (D) *have optimal solutions; moreover,* val(P) $=$ val(D) $\Leftrightarrow \alpha \in]0,1] \Leftrightarrow \partial v(\chi_{I\cup J}) \neq \emptyset$.

Proof. Clearly, if $(x,r) \in P$ is feasible, then $\int_t^1 x(s)ds + r \geq 1$ for every $t \in [\gamma, 1[$; because $\lim_{[\gamma,1[\ni t \to 1} \int_t^1 x(s)ds = 0$, one gets $r \geq 1$. Because $(0,1)$ is feasible for (P), one has that 0 is an optimal solution for (P) and val(P) $= \alpha$.

Observe that for $z \geq 0$ with $\int_0^t z(s)ds \leq t$ for $t \in [0,1]$ one has $\int_0^1 z(s)ds \leq 1$, and so, when z is feasible for (D) one has $\int_0^1 \chi_{[\gamma,1]} z = \int_\gamma^1 z \leq \int_0^1 z \leq \min\{1,\alpha\}$. Hence $0 \leq$ val(D) $\leq \min\{1,\alpha\} =: \mu$. Take $\eta \in [\gamma, 1[$ and $z := \mu(1-\eta)^{-1}\chi_{[\eta,1]}$ (≥ 0); then $\int_0^t z(s)ds = 0$ for $t \in [0,\eta]$ and $\int_0^t z(s)ds = \mu(1-\eta)^{-1}\int_\eta^t 1ds = \mu\frac{t-\eta}{1-\eta} \leq \mu t \leq t$ for $t \in [\eta, 1]$ and so z is feasible for (D). Moreover, $\int_0^1 z(t)dt = \mu$, and so z is an optimal solution for (D), whence val(D) $= \min\{1,\alpha\}$. Consequently, both problems have optimal solutions, and $\partial v(\chi_{[\gamma,1]}) \neq \emptyset$ if and only if $\alpha \in]0,1]$. □

Taking $\alpha := 2$ and $\delta := \gamma = 0$ one (re)obtains (as already mentioned) the example from [7, Examp. 5.1], as well as the one from [1, p. 270] and the conclusions from there, that is, both problems have optimal solutions, but there is a (positive) duality gap.

Consequently, the previous example shows not only that $v|_{\operatorname{dom} v}$ is not locally Lipschitz, but also that $v|_{\operatorname{dom} v}$ is not l.s.c. on its domain; therefore, [1, Examp. 1] provides a counterexample to [1, Lem. 1].

Proposition 5.3. *Take $\alpha \in \mathbb{P}$ and $b := \chi_{[0,\delta]}$ with $\delta \in]0,1[$. Then* val(P) = val(D) = $\min\{\delta, \alpha\}$ *and* (D) *has optimal solutions; consequently, $\partial v(\chi_{[0,\delta]}) \neq \emptyset$. Furthermore,* (P) *has optimal solutions iff $\alpha \leq \delta$.*

Proof. First observe that for $(x,r) \in P$, the following assertions are equivalent: (x,r) is feasible for (P); $(x \cdot \chi_{[\delta,1]}, r)$ is feasible for (P); $\int_\delta^1 x(s)ds + r \geq 1$; $r \geq \left(1 - \int_\delta^1 x(s)ds\right)_+$. Set

$$F_1 := \left\{x \in L_+^2 \mid \int_\delta^1 x(s)ds \geq 1\right\}, \quad F_2 := \left\{x \in L_+^2 \mid \int_\delta^1 x(s)ds \leq 1\right\}.$$

Notice that $F_1 \cap F_2 \neq \emptyset$ and $0 \in F_2$; moreover, $(x,0)$ is feasible when $x \in F_1$ and $\left(x, 1 - \int_\delta^1 x(s)ds\right)$ is feasible when $x \in F_2$. It follows that val(P) $= \min\{v_1, v_2\}$, where $v_1 := \inf_{x \in F_1} \int_0^1 tx(t)dt$ and

$$\begin{aligned} v_2 :=& \inf_{x \in F_2} \left(\int_0^1 tx(t)dt + \alpha - \alpha \int_\delta^1 x(t)dt \right) = \inf_{x \in F_2} \left(\alpha + \int_\delta^1 (t-\alpha)x(t)dt\right) \\ =& \alpha - \sup_{x \in F_2} \int_\delta^1 (\alpha - t)x(t)dt \leq \alpha. \end{aligned}$$

For $x \in L_+^2$ and $t \in [\delta,1]$ one has $(\alpha - t)x(t) \leq (\alpha - \delta)x(t)$, and so $\int_\delta^1 (\alpha - t)x(t)dt \leq (\alpha - \delta)\int_\delta^1 x(t)dt$, with equality iff $x \cdot \chi_{[\delta,1]} = 0$. Assume that $x \in F_2$; for $\alpha > \delta$ one has $\int_\delta^1 (\alpha - t)x(t)dt \leq \alpha - \delta$, while for $\alpha \leq \delta$ one has $\int_\delta^1 (\alpha - t)x(t)dt \leq 0$. Therefore, $v_2 \geq \delta$ if $\alpha \geq \delta$ and $v_2 = \alpha$ if $\alpha \leq \delta$, v_2 being attained for $x = 0$ in the latter case. In what concerns v_1, one has

$$v_1 = \inf_{x \in F_1} \left(\int_0^\delta tx(t)dt + \int_\delta^1 tx(t)dt \right) = \inf_{x \in F_1} \int_\delta^1 tx(t)dt \geq \delta \inf_{x \in F_1} \int_\delta^1 x(t)dt \geq \delta.$$

For $\varepsilon \in]0, 1-\delta[$ and $x := \varepsilon^{-1}\chi_{[\delta,\delta+\varepsilon]}$, one has $x \in F_1$ and $\int_0^1 tx(t)dt = \varepsilon^{-1}\int_\delta^{\delta+\varepsilon} tdt = \delta + \varepsilon/2$, and so $v_1 = \delta$. Consequently, val(P) $= \min\{\alpha, \delta\}$. Moreover, if $\alpha > \delta$ then (P) has not optimal solutions, and $x = 0$ is solution of (P) if $\alpha \leq \delta$. If z is feasible for (D), then $\int_0^\delta z(t)dt \leq \delta$ and $\int_0^\delta z(t)dt \leq \int_0^1 z(t)dt \leq \alpha$, and so val(D) $\leq \min\{\delta, \alpha\}$. Clearly, $z := \chi_{[0,\min\{\alpha,\delta\}]}$ is an optimal solution of (D), and so val(D) $= \min\{\delta, \alpha\}$. Therefore, val(P) = val(D) $= \min\{\delta, \alpha\}$ and (D) has optimal solutions. Hence $\partial v(\chi_{[0,\delta]}) \neq \emptyset$. Furthermore, (P) has optimal solutions iff $\alpha \leq \delta$. □

Corollary 5.1. *Let $\alpha \in]1,\infty[$ and $\delta \in]0,1[$, and consider the problems*

(P_y) minimize $\int_0^1 tx(t)dt + \alpha r$ *s.t.* $x \geq 0$, $r \geq 0$, $\int_t^1 x(s)ds + r \geq y(t)$ *a.e.* $t \in [0,1]$,

(D_y) maximize $\int_0^1 y(t)z(t)dt$ *s.t.* $z \geq 0$, $\int_0^t z(s)ds \leq t$ *a.e.* $t \in [0,1]$, $\int_0^1 z(s)ds \leq \alpha$.

Then $\partial v(\chi_{[0,\delta]}) \neq \emptyset$ and $v|_{\mathrm{dom}\, v}$ is not continuous at $\chi_{[0,\delta]}$.

Proof. By Proposition 5.3, one has that $v(\chi_{[0,\delta]}) = \delta$ and $\partial v(\chi_{[0,\delta]}) \neq \emptyset$, while from Proposition 5.2 one has that $v(\chi_{[0,\delta]\cup[\gamma,1]}) = \alpha$ for every $\gamma \in]\delta,1[$. Because $\|\chi_{[0,\delta]\cup[\gamma,1]} - \chi_{[0,\delta]}\|_2 = \|\chi_{[\gamma,1]}\|_2 = (1-\gamma)^{1/2} \to 0$ for $\gamma \to 1$, clearly $v|_{\mathrm{dom}\, v}$ is not continuous at $\chi_{[0,\delta]}$. □

In the paragraph before [1, Examp. 1, p. 269], one says:

Q7 – "We give an example of a convex function which is subdifferentiable but not locally Lipschitz by exhibiting a linear programming problem for which the value function has this property. The example takes place in the Banach lattice $L^2[0,1]$ (a space for which the positive cone has empty interior) and *for which* $\mathrm{dom}\, v \supseteq L^2[0,1]_+$". (Our emphasis.)

This text is completed by the following ones from [1, p. 270]:

Q8 – "On the other hand, we will establish that v is not locally Lipschitz at b_0; in fact, v is not even continuous there (*or anywhere*)." (Our emphasis.)

Q9 – "*Similar perturbations* show that v is not continuous anywhere on $L^2[0,1]_+$. (*Of course, v is lower semicontinuous*.)" (Our emphasis.)

As seen in Lemma 5.1, one has $L_+^\infty \subseteq L^\infty \subseteq \operatorname{dom} v \not\supseteq L_+^2 \supseteq L_+^\infty$, which shows that the inclusion $\operatorname{dom} v \supseteq L^2[0,1]_+$, mentioned in Q7, is not true.

Having in view the texts from Q7, Q8 and Q9, one may wonder what is meant in [1] by continuity and lower semicontinuity of v at some point in Y, as well as by local Lipschitzness and subdifferentiability.

In what concerns the local Lipschitzness, it is quite clear that this is meant in the sense from condition 2. in Q4; related to "subdifferentiability", this is not at any point b with $v(b) \in \mathbb{R}$ as suggested by Q7, but just at a certain point b as in Q8. As seen in Section 2, the are important differences among the continuity properties of g and $g|_{\operatorname{dom} g}$ at points from $\operatorname{dom} g$. In fact, inspecting the proof of [1, Lem. 1] and the discussion of the modified version of [1, Examp. 1], in [1] one has in view the continuity and the lower semicontinuity of $v|_{\operatorname{dom} v}$ at points in $\operatorname{dom} v$.[5]

Having in view Proposition 5.1, we agree with the remark "v is not even continuous there (or anywhere)" from Q8. In what concerns Q9, on one hand, we would like to see those "similar perturbations" which "show that v is not continuous anywhere on" $L^2[0,1]_+ \cap \operatorname{dom} v$; on the other hand, as already mentioned, we do not agree with the remark "Of course, v is lower semicontinuous", which is surely based on [1, Lem. 1].

6. Some Comments on Proposition 2 from [1]

In Section 6 of [1], one establishes two results on the Lipschitzness of the value function in infinite-dimensional linear programming; the second one, Proposition 2, is applied to the assignment model in [1, Sect. 7]. Our aim is to discuss the proof of [1, Prop. 2]. For easy reference, we quote its statement and proof, as well as its preamble:

Q10 – "Another structural condition is useful for application to the assignment model. We will use the condition in the context of a maximization problem and will state it as such.

PROPOSITION 2. Let X and Y be *Banach lattices* and let A, b, and c be the data for an LP maximization problem.[6] Assume that

- A is a positive operator which maps the positive cone X_+ onto Y_+;
- the order interval $[0, x_0]$ is mapped onto the order interval $[0, Ax_0]$ for every $x_0 \geq 0$;
- A is bounded below on the positive cone X_+, i.e. there exists a constant $M > 0$ such that $\|Ax\| \geq M\|x\|$ for all $x \geq 0$.

Then the value function is Lipschitz on the positive cone X_+.[7]

Proof. Start with $b_1 \geq 0$ and $b_2 \geq 0$. First, consider the case that $b_2 \leq b_1$. Given $\varepsilon > 0$, there is an almost optimal x_1 for b_1, viz. *there is* $x_1 \geq 0$ *with* $Ax_1 = b_1$, and $c^*(x_1) + \varepsilon > v(b_1)$. Since $0 \leq b_2 \leq b_1 = Ax_1$ and since the positive operator A maps $[0, x_1]$ onto $[0, Ax_1]$, there is x_2 such that $0 \leq x_2 \leq x_1$ with $Ax_2 = b_2$. Clearly, x_2 is feasible for b_2; hence, $v(b_2) \geq c^*(x_2)$.

We compute

$$v(b_1) - v(b_2) \leq c^*(x_1) + \varepsilon - c^*(x_2) \leq \|c^*\| \, \|x_1 - x_2\| + \varepsilon \leq \|c^*\| \, \tfrac{1}{M} \, \|Ax_1 - Ax_2\| + \varepsilon$$

[5] Recall that $v|_{\operatorname{dom} v}$ is not l.s.c. at every $y \in \mathbb{R} \times \mathbb{P} \times \{0\}$ in Example 4.1, and $v|_{\operatorname{dom} v}$ is not l.s.c. at every $y \in \operatorname{dom} v$ in Example 4.2.

[6] Of course, it is c^* instead of c.

[7] In fact, it is Y_+ instead of X_+.

$\leq \|c^*\| \frac{1}{M} \|b_1 - b_2\| + \varepsilon$

Since this true for arbitrary $\varepsilon > 0$, we have that $v(b_1) - v(b_2) \leq \frac{1}{M} \|c^*\| \|b_1 - b_2\|$
Switching the roles of b_1 *and* b_2 gives us $|v(b_1) - v(b_2)| \leq \frac{1}{M} \|c^*\| \|b_1 - b_2\|$ as desired.

For the general case in which *we do not assume any order dominance between* x_1 *and* x_2, define $x_3 = x_1 \wedge x_2$. Then $b_3 = b_1 - (b_1 - b_2)^+$; i.e., $b_1 - b_3 = (b_1 - b_2)^+$. Consequently,
$\|b_1 - b_3\| \leq \|(b_1 - b_2)^+\| \leq \|b_1 - b_2\|$.
Since $0 \leq b_3 \leq b_2$ and v *is an increasing function*, we have that
$v(b_1) - v(b_2) \leq v(b_1) - v(b_3) \leq c \leq \|c^*\| \frac{1}{M} \|b_1 - b_2\|$
The same x_3 works for $v(b_2) - v(b_1)$ and we have shown that v is Lipschitz on Y.[8]" (Our emphasis.)

Remarks:

1) Even if not clearly stated, the considered problem is: maximize $c^*(x)$ s.t. $Ax \leq b$ and $x \geq 0$; compare with problem (P) on [1, p. 273] to which [1, Prop. 2] is applied; this is also confirmed by the argument "Since $0 \leq b_3 \leq b_2$ and v *is an increasing function*, we have that ..." from the end of the proof.

Set $F(b) := \{x \in X \mid x \geq 0, Ax \leq b\}$ (the feasible set corresponding to $b \in Y_+$).

2) (One had to) Observe first that $F(b)$ is bounded, and so $v(b) \in \mathbb{R}_+$, for every $b \in Y_+$

3) By 2) and the definition of $v(b_1)$, for each $\varepsilon > 0$ there exists $x_1 \in F(b_1)$ such that $c^*(x_1) + \varepsilon > v(b_1)$; hence $x_1 \geq 0$ and $Ax_1 \leq b_1$.

So, why $Ax_1 = b_1$? Without having $Ax_1 = b_1$ one cannot find (using the hypotheses) $x_2 \in [0, x_1]$ such that $Ax_2 = b_2$ because b_2 could be outside $[0, Ax_1]$. How is the argument continued?

4) Assume that for each $\varepsilon > 0$ one finds $x_1 \in F(b_1)$ such that $Ax_1 = b_1$ and $c^*(x_1) + \varepsilon > v(b_1)$. "Switching the roles of b_1 and b_2", will b_2 have the same property, that is, for each $\varepsilon > 0$ one finds $x_2 \in F(b_2)$ such that $Ax_2 = b_2$ and $c^*(x_2) + \varepsilon > v(b_2)$? *If so, we agree with the estimate* $|v(b_1) - v(b_2)| \leq \frac{1}{M} \|c^*\| \|b_1 - b_2\|$.

5) 5a) The particular case was the one in which $(0 \leq)$ $b_2 \leq b_1$, that is, the case in which b_1 and b_2 are comparable.

5b) Hence, the general case must be "the one in which we do not assume any order dominance between" b_1 and b_2.

5c) Under 5b), which are x_1 and x_2 here? and which is b_3? is it Ax_3?

5d) We agree with $x_3 = x_1 \wedge x_2 \Rightarrow x_3 = x_1 - (x_1 - x_2)^+$. Assume that $b_k = Ax_k$ for $k \in \{1, 2, 3\}$ (which could be envisaged because one had already $b_k = Ax_k$ for $k \in \{1, 2\}$). Because $b_3 = b_1 - (b_1 - b_2)^+ = b_1 \wedge b_2$, one must have $A(x_1 \wedge x_2) = (Ax_1) \wedge (Ax_2)$ for $x_1, x_2 \in X_+$ (or, equivalently, for $x_1, x_2 \in X$). Do the imposed conditions on the data of [1, Prop. 2] ensure that A is a homomorphism of Banach lattices?

6) Probably, c from the inequality $v(b_1) - v(b_3) \leq c$ is $\|c^*\| \frac{1}{M} \|b_1 - b_3\|$, gotten because $0 \leq b_3 \leq b_1$.

Having in view the above remarks, we consider that the proof of [1, Prop. 2] needs several clarifications.

So, in our opinion, the authors of [1] did not succeed to accomplish their goal that emerges from the following text taken from the beginning of Section 2 of [1]:

[8] Of course, it must be Y_+ instead of Y.

Q11 – "*The present study was motivated by the problem of showing that there was no gap in the* infinite-dimensional linear programming *problem that arose in* our studies of *the continuum assignment problem in* [3]. *The no-gap argument given there was incomplete; the current paper rectifies that omission*." (Our emphasis.)

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