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THE STABILITY OF THE PARAMETRIC CAUCHY PROBLEM OF INITIAL-VALUE ORDINARY DIFFERENTIAL EQUATIONS REVISITED

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Abstract. In this paper, given a function $f : I \times V \to \mathbb{R}^m$, where V is an open subset of $\mathbb{R}^m$, $x_0 \in V$, and $I = [0,T]$ is the interval of interest, we consider the Cauchy ordinary differential equation initial-value problem $\dot{x}(f,x_0) = f(t,x(t)), x(0) = x_0$. We first present a new quantitative stability result under a partial and/or global variation of the data of the problem by involving exact and/or approximate fixed points for which we apply Lim's Lemma either in its exact format or in its very recent approximate version. Our main result is then applied to parametric linear control systems. Finally, we demonstrate that our treatment is coherent with the management of perturbations generated in the classic one-step numerical method. A numerical example written in Scilab 6.1 illustrates the obtained stability.

Keywords. Cauchy problem; Exact solutions; Initial-value problem; Parametric perturbation; Quantitative stability.

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1. Introduction

The principal aim of this paper is to provide a generalization of the results of [1] on the quantitative stability of exact solutions to perturbed initial-value problems of ordinary differential equations to the case of approximate solutions. In [1], the main results are based on direct computations with the use of Gronwall Lemma. Here, the key idea is to involve stability of fixed points using the famous Lim's Lemma [2], which provides an estimate between exact fixed points of two contracting maps. Naturally, such a result requires that the dependence on a parameter, say λ, of the function, say f, of the underlying Cauchy problem to be of a Lipschitzian type. In this way, we obtain a stability with respect to variation on the data of a quantitative aspect, which measures the distance between solutions. When this dependence is just continuous, we rather obtain a qualitative stability in the sense that a net, say x_λ, of the solutions to a net of Cauchy problems, converges to the solution $x_{\overline{\lambda}}$ of a nominal Cauchy problem as λ converges to a nominal value $\overline{\lambda}$ of the parameter. A further new key observation of the present paper is to involve the approximate version of Lim's Lemma established recently by Ait Mansour *el al* in [3], which leads to stability assertions not only for exact solutions but also for approximate

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solutions. In this respect, we essentially establish stability results with respect either to a partial parametric perturbation or a global variation of the function of the corresponding Cauchy problem. Accordingly, we generalize the results of [1] to approximate solutions under parametric perturbations without recourse to Gronwall's techniques. Thereafter, we refine the stability of the observability variable in parametric linear control systems considered in [1] which we support here by an additional numerical illustration to highlight the convergence and estimation rates written with Scilab 6.1.

The organization of the paper is detailed as follows. In Section 2, we recall the necessary preliminaries related to the Cauchy problem of initial-value ordinary differential equations. This section also includes the basic results of stability of fixed points, and the so-called Lim's lemma is underlined in its exact and approximate formats. Sections 3 is devoted to our stability results. First, with respect to parametric perturbations, we present a variant of the main stability Theorem of [1] under slightly general assumptions (Theorem 3.2). Consequently, we cover an interesting qualitative result in Theorem 3.3. The extension to the case of global variation of the main function of the Cauchy problem under treatment is done in Theorem 3.4. Accordingly, we find again with different but comparable error bounds the stability estimates of the parametric case in Corollary 3.1. In Section 4, we discuss an application to the observability equation in linear control systems by refining the result of [1] (see Theorem 4.1). In Section 5, we obtain an extension of our quantitative stability to the case of approximate solutions, Theorem 5.2. In Section 6, we emphasize that our quantitative stability when applied to ordinary differential equations conducts to a coherent conclusion for the parametric perturbations arising in numerical methods, which we clarify for one-step methods. Finally, Section 7 focuses the attention on a numerical example related to linear control systems wherein we stress the linear rate of convergence of our stability results for the observability variable.

2. The Cauchy Problem and Preliminaries

2.1. **The Cauchy problem of initial-value ordinary differential equation.** Throughout this paper, unless otherwise is specified, the Euclidian space $\mathbb{R}^n$ is equipped with supremum norm $\|.\|$. For a given point $x_0 \in \mathbb{R}^n$ and a nonnegative real-number $r > 0$, we denote by $B(x_0, r)$ the ball with center x_0 and radius r and consider a real-valued function $f : [0,T] \times B(x_0, r) \longrightarrow \mathbb{R}^n$, where T is a nonnegative real number standing for the final time of the interval of interest. Then, the corresponding Cauchy problem of initial-value ordinary differential equation associated with these data is as follows:

$$S(f,x_0) \begin{cases} x'(t) = f(t,x(t)), \text{ for } a.e\; t \in [0,T] \\ x(0) = x_0. \end{cases}$$

Let U be an open subset of $\mathbb{R}^n$ and $T > 0$. We recall that a function $f : [0,T] \times U \longrightarrow \mathbb{R}^n$ is said to be L^1-Carathéodory if the following conditions hold:

- (H_1) the map $t \longmapsto f(t,x)$ is a measurable for each $x \in U$;
- (H_2) the map $x \longmapsto f(t,x)$ is continuous for almost all $t \in [0,T]$;
- (H_3) there exists a function ,$m \in L^1([0,T], \mathbb{R}^+)$ such that $\|f(t,x)| \leq m(t)$ for almost all $t \in [0,T]$.

Theorem 2.1 ([4, Theorem 1])**.** *Let $T > 0$, $x_0 \in \mathbb{R}^n$ and let $f : [0,T] \times B(x_0, r) \longrightarrow \mathbb{R}^n$ be a L^1-Carathéodory function. Then, for any real number d such that $0 < d \leq T$, $\int_0^d m(s)ds \leq r$, the*

Cauchy problem $x' = f(t,x)$ subject to the initial condition $x(0) = x_0$ admits a unique solution on $[0,d]$.

Theorem 2.2 ([4, Theorem 2]). *Assume that there exists an integrable function l such that*

$$\|f(t,x) - f(t,y)\| \leq l(t)\|x-y\| \quad \text{for all } (t,x) \text{ and } (t,y) \in [0,T] \times B(x_0,r).$$

Then Cauchy problem $x' = f(t,x),\ x(0) = x_0$ admits at most one solution on $[0,T] \times B(x_0,r)$.

2.2. **Exact and approximate versions of Lim's Lemma.** Let X and Y be two normed vector spaces whose norm is denoted by $\|.\|$. For any nonempty subset A of X and any point $x \in X$, $d(x,A) = \inf\{\|x-y\| : y \in A\}$ stand for the distance from x to A whereas $d(x,\emptyset) = \infty$. If B is another nonempty subset of X, $e(A,B)$ denotes the excess of A on B given by $e(A,B) = \sup\{d(a,B) :\ a \in A\}$. We adopt the convention $e(\emptyset,A) = 0$ for any subset $\emptyset \neq A \subset X$ and $e(A,\emptyset) = +\infty$. The *extended Hausdorff distance* between two subsets A and B of X is given by

$$h(A,B) = \max\{e(A,B), e(B,A)\}.$$

Notice that the word "extended" refers to the possibility of the distance to be ∞. The minimal distance between two nonempty subsets A, B of X is denoted and given by

$$d(A,B) = \inf\{\|x-y\| \ :\ (x,y) \in A \times B\}.$$

When one of the sets A and B is empty, we set $d(A,B) = h(A,B) = +\infty$. For a given map $\Phi : X \rightrightarrows X$, for every $\varepsilon \geq 0$, we consider the notation $\varepsilon\text{-Fix}(\Phi) := \{x \in X \mid d(x,\Phi(x)) \leq \varepsilon\}$ to refer to the set of ε-approximate fixed points of Φ while we write $\text{Fix}(\Phi)$ to stand for fixed points of Φ, i.e., $x \in \text{Fix}(\Phi)$ if and only if $x \in \Phi(x)$.

Theorem 2.3 ([3, Theorem 15]). *Let X be a metric space, and let $T_1 : X \rightrightarrows X$ and $T_2 : X \rightrightarrows X$. Suppose that both T_1 and T_2 are Lipschitz continuous on X with the same Lipschitz constant $\lambda \in [0,1)$. Then, for every $\varepsilon > 0$, the set of ε-approximate fixed points of T_i, $i = 1,2$, is nonempty i.e., $\varepsilon\text{-Fix}(T_i) \neq \emptyset$. Moreover,*

$$h(\varepsilon\text{-Fix}(T_1), \varepsilon\text{-Fix}(T_2)) \leq \frac{\varepsilon}{1-\lambda} + \frac{1}{1-\lambda} \sup_{x \in X} h(T_1(x), T_2(x)).$$

If X is in addition complete, then Theorem 2.3 implies Lim's Lemma.

Theorem 2.4 ([2, Lemma 1]). *Let X be a complete metric space, and let T_1 and T_2 map X into the family of nonempty closed subsets of X. Suppose that both T_1 and T_2 are Lipschitz continuous on X with the same Lipschitz constant $\lambda \in [0,1)$. Then*

$$h(\text{Fix}(T_1), \text{Fix}(T_2)) \leq \frac{1}{1-\lambda} \sup_{x \in X} h(T_1(x), T_2(x)).$$

Theorem 2.5 ([5, Theorem 3.1]). *Let M and N be closed subsets of a metric space endowed with a metric d. Assume that N satisfies the Bolzano-Weierstrass condition, i.e., each bounded sequence in N has a convergent subsequence. Then, for all $x \in M$, there exists $y \in N$ such that*

$$d(x,y) \leq h(M,N).$$

3. THE STABILITY RESULTS

We consider perturbed formats of the system $S(f,x_0)$, which involves an external parameter λ that belongs to another space. Precisely, λ is localized in some open subset U of a some normed space $(\Lambda, \|\,.\,\|)$. The parametric Cauchy problem under consideration is as follows :

$$S(f_\lambda, x_\lambda^0) \begin{cases} x'_\lambda(t) = f_\lambda(t, x_\lambda(t)) \\ x_\lambda(0) = x_\lambda^0 \end{cases}$$

where $f_\lambda : [0,T] \times B(x_0, r) \longrightarrow \mathbb{R}^n$. The initial value of the parameter λ is denoted by $\overline{\lambda}$: $f_{\overline{\lambda}} = f$, $x_{\overline{\lambda}} = x$ and $x_{\overline{\lambda}}(0) = x_0$.

Using direct computations based on the famous Gronwall Lemma, the author of [1] proved the following:

Theorem 3.1 ([1, Theorem 2])**.** *Assume that for some $L > 0, L' > 0$ the following conditions hold:*

(h_1) *f is L-Lipschitz continuous w.r to x, uniformly in t and λ;*
(h_2) *f is L'-Lipschitz continuous w.r to λ, uniformly in t and x.*

Then, for all $t \in [0,T]$, the following estimate is satisfied

$$\|x(t) - x_\lambda(t)\| \le e^{Lt}\|x_0 - x_{\lambda_0}\| + \frac{L'}{L}(e^{Lt} - 1)\|\,\lambda - \overline{\lambda}\|.$$

Next, we state and prove a variant of Theorem 3.1 by means of a different argument based on Lim's Lemma (Theorem 2.4).

Theorem 3.2. *Suppose the following conditions hold:*

1) *the map $t \longmapsto f_\lambda(t,x)$ is a measurable for each $x \in \mathbb{R}^n$ and $\lambda \in U$;*
2) *there exists a function $m \in L^1([0,T], \mathbb{R}^+)$ such that $\|f_\lambda(t,x)\| \le m(t)$ for almost all $t \in [0,T]$;*
3) *there exists a integrable function l such that*

$$\|f_\lambda(t,x) - f_\lambda(t,y)\| \le l(t)\|x-y\| \text{ for all } (t,x); \text{and } (t,y) \in [0,T] \times \mathbb{R}^n;$$

4) *the map f_λ is L-Lipschitz in λ uniformly in t and x.*

Let x_λ the unique solution to Cauchy problem $S(f_\lambda, x_\lambda^0)$. Then,

$$\|x - x_\lambda\|_L \le \; e^{\gamma(T)} \left(\|x_0 - x_{\lambda_0}\| + LT\|\lambda - \overline{\lambda}\| \right),$$

with $\gamma(T) = \int_0^T l(s)ds$.

<u>**Notation**</u>: In the Banach space $C([0,T], \mathbb{R}^n)$ we introduce the norm $\|\,.\,\|_L$ given for a function x by $\|x\|_L = \sup_{t\in[0,T]} e^{-\gamma(t)}\|x(t)\|$ where $\gamma(t) = \int_0^t l(s)ds$.

Proof of Theorem 3.2. Consider the two maps Γ_{f,x_0} and $\Gamma_{f_\lambda, x_\lambda^0}$ defined respectively by

$$\begin{aligned} \Gamma_{f,x_0} : C([0,T], \mathbb{R}^n) &\longrightarrow C([0,T], \mathbb{R}^n) \\ x &\longmapsto x_0 + \int_0^t f(s, x(s))ds \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \Gamma_{f_\lambda, x_\lambda^0} : C([0,T], \mathbb{R}^n) &\longrightarrow C([0,T], \mathbb{R}^n) \\ x &\longmapsto x_\lambda^0 + \int_0^t f_\lambda(s, x(s))ds. \end{aligned}$$

It is easy to observe that these two mappings are contractions with the same constant $1 - e^{-\gamma(T)}$. Let $x \in C([0,T], \mathbb{R}^n)$.

$$\begin{aligned} h(\Gamma_{f,x_0}(x), \Gamma_{f_\lambda,x_\lambda^0}(x)) &= \|\Gamma_{f,x_0}(x) - \Gamma_{f_\lambda,x_\lambda^0}(x)\|_L \\ &\leq \|x_0 - x_\lambda^0\| + \sup_{t\in[0,T]} e^{-\gamma(t)} \| \int_0^t (f(s,x(s)) - f_\lambda(s,x(s)))ds\| \\ &\leq \|x_0 - x_\lambda^0\| + \sup_{t\in[0,T]} e^{-\gamma(t)} \| \int_0^t (f(s,x(s)) - f_\lambda(s,x(s)))ds\|. \end{aligned}$$

In view of $\mathrm{Fix}(\Gamma_{f,x_0}) = \{x\}$ and $\mathrm{Fix}(\Gamma_{f_\lambda,x_\lambda^0}) = \{x_\lambda\}$, we obtain by applying Theorem 2.4 that

$$\|x - x_\lambda\|_L \leq e^{\gamma(T)} \left(\|x_0 - x_\lambda^0\| + \sup_{x\in C([0,T],\mathbb{R}^n)} \sup_{t\in[0,T]} e^{-\gamma(t)} \| \int_0^t (f(s,x(s)) - f_\lambda(s,x(s)))ds\| \right).$$

Since f_λ is L-Lipschitz in λ, one has

$$\|x - x_\lambda\|_L \leq e^{\gamma(T)} \left(\|x_0 - x_\lambda^0\| + LT\|\lambda - \overline{\lambda}\| \right).$$

This completes the proof. □

By replacing the Lipschitzian character by the continuity in λ, we obtain the following qualitative stability.

Theorem 3.3. *Suppose that the following conditions are satisfied:*

1) the map $t \longmapsto f_\lambda(t,x)$ is a measurable for each $x \in \mathbb{R}^n$ and $\lambda \in U$

2) there exists $m \in L^1([0,T], \mathbb{R}^+)$ such that $\|f_\lambda(t,x)\| \leq m(t)$ for almost all $t \in [0,T]$.

3) there exists an integrable function l such that

$$\|f_\lambda(t,x) - f_\lambda(t,y)\| \leq l(t)\|x - y\| \text{ for all } (t,x) \text{ and } (t,y) \in [0,T] \times \mathbb{R}^n.$$

4) for almost all $t \in [0,T]$, the function $\lambda \longmapsto f_\lambda(t,x)$ be continuous in $\overline{\lambda}$.

Let x_λ be the unique solution to Cauchy problem $S(f_\lambda, x_\lambda^0)$. Then, x_λ converges uniformly to $x = x_{\overline{\lambda}}$ as $\lambda \to \overline{\lambda}$ and $x_\lambda^0 \to x_0$.

Proof. Following the proof of the previous Theorem at the level of (3), by using assumptions 3) and 5) (as well as the continuity theorem under integral sign) we see that $\int_0^t f_\lambda(s,x(s))ds$ converges to $\int_0^t f(s,x(s))ds$ when λ converges to $\overline{\lambda}$, uniformly in t and x. This completes the proof. □

Now, we present a generalization of Theorem 3.2 with a global variation on the function of the Cauchy problem. To do that, let $x_0, \tilde{x_0} \in X$ be two initial conditions, and let $f : [0,T] \times \overline{B}(x_0,r) \to X$ and $g : [0,T] \times \overline{B}(\tilde{x_0},r) \to X$ be two real-valued functions corresponding to two Cauchy problems of differential equations. Let us set

$$M_f = \sup\{|f(t,x)|,\ (t,x) \in [0,T] \times \overline{B}(x_0,r)\}$$

and

$$M_g = \sup\{|g(t,x)|,\ (t,x) \in [0,T] \times \overline{B}(\tilde{x_0},r)\}.$$

We assume that $(H) : \|x_0 - \tilde{x_0}\| \leq \frac{r}{2}$ and fix the following notation

$$\begin{cases} M = \max(M_f, M_g) \\ \tau = \min\left(T, \dfrac{r}{2M}\right) \\ B = \overline{B}(x_0,r) \cap \overline{B}(\tilde{x_0},r). \end{cases}$$

The Banach space $C([0,\tau],B)$ is equipped with the norm $\|x\|_{\infty,B} = \sup\{\|x(t)\|,\ t \in [0,\tau]\}$.

Theorem 3.4. *Let f and g be Lipschitz continuous functions with respect to x uniformly in t with the same Lipschitz constant $L > 0$ with $L\tau < 1$. Let x_f (resp x_g) be the unique solution to $S(f,x_0)$ (resp $S(g,\tilde{x_0})$) on the interval $[0,\tau]$. Then*

$$\|x_f - x_g\|_{\infty,B} \leq \frac{1}{1-L\tau}\|x_0 - \tilde{x_0}\| + \frac{\tau}{1-L\tau}\|f-g\|_\infty. \tag{3.2}$$

Proof. Consider the maps Γ_{f,x_0} and $\Gamma_{g,\tilde{x_0}}$ defined as in (3.1). They are the contractions of the Banach space $C([0,\tau],B)$ with the same constant $L\tau$. Since $\mathrm{Fix}(\Gamma_{f,x_0}) = \{x_f\}$ and $\mathrm{Fix}(\Gamma_{g,\tilde{x_0}}) = \{x_g\}$, then it follows that

$$h(\mathrm{Fix}(\Gamma_{f,x_0}), \mathrm{Fix}(\Gamma_{g,\tilde{x_0}})) = \|x_f - x_g\|_{\infty,B}.$$

Fix a point $x \in C([0,\tau],B)$, and let $y_1 = \Gamma_{f,x_0}(x)$ and $y_2 = \Gamma_{g,\tilde{x_0}}(x)$. Then

$$y_1(t) = x_0 + \int_0^t f(s,x(s))ds, \text{ and } y_2(t) = \tilde{x_0} + \int_0^t g(s,x(s))ds \text{ for all } t \in [0,\tau].$$

Therefore,

$$\|y_1 - y_2\|_\infty = \sup_{t\in[0,\tau]} \|y_1(t) - y_2(t)\| = \sup_{t\in[0,\tau]} \|x_0 - \tilde{x_0} + \int_0^t (f(s,x(s)) - g(s,x(s)))ds\|.$$

On the other hand, given that the functions f and g are continuous on the compact $[0;\tau] \times B$ it follows that

$$\begin{aligned} \|y_1 - y_2\|_\infty &\leq \|x_0 - \tilde{x_0}\| + \sup_{t\in[0,\tau]} \int_0^t \|f(s,x(s)) - g(s,x(s))\|ds \\ &\leq \|x_0 - \tilde{x_0}\| + \tau\|f-g\|_\infty. \end{aligned}$$

Accordingly, for all $x \in C([0,\tau],B)$,

$$h(\Gamma_{f,x_0}(x), \Gamma_{g,\tilde{x_0}}(x)) \leq \|x_0 - \tilde{x_0}\| + \tau\|f-g\|_\infty.$$

Thus, from Theorem 2.4, it results that

$$h(\mathrm{Fix}(\Gamma_{f,x_0}), \mathrm{Fix}(\Gamma_{g,\tilde{x_0}})) \leq \frac{1}{1-L\tau} \sup_{x\in C([0,\tau],B)} h(\Gamma_{f,x_0}(x), \Gamma_{g,\tilde{x_0}}(x)).$$

This concludes the required estimate, i.e.,

$$\|x_f - x_g\|_{\infty,B} \leq \frac{1}{1-L\tau}\|x_0 - \tilde{x_0}\| + \frac{\tau}{1-L\tau}\|f-g\|_\infty.$$

□

Obviously, from Theorem 3.4, we immediately obtain the quantitative stability of the parametric case as follows.

Corollary 3.1. *Assume that*

(1) *the functions f, f_λ are Lipschitz continuous in x uniformly in t of same Lipschitz constant $L > 0$ with $L\tau < 1$ for all $\lambda \in U$;*
(2) *the functions f_λ is L'-Lipschitz in λ uniformly in t and x for all $\lambda \in U$.*

Let x (resp x_λ) be the unique solution to $S(f,x_0)$ (resp $S(f_\lambda, x_\lambda^0)$). Then

$$\|x - x_\lambda\|_{\infty,B} \leq \frac{1}{1-L\tau}\|x_0 - x_\lambda^0\| + \frac{\tau L'}{1-L\tau}\|\lambda - \overline{\lambda}\|. \tag{3.3}$$

Proof. Let us take in Theorem 3.4 the functions $g = f_\lambda,\ \tilde{x_0} = x_\lambda^0,\ f = f_{\overline{\lambda}}$, and $x_0 = x_{\overline{\lambda}}(0.)$ Then, in this case, the inequality (3.2) turns out to be

$$\|x - x_\lambda\|_{\infty} \leq \frac{1}{1-L\tau}\|x_0 - x_\lambda^0\| + \frac{\tau}{1-L\tau}\|f - f_\lambda\|_\infty.$$

But f_λ is L'-Lipschitz in λ. It follows that

$$\|x - x_\lambda\|_{\infty,B} \leq \frac{1}{1-L\tau}\|x_0 - x_\lambda^0\| + \frac{\tau L'}{1-L\tau}\|\lambda - \overline{\lambda}\|.$$

□

Remark 3.1. If the initial condition is not perturbed, then the solution x_λ of the parametric system converges uniformly to the solution x_f when λ converges to $\overline{\lambda}$.

4. Application to Linear Control Systems

Now we give an application in linear control systems. For given 2×2 matrices A,B,C,D, and a initial condition x_0, we consider the following standard automatic system:

$$S(A,B,u,C,D,x_0): \begin{cases} x'(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \\ x(0) = x_0 \quad , & y_0 = Cx_0 + Du(0), \end{cases}$$

where $x(t)$ is the state variable, $y(t)$ is the observability variable, while $u(t)$ denotes the control of the system. In this example of applications, the function f is nothing but $f(t,x(t)) = Ax(t) + Bu(t)$. We consider here a linear perturbation of the matrix A, i.e., $A = (a_{i,j})_{i,j=1,2}$ and

$$A_\lambda = \begin{pmatrix} a_{11} + \lambda - \overline{\lambda} & a_{12} \\ a_{21} & a_{21} + \lambda - \overline{\lambda} \end{pmatrix},$$

with $\overline{\lambda}$ the initial value of λ. Define therefore f_λ as follows

$$f_\lambda(t,x(t)) = A_\lambda x(t) + Bu(t).$$

Of course, f_λ is $\|A_\lambda\|$-Lipschitz in x, where $\|A_\lambda\|$ is the norm of the matrix A_λ. We suppose that there exist $r > 0$ and a neighborhood $\nu(\overline{\lambda})$ of $\overline{\lambda}$ such that $x \in B(x_0,r) \cap B(x_\lambda^0, r)$, and $x_\lambda \in B(x_0,r) \cap B(x_\lambda^0, r)$ for all $\lambda \in \nu(\overline{\lambda})$. Then

$$\|A_\lambda x - A_{\lambda'} x\| \leq (r + \|x_0\|)|\lambda - \lambda'|$$

and f_λ is $L' = r + \|x_0\|$-Lipschitz in λ. It is further assumed that on the same neighborhood $\nu(\overline{\lambda})$ the norm $\|A_\lambda\|$ is independently bounded in λ.

Theorem 4.1. *Under the notation above, the following estimate is satisfied*

$$\|y - y_\lambda\|_{\infty,B} \leq \frac{\|C\|}{1 - L\tau}\left(\|x_0 - x_\lambda^0\| + L'\tau|\lambda - \overline{\lambda}|\right), \quad \forall \lambda \in \nu(\overline{\lambda}). \tag{4.1}$$

Proof. From estimate (3.3), we have the desired conclusion immediately. □

5. Extension of Stability to Approximate Solutions

Let us now introduce the concept of approximate solutions to our Cauchy problem.

Definition 5.1. We say that a function x from $C^1(0,T,X)$ is an ε-approximate solution to $S(f,x_0)$ if and only if it is an ε-fixed point of Γ_{f,x_0} i.e, $x \in \varepsilon - \mathrm{Fix}(\Gamma_{f,x_0})$.

We denote the set $\varepsilon - \mathrm{Fix}(\Gamma_{f,x_0})$ by $S^\varepsilon(f,x_0)$ next.

Example 5.1. We consider the Cauchy problem $\dot{x}(t) = Ax(t),\ x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, where $A = \begin{pmatrix} 0 & -3 \\ 1 & -4 \end{pmatrix}$. Then

$$\begin{aligned} x_\varepsilon \in S^\varepsilon(f,x_0) &\iff \|x_\varepsilon - \Gamma_{f,x_0}(x_\varepsilon)\|_\infty \leq \varepsilon \\ &\iff \|x_\varepsilon(t) - x_\varepsilon(0) - \int_0^t Ax_\varepsilon(s)ds\| \leq \varepsilon \quad \forall t \in [0,T]. \end{aligned} \tag{5.1}$$

The ε-approximate solution x_ε can be found in the form: $x_\varepsilon(t) = x(t) + \begin{pmatrix} \varepsilon' \\ \varepsilon' \end{pmatrix}$, where x is the exact solution to the corresponding Cauchy problem and $\varepsilon' > 0$. The last inequality (5.1) gives us:

$$x_\varepsilon(t) = \begin{pmatrix} \dfrac{3e^{-t} - e^{-3t}}{2} + \varepsilon' \\ \dfrac{e^{-t} - e^{-3t}}{2} + \varepsilon' \end{pmatrix}$$

with the condition $\varepsilon' \leq \dfrac{\varepsilon}{3T}$.

In the following, we present a qualitative stability result for the sets of approximate solutions to Cauchy problem.

Theorem 5.1. *Let X be a normed space. Let $f_\lambda : [0,T] \times X \to X$ be a function. Assume that the following conditions hold :*
1) *the function $t \longmapsto f(t,x)$ is measurable for all $x \in X$;*
2) *the function $x \longmapsto f(t,x)$ is continuous for almost all $t \in [0,T]$;*
3) *f_λ is Lipschitz in x uniformly in t and λ of same Lipschitz constant $L > 0$;*

Then, for all $\varepsilon > 0$ and for all $\lambda,\ \lambda' \in U$, sets $S^\varepsilon(f_\lambda, x_0^\lambda)$, and $S^\varepsilon(f_{\lambda'}, x_0^{\lambda'})$ are non-empty. Moreover,

$$h\left(S^\varepsilon(f_\lambda, x_0^\lambda), S^\varepsilon(f_{\lambda'}, x_0^{\lambda'})\right) \leq \varepsilon e^{LT} + e^{LT}\left(\|x_0^{\lambda'} - x_0^\lambda\| + LT\|\lambda - \lambda'\|\right).$$

In particular,

$$h\left(S^\varepsilon(f_\lambda, x_0^\lambda), S^\varepsilon(f_{\lambda'}, x_0^{\lambda'})\right) \to 0, \text{ where } \lambda' \to \lambda,\ x_0^\lambda \to x_0^{\lambda'} \text{ and } \varepsilon \to 0.$$

Proof. We consider the space $C([0,T],X)$ equipped again with the the same norm $\|.\|_L$ given for a function x by

$$\|x\|_L = \sup_{t\in[0,T]} e^{-LT}\|x(t)\|_X.$$

Let λ, $\lambda' \in U$. Remarking that $\Gamma_{f_\lambda,x_0^\lambda}$ and $\Gamma_{f_{\lambda'},x_0^{\lambda'}}$ are contraction with the same constant $1-e^{-LT}$, by applying Theorem 2.3, we obtain

$$h\left(S^\varepsilon(f_\lambda,x_0^\lambda),S^\varepsilon(f_{\lambda'},x_0^{\lambda'})\right) \le \varepsilon e^{LT} + e^{LT} \sup_{x\in C([0,T],X)} h(\Gamma_{f_\lambda,x_0^\lambda}(x),\Gamma_{f_{\lambda'},x_0^{\lambda'}}(x))$$

Fixing $x\in C([0,T],X)$, one has

$$\begin{aligned} h\left(S^\varepsilon(f_\lambda,x_0^\lambda),S^\varepsilon(f_{\lambda'},x_0^{\lambda'})\right) &= \|\Gamma_{f_\lambda,x_0^\lambda}(x)-\Gamma_{f_{\lambda'},x_0^{\lambda'}}(x))\|_L \\ &\le \|x_0^\lambda - x_0^{\lambda'}\| + \sup_{t\in[0,T]} \|\int_0^t (f_\lambda(s,x(s))-f_{\lambda'}(s,x(s)))ds\|. \end{aligned}$$

As a result,

$$\begin{aligned} &\sup_{x\in C([0,T],X)} h\left(S^\varepsilon(f_\lambda,x_0^\lambda),S^\varepsilon(f_{\lambda'},x_0^{\lambda'})\right) \\ &\le \|x_0^\lambda - x_0^{\lambda'}\| + \sup_{x\in C([0,T],X)}\sup_{t\in[0,T]} \|\int_0^t (f_\lambda(s,x(s))-f_{\lambda'}(s,x(s)))ds\|. \end{aligned}$$

Then

$$\begin{aligned} &h\left(S^\varepsilon(f_\lambda,x_0^\lambda),S^\varepsilon(f_{\lambda'},x_0^{\lambda'})\right) \\ &\le \varepsilon e^{LT} + e^{LT}\left(\|x_0^{\lambda'} - x_0^{\lambda}\| + \sup_{x\in C([0,T],X)}\sup_{t\in[0,T]} \|\int_0^t (f_\lambda(s,x(s))-f_{\lambda'}(s,x(s)))ds\|\right). \end{aligned}$$

Since f_λ is L lipschitz in λ with respect x and t, we obtain

$$\sup_{x\in C([0,T],X)}\sup_{t\in[0,T]} \|\int_0^t (f_\lambda(s,x(s))-f_{\lambda'}(s,x(s)))ds\| \le LT\|\lambda-\lambda'\|$$

This proves the intended inequality

$$h\left(S^\varepsilon(f_\lambda,x_0^\lambda),S^\varepsilon(f_{\lambda'},x_0^{\lambda'})\right) \le \varepsilon e^{LT} + e^{LT}\left(\|x_0^{\lambda'} - x_0^{\lambda}\| + LT\|\lambda-\lambda'\|\right).$$

This completes the proof. □

The quantitative aspect of sets of approximate solutions is given as follows.

Theorem 5.2. *Let X be a normed space and $x_0, \tilde{x_0} \in X$. Let $f : [0,T]\times\overline{B}(x_0,r) \to X$ and $g : [0,T]\times\overline{B}(\tilde{x_0},r) \to X$ be two continuous functions Lipschitz in x uniformly in t of same Lipschitz constant $L>0$. Assume that $L\tau<1$. Then both of the sets of approximate solutions $S^\varepsilon(f,x_0)$ and $S^\varepsilon(g,\tilde{x_0}))$ are non-empty. Moreover*

$$h(S^\varepsilon(f,x_0),S^\varepsilon(g,\tilde{x_0}))) \le \frac{1}{1-L\tau}\|x_0-\tilde{x_0}\| + \frac{\tau}{1-L\tau}\|f-g\| + \frac{\varepsilon}{1-L\tau}.$$

Proof. Fix $x \in C([0,\tau],X)$, and let $y_1 = \Gamma_{f,x_0}(x)$ and $y_2 = \Gamma_{g,\tilde{x_0}}(x)$. Then,

$$\|y_1 - y_2\|_{\infty,B} \leq \|x_0 - \tilde{x_0}\| + \tau\|f-g\|.$$

Hence,

$$\sup_{x\in C([0,T],X)} h(\Gamma_{f,x_0}(x), \Gamma_{g,\tilde{x_0}}(x)) \leq \|x_0 - \tilde{x_0}\| + T\|f-g\|_\infty.$$

As $L\tau < 1$, it follows that Γ_{f,x_0} and $\Gamma_{g,\tilde{x_0}}$ are contractions with the same constant $L\tau$. Thus, by Theorem 2.3, it results that

$$\begin{aligned} h(S^\varepsilon(f,x_0), S^\varepsilon(g,\tilde{x_0}))) &\leq \frac{\varepsilon}{1-L\tau} + \frac{1}{1-L\tau}\sup_{x\in C([0,\tau],X)} h(\Gamma_{f,x_0}(x), \Gamma_{g,\tilde{x_0}}(x)) \\ &\leq \frac{1}{1-L\tau}\|x_0 - \tilde{x_0}\| + \frac{T}{1-L\tau}\|f-g\| + \frac{\varepsilon}{1-L\tau}. \end{aligned}$$

□

Remark 5.1. Fix $\varepsilon > 0$. Let $x_f^\varepsilon \in S^\varepsilon(f,x_0)$. Then there exists $x_g^\varepsilon \in S^\varepsilon(g,\tilde{x_0}))$ such that

$$\|x_f^\varepsilon - x_g^\varepsilon\|_\infty \leq \varepsilon + h(S^\varepsilon(f,x_0), S^\varepsilon(g,\tilde{x_0}))).$$

Corollary 5.1. *Let X be a normed space and $x_0, \tilde{x_0} \in X$. Let $f : [0,T]\times\overline{B}(x_0,r) \to X$ and $g : [0,T]\times\overline{B}(\tilde{x_0},r) \to X$ be two continuous functions Lipschitz in x uniformly in t of same Lipschitz constant $L > 0$. Assume that $L\tau < 1$. Let x_{f,x_0} be the unique (maximal) solution of the Cauchy problem $S(f,x_0)$ on the interval $[0,\tau]$. Then, for all $\varepsilon > 0$, there exists $x^\varepsilon_{f,\tilde{x_0}} \in S^\varepsilon(f,\tilde{x_0})$ such that*

$$\|x_{f,x_0} - x^\varepsilon_{g,\tilde{x_0}}\|_{\infty,B} \leq \varepsilon\left(1 + \frac{1}{1-L\tau}\right) + \frac{1}{1-L\tau}\|x_0 - \tilde{x_0}\| + \frac{\tau}{1-L\tau}\|f-g\|.$$

Proof. The proof is an immediate result from Remark 5.1 and Theorem 5.2. □

Corollary 5.2. *Let X be a normed space and $x_0, \tilde{x_0} \in X$. Let $f : [0,T]\times\overline{B}(x_0,r) \to X$ be a continuous functions Lipschitz in x uniformly in t of same Lipschitz constant $L > 0$. Assume that $L\tau < 1$. Let x_f be the unique maximal solution to Cauchy problem $S(f.x_0)$ on the interval $[0,\tau]$. Then there exists $x^\varepsilon_{f,\tilde{x_0}} \in S^\varepsilon(f,\tilde{x_0})$ such that*

$$\|x_{f,x_0} - x^\varepsilon_{f,x_0}\|_{\infty,B} \leq \varepsilon\left(1 + \frac{1}{1-L\tau}\right).$$

In particular, $\left(x^\varepsilon_{f,x_0}\right)_{\varepsilon>0}$ converges uniformly to x_{f,x_0} when ε tends to 0.

Corollary 5.3. *Let X be a normed space and $x_0, \tilde{x_0} \in X$. Let $f : [0,T]\times\overline{B}(x_0,r) \to X$ and $g : [0,T]\times\overline{B}(\tilde{x_0},r) \to X$ be two continuous functions Lipschitz in x uniformly in t of same Lipschitz constant $L > 0$. Assume that $L\tau < 1$, and the set $S^\varepsilon(g,\tilde{x_0}))$ is compact. Then there exists $x^\varepsilon_{g,\tilde{x_0}} \in S^\varepsilon(g,\tilde{x_0})$ such that*

$$\|x_{f,x_0} - x^\varepsilon_{g,\tilde{x_0}}\| \leq \frac{1}{1-L\tau}\|x_0 - \tilde{x_0}\| + \frac{\tau}{1-L\tau}\|f-g\| + \frac{\varepsilon}{1-L\tau}.$$

Proof. Observe that the conditions of Theorem 2.5 are satisfied. Indeed, the set $S^{\varepsilon}(f,x_0)$ is closed because it is the reciprocal image of the closed interval $]-\infty,\varepsilon]$ by a continuous application $x \longmapsto \|x-\Gamma_{f,x_0}(x)\|$. Since the set $S^{\varepsilon}(g,\tilde{x_0})$ is compact by assumption, then it verifies the Bolzano-Weierstrass condition. Then from this theorem, there exists $x^{\varepsilon}_{g,\tilde{x_0}} \in S^{\varepsilon}(g,\tilde{x_0})$ such that

$$\|x_{f,x_0}-x^{\varepsilon}_{g,\tilde{x_0}}\| \leq h\left(S^{\varepsilon}(f,x_0),S^{\varepsilon}(g,\tilde{x_0})\right)).$$

By applying Theorem 5.2, we conclude that

$$\|x_{f,x_0}-x^{\varepsilon}_{g,\tilde{x_0}}\| \leq \frac{1}{1-L\tau}\|x_0-\tilde{x_0}\| + \frac{\tau}{1-L\tau}\|f-g\| + \frac{\varepsilon}{1-L\tau}.$$

□

Remark 5.2. If the space X is of finite dimension, then set $S^{\varepsilon}(f,x_0)$ is compact and the result follows.

6. Stability of Solutions in One-Step Methods under Parametric Perturbation

The aim of this section is to demonstrate that our stability results are coherent with parametric perturbation arising in numerical methods. For instance, we consider one-step methods for the same Cauchy problem $S(f,x_0)$ given by

$$\begin{cases} x'(t) = f(t,x(t)),\ t\in[0,T], \\ x(0) = x_0. \end{cases}$$

Given an integer $N \in \mathbb{N}^*$, we define a time step $h=\frac{T}{N}$ and consider the subdivision of the interval $[0,T]:\ t_n = nh$ pour $n=0,\ldots,N$. One-step methods use only information gathered at the current mesh point, t_n. Indeed, given an iterate x_n, the next one is expressed by

$$x_{n+1} = x_n + h\Phi(t_n,x_n,h).$$

Similarly, for the perturbed system $S(f_\lambda,x^0_\lambda)$

$$\begin{cases} x'_\lambda(t) = f_\lambda(t,x_\lambda(t)), \\ x_\lambda(0) = x^0_\lambda, \end{cases}$$

where $f_\lambda : [0,T]\times B(x^0_\lambda,r) \longrightarrow X$, we have $x_{n+1,\lambda} = x_{n,\lambda} + h\Phi(t_n,x_{n,\lambda},h)$. The initial value of the parameter λ is still denoted by $\overline{\lambda}$. The following notation is adopted below

$$\begin{cases} f_{\overline{\lambda}} = f, \\ x_{\overline{\lambda}} = x, \\ x_{\overline{\lambda}}(0) = x_0, \\ \Phi(t,x,h,\overline{\lambda}) = \Phi(t,x,h). \end{cases}$$

Lemma 6.1. *Consider the one-step method with a constant step. Suppose that the function* $(t,x,h,\lambda) \longrightarrow \Phi(t,x,h,\lambda)$ *is*
(H_1) *L-Lipschitz in x, uniformly with respect to h, t, and* λ*;*
(H_2) *L'-Lipschitz in λ, uniformly with respect to x, h, and t.*

Let (x_n) and $(x_{n,\lambda})$ be the sequences given by: $x_{n+1} = x_n + h\Phi(t_n, x_n, h)$ and $x_{n+1,\lambda} = x_{n,\lambda} + h\Phi(t_n, x_{n,\lambda}, h, \lambda)$. Then the following estimate is satisfied

$$\|x_{n+1,\lambda} - x_{n+1}\| \leq e^{LT}\|x_\lambda^0 - x_0\| + \frac{L'}{L}(e^{LT} - 1)\|\lambda - \overline{\lambda}\|.$$

Proof. Observe that

$$\begin{aligned} & x_{n+1,\lambda} - x_{n+1} \\ & = x_{n,\lambda} - x_n + h\left(\Phi(t_n, x_{n,\lambda}, h, \lambda) - \Phi(t_n, x_n, h)\right) \\ & = x_{n,\lambda} - x_n + h\Big(\Phi(t_n, x_{n,\lambda}, h, \lambda) - \Phi(t_n, x_{n,\lambda}, h, \overline{\lambda}) + \Phi(t_n, x_{n,\lambda}, h, \overline{\lambda}) - \Phi(t_n, x_n, h)\Big). \end{aligned}$$

Then, using the fact that Φ is Lipschitz in x of constant L and L'-Lipschitz in λ, we infer

$$\|x_{n+1,\lambda} - x_{n+1}\| \leq (1 + hL)\|x_{n,\lambda} - x_n\| + hL'\|\lambda - \overline{\lambda}\|.$$

A simple induction yields

$$\|x_{n+1,\lambda} - x_{n+1}\| \leq (1 + hL)^{n+1}\|x_\lambda^0 - x_0\| + hL'(\frac{e^{(n+1)hL} - 1}{hL})\|\lambda - \overline{\lambda}\|.$$

Hence, it follows from $(n+1)h \leq T$ that

$$\|x_{n+1,\lambda} - x_{n+1}\| \leq e^{LT}\|x_\lambda^0 - x_0\| + \frac{L'}{L}(e^{LT} - 1)\|\lambda - \overline{\lambda}\|.$$

□

The error at time t_n is defined by $e_n = x(t_n) - x_n$. In the same way, the error at time t_n for the perturbed system is defined by $e_{\lambda,n} = x_\lambda(t_n) - x_{n,\lambda}$.

Theorem 6.1. *If the assumptions of Corollary 3.1 and Lemma 6.1 are satisfied, then*

$$\begin{aligned} & \|e_{\lambda,n+1} - e_{n+1}\| \\ & \leq \left(e^{LT}(1 + hL) + \frac{1}{1 - L\tau}\right)\|x_0 - x_\lambda^0\| + \left(\frac{\tau L'}{1 - L\tau} + (1 + Lh)\frac{L'}{L}(e^{LT} - 1)\right)\|\lambda - \overline{\lambda}\|. \end{aligned}$$

Proof. Observe that

$$e_{\lambda,n+1} - e_{n+1} = x_\lambda(t_{n+1}) - x(t_{n+1}) + x_{n,\lambda} - x_n + h\left(\Phi(t_n, x_{n,\lambda}, h, \lambda) - \Phi(t_n, x_n, h)\right).$$

Thanks to Corollary 3.1, we obtain that

$$|x_\lambda(t_{n+1}) - x(t_{n+1})| \leq \frac{1}{1 - L\tau}\|x_0 - x_\lambda^0\| + \frac{\tau L'}{1 - L\tau}\|\lambda - \overline{\lambda}\|.$$

On the other hand, as Φ is L-Lipschitz in x, it results from Lemma 6.1 that

$$\|\Phi(t_n, x_{n,\lambda}, h, \lambda) - \Phi(t_n, x_n, h)\| \leq Le^{LT}\|x_0 - x_\lambda^0\| + L'(e^{LT} - 1)\|\lambda - \overline{\lambda}\|.$$

Hence,

$$\begin{aligned} & \|e_{\lambda,n+1} - e_{n+1}\| \\ & \leq \left(e^{LT}(1 + hL) + \frac{1}{1 - L\tau}\right)\|x_0 - x_\lambda^0\| + \left(\frac{\tau L'}{1 - L\tau} + (1 + Lh)\frac{L'}{L}(e^{LT} - 1)\right)\|\lambda - \overline{\lambda}\|. \end{aligned}$$

□

7. Numerical Experiments

We consider the following example: $\dot{x}(t) = Ax(t),\ x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, where $A = \begin{pmatrix} 0 & -3 \\ 1 & -4 \end{pmatrix}$ and

$$A_\lambda = \begin{pmatrix} \lambda & -3 \\ 1 & -4+\lambda \end{pmatrix}.$$

The objective of this section is to justify the validity of Theorem 4.1 since the parametric functions satisfy the conditions of the cited theorem. Figure 1 below describes the behavior of the norm of the difference between the observed solutions y_λ of the parametric system

$$y_\lambda(t) = Cx_\lambda(t),\ x'(t) = f_\lambda(t, x(t)) = A_\lambda x(t),$$

where x is the solution for $\overline{\lambda} = 0$.

Actually, the linear system in question and its perturbed format are precisely as follows:

$$S(A, C, x_0): \begin{cases} x'(t) &= Ax(t), \\ y(t) &= Cx(t), \\ x(0) = x_0 &, \quad y_0 = Cx_0, \end{cases}$$

$$S_\lambda(A_\lambda, C, x_\lambda^0 = x_0): \begin{cases} x'_\lambda(t) &= A_\lambda x(t), \\ y_\lambda(t) &= Cx_\lambda(t), \\ x_\lambda(0) = x_0 &, \quad y_0 = Cx_0. \end{cases}$$

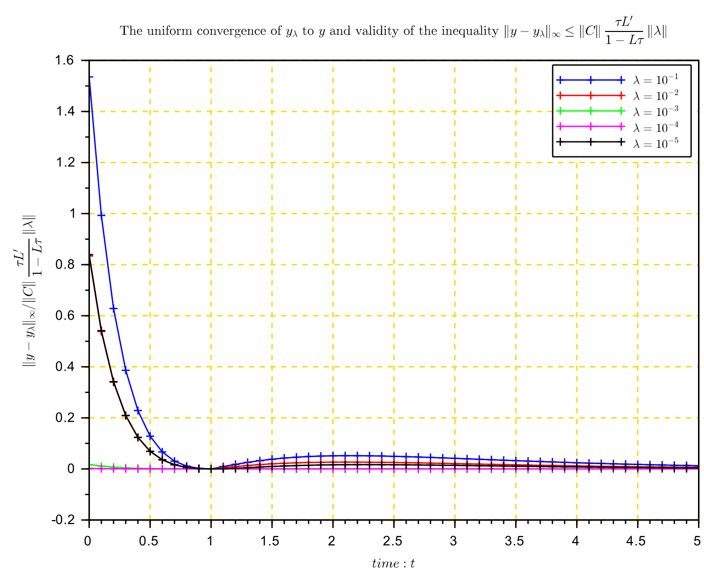

FIGURE 1. The uniform convergence of y_λ to y and validity of the inequality $\|y - y_\lambda\|_\infty \le \|C\| \frac{\tau L'}{1-L\tau} \|\lambda\|$.

In Figure 1 above, we check the validity of inequality (4.1):

$$\|y - y_\lambda\|_{\infty, B} \le \frac{\|C\|}{1 - L\tau} \left(\|x_0 - x_\lambda^0\| + L'\tau|\lambda - \overline{\lambda}| \right), \quad \forall \lambda \in \nu(\overline{\lambda})$$

for the following values of the parameter λ :
$\lambda = 10^{-1}$ $rp = 1.07649091$
$\lambda = 10^{-2}$ $rp = 1.01564810$
$\lambda = 10^{-3}$ $rp = 1.00978716$
$\lambda = 10^{-4}$ $rp = 1.00920324$
$\lambda = 10^{-5}$ $rp = 1.00914487$.
This clearly indicates that the norm of the difference $\|y - y_\lambda\|_{\infty,B}$ (of observability solutions) decreases linearly with respect to the parameter λ. In conclusion, the numerical example above demonstrates that our quantitative stability results are meaningful and still deserve further attention in quite different contexts of applied physical models.

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