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ON NONSMOOTH MULTIOBJECTIVE SEMI-INFINITE PROGRAMMING WITH SWITCHING CONSTRAINTS

MOHSINE JENNANE[1], EL MOSTAFA KALMOUN[2,*], LAHOUSSINE LAFHIM[1]

[1]*LASMA, FSDM, Department of Mathematics, Sidi Mohamed Ben Abdellah University, Fez, Morocco*
[2]*School of Science and Engineering, Al Akhawayn University in Ifrane, PO Box 104, Ifrane 53000, Morocco*

Abstract. In this paper, we establish optimality conditions for a nonsmooth multiobjective semi-infinite programming problem subject to switching constraints. In particular, we employ a surrogate problem and a suitable constraint qualification to state necessary M-stationary conditions in terms of Clarke subdifferentials. Moreover, we demonstrate that in different cases these M-stationary conditions becomes sufficient as well. Finally, we also present Weak and strong duality results of Wolfe an Mond-Weir types.

Keywords. Multiobjective semi-infinite programming; Optimality conditions; Switching constraints; Stationarity; Weak and strong duality.

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1. INTRODUCTION

We take up the following nonsmooth multiobjective semi-infinite programming problem subject to switching constraints, NMPSC for short

$$\begin{cases} \min & f(x) = (f_1(x), ..., f_m(x)), \\ s.t. & g_t(x) \leq 0, \ \forall t \in T, \\ & h_k(x) = 0, \ \forall k \in K = \{1, ..., q\}, \\ & G_i(x)H_i(x) = 0, \ \forall i \in I = \{1, ..., l\}, \end{cases} \tag{1.1}$$

where the index set T is an arbitrary nonempty set, not necessary finite. The real-valued functions f_j, $j \in J = \{1, ..., m\}$, g_t, $t \in T$, h_k, $k \in K$, G_i and H_i, $i \in I$ are defined on $\mathbb{R}^n$ and not necessary convex nor differentiable. The feasible region of (1.1) is given by

$$\Pi := \{x \in \mathbb{R}^n : g_t(x) \leq 0, t \in T, h_k(x) = 0, k \in K, G_i(x)H_i(x) = 0, i \in I\}.$$

The terminology "switching constraints" originates from the fact that if the product of two functions is equal to zero, then at least one of them must be equal to zero. Problems under the form (1.1) were recently introduced to investigate the discretization of optimal control problems with switching constraints [1, 2, 3], and to study mathematical programs with

*Corresponding author.
E-mail addresses: mohsine.jennane@usmba.ac.ma (M. Jennane), E.Kalmoun@aui.ma (E.M. Kalmoun), lahoussine.lafhim@usmba.ac.ma (L. Lafhim).

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either-or-constraints [4, 5, 6]. Moreover, NMPSC can be seen as an extension of another class of optimization problems, namely mathematical programming with equilibrium constraints (MPEC) [7, 8, 9], which has the same form as NMPSC subject to an additional condition "$G_i(x) \geq 0$ and $H_i(x) \geq 0$ for all $i \in I$". Although the latter condition does not appear in the problems studied in many papers of optimal control and related fields, we find that the published results on NMPSC are very few where compared to the MPEQ, which motivated us to deal with this type of problems as they provide less restrictive setting.

The major difficulty in solving (1.1) is that it typically violates the majority of classical constraint qualifications (such as Mangasarian-Fromovitz constraint qualification, linear independence constraint qualification), and hence the standard KKT conditions are not relevant in the context of mathematical programming with switching constraints (MPSC). This led to introduce various stationarity concepts (weak, Mordukhovich, and strong stationarity) for MPSC and to derive some associated constraint qualifications [4]. Kanzow et al. [10] proposed several relaxation methods from the numerical treatment of MPEC to MPSC. Li and Guo extended some weak and verifiable constraint qualifications for nonlinear programs to MPSC in [11]. Later, Mehlitz investigated a second-order optimality conditions for MPSC in [12]. It is worth noting that (NMPSC) is closely related to mathematical programs with vanishing constraints and that these programs can be treated within the general framework of conditional vector optimization problems recently introduced in [13]. Very recently, Liang and Ye [14] survey recent results on constraint qualifications and optimality conditions for mathematical programs with disjunctive constraints, and apply them to MPSC. Moreover, they provide two types of sufficient conditions for the local error bound and exact penalty results for MPSC.

In this paper, we are concerned with a nonsmooth, multiobjective and semi-infinite version of MPSC, and introduce a constraint qualification of a surrogate problem, which will guarantee an optimality condition, called M-stationarity, to hold at a local minimum. We are also interested in establishing weak and strong duality results of Wolfe and Mond-Weir types. We claim that this paper is the first work that treats the nonsmooth case for multiobjective semi-infinite programming with switching constraints.

The organization of the paper is as follows. In the next section, we present the used notations and recall some definitions. In Sections 3 and 4, we respectively propose necessary and sufficient M-stationary conditions for (weak) efficient solutions of (1.1) involving an appropriate constraint qualification of a surrogate problem. Section 5 is devoted to formulate the Wolfe and Mond-Weir type dual models for the problem so that to derive weak and strong duality results. Finally, a conclusion is given in Section 6.

2. PRELIMINARIES

From now on, we take the following order in the Euclidean space: $a, b \in \mathbb{R}^m$ satisfies

- $a \leq b$ if and only if $a_i \leq b_i$ for all $i = 1, 2, \ldots, m$ with strict inequality for at least one i.
- $a < b$ if and only if $a_i < b_i$.

Given a nonempty subset $\mathscr{S}$ of $\mathbb{R}^n$, $co\mathscr{S}$ and $cl\mathscr{S}$ denote the convex hull, and closure of $\mathscr{S}$, respectively. Also, the polar cone, the strictly negative polar cone and the orthogonal complement of $\mathscr{S}$ are respectively defined by

$$\begin{array}{rcl}\mathscr{S}^{\circ} & = & \{x \in \mathbb{R}^n : \langle x, d\rangle \leq 0, \forall d \in \mathscr{S}\},\\ \mathscr{S}^{s} & = & \{x \in \mathbb{R}^n : \langle x, d\rangle < 0, \forall d \in \mathscr{S} \setminus \{0\}\},\\ \mathscr{S}^{\perp} & = & \{x \in \mathbb{R}^n : \langle x, d\rangle = 0, \forall d \in \mathscr{S}\}.\end{array}$$

It can easily be shown that $\mathscr{S}^{\perp} = \mathscr{S}^{\circ} \cap (-\mathscr{S})^{\circ}$. Moreover, at $\bar{x} \in cl\mathscr{S}$, the tangent cone, the convex cone generated by $\mathscr{S}$ and the linear hull of $\mathscr{S}$ are respectively given by

$$T(\mathscr{S}, \bar{x}) = \left\{ \upsilon \in \mathbb{R}^n : \exists t_n \downarrow 0, \ \exists \upsilon_n \to \upsilon, \ \bar{x} + t_n \upsilon_n \in \mathscr{S} \right\},$$

$$cone(\mathscr{S}) = \left\{ y = \sum_{i=1}^{k} \lambda_i y_i : k \in \mathbb{N}, \lambda_i \geq 0, \ y_i \in \mathscr{S}, i = 1, 2, \ldots, k \right\},$$

$$lin(\mathscr{S}) = \left\{ y = \sum_{i=1}^{k} \lambda_i y_i : k \in \mathbb{N}, \lambda_i \in \mathbb{R}, \ y_i \in \mathscr{S}, i = 1, 2, \ldots, k \right\}.$$

Further details on tangent cones as well as several concepts of this section can be found in [15]. Recall also that for any two sets $\mathscr{S}_1$ and $\mathscr{S}_2$ in $\mathbb{R}^n$ one has $lin(\mathscr{S}_1 \cup \mathscr{S}_2) = lin(\mathscr{S}_1) + lin(\mathscr{S}_2)$.

Definition 2.1 ([16])**.** A function $\phi : X \to \mathbb{R}$ is said to be locally Lipschitz at $x_0 \in X$ if there are positive constants k and δ satisfying, for all $x, y \in B(x_0, \delta) \cap X$,

$$|\phi(x) - \phi(y)| \leq k\|x - y\|.$$

It is said to be locally Lipschitz on X if it is so at each $x_0 \in X$.

Definition 2.2 ([16])**.** Let $\phi : X \to \mathbb{R}$ be a locally Lipschitz function at $x \in X$. The Clarke generalized subdifferential of ϕ at x is defined as

$$\partial_C \phi(x) := \{y \in \mathbb{R}^n : \phi^{\circ}(x; v) \geq \langle y, v\rangle, \forall v \in \mathbb{R}^n\},$$

where $\phi^{\circ}(x; v)$ is Clarke's generalized directional derivative of ϕ along $v \in \mathbb{R}^n$ at $x \in X$, which is defined as

$$\phi^{\circ}(x, v) := \limsup_{\substack{t \downarrow 0 \\ y \to x}} \frac{\phi(y + tv) - \phi(y)}{t}.$$

It is worth noting that we have the following properties:

- $\partial_C \phi(x)$ is a nonempty convex compact set in $\mathbb{R}^n$.
- The set-valued mapping $x \to \partial_C \phi(x)$ is upper semicontinuous.
- $\partial_C \big(\lambda \phi(x)\big) = \lambda \partial_C \phi(x)$ for all $\lambda \in \mathbb{R}$.

Definition 2.3. Let Ω and X two subsets of $\mathbb{R}^n$ such that $\Omega \subset X$. Suppose that we are given a function $\varphi : X \to \mathbb{R}$ locally Lipschitz at $\bar{x} \in \mathbb{R}^n$ and a mapping $\eta : X \times X \to \mathbb{R}^n$. φ is called

- invex (resp. strictly invex) on Ω at $\bar{x}$ w.r.t. η iff

$$\langle \bar{x}^*, \eta(x, \bar{x})\rangle \leq \ (resp. \ <) \ \varphi(x) - \varphi(\bar{x}), \ \forall x \in \Omega, \forall \bar{x}^* \in \partial_C \varphi(\bar{x}).$$

- pseudo-invex on Ω at $\bar{x}$ w.r.t. η iff

$$\varphi(x) < \varphi(\bar{x}) \Rightarrow \langle \bar{x}^*, \eta(x, \bar{x})\rangle < 0, \ \forall x \in \Omega, \forall \bar{x}^* \in \partial_C \varphi(\bar{x}).$$

- strictly pseudo-invex on Ω at $\bar{x}$ w.r.t. η iff

$$\varphi(x) \leq \varphi(\bar{x}) \Rightarrow \langle \bar{x}^*, \eta(x, \bar{x})\rangle < 0, \ \forall x \in \Omega \setminus \{\bar{x}\}, \forall \bar{x}^* \in \partial_C \varphi(\bar{x}).$$

- quasi-invex on Ω at $\bar{x}$ w.r.t. η iff

$$\varphi(x) \leq \varphi(\bar{x}) \Rightarrow \langle \bar{x}^*, \eta(x,\bar{x}) \rangle \leq 0, \ \forall x \in \Omega, \forall \bar{x}^* \in \partial_C \varphi(\bar{x}).$$

Note that

- In what follows, if $\Omega = X$, the expression "on Ω" in the above definition will be omitted.
- φ is invex at $\bar{x}$ w.r.t. $\eta \Rightarrow \varphi$ is pseudo-invex at $\bar{x}$ w.r.t. $\eta \Rightarrow \varphi$ is quasi-invex at $\bar{x}$ w.r.t. η.
- When $\eta(x,y) = y - x$, we recover the corresponding definitions of convexity instead of invexity.

Hereafter, we assume that $\bar{x} \in \Pi$, f_j, $j \in J$, g_t, $t \in T$, h_k, $k \in K$, G_i and H_i, $i \in I$ are locally Lipschitz at $\bar{x}$. We say that $\bar{x}$ is a local (weak) efficient solution to (1.1) if there is a neighbourhood V of $\bar{x}$ such that for each $y \in V \cap \Pi$ the inequality $f(y) \leq (<) f(\bar{x})$ does not hold. It is straightforward to check that every local efficient solution for (1.1) is local weak efficient. When $V = \mathbb{R}^n$, the word "local" will be omitted.

We denote by $\mathbb{R}_+^{|T|}$ the collection of all functions $\lambda : T \to \mathbb{R}$ taking positive values λ_t only at finitely many points of T, and zero otherwise. For $\bar{x} \in \Pi$, we let $T(\bar{x}) := \{t \in T \mid g_t(\bar{x}) = 0\}$ be the index set of all active constraints at $\bar{x}$ and $A(\bar{x}) := \{\lambda \in \mathbb{R}_+^{|T|} \mid \lambda_t g_t(\bar{x}) = 0, \forall t \in T\}$ be that of active constraint multipliers at $\bar{x}$. Notice that $\lambda \in A(\bar{x})$ if there exists a finite index set $R \subset T(\bar{x})$ such that $\lambda_t > 0$ for all $t \in R$ and $\lambda_t = 0$ for all $t \in T \setminus R$. Let us also define

$$I_G = I_G(\bar{x}) := \{i \in I \mid G_i(\bar{x}) = 0, H_i(\bar{x}) \neq 0\},$$
$$I_H = I_H(\bar{x}) := \{i \in I \mid G_i(\bar{x}) \neq 0, H_i(\bar{x}) = 0\},$$

and

$$I_{GH} = I_{GH}(\bar{x}) := \{i \in I \mid G_i(\bar{x}) = 0, H_i(\bar{x}) = 0\}.$$

We suppose that I_{GH} is a nonempty set and denote by $\mathscr{P}(I_{GH})$ the set of all bipartitions of I_{GH}; i.e., $\mathscr{P}(I_{GH}) = \{(B_1, B_2) : \ B_1 \cup B_2 = I_{GH}, \ B_1 \cap B_2 = \emptyset\}$.

The point $\bar{x}$ is called weakly stationary, W-stationary for short if there exist multipliers solving the system

$$\begin{aligned} 0 \in & \sum_{j \in J} \lambda_j \partial_C f_j(\bar{x}) + \sum_{t \in T(\bar{x})} \lambda_t^g \partial_C g_t(\bar{x}) + \sum_{k \in K} \lambda_k^h \partial_C h_k(\bar{x}) \\ & + \sum_{i \in I} \lambda_i^G \partial_C G_i(\bar{x}) + \sum_{i \in I} \lambda_i^H \partial_C H_i(\bar{x}), \\ & \forall j \in J : \lambda_j \geq 0, \quad \forall t \in T(\bar{x}) : \lambda_t^g \geq 0, \\ & \forall i \in I_H(\bar{x}) : \lambda_i^G = 0, \quad \forall i \in I_G(\bar{x}) : \lambda_i^H = 0. \end{aligned} \tag{2.1}$$

It is called Mordukhovich-stationary, M-stationary for short, if in addition to (2.1), $\lambda_i^G \lambda_i^H = 0$ for all $i \in I_{GH}(\bar{x})$. Finally, it is strongly stationary, S-stationary for short, if in addition to (2.1), $\lambda_i^G = 0$ *and* $\lambda_i^H = 0$ for all $i \in I_{GH}(\bar{x})$. Clearly, S-stationarity yields M-stationarity, which yields W-stationarity.

Now, we present a useful lemmas and theorem which we need to prove our main result.

Lemma 2.1 ([17]). *Let $\{\mathscr{S}_j | j \in J\}$ be a family of nonempty convex sets in $\mathbb{R}^n$. Then, every nonzero vector of $\mathscr{C} = cone(\cup_{j \in J} \mathscr{S}_j)$ can be written as a non-negative linear combination of at most n linear independent vectors, each belonging to a different $\mathscr{S}_j$.*

Lemma 2.2 ([18])**.** *Let S, T and P be three arbitrary index sets (possibly infinite). Consider the maps $\varphi : S \to \mathbb{R}^n$, $\phi : T \to \mathbb{R}^n$ and $\psi : P \to \mathbb{R}^n$. If the set $co\{\varphi(s), t \in T\} + cone\{\phi(t), t \in T\} + lin\{\psi(p), p \in P\}$ is closed, the following two assertions are equivalent:*

(i) $\begin{cases} \langle \varphi(s), d\rangle < 0, \ t \in T, S \neq \emptyset, \\ \langle \phi(t), d\rangle \leq 0, \ t \in T, \\ \langle \psi(p), d\rangle = 0, \ p \in P, \end{cases}$ *has no solution $d \in \mathbb{R}^n$;*

(ii) $0 \in co\{\varphi(s), t \in T\} + cone\{\phi(t), t \in T\} + lin\{\psi(p), p \in P\}$.

Theorem 2.1. *(Lebourge mean-value Theorem) Assume that $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function and $x, y \in \mathbb{R}^n$. Then there exists a point u in the open line segment $\{tx + (1-t)y \mid 0 < t < 1\}$, and $u^* \in \partial_C \varphi(u)$ such that $\varphi(y) - \varphi(x) = \langle u^*, y - x\rangle$.*

3. M-STATIONARY CONDITIONS FOR WEAK EFFICIENT SOLUTIONS

In this section, we derive M-stationary conditions for weak efficient solutions of (1.1). To proceed, we consider the following nonlinear programming problem with respect to a partition (B_1, B_2) of I_{GH}

$$\begin{cases} \min & f(x) = (f_1(x), \dots, f_m(x)), \\ s.t. & g_t(x) \leq 0, \ \forall t \in T, \\ & h_k(x) = 0, \ \forall k \in K, \\ & G_i(x) = 0, \ \forall i \in I_G \cup B_1, \\ & H_i(x) = 0, \ \forall i \in I_H \cup B_2. \end{cases} \tag{3.1}$$

Note that the feasible set of (1.1) contains that of (3.1). Moreover, one has the union of all the feasible sets of (3.1) over $(B_1, B_2) \in \mathscr{P}(I_{GH})$ corresponds to the feasible set of (1.1) locally around x.

In the following, we mention a result by Mehlitz [4] showing the relationship between problems (1.1) and (3.1).

Lemma 3.1 ([4], Lemma 4.2)**.** *Let $\bar{x}$ be a feasible point to (1.1). $\bar{x}$ is an M-stationary point of (1.1) if and only if there is a partition $(B_1, B_2) \in \mathscr{P}(I_{GH})$ such that $\bar{x}$ is a KKT point of (3.1).*

Let us define the following type of Abadie constraint qualifications (ACQs):

$$\partial_C\text{-ACQ}(B_1, B_2) \ : \ \Lambda_{(B_1,B_2)}(\bar{x}) \subseteq T(\Pi_{(B_1,B_2)}, \bar{x}),$$

where

$$\Lambda_{(B_1,B_2)}(\bar{x}) = (\bigcup_{t \in T} \partial_C g_t(\bar{x}))^- \cap (\bigcup_{k \in K} \partial_C h_k(\bar{x}))^\perp \\ \cap (\bigcup_{i \in I_G \cup B_1} \partial_C G_i(\bar{x}))^\perp \cap (\bigcup_{i \in I_H \cup B_2} \partial_C H_i(\bar{x}))^\perp,$$

and $\Pi_{(B_1,B_2)}$ is the feasible set of (3.1) corresponding to (B_1, B_2).

We are now in a position to give necessary optimality conditions for local efficient solutions of (1.1).

Theorem 3.1. *Let $\bar{x}$ be a weak efficient solution to (1.1). Assume that there exists a partition $(B_1, B_2) \in \mathscr{P}(I_{GH})$ such that ∂_C-ACQ(B_1, B_2) holds for $\bar{x}$ and*

$$D = cone\left(\bigcup_{t\in T} \partial_C g_t(\bar{x})\right) + lin\left(\bigcup_{k\in K} \partial_C h_k(\bar{x}) \cup \bigcup_{i\in I_G\cup I_{B_1}} \partial_C G_i(\bar{x}) \cup \bigcup_{i\in I_H\cup I_{B_2}} \partial_C H_i(\bar{x})\right) \tag{3.2}$$

is closed. Then $\bar{x}$ is an M-stationary point of (1.1).

Proof. We claim that

$$\Big(\bigcup_{j\in J} \partial_C f_j(\bar{x})\Big)^s \cap T(\Pi, \bar{x}) = \emptyset. \tag{3.3}$$

Indeed, suppose that there exists $\omega \in \big(\bigcup_{j\in J} \partial_C f_j(\bar{x})\big)^s \cap T(\Pi, \bar{x})$. Then, from $\omega \in \big(\bigcup_{j\in J} \partial_C f_j(\bar{x})\big)^s$, it follows that

$$\langle x^*, \omega\rangle < 0, \quad \forall x^* \in \partial_C f_j(\bar{x}),\ \forall j \in J, \tag{3.4}$$

and from $\omega \in T(\Pi, \bar{x})$, there is $t_n \downarrow 0$ and $\omega_n \to \omega$ satisfying $\bar{x} + t_n\omega_n \in \Pi$ for all n.

According to Theorem 2.1, one has that, for each $n \in \mathbb{N}$, there is u_n in the open line segment $(\bar{x}, \bar{x} + t_n\omega_n)$ and $u_n^* \in \partial_C f_1(u_n)$ such that

$$f_1(\bar{x} + t_n\omega_n) - f_1(\bar{x}) = t_n\langle u_n^*, \omega_n\rangle. \tag{3.5}$$

Because the mapping $x \to \partial_C f_1(x)$ is upper semicontinuous and $u_n \to \bar{x}$, there exits a subsequence $u_{n_p}^*$ of u_n^*, such that $u_{n_p}^* \to \widehat{u}$ and $\widehat{u} \in \partial_C f_1(\bar{x})$. By (3.4) and (3.5), we have

$$\langle \widehat{u}, \omega\rangle < 0 \quad \text{and} \quad f_1(\bar{x} + t_{n_p}\omega_{n_p}) - f_1(\bar{x}) = t_{n_p}\langle u_{n_p}^*, \omega_{n_p}\rangle < 0.$$

Hence, by taking into account that $\langle u_{n_p}^*, \omega_{n_p}\rangle \to \langle \widehat{u}, \omega\rangle$ and $t_{n_p} > 0$, we deduce that

$$f_1(\bar{x} + t_{n_p}\omega_{n_p}) < f_1(\bar{x}), \qquad \text{for p large enough.}$$

By denoting $\{\bar{x} + t_n\omega_n\}$ by $\{\bar{x} + t_n^{(1)}\omega_n^{(1)}\}$ and reproducing the same arguments above for f_2, we deduce that there exist a subsequence $\{\bar{x} + t_n^{(2)}\omega_n^{(2)}\}$ of $\{\bar{x} + t_n^{(1)}\omega_n^{(1)}\}$ such that for large enough indexes we have

$$\begin{cases} f_1(\bar{x} + t_n^{(2)}\omega_n^{(2)}) < f_1(\bar{x}) \\ f_2(\bar{x} + t_n^{(2)}\omega_n^{(2)}) < f_2(\bar{x}). \end{cases}$$

And so on, we obtain a subsequence $\{\bar{x} + t_n^{(m)}\omega_n^{(m)}\}$ of $\{\bar{x} + t_n\omega_n\}$ such that

$$\begin{cases} f_1(\bar{x} + t_n^{(m)}\omega_n^{(m)}) < f_1(\bar{x}) \\ f_2(\bar{x} + t_n^{(m)}\omega_n^{(m)}) < f_2(\bar{x}) \\ . \\ . \\ . \\ f_m(\bar{x} + t_n^{(m)}\omega_n^{(m)}) < f_m(\bar{x}). \end{cases}$$

Using the fact that $\{\bar{x}+t_n^{(m)}\omega_n^{(m)}\}\subset\Pi$ and the above relations we get a contradiction with the weak efficiency of $\bar{x}$, and consequently, (3.3) is fulfilled. Since $\Pi_{(B_1,B_2)}\subseteq\Pi$, then $T(\Pi_{(B_1,B_2)},\bar{x})\subseteq T(\Pi,\bar{x})$. Hence

$$\Big(\bigcup_{j\in J}\partial_C f_j(\bar{x})\Big)^s\cap T(\Pi_{(B_1,B_2)},\bar{x})=\emptyset.$$

On the basis of ∂_C-ACQ(B_1,B_2), we have

$$\left(\bigcup_{j\in J}(\partial_C f_j(\bar{x})\right)^s\cap\left(\bigcup_{t\in T(\bar{x})}\partial_C g_t(\bar{x})\right)^\circ\cap\left(\bigcup_{k\in K}\partial_C h_k(\bar{x})\right)^\perp$$
$$\cap\left(\bigcup_{i\in I_G\cup B_1}\partial_C G_i(\bar{x})\right)^\perp\cap\left(\bigcup_{i\in I_H\cup B_2}\partial_C H_i(\bar{x})\right)^\perp=\emptyset.$$

Then, we see that the system

$$\begin{cases}\langle\zeta_j,y^*\rangle<0, & \forall j\in J,\ \forall\zeta_j\in\partial_C f_j(\bar{x}),\\ \langle\vartheta_t,y^*\rangle\le 0, & \forall t\in T,\ \forall\vartheta_t\in\partial_C g_t(\bar{x}),\\ \langle\eta_k,y^*\rangle=0, & \forall k\in K,\ \forall\eta_k\in\partial_C h_k(\bar{x})\\ \langle\theta_i,y^*\rangle=0, & \forall i\in I_G\cup B_1,\ \forall\theta_i\in\partial_C G_i(\bar{x}),\\ \langle\xi_i,y^*\rangle=0, & \forall i\in I_H\cup B_2,\ \forall\xi_i\in\partial_C H_i(\bar{x}),\end{cases}$$

has no solution $y^*\in\mathbb{R}^n$. On the other hand, since $\partial_C f_j(\bar{x})$ is compact for all $j\in J$, the set $\bigcup_{j=1}^m\partial_C f_j(\bar{x})$ is also compact, and hence $\bigcup_{j=1}^m\partial_C f_j(\bar{x})+D$ is closed because so is D. Thus, by virtue of Lemma 2.2, we are led to

$$0\in co\left(\bigcup_{j\in J}\partial_C f_j(\bar{x})\right)+cone\left(\bigcup_{t\in T(\bar{x})}\partial_C g_t(\bar{x})\right)+lin\left(\bigcup_{k\in K}\partial_C h_k(\bar{x})\right)$$
$$+lin\left(\bigcup_{i\in I_G\cup B_1}\partial_C G_i(\bar{x})\right)+lin\left(\bigcup_{i\in I_H\cup B_2}\partial_C H_i(\bar{x})\right).$$

On the basis of Lemma 2.1, we deduce that there exist $\lambda=(\lambda_1,\ldots,\lambda_m)\in\mathbb{R}^m_+$ with $\sum_{j=1}^m\lambda_j=1$, $\lambda^g\in A(\bar{x})$, $\lambda^h=(\lambda_1^h,\ldots,\lambda_q^h)\in\mathbb{R}^q$, $\rho=(\rho_1,\ldots,\rho_l)\in\mathbb{R}^l$ and $\sigma=(\sigma_1,\ldots,\sigma_l)\in\mathbb{R}^l$ such that

$$0\in\sum_{j\in J}\lambda_j\partial_C f_j(\bar{x})+\sum_{t\in T(\bar{x})}\lambda_t^g\partial_C g_t(\bar{x})+\sum_{k\in K}\lambda_k^h\partial_C h_k(\bar{x})$$
$$+\sum_{i\in I_G\cup B_1}\rho_i\partial_C G_i(\bar{x})+\sum_{i\in I_H\cup B_2}\sigma_i\partial_C H_i(\bar{x}).$$

By taking

$$\lambda_i^G=\begin{cases}\rho_i, & i\in I_G(\bar{x})\cup B_1,\\ 0, & i\in I_H(\bar{x})\cup B_2,\end{cases}\qquad\qquad \lambda_i^H=\begin{cases}0, & i\in I_G(\bar{x})\cup B_1,\\ \sigma_i, & i\in I_H(\bar{x})\cup B_2,\end{cases}$$

and using the fact that for all $t\in T(\bar{x}):\lambda_t^g\ge 0$, we deduce that $\bar{x}$ is an M-stationary point of (1.1). □

To illustrate Theorem 3.1, we present the following example of (1.1).

Example 3.1. Consider the functions $f=(f_1,f_2):\mathbb{R}^2\to\mathbb{R}^2$, $g_t:\mathbb{R}^2\to\mathbb{R}$, $\forall t\in T=[0,1]$, $h:\mathbb{R}^2\to\mathbb{R}$, $G=(G_1,G_2,G_3):\mathbb{R}^2\to\mathbb{R}^3$, $H=(H_1,H_2,H_3):\mathbb{R}^2\to\mathbb{R}^3$ defined by

$$f_1(x,y)=y,\quad f_2(x,y)=x+|y|,$$

$$g_t(x,y)=|x|+(1-t)|y-1|-(1-t)\ (t\in T),\quad h(x,y)=\begin{cases}0, & y\le 1,\\ 1, & otherwise,\end{cases}$$

$$G_1(x,y)=x,\ \ G_2(x,y)=x,\ \ G_3(x,y)=1$$
$$H_1(x,y)=x,\ \ H_2(x,y)=1,\ \ H_3(x,y)=x.$$

We have that $\Pi=\{0\}\times[0,1]$ and $\bar{x}=(0,0)\in\Pi$ is a local efficient solution of (1.1). It is easily seen that

$$\partial_C f_1(\bar{x})=\{(0,1)\},\quad \partial_C f_2(\bar{x})=\{1\}\times[-1,1],$$
$$\partial_C g_t(\bar{x})=[-1,1]\times\{t-1\}\ \forall t\in T,\quad \bigcup_{t\in T(\bar{x})}\partial_C g_t(\bar{x})=[-1,1]\times[-1,0],$$
$$\partial_C h(\bar{x})=\partial_C G_3(\bar{x})=\partial_C H_2(\bar{x})=\{(0,0)\},$$
$$\partial_C G_1(\bar{x})=\partial_C G_2(\bar{x})=\partial_C H_1(\bar{x})=\partial_C H_3(\bar{x})=\{(1,0)\},$$
$$T(\Pi,\bar{x})=\{0\}\times\mathbb{R}_+,\ I_{GH}(\bar{x})=\{1\},\ I_G(\bar{x})=\{2\},\ I_H(\bar{x})=\{3\}.$$

In choosing $B_1=\emptyset$ and $B_2=I_{GH}(\bar{x})$, we can easily check that the qualification constraint ∂_C-ACQ(B_1,B_2) holds at $\bar{x}$ and that D, defined by (3.2), is closed. Consequently, $\bar{x}$ satisfies the assumptions of Theorem 3.1. In taking $\lambda_1=\lambda_2=1,\lambda^g=0,\ \lambda^h=1,\ \lambda_1^G=\lambda_2^G=\lambda_3^H=-\frac{1}{3}$ and $\lambda_3^G=\lambda_1^H=\lambda_2^H=0$, condition (2.1) is verified with $\lambda_1^G\lambda_1^H=\lambda_2^G\lambda_2^H=\lambda_3^G\lambda_3^H=0$, which means that $\bar{x}$ is an M-stationary point of (1.1).

4. Sufficient Optimality Conditions

In order to provide sufficient optimality conditions for initial problem (1.1), we introduce the following notations:

$$G^+=\{x\in\Pi:G_i(x)\ge 0\text{ for all }i\in I\},G^-=\{x\in\Pi:G_i(x)\le 0\text{ for all }i\in I\},$$
$$H^+=\{x\in\Pi:H_i(x)\ge 0\text{ for all }i\in I\},H^-=\{x\in\Pi:H_i(x)\le 0\text{ for all }i\in I\},$$
$$K^+=\{k\in K:\lambda_k^h>0\},K^-=\{k\in K:\lambda_k^h<0\},$$
$$I_G^+=\{i\in I_G:\lambda_i^G>0\},I_G^-=\{i\in I_G:\lambda_i^G<0\},$$
$$I_H^+=\{i\in I_H:\lambda_i^H>0\},I_H^-=\{i\in I_H:\lambda_i^H<0\},$$
$$I_{GH}^{+0}=\{i\in I_{GH}:\lambda_i^G>0,\lambda_i^H=0\},I_{GH}^{+-}=\{i\in I_{GH}:\lambda_i^G>0,\lambda_i^H<0\},$$
$$I_{GH}^{0+}=\{i\in I_{GH}:\lambda_i^G=0,\lambda_i^H>0\},I_{GH}^{-0}=\{i\in I_{GH}:\lambda_i^G<0,\lambda_i^H=0\}.$$

Our next step is to investigate sufficient optimality conditions for efficient solutions of (1.1). We always suppose that $f_j,\ j\in J$, $g_t,\ t\in T$, $h_k,\ k\in K$, G_i and H_i, $i\in I$ are all locally Lipschitz at a reference point $\bar{x}\in\Pi$.

Theorem 4.1. *Suppose that $\bar{x} \in \Pi$ is M-stationary point of* (1.1) *such that $f_j,\ j \in J$, is strictly pseudo-invex at $\bar{x}$ and g_t, $t \in T(\bar{x})$, h_k, $k \in K^+$, $-h_k$, $k \in K^-$, are quasi-invex at $\bar{x}$ w.r.t. η. If one of the following cases holds*

(i) $G^- \cap H^- \neq \emptyset$, $G_i\,(i \in I_G \cup I_{GH})$, $H_i\,(i \in I_H \cup I_{GH})$, *are quasi-invex on* $G^- \cap H^-$ *at* $\bar{x}$ *w.r.t.* η, *and* $I_G^- \cup I_H^- \cup I_{GH}^{-0} \cup I_{GH}^{0-} = \emptyset$.

(ii) $G^+ \cup H^+ \neq \emptyset$, $-G_i\,(i \in I_G \cup I_{GH})$, $-H_i\,(i \in I_H \cup I_{GH})$, *are quasi-invex on* $G^+ \cap H^+$ *at* $\bar{x}$ *w.r.t.* η, *and* $I_G^+ \cup I_H^+ \cup I_{GH}^{+0} \cup I_{GH}^{0+} = \emptyset$.

(iii) $G^+ \cup H^- \neq \emptyset$, $-G_i\,(i \in I_G \cup I_{GH})$, $H_i\,(i \in I_H \cup I_{GH})$, *are quasi-invex on* $G^+ \cap H^-$ *at* $\bar{x}$ *w.r.t.* η, *and* $I_G^+ \cup I_H^- \cup I_{GH}^{+0} \cup I_{GH}^{0-} = \emptyset$.

(iv) $G^- \cup H^+ \neq \emptyset$, $G_i\,(i \in I_G \cup I_{GH})$, $-H_i\,(i \in I_H \cup I_{GH})$, *are quasi-invex on* $G^- \cap H^+$ *at* $\bar{x}$ *w.r.t.* η, *and* $I_G^- \cup I_H^+ \cup I_{GH}^{-0} \cup I_{GH}^{0+} = \emptyset$,

then $\bar{x}$ is an efficient solution to (1.1).

Proof. Contrary to our claim, suppose that $\bar{x}$ is not an efficient solution. This means that we can find a vector $x \in \Pi$ satisfying $f_j(x) \leq f_j(\bar{x})$, for all $j \in J$ with strict inequality for at least one i. Since $f_j,\ j \in J$, is strictly pseudo-invex at $\bar{x}$ w.r.t. η , then $\langle x_j^*, \eta(x,\bar{x})\rangle < 0$, $\forall x_j^* \in \partial_C f_j(\bar{x})$. In multiplying both sides of this inequality by λ_j and summing over i, we are led to

$$\langle \sum_{j\in J} \lambda_j x_j^*, \eta(x,\bar{x})\rangle < 0, \quad \forall x_j^* \in \partial_C f_j(\bar{x}) \tag{4.1}$$

On the other side, since $\bar{x}$ is M-stationary point of (1.1), then by (2.1) there exist $x_j^* \in \partial_C f_j(\bar{x})$, $i = 1,\dots,m$, $y_t^* \in \partial_C g_t(\bar{x})$, $t \in T(\bar{x})$, $z_k^* \in \partial_C h_k(\bar{x})$, $k = 1,\dots,q$, $v_i^* \in \partial_C G_i(\bar{x})$ and $w_i^* \in \partial_C H_i(\bar{x})$, $i \in I$ satisfying

$$\sum_{j\in J} \lambda_j x_j^* + \sum_{t\in T(\bar{x})} \lambda_t^g y_t^* + \sum_{k\in K} \lambda_k^h z_k^* + \sum_{i\in I} \lambda_i^G v_i^* + \sum_{i\in I} \lambda_i^H w_i^* = 0,$$
$$\forall t \in T(\bar{x}) : \lambda_t^g \geq 0, \quad \forall i \in I_H(\bar{x}) : \lambda_i^G = 0, \quad \forall i \in I_G(\bar{x}) : \lambda_i^H = 0,$$

with $\lambda_i^G \lambda_i^H = 0$ for all $i \in I_{GH}(\bar{x})$. Hence

$$\sum_{t\in T(\bar{x})} \lambda_t^g y_t^* + \sum_{k\in K} \lambda_k^h z_k^* + \sum_{i\in I} \lambda_i^G v_i^* + \sum_{i\in I} \lambda_i^H w_i^* = -\sum_{j\in J} \lambda_j x_j^*.$$

Multiplying both sides of the above equality by $\eta(x,\bar{x})$, it follows that

$$\langle \sum_{t\in T(\bar{x})} \lambda_t^g y_t^* + \sum_{k\in K} \lambda_k^h z_k^* + \sum_{i\in I} \lambda_i^G v_i^* + \sum_{i\in I} \lambda_i^H w_i^*, \eta(x,\bar{x})\rangle = -\langle \sum_{j\in J} \lambda_j x_j^*, \eta(x,\bar{x})\rangle.$$

According to (4.1), one has

$$\langle \sum_{t\in T(\bar{x})} \lambda_t^g y_t^* + \sum_{k\in K} \lambda_k^h z_k^* + \sum_{i\in I} \lambda_i^G v_i^* + \sum_{i\in I} \lambda_i^H w_i^*, \eta(x,\bar{x})\rangle > 0 \tag{4.2}$$

In the remainder of the proof, we confine ourselves to the first case of the theorem say (i), and the other cases can be proved in the same way.

Since in the case (i) we have $G^- \cap H^- \neq \emptyset$, we have that there exists $\widehat{x} \in G^- \cap H^-$. We have the following inequalities:

$$\begin{aligned}
g_t(\widehat{x}) &\leq 0 = g_t(\bar{x}), \quad \forall t \in T(\bar{x}),\\
-h_k(\widehat{x}) &\leq 0 = -h_k(\bar{x}), \quad \forall k \in K^-,\\
h_k(\widehat{x}) &\leq 0 = h_k(\bar{x}), \quad \forall k \in K^+,\\
G_i(\widehat{x}) &\leq 0 = G_i(\bar{x}), \quad \forall i \in I_G \cup I_{GH},\\
H_i(\widehat{x}) &\leq 0 = H_i(\bar{x}), \quad \forall i \in I_H \cup I_{GH}.
\end{aligned}$$

By the quasi-invexity at $\bar{x}$ w.r.t. η of g_t, $t \in T(\bar{x})$, h_k, $k \in K$, $-h_k$, $k \in K^-$, the assumption of case (i), and the fact that $\partial_C\big(-\phi(.)\big) = -\partial_C\phi(.)$, we deduce that

$$\begin{aligned}
\langle y_t^*, \eta(\widehat{x},\bar{x})\rangle &\leq 0, \quad \forall t \in T(\bar{x}),\\
\langle z_k^*, \eta(\widehat{x},\bar{x})\rangle &\geq 0, \quad \forall k \in K^-.\\
\langle z_k^*, \eta(\widehat{x},\bar{x})\rangle &\leq 0, \quad \forall k \in K^+.\\
\langle v_i^*, \eta(\widehat{x},\bar{x})\rangle &\leq 0, \quad \forall i \in I_G \cup I_{GH},\\
\langle w_i^*, \eta(\widehat{x},\bar{x})\rangle &\leq 0, \quad \forall i \in I_H \cup I_{GH}.
\end{aligned}$$

Multiplying both sides of the above inequalities respectively by λ_t^g, λ_k^h, λ_i^G, and λ_i^H, we obtain under the condition $I_G^- \cup I_H^- \cup I_{GH}^{-0} \cup I_{GH}^{0-} = \emptyset$ that

$$\begin{aligned}
\langle \lambda_t^g y_t^*, \eta(\widehat{x},\bar{x})\rangle &\leq 0, \quad \forall t \in T(\bar{x}),\\
\langle \lambda_k^h z_k^*, \eta(\widehat{x},\bar{x})\rangle &\leq 0, \quad \forall k \in K.\\
\langle \lambda_i^G v_i^*, \eta(\widehat{x},\bar{x})\rangle &\leq 0, \quad \forall i \in I_G \cup I_{GH},\\
\langle \lambda_i^H w_i^*, \eta(\widehat{x},\bar{x})\rangle &\leq 0, \quad \forall i \in I_H \cup I_{GH}.
\end{aligned}$$

By summing the above inequalities while taking into account that $\lambda_i^G = 0$ for all $i \in I_H$ and $\lambda_i^H = 0$ for all $i \in I_G$, one obtains

$$\langle \sum_{t\in T(\bar{x})} \lambda_t^g y_t^* + \sum_{k\in K} \lambda_k^h z_k^* + \sum_{i\in I} \lambda_i^G v_i^* + \sum_{i\in I} \lambda_i^H w_i^*, \eta(\widehat{x},\bar{x})\rangle \leq 0,$$

which presents a contradiction to (4.2). □

Remark 4.1. Assume that in the previous theorem one of these two conditions is verified

- f_j, $j \in J$, are just pseudo-invex at $\bar{x}$ w.r.t. η (instead of being strictly pseudo-invex at $\bar{x}$ w.r.t. η).
- all the involved functions are supposed invex at $\bar{x}$ w.r.t. η.

Then by making use of identical proof steps, we can prove that $\bar{x}$ is a weak efficient solution to (1.1).

Remark 4.2. Since the nonsmooth multiobjective semi-infinite mathematical programming with equilibrium constraints falls under the second case of the previous theorem, the latter generalizes the main result in [19].

5. Duality

Related to (1.1), we investigate in this section dual problems according to Wolfe [20] and Mond-Weir [21], and explore weak, strong and converse duality relations.

5.1. **The Wolfe duality.** Let $(u,\lambda,\lambda^g,\lambda^h,\lambda^G,\lambda^H)\in\mathbb{R}^n\times\mathbb{R}^m_+\times\mathbb{R}^{|S|}_+\times\mathbb{R}^q\times\mathbb{R}^l\times\mathbb{R}^l$ with $\sum_{i=1}^m\lambda_i = 1$ and $\tau=(1,...,1)\in\mathbb{R}^m$. We first formulate the Wolfe type dual problem for (1.1)

$$
\text{(WP)}\begin{cases}
\max & \widetilde{f}(u,\lambda,\lambda^g,\lambda^h,\lambda^G,\lambda^H)=f(u)+\Big(\sum_{t\in T(u)}\lambda_t^g g_t(u)+\sum_{k\in K}\lambda_k^h h_k(u)\\
& \quad +\sum_{i\in I}\lambda_i^G G_i(u)+\sum_{i\in I}\lambda_i^H H_i(u)\Big)\tau,\\
\text{s.t.} & 0\in\sum_{j\in J}\lambda_j\left(\partial_C f_j(u)\right)+\sum_{t\in T(u)}\lambda_t^g\left(\partial_C g_t(u)\right)+\sum_{k\in K}\lambda_k^h\left(\partial_C h_k(u)\right)\\
& \quad +\sum_{i\in I}\lambda_i^G\left(\partial_C G_i(u)\right)+\sum_{i\in I}\lambda_i^H\left(\partial_C H_i(u)\right),\\
& \sum_{i=1}^m\lambda_i=1,\ \lambda_i^G=0,\forall i\in I_H(u),\ \lambda_i^H=0,\forall i\in I_G(u),\ \lambda_i^G\lambda_i^H=0,\forall i\in I_{GH}(u)\\
& (u,\lambda,\lambda^g,\lambda^h,\lambda^G,\lambda^H)\in\mathbb{R}^n\times\mathbb{R}^m_+\times\mathbb{R}^{|S|}_+\times\mathbb{R}^q\times\mathbb{R}^l\times\mathbb{R}^l.
\end{cases}
$$

Its feasible set is given by

$$
\begin{aligned}
\Pi_W=\Big\{&(u,\lambda,\lambda^g,\lambda^h,\lambda^G,\lambda^H)\in\mathbb{R}^n\times\mathbb{R}^m_+\times\mathbb{R}^{|S|}_+\times\mathbb{R}^q\times\mathbb{R}^l\times\mathbb{R}^l:\\
&0\in\sum_{j\in J}\lambda_j\left(\partial_C f_j(u)\right)+\sum_{t\in T(u)}\lambda_t^g\left(\partial_C g_t(u)\right)+\sum_{k\in K}\lambda_k^h\left(\partial_C h_k(u)\right)+\sum_{i\in I}\lambda_i^G\left(\partial_C G_i(u)\right)\\
&+\sum_{i\in I}\lambda_i^H\left(\partial_C H_i(u)\right),\quad \sum_{i=1}^m\lambda_i=1,\ \lambda_i^G=0,\forall i\in I_H(u),\quad \lambda_i^H=0,\forall i\in I_G(u),\\
&\lambda_i^G\lambda_i^H=0,\forall i\in I_{GH}(u)\Big\}.
\end{aligned}
$$

Let us start with a weak Wolfe duality theorem.

Theorem 5.1. *(Weak duality) Assume that $(u,\lambda,\lambda^g,\lambda^h,\lambda^G,\lambda^H)\in\Pi_W$. If $f_j,\ j\in J$, g_t, $t\in T(u)$, $\pm h_k$, $k\in K$, $\pm G_i$, $i\in I_G(u)\cup I_{GH}(u)$, $\pm H_i$, $i\in I_H(u)\cup I_{GH}(u)$, are all invex at u w.r.t. η. Then there exists $r>0$ such that, for each $x\in B(u,r)\cap\Pi$, we do not have $f(x)<\widetilde{f}(u,\lambda,\lambda^g,\lambda^h,\lambda^G,\lambda^H)$.*

Proof. Let $(u,\lambda,\lambda^g,\lambda^h,\lambda^G,\lambda^H)\in\Pi_W$. Then there exist $x_j^*\in\partial_C f_j(u)$, $i=1,...,m$, $y_t^*\in\partial_C g_t(u)$, $t\in T(u)$, $z_k^*\in\partial_C h_k(u)$, $k=1,...,q$, $v_i^*\in\partial_C G_i(u)$ and $w_i^*\in\partial_C H_i(u)$, $i\in I$ satisfying

$$
\sum_{j\in J}\lambda_j x_j^*+\sum_{t\in T(u)}\lambda_t^g y_t^*+\sum_{k\in K}\lambda_k^h z_k^*+\sum_{i\in I}\lambda_i^G v_i^*+\sum_{i\in I}\lambda_i^H w_i^*=0, \tag{5.1}
$$

By contradiction, assume that, for each $r>0$, there is $x\in B(u,r)\cap\Pi$ such that

$$
f(x)<\widetilde{f}(u,\lambda,\lambda^g,\lambda^h,\lambda^G,\lambda^H).
$$

Then $\sum_{j=1}^m\ \lambda_j\big(f_j(x)-\widetilde{f}_j(u,\lambda,\lambda^g,\lambda^h,\lambda^G,\lambda^H)\big)<0$. Hence,

$$
\sum_{j=1}^m\ \lambda_j\Bigg(f_j(x)-f_j(u)-\Big(\sum_{t\in T(u)}\lambda_t^g g_t(u)+\sum_{k\in K}\lambda_k^h h_k(u)+\sum_{i\in I}\lambda_i^G G_i(u)+\sum_{i\in I}\lambda_i^H H_i(u)\Big)\Bigg)<0.
$$

Thus,

$$
\sum_{j=1}^m\ \lambda_j\Big(f_j(x)-f_j(u)\Big)-\sum_{i=1}^m\ \lambda_j\Big(\sum_{t\in T(u)}\lambda_t^g g_t(u)+\sum_{k\in K}\lambda_k^h h_k(u)+\sum_{i\in I}\lambda_i^G G_i(u)+\sum_{i\in I}\lambda_i^H H_i(u)\Big)<0.
$$

From $\sum_{j=1}^{m} \lambda_j = 1$, it follows that

$$\sum_{j=1}^{m} \lambda_j \Big(f_j(x) - f_j(u) \Big) - \Big(\sum_{t \in T(u)} \lambda_t^g g_t(u) + \sum_{k \in K} \lambda_k^h h_k(u) + \sum_{i \in I} \lambda_i^G G_i(u) + \sum_{i \in I} \lambda_i^H H_i(u) \Big) < 0. \tag{5.2}$$

However, by virtue of the invexity at u w.r.t. η of $f_j,\ j \in J$, $g_t, t \in T(u)$, $\pm h_k, k \in K$, $\pm G_i, i \in I$, $\pm H_i,\ i \in I$, one sees that

$$\begin{array}{rcll} f_j(x) - f_j(u) & \geq & \langle x_j^*, \eta(x,u) \rangle, & \forall x_j^* \in \partial_C f_j(u),\ \forall\ j \in J, \\ g_t(x) - g_t(u) & \geq & \langle y_t^*, \eta(x,u) \rangle, & \forall y_t^* \in \partial_C g_t(u),\ \forall\ t \in T(u), \\ h_k(x) - h_k(u) & \geq & \langle z_k^*, \eta(x,u) \rangle, & \forall z_k^* \in \partial_C h_k(u),\ \forall\ k \in K, \\ (-h_k)(x) - (-h_k)(u) & \geq & \langle \widehat{z}_k^*, \eta(x,u) \rangle, & \forall \widehat{z}_k^* \in \partial_C (-h_k)(u),\ \forall\ k \in K, \\ G_i(x) - G_i(u) & \geq & \langle v_i^*, \eta(x,u) \rangle, & \forall v_i^* \in \partial_C G_i(u),\ \forall\ i \in I_G(u) \cup I_{GH}(u), \\ (-G_i)(x) - (-G_i)(u) & \geq & \langle \widehat{v}_i^*, \eta(x,u) \rangle, & \forall \widehat{v}_i^* \in \partial_C (-G_i)(u),\ \forall\ i \in I_G(u) \cup I_{GH}(u), \\ H_i(x) - H_i(u) & \geq & \langle w_i^*, \eta(x,u) \rangle, & \forall w_i^* \in \partial_C H_i(u),\ \forall\ i \in I_H(u) \cup I_{GH}(u), \\ (-H_i)(x) - (H_i)(u) & \geq & \langle \widehat{w}_i^*, \eta(x,u) \rangle, & \forall \widehat{w}_i^* \in \partial_C (-H_i)(u),\ \forall\ i \in I_H(u) \cup I_{GH}(u). \end{array}$$

By taking $\widehat{z}_k^* = -z_k^*,\ \forall\ k \in K$, $\widehat{v}_i^* = -v_i^*,\ \forall\ i \in I_G(u) \cup I_{GH}(u)$ and $\widehat{w}_i^* = -w_i^*,\ \forall\ I_H(u) \cup I_{GH}(u)$ one has

$$\begin{array}{rcll} f_j(x) - f_j(u) & \geq & \langle x_j^*, \eta(x,u) \rangle, & \forall x_j^* \in \partial_C f_j(u),\ \forall\ j \in J, \\ g_t(x) - g_t(u) & \geq & \langle y_t^*, \eta(x,u) \rangle, & \forall y_t^* \in \partial_C g_t(u),\ \forall\ t \in T(u), \\ h_k(x) - h_k(u) & = & \langle z_k^*, \eta(x,u) \rangle, & \forall z_k^* \in \partial_C h_k(u),\ \forall\ k \in K, \\ G_i(x) - G_i(u) & = & \langle v_i^*, \eta(x,u) \rangle, & \forall v_i^* \in \partial_C G_i(u),\ \forall\ i \in I_G(u) \cup I_{GH}(u), \\ H_i(x) - H_i(u) & = & \langle w_i^*, \eta(x,u) \rangle, & \forall w_i^* \in \partial_C H_i(u),\ \forall\ i \in I_H(u) \cup I_{GH}(u). \end{array}$$

Multiplying both sides of the above inequalities respectively by λ_j, λ_t^g, λ_k^h, λ_i^G and λ_i^H while considering that $g_t(x) \leq 0\ (t \in T(u))$, $h_k(x) = 0\ (k \in K)$, $\lambda_i^G = 0\ (i \in I_H(u))$ and $\lambda_i^H = 0\ (i \in I_G(u))$, we obtain

$$\begin{array}{rcll} \lambda_j(f_j(x) - f_j(u)) & \geq & \langle \lambda_j x_j^*, \eta(x,u) \rangle, & \forall\ j \in J, \\ -\lambda_t^g g_t(u) & \geq & \langle \lambda_t^g y_t^*, \eta(x,u) \rangle, & \forall\ t \in T(u), \\ -\lambda_k^h h_k(u) & = & \langle \lambda_k^h z_k^*, \eta(x,u) \rangle, & \forall\ k \in K, \\ -\lambda_i^G G_i(u) & = & \langle \lambda_i^G v_i^*, \eta(x,u) \rangle, & \forall\ i \in I, \\ -\lambda_i^G H_i(u) & = & \langle \lambda_i^G w_i^*, \eta(x,u) \rangle, & \forall\ i \in I. \end{array}$$

In combining the above inequalities with (5.1), we deduce that

$$\begin{aligned} 0 = & \langle \sum_{j \in J} \lambda_j x_j^* + \sum_{t \in T(u)} \lambda_t^g y_t^* + \sum_{k \in K} \lambda_k^h z_k^* + \sum_{i \in I} \lambda_i^G v_i^* + \sum_{i \in I} \lambda_i^H w_i^*, \eta(x,\bar{x}) \rangle \\ & \leq \sum_{j=1}^{m} \lambda_j \Big(f_j(x) - f_j(u) \Big) - \Big(\sum_{t \in T(u)} \lambda_t^g g_t(u) + \sum_{k \in K} \lambda_k^h h_k(u) + \sum_{i \in I} \lambda_i^G G_i(u) + \sum_{i \in I} \lambda_i^H H_i(u) \Big), \end{aligned}$$

which stands in contradiction to (5.2). □

Now, we establish the following strong Wolfe duality theorem.

Theorem 5.2. *(Strong duality) Let u be a weak efficient solution to (1.1), D in (3.2) be closed, and $\partial_C ACQ(B_1,B_2)$ hold at u. Then, there is $(\lambda,\lambda^g,\lambda^h,\lambda^G,\lambda^H)\in\mathbb{R}^m_+\times\mathbb{R}^{|S|}_+\times\mathbb{R}^q\times\mathbb{R}^l\times\mathbb{R}^l$ such that $(u,\lambda,\lambda^g,\lambda^h,\lambda^G,\lambda^H)\in\Pi_W$ and $f(u)=\widetilde{f}(u,\lambda,\lambda^g,\lambda^h,\lambda^G,\lambda^H)$. Moreover, if $f_j,\ j\in J$, g_t, $t\in T(u)$, $\pm h_k$, $k\in K$, $\pm G_i$, $i\in I$, $\pm H_i$, $i\in I$, are all invex at u w.r.t. η, then (u,λ,μ,ν,ρ) is a local weak efficient solution to (WP).*

Proof. From Theorem 3.1, there exist $(\lambda,\lambda^g,\lambda^h,\lambda^G,\lambda^H)\in\mathbb{R}^m_+\times\mathbb{R}^{|S|}_+\times\mathbb{R}^q\times\mathbb{R}^l\times\mathbb{R}^l$ satisfying

$$0\in\sum_{j\in J}\lambda_j\left(\partial_C f_j(u)\right)+\sum_{t\in T(u)}\lambda^g_t\left(\partial_C g_t(u)\right)+\sum_{k\in K}\lambda^h_k\left(\partial_C h_k(u)\right)$$
$$+\sum_{i\in I}\lambda^G_i\left(\partial_C G_i(u)\right)+\sum_{i\in I}\lambda^H_i\left(\partial_C H_i(u)\right),$$

with $\sum_{i=1}^m\lambda_i=1$, $\lambda^G_i=0,\forall i\in I_H(u)$, $\lambda^H_i=0,\forall i\in I_G(u)$, $\lambda^G_i\lambda^H_i=0,\forall i\in I_{GH}(u)$, which means that $(u,\lambda,\lambda^g,\lambda^h,\lambda^G,\lambda^H)\in\Pi_W$. On the basis of $\lambda^g_t g_t(u)=0$ for any $t\in T(u)$, $h_k(u)=0$ for any $k\in K$, and $\lambda^G_i G_i(u)=\lambda^H_i H_i(u)=0$ for any $i\in I$, we see that

$$\sum_{t\in T(u)}\lambda^g_t g_t(u)+\sum_{k\in K}\lambda^h_k h_k(u)+\sum_{i\in I}\lambda^G_i G_i(u)+\sum_{i\in I}\lambda^H_i H_i(u)=0.$$

Consequently, $f(u)=\widetilde{f}(u,\lambda,\lambda^g,\lambda^h,\lambda^G,\lambda^H)$. Furthermore, since $f_j,\ j\in J$, g_t, $t\in T(u)$, $\pm h_k$, $k\in K$, $\pm G_i$, $i\in I$, $\pm H_i$, $i\in I$, are all invex at u w.r.t. η, we see from Theorem 5.1 that we can find $r>0$ such that for each $x\in B(u,r)\cap\Pi$ we do not have $f(x)<\widetilde{f}(u,\lambda,\lambda^g,\lambda^h,\lambda^G,\lambda^H)$. Hence, we conclude that $(u,\lambda,\lambda^g,\lambda^h,\lambda^G,\lambda^H)$ is a local weak efficient solution to (WP). □

Remark 5.1. We can easily verify that, if we require the function f to be strictly invex in Theorem 5.1, then we will obtain that there exists $r>0$ such that, for each $x\in B(u,r)\cap\Pi$, the inequality $f(x)\le\widetilde{f}(u,\lambda,\lambda^g,\lambda^h,\lambda^G,\lambda^H)$ does not hold. This stronger assumption when added in Theorem 5.2 will yield that $(u,\lambda,\lambda^g,\lambda^h,\lambda^G,\lambda^H)$ is a local efficient solution of (WP).

5.2. **The Mond-Weir Duality.** We are concerned here with the Mond-Weir duality type for (1.1). To proceed, for an arbitrary $(u,\lambda,\lambda^g,\lambda^h,\lambda^G,\lambda^H)\in\mathbb{R}^n\times\mathbb{R}^m_+\times\mathbb{R}^{|S|}_+\times\mathbb{R}^q\times\mathbb{R}^l\times\mathbb{R}^l$ with $\sum_{i=1}^m\lambda_i=1$, we consider

$$(\text{MWP})\begin{cases}\max & f(u),\\ \text{s.t.} & 0\in\sum_{j\in J}\lambda_j\left(\partial_C f_j(u)\right)+\sum_{t\in T(u)}\lambda^g_t\left(\partial_C g_t(u)\right)+\sum_{k\in K}\lambda^h_k\left(\partial_C h_k(u)\right)\\ & +\sum_{i\in I}\lambda^G_i\left(\partial_C G_i(u)\right)+\sum_{i\in I}\lambda^H_i\left(\partial_C H_i(u)\right),\\ & \sum_{t\in T(u)}\lambda^g_t g_t(u)+\sum_{k\in K}\lambda^h_k h_k(u)+\sum_{i\in I}\lambda^G_i G_i(u)+\sum_{i\in I}\lambda^H_i H_i(u)\ge 0,\\ & u\in\mathbb{R}^n,\ (\lambda,\lambda^g,\lambda^h,\lambda^G,\lambda^H)\in\mathbb{R}^m_+\times\mathbb{R}^{|S|}_+\times\mathbb{R}^q\times\mathbb{R}^l\times\mathbb{R}^l,\\ & \sum_{i=1}^m\lambda_i=1,\ \lambda^G_i=0,\forall i\in I_H(u),\ \lambda^H_i=0,\forall i\in I_G(u),\ \lambda^G_i\lambda^H_i=0,\forall i\in I_{GH}(u),\end{cases}$$

which has a feasible set given by

$$\Pi_D = \Big\{ (u,\lambda,\mu,\nu,\rho) \in \mathbb{R}^n \times \mathbb{R}^m_+ \times \mathbb{R}^p_+ \times \mathbb{R}^q \times int(\mathbb{R}^m_+) \ : \ \sum_{i=1}^m \lambda_i = 1,$$
$$0 \in \sum_{j\in J} \lambda_j \big(\partial_C f_j(u)\big) + \sum_{t\in T(u)} \lambda_t^g \left(\partial_C g_t(u)\right) + \sum_{k\in K} \lambda_k^h \left(\partial_C h_k(u)\right)$$
$$+ \sum_{i\in I} \lambda_i^G \left(\partial_C G_i(u)\right) + \sum_{i\in I} \lambda_i^H \left(\partial_C H_i(u)\right),$$
$$\sum_{t\in T(u)} \lambda_t^g g_t(u) + \sum_{k\in K} \lambda_k^h h_k(u) + \sum_{i\in I} \lambda_i^G G_i(u) + \sum_{i\in I} \lambda_i^H H_i(u) \geq 0,$$
$$\lambda_i^G = 0, \forall i \in I_H(u), \quad \lambda_i^H = 0, \forall i \in I_G(u), \quad \lambda_i^G \lambda_i^H = 0, \forall i \in I_{GH}(u), \Big\}.$$

The following theorem presents a weak Mond-Weir duality result.

Theorem 5.3. *(Weak duality) If $(u,\lambda,\lambda^g,\lambda^h,\lambda^G,\lambda^H) \in \Pi_D$ and f_j, $j \in J$, g_t, $t \in T(u)$, $\pm h_k$, $k \in K$, $\pm G_i$, $i \in I_G(u) \cup I_{GH}(u)$, $\pm H_i$, $i \in I_H(u) \cup I_{GH}(u)$, are all invex at u w.r.t. η, then we can find $r > 0$ such that for each $x \in B(u,r) \cap \Pi$ the inequality $f(x) < f(u)$ does not hold.*

Proof. Since $(u,\lambda,\lambda^g,\lambda^h,\lambda^G,\lambda^H)$ is a feasible point for (MWP), there exist there exist $x_j^* \in \partial_C f_j(u)$, $i = 1,...,m$, $y_t^* \in \partial_C g_t(u)$, $t \in T(u)$, $z_k^* \in \partial_C h_k(u)$, $k = 1,...,q$, $v_i^* \in \partial_C G_i(u)$ and $w_i^* \in \partial_C H_i(u)$, $i \in I$ satisfying

$$\sum_{j\in J} \lambda_j x_j^* + \sum_{t\in T(u)} \lambda_t^g y_t^* + \sum_{k\in K} \lambda_k^h z_k^* + \sum_{i\in I} \lambda_i^G v_i^* + \sum_{i\in I} \lambda_i^H w_i^* = 0, \tag{5.3}$$

and

$$\sum_{t\in T(u)} \lambda_t^g g_t(u) + \sum_{k\in K} \lambda_k^h h_k(u) + \sum_{i\in I} \lambda_i^G G_i(u) + \sum_{i\in I} \lambda_i^H H_i(u) \geq 0. \tag{5.4}$$

By contradiction, we assume that for each $r > 0$ there is $x \in B(u,r) \cap \Pi$ such that $f(x) < f(u)$. Then

$$\sum_{i=1}^m \lambda_i \big(f_i(x) - f_i(u)\big) < 0. \tag{5.5}$$

By virtue of the invexity at u w.r.t. η of f_j, $j \in J$, g_t, $t \in T(u)$, $\pm h_k$, $k \in K$, $\pm G_i$, $i \in I$, $\pm H_i$, $i \in I$, one has

$$\begin{array}{rcll}
f_j(x) - f_j(u) & \geq & \langle x_j^*, \eta(x,u)\rangle, & \forall x_j^* \in \partial_C f_j(u),\ \forall\, j \in J, \\
g_t(x) - g_t(u) & \geq & \langle y_t^*, \eta(x,u)\rangle, & \forall y_t^* \in \partial_C g_t(u),\ \forall\, t \in T(u), \\
h_k(x) - h_k(u) & \geq & \langle z_k^*, \eta(x,u)\rangle, & \forall z_k^* \in \partial_C h_k(u),\ \forall\, k \in K, \\
(-h_k)(x) - (-h_k)(u) & \geq & \langle \widehat{z}_k^*, \eta(x,u)\rangle, & \forall \widehat{z}_k^* \in \partial_C(-h_k)(u),\ \forall\, k \in K, \\
G_i(x) - G_i(u) & \geq & \langle v_i^*, \eta(x,u)\rangle, & \forall v_i^* \in \partial_C G_i(u),\ \forall\, i \in I_G(u) \cup I_{GH}(u), \\
(-G_i)(x) - (-G_i)(u) & \geq & \langle \widehat{v}_i^*, \eta(x,u)\rangle, & \forall \widehat{v}_i^* \in \partial_C(-G_i)(u),\ \forall\, i \in I_G(u) \cup I_{GH}(u), \\
H_i(x) - H_i(u) & \geq & \langle w_i^*, \eta(x,u)\rangle, & \forall w_i^* \in \partial_C H_i(u),\ \forall\, i \in I_H(u) \cup I_{GH}(u), \\
(-H_i)(x) - (H_i)(u) & \geq & \langle \widehat{w}_i^*, \eta(x,u)\rangle, & \forall \widehat{w}_i^* \in \partial_C(-H_i)(u),\ \forall\, i \in I_H(u) \cup I_{GH}(u).
\end{array}$$

By taking $\widehat{z}_k^* = -z_k^*$, $\forall\, k \in K$, $\widehat{v}_i^* = -v_i^*$, $\forall\, i \in I_G(u) \cup I_{GH}(u)$ and $\widehat{w}_i^* = -w_i^*$, $\forall\, I_H(u) \cup I_{GH}(u)$ one has

$$\begin{array}{rcll} f_j(x) - f_j(u) & \geq & \langle x_j^*, \eta(x,u)\rangle, & \forall x_j^* \in \partial_C f_j(u),\ \forall\, j \in J, \\ g_t(x) - g_t(u) & \geq & \langle y_t^*, \eta(x,u)\rangle, & \forall y_t^* \in \partial_C g_t(u),\ \forall\, t \in T(u), \\ h_k(x) - h_k(u) & = & \langle z_k^*, \eta(x,u)\rangle, & \forall z_k^* \in \partial_C h_k(u),\ \forall\, k \in K, \\ G_i(x) - G_i(u) & = & \langle v_i^*, \eta(x,u)\rangle, & \forall v_i^* \in \partial_C G_i(u),\ \forall\, i \in I_G(u) \cup I_{GH}(u), \\ H_i(x) - H_i(u) & = & \langle w_i^*, \eta(x,u)\rangle, & \forall w_i^* \in \partial_C H_i(u),\ \forall\, i \in I_H(u) \cup I_{GH}(u). \end{array}$$

Multiplying both sides of the above inequalities respectively by λ_j, λ_t^g, λ_k^h, λ_i^G and λ_i^H while considering that $g_t(x) \leq 0$ $(t \in T(u))$, $h_k(x) = 0$ $(k \in K)$, and $\lambda_i^G G_i(x) = \lambda_i^H H_i(x) = 0$ for any $i \in I$, we obtain

$$\begin{array}{rcll} \lambda_j(f_j(x) - f_j(u)) & \geq & \langle \lambda_j x_j^*, \eta(x,u)\rangle, & \forall\, j \in J, \\ -\lambda_t^g g_t(u) & \geq & \langle \lambda_t^g y_t^*, \eta(x,u)\rangle, & \forall\, t \in T(u), \\ -\lambda_k^h h_k(u) & = & \langle \lambda_k^h z_k^*, \eta(x,u)\rangle, & \forall\, k \in K, \\ -\lambda_i^G G_i(u) & = & \langle \lambda_i^G v_i^*, \eta(x,u)\rangle, & \forall\, i \in I, \\ -\lambda_i^G H_i(u) & = & \langle \lambda_i^G w_i^*, \eta(x,u)\rangle, & \forall\, i \in I. \end{array}$$

The combination of the above inequalities with (5.3) tells us that

$$\begin{aligned} 0 =& \langle \sum_{j\in J} \lambda_j x_j^* + \sum_{t\in T(u)} \lambda_t^g y_t^* + \sum_{k\in K} \lambda_k^h z_k^* + \sum_{i\in I} \lambda_i^G v_i^* + \sum_{i\in I} \lambda_i^H w_i^*, \eta(x,\bar{x})\rangle \\ \leq & \sum_{j=1}^{m} \lambda_j \Big(f_j(x) - f_j(u) \Big) - \Big(\sum_{t\in T(u)} \lambda_t^g g_t(u) + \sum_{k\in K} \lambda_k^h h_k(u) + \sum_{i\in I} \lambda_i^G G_i(u) + \sum_{i\in I} \lambda_i^H H_i(u) \Big), \end{aligned}$$

From (5.4), we see that

$$\sum_{i=1}^{m} \lambda_i \big(f_i(x) - f_i(u) \big) \geq 0,$$

which stands in contradiction to (5.5). □

Next, we establish the following strong Mond-Weir duality result.

Theorem 5.4. *(Strong duality)Let u be a weak efficient solution of* (1.1)*, D in* (3.2) *be closed and $\partial_C ACQ(B_1, B_2)$ holds at u. Then, there is $(\lambda, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}_+^m \times \mathbb{R}_+^{|S|} \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l$ such that $(u, \lambda, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \Pi_D$. Moreover, if f_j, $j \in J$, g_t, $t \in T(u)$, $\pm h_k$, $k \in K$, $\pm G_i$, $i \in I$, $\pm H_i$, $i \in I$, are all invex at u w.r.t. η, then, one has $(u, \lambda, \mu, \nu, \rho)$ is a local weak efficient solution of (MWP), and the objective values of* (1.1) *and (MWP) are equal.*

Proof. According to Theorem 3.1, there exists $(\lambda, \lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}_+^m \times \mathbb{R}_+^{|S|} \times \mathbb{R}^q \times \mathbb{R}^l \times \mathbb{R}^l$ satisfying

$$\begin{aligned} 0 \in & \sum_{j\in J} \lambda_j \left(\partial_C f_j(u) \right) + \sum_{t\in T(u)} \lambda_t^g \left(\partial_C g_t(u) \right) + \sum_{k\in K} \lambda_k^h \left(\partial_C h_k(u) \right) \\ & + \sum_{i\in I} \lambda_i^G \left(\partial_C G_i(u) \right) + \sum_{i\in I} \lambda_i^H \left(\partial_C H_i(u) \right), \end{aligned}$$

with $\sum_{i=1}^m \lambda_i = 1$, $\lambda_i^G = 0$, $\forall i \in I_H(u)$, $\lambda_i^H = 0$, $\forall i \in I_G(u)$, $\lambda_i^G \lambda_i^H = 0$, $\forall i \in I_{GH}(u)$, From $\lambda_t^g g_t(u) = 0$ for any $t \in T(u)$, $h_k(u) = 0$ for any $k \in K$, and $\lambda_i^G G_i(u) = \lambda_i^H H_i(u) = 0$ for any

$i \in I$, we see that

$$\sum_{t\in T(u)} \lambda_t^g g_t(u) + \sum_{k\in K} \lambda_k^h h_k(u) + \sum_{i\in I} \lambda_i^G G_i(u) + \sum_{i\in I} \lambda_i^H H_i(u) = 0.$$

Consequently $(u,\lambda,\lambda^g,\lambda^h,\lambda^G,\lambda^H) \in \Pi_D$. As $f_j,\ j\in J$, g_t, $t\in T(u)$, $\pm h_k$, $k\in K$, $\pm G_i$, $i\in I$, $\pm H_i$, $i\in I$, are all invex at u w.r.t. η, Theorem 5.3 tells us that there exists $r>0$ such that for each $x\in B(u,r)\cap\Pi$ we do not have $f(x)<f(u)$. Hence, we obtain that $(u,\lambda,\lambda^g,\lambda^h,\lambda^G,\lambda^H)$ a local weak efficient solution of (WP). It is easy to see that the objective values of (1.1) and (MWP) are equal to $f(u)$. □

Remark 5.2. Again, we can easily verify that if we require the function f to be strictly invex in Theorem 5.3, then we will obtain that there exists $r>0$ such that for each $x\in B(u,r)\cap\Pi$, the inequality $f(x)\leq f(u)$ does not hold. This stronger assumption when added in Theorem 5.4 will yield that $(u,\lambda,\lambda^g,\lambda^h,\lambda^G,\lambda^H)$ is a local efficient solution of (MWP).

6. Conclusion

In this work, we established necessary M-stationary conditions for a nonsmooth multiobjective semi-infinite programming with switching constraints by using an equivalent problem and Clarke's generalized subdifferentials. Moreover, we h employed the ACQs which are weaker than most of known nonsmooth constraint qualification (Slater, Cottle, Zangwill, etc). We also established the sufficient M-stationary conditions for (1.1), formulate its Mond-Weir and Wolf dual problems and prove weak and strong duality results. To the best of our knowledge, there is no paper dealing with nonsmooth case for multiobjective semi-infinite programming with switching constraints. For future research, we can derive optimality conditions for the same problem we studied by using weaker subdifferentials such as tangential subdifferentials and convexificators.

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