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STRUCTURAL CONVEXITY AND RAVINES OF QUADRATIC FUNCTIONS

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Abstract. E.G. Belousov and V.G. Andronov [Solvability and Stability of Problems of Polynomial Programming (in Russian), Publishing House of the Moscow University, Moscow, p. 4, 1993] have observed that the notions of *structural convexity* and *ravine* of a polynomial function, which were introduced by themselves, are useful for studying stability and solvability of convex polynomial mathematical programming problems, as well as for the investigation of the distribution of integer points in convex sets. The results given in Chapters 3–5 of the book justify their observations. This paper presents some facts about structural convexity and ravines of quadratic functions. Among other things, we obtain a verifiable criterion for structural convexity of a quadratic function and show that such a function cannot have ravines along linear subspaces.

Keywords. Polynomial function; Structural convexity; Ravine; Convex function; Quadratic function.

1. Introduction

In this paper, we are interested in studying the notions of of *structural convexity* and *ravine* of a polynomial function, which are due to Belousov and Andronov (see [1] and the references therein).

What is structural convexity and why the notion deserves a careful consideration? It is well known [2, Theorem 4.5] that a twice continuously differentiable function $f:\mathbb{R}^n\to\mathbb{R}$ is *convex* on $\mathbb{R}^n$ if and only if the Hessian matrix $\nabla^2 f(x)=\left(\dfrac{\partial^2 f(x)}{\partial x_j\partial x_i}\right)$ of f at any point $x\in\mathbb{R}^n$ is positive semidefinite, i.e.,

$$\sum_{j=1}^{n}\sum_{i=1}^{n} f''_{i,j}(x)v_jv_j\geq 0,\quad \forall x\in\mathbb{R}^n,\ \forall v=(v_1,\ldots,v_n)\in\mathbb{R}^n, \tag{1.1}$$

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where $f''_{i,j}(x) := \dfrac{\partial^2 f(x)}{\partial x_j \partial x_i}$ for $i, j \in I := \{1, \ldots, n\}$ are the second-order partial derivatives of f at x. This amounts to saying that, for any $x \in \mathbb{R}^n$, all the principal minors of the matrix $\big(f''_{i,j}(x)\big)$ are nonnegative. In particular, the first-order and second-order principal minors must be nonnegative, i.e., one must have

$$f''_{i,i}(x) \geq 0, \quad f''_{i,i}(x) f''_{j,j}(x) \geq \big(f''_{i,j}(x)\big)^2, \quad \forall i, j \in I,\ j \neq i. \tag{1.2}$$

As it has been noted in [1, pp. 5–6], both verification of the necessary and sufficient conditions for convexity (1.1) and verification of the necessary conditions for convexity (1.2) are difficult tasks. This is because one has to consider a system of nonlinear inequalities. Moreover, the tasks are still difficult if f is a polynomial function, where the number n of the variables is larger or equal to 2 and the degree of the polynomial $f(x)$ is larger or equal to 3. In addition, according to [1], effective algorithms for verifying the convexity of an arbitrary polynomial function are still unavailable. This situation gave birth to the concept of structural convexity, whose exact formulation will be recalled in the next section. Structural convexity and its characterizations are very useful for the investigations of convex polynomial mathematical programming problems [1, Chapter 4]. We only observe that if a polynomial function is convex, then it is structurally convex; but the converse is not true in general. So, any criterion for the structural convexity of a polynomial function can be used as a tool for a preliminary verification of the convexity of the objective function or a constraint function of a polynomial mathematical programming problem (if the criterion is violated for one of these functions, then we have deal with a nonconvex optimization problem).

Observe that the notion of ravine of a function was introduced by Belousov and Andronov in the papers [3, 4] and studied in detail in the book [1]. The notion has a clear geometrical meaning [1, p. 4] and is very useful for analyzing stability of convex polynomial mathematical programming problems [1, Chapter 4] and the distribution of integer points in convex sets [1, Chapter 3] (see, for instance, Theorem 0 on pp. 121-122). Roughly speaking, ravine of a continuous function is a sequence of points having the norms tending to $+\infty$ around which the function has a very bad behavior (see Remark 2.2 in the next section).

The results and techniques given in the book [1] have been applied or developed further by many authors. For instance, the theorems on the existence of solutions and stability of optimization problems involving polynomial functions from [1, Chapter 5] and [5] have inspired the studies of Belousov and Klatte [6] on Frank-Wolfe type theorems for convex polynomial programs and of Klatte [7] on a Frank-Wolfe type theorem in cubic optimization. The results of [1] on the asymptotical behavior of functions have been used by Klatte and Li [8] in constructing various asymptotic constraint qualifications for the existence of global error bounds for approximate solutions of convex inequalities. The investigations of [1] have tight connections with reverse convex and convex mixed-integer programming [9], global error bounds for piecewise convex polynomials based on the recession properties of convex polynomials [10], Hausdorff discontinuity from above of set-valued functions associated to a finite system of convex inequalities under right-hand-side perturbations [11], the nonemptiness of the intersection of sets and the existence of solutions in nonlinear programming based on the asymptotic directions of closed sets [12].

Due to the importance of quadratic programs in optimization theory (see, e.g., [13] and the references therein), we think that it is of interest to know what do structural convexity and ravine mean for quadratic functions. Our aim in this paper is to present some facts about structural convexity and ravines of quadratic functions. Among other things, we will establish a verifiable criterion for structural convexity of a quadratic function and prove that such a function has no ravines along any linear proper subspace.

Section 2 recalls some notions and results from [1]. Structural convexity of quadratic functions is studied in Section 3. The result saying that quadratic functions cannot have ravines along linear subspaces is established in Section 4.

2. Preliminaries

This section is devoted to the notions of structural convexity of a polynomial function and ravine of a real-valued function, which were proposed by Belousov and Andronov (see [1, Chapters 1 and 2]).

2.1. **Structural convexity of polynomial functions.**

Definition 2.1. (See [1, p. 6]) A polynomial function $f(x)$, $x=(x_1,\dots,x_n)$, of n real variables $x_1,\dots,x_n$ with real coefficients is said to be *structurally convex* if the nonzero coefficients can be changed so that the obtained polynomial function is convex, and all the nonzero coefficients remain nonzero as before.

The functions $g_1(x) := -x^2$ and $g_2(x) := -x^4+x^2-x$, $x\in\mathbb{R}$, are structurally convex, because the functions $\widetilde{g}_1(x) := x^2$ and $\widetilde{g}_2(x) := x^4+x^2-x$ are convex. Meanwhile, by using Definition 2.1 one can easily verify that the functions $h_1(x) := x^3$ and $h_2(x) := x^3+x$, $x\in\mathbb{R}$, are not structurally convex.

2.2. **Ravines of real-valued functions.** Let $f:\mathbb{R}^n\to\mathbb{R}$ be a real-valued function defined on the Euclidean space $\mathbb{R}^n$ and let $L\subset\mathbb{R}^n$ be a proper linear subspace. As usual, $\langle\cdot,\cdot\rangle$ stands for the scalar product in $\mathbb{R}^n$ and $d(z,\Omega) := \inf\limits_{x\in\Omega}\|z-x\|$ denotes the distance from $z\in\mathbb{R}^n$ to a subset $\Omega\subset\mathbb{R}^n$.

Definition 2.2. (See [1, p. 34]) We say that f has a *ravine along the subspace* L or, shorter, *L-ravine*, if there exists a sequence $\{x^k\}$, called an *L-ravine sequence*, such that for any positive numbers δ and ε, and for any sequences $\{y^k\}$ and $\{z^k\}$ satisfying

$$\begin{cases} \|x^k-y^k\|<\delta, & y^k\in x^k+L,\\ \|x^k-z^k\|<\delta, & d(z^k,x^k+L)\ >\varepsilon, \end{cases} \tag{2.1}$$

the equality $\lim\limits_{k\to\infty}\left[f(z^k)-f(y^k)\right]=+\infty$ is fulfilled.

Many examples of functions having ravines can be found in [3] and [1, Section 2.2]. To be precise, four series of 28 functions having ravines were given in [1, Section 2.2]. Let us have a closer look at one of these functions.

Example 2.1. (See [1, Example 1, p. 48]) Consider the polynomial function $f(x)=x_1^2x_2$ of the variable $x=(x_1,x_2)\in\mathbb{R}^2$ and let $L=\{0\}\times\mathbb{R}$. Choosing $x^{(k)}=(0,t_k)$, where t_k tends to $+\infty$ as $k\to\infty$. Suppose that the constants $\delta>0$ and $\varepsilon>0$ are given arbitrarily. If $\{y^{(k)}\}$ and

$\{z^{(k)}\}$ are two sequences of vectors in $\mathbb{R}^2$ satisfying the conditions in (2.1). Then one must have $y^{(k)} = (0, t_k + \alpha_k)$ and $z^{(k)} = (\xi_k, t_k + \beta_k)$ with $|\alpha_k| < \delta$, $\left(\xi_k^2 + \beta_k^2\right)^{1/2} < \delta$, and $|\xi_k| > \varepsilon$ for all k. Therefore,

$$f(z^{(k)}) - f(y^{(k)}) = f(z^{(k)}) = \xi_k^2(t_k + \beta_k) \to +\infty \text{ as } k \to \infty.$$

This shows that $\{x^{(k)}\}$ is an L-ravine sequence of f.

The geometrical concept in Definition 2.2 has proved to be very useful in the analysis of stability of mathematical programming problems and in the study of the distribution of integer points in convex sets; see [1].

If f is continuous on $\mathbb{R}^n$ and $\{x^k\}$ is an L-ravine sequence, then one must have

$$\lim_{k\to\infty} \|x^k\| = +\infty. \tag{2.2}$$

Indeed, if (2.2) does not hold, then we can extract from $\{x^k\}$ a bounded subsequence $\{x^{k_i}\}$. Fix any positive numbers δ and ε with $\varepsilon < \delta$. Let $\{y^k\}$ be any sequence satisfying the conditions $\|x^k - y^k\| < \delta$ and $y^k \in x^k + L$ for all k. (One can choose $y^k = x^k$ for all k.) Since L is a proper linear subspace of $\mathbb{R}^n$ and $\varepsilon < \delta$, for each natural number k, it is possible to select a vector $z^k \in \mathbb{R}^n$ such that $\|x^k - z^k\| < \delta$ and $d(z^k, x^k + L) > \varepsilon$. Thus, the sequences $\{y^k\}$ and $\{z^k\}$ satisfy the conditions in (2.1). Since $\{x^k\}$ is an L-ravine sequence, one has

$$\lim_{k\to\infty} \left[f(z^k) - f(y^k)\right] = +\infty. \tag{2.3}$$

From (2.1) it follows that $\{y^{k_i}\}$ and $\{z^{k_i}\}$ are bounded sequences. Then, by using the continuity of f, one can prove that $\{f(y^{k_i})\}$ and $\{f(z^{k_i})\}$ are bounded sequences. This contradicts (2.3). Thus, (2.2) must hold.

Remark 2.1. If $\{x^k\}$ is an L-ravine sequence of f, then there exist sequences satisfying (2.1) only if $\varepsilon < \delta$.

Remark 2.2. If $\{x^k\}$ is an L-ravine sequence of f, then for any positive numbers δ and ε, and for any sequence $\{z^k\}$ satisfying $\|x^k - z^k\| < \delta$ and $d(z^k, x^k + L) > \varepsilon$ for all k, we have $\lim_{k\to\infty} \left[f(z^k) - f(x^k)\right] = +\infty$. This fact follows from Definition 2.2 if one chooses $y^k = x^k$ for all k. Thus, noting that such sequences $\{z^k\}$ do exist for any positive numbers δ and ε with $\varepsilon < \delta$ (because L is a proper linear subspace of $\mathbb{R}^n$), one can say that the function f has a very bad behavior along the sequence $\{x^k\}$.

The arbitrariness of the sequences $\{y^k\}$ and $\{z^k\}$ in Definition 2.2 is a crucial requirement. Namely, if the equality $\lim_{k\to\infty} \left[f(z^k) - f(y^k)\right] = +\infty$ holds for some sequences $\{y^k\}$ and $\{z^k\}$, but it fails to hold for a certain pair of sequences, then $\{x^k\}$ is not an L-ravine sequence of f. The following example gives an illustration of this situation.

Example 2.2. Consider the polynomial function $f(x) = -x^2$, $x \in \mathbb{R}$, and let $L = \{0\}$. Fix any sequence $\{x^k\}$ with $x^k = t_k$, where $|t_k|$ tends to $+\infty$ as $k \to \infty$. To show that $\{x^k\}$ is not an L-ravine sequence of f, let $\delta > 0$ and $\varepsilon > 0$ be such that $\varepsilon < \delta$. Choose $y^k = t_k$ and $z^k = t_k + \beta$ if $t_k > 0$, and $z^k = t_k + \beta$ if $t_k < 0$, where $\beta \in (\delta, \varepsilon)$ is a fixed number. Clearly, the chosen sequences $\{y^k\}$ and $\{z^k\}$ satisfy the conditions in (2.1). Since $f(z^k) - f(y^k) < 0$ for all k, the equality $\lim_{k\to\infty} \left[f(z^k) - f(y^k)\right] = +\infty$ does not hold. Nevertheless, the last equality is fulfilled when one chooses $y^k = t_k$ and $z^k = t_k - \beta_k$ if $t_k > 0$ and $z^k = t_k + \beta_k$ if $t_k < 0$, where $\beta_k \in (\varepsilon, \delta)$ can vary along with k.

Under some additional conditions, convex polynomial functions cannot have ravines. Namely, the following theorems were obtained by Belousov and Andronov [1].

Theorem 2.1. ([1], Theorem 1, p. 45) *Convex polynomial functions of two variables cannot have a ravine along any linear subspace.*

Theorem 2.2. ([1], Theorem 2, p. 45) *Convex polynomial functions of an arbitrary number of variables cannot have a ravine along any linear subspace of dimension less than or equal to* 1.

3. Structural Convexity of Quadratic Functions

Consider a quadratic function of the form

$$f(x) = \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_ix_j + \sum_{k=1}^{n} b_kx_k + \gamma, \tag{3.1}$$

where a_{ij}, b_i, and γ are real coefficients. It is assumed that $a_{ij} = a_{ji}$ for any indexes $i, j \in I$ where, as before, $I = \{1, \ldots, n\}$. In the sequel, by A we denote the symmetric matrix with the elements a_{ij}. Note that $\nabla^2 f(x) = A$.

Theorem 3.1. *If $n = 1$, then the quadratic function $f(x)$ in (3.1) has the structural convexity. If $n \geq 2$, then the function $f(x)$ in (3.1) is structurally convex if and only if*

$$\forall i, j \in I,\ i \neq j:\ \ a_{ij} \neq 0 \implies a_{ii} \neq 0 \ \text{and}\ a_{jj} \neq 0. \tag{3.2}$$

Proof. If $n = 1$, then using Definition 2.1 one can easily show that the one-variable quadratic function $f(x)$ in (3.1) has the structural convexity. Now, suppose that $n \geq 2$. Denote by P_{ij} the principal submatrix $\begin{pmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{pmatrix}$ of A.

To prove the assertion "only if" of the theorem, suppose on the contrary that $f(x)$ has the structural convexity, but property (3.2) fails to hold. Then, there exist $i_0, j_0 \in \{1, \ldots, n\}$, $i_0 \neq j_0$, such that $a_{i_0j_0} \neq 0$, but $a_{i_0i_0} = 0$ or $a_{j_0j_0} = 0$. Since

$$\frac{\partial^2 f(x)}{\partial x_{i_0}\partial x_{j_0}} = a_{i_0j_0}, \quad \frac{\partial^2 f(x)}{\partial x_{i_0}^2} = a_{i_0i_0}, \quad \frac{\partial^2 f(x)}{\partial x_{j_0}^2} = a_{j_0j_0},$$

one has

$$\det P_{i_0j_0} = a_{i_0i_0}a_{j_0j_0} - a_{i_0j_0}^2 < 0. \tag{3.3}$$

Let the coefficients a_{ij}, b_k, and γ of the polynomial function $f(x)$ in (3.1) be changed into $\widetilde{a}_{ij}$, $\widetilde{b}_k$, and $\widetilde{\gamma}$ in an arbitrary way, provided that, for all i, j, and k, one has

- $\widetilde{a}_{ij} \neq 0$ if and only if $a_{ij} \neq 0$;
- $\widetilde{b}_k \neq 0$ if and only if $b_k \neq 0$;
- $\widetilde{\gamma} \neq 0$ if and only if $\gamma \neq 0$.

The obtained function is abbreviated to $\widetilde{f}(x)$. Denote by $\widetilde{P}_{ij}$ the principal submatrix $\begin{pmatrix} \widetilde{a}_{ii} & \widetilde{a}_{ij} \\ \widetilde{a}_{ji} & \widetilde{a}_{jj} \end{pmatrix}$ of the Hessian matrix $\nabla^2 \widetilde{f}(x)$ of $\widetilde{f}$ at x. Since $\widetilde{a}_{i_0j_0} \neq 0$ and $\widetilde{a}_{i_0i_0} = 0$, or $\widetilde{a}_{j_0j_0} = 0$, by (3.3) we have

$$\det \widetilde{P}_{i_0j_0} = \widetilde{a}_{i_0i_0}\widetilde{a}_{j_0j_0} - \widetilde{a}_{i_0j_0}^2 < 0.$$

So, for any given $x \in \mathbb{R}^n$, the matrix $\nabla^2 \widetilde{f}(x)$ cannot be positive semidefinite [14, p. 566]. Hence, by [2, Theorem 4.5], $\widetilde{f}(x)$ is a nonconvex function on $\mathbb{R}^n$. This contradicts our assumption that $f(x)$ has the structural convexity.

Next, to justify the assertion "if" of the theorem, suppose that (3.2) holds. The structural convexity of $f(x)$ will be proved by induction on the number n of the variables of the function $f(x)$. For $n=1$, formula (3.1) gives $f(x) = \frac{1}{2}a_{11}x_1^2 + b_1x_1 + \gamma$ for all $x = x_1 \in \mathbb{R}^1$. The fact that this function is structurally convex follows from Definition 2.1 and we have mentioned it at the beginning of this proof. Now, assume that the assertion "(3.2) *implies the structural convexity of* $f(x)$" is true for all $n \le k$, where $k \ge 1$. We will show that the assertion holds for $n = k+1$. To do so, observe that the formula for $f(x)$ can be rewritten as

$$f(x) = f_k(x) + \sum_{p=1}^{k} a_{p,k+1}x_px_{k+1} + a_{k+1,k+1}x_{k+1}^2 + \Psi(x_1,\dots,x_{k+1}), \tag{3.4}$$

where $f_k(x) := \dfrac{1}{2}\displaystyle\sum_{j=1}^{k}\sum_{i=1}^{k} a_{ij}x_ix_j$ and $\Psi(x_1,\dots,x_k)$ is an affine function of the variables $x_1,\dots,x_{k+1}$. Applying (3.2) and the induction hypothesis to the quadratic function $f_k(x)$ of the variables $x_1,\dots,x_k$, for every pair $(i,j) \in \{1,\dots,k\}\times\{1,\dots,k\}$ we can find a real number a_{ij}^1 such that $a_{ij}^1 \ne 0$ if and only if $a_{ij} \ne 0$ and the function $f_k^1(x) := \dfrac{1}{2}\displaystyle\sum_{j=1}^{k}\sum_{i=1}^{k} a_{ij}^1x_ix_j$ is convex. Without loss of generality, we may assume that $a_{ij}^1 = a_{ji}^1$ for all $i,j \in \{1,\dots,k\}$. Denote by P the set of the indexes $p \in \{1,\dots,k\}$ having the property $a_{p,k+1} \ne 0$.

If $P = \emptyset$ then, by putting $\widetilde{a}_{ij} = a_{ij}^1$ for all i, j from the set $\{1,\dots,k\}$,

$$\widetilde{a}_{p,k+1} = \widetilde{a}_{k+1,p} = a_{p,k+1} = 0, \quad \forall p \in \{1,\dots,k\},$$

and $\widetilde{a}_{k+1,k+1} = |a_{k+1,k+1}|$, one gets from the expression (3.4) the function

$$\begin{aligned}\widetilde{f}(x) &:= \widetilde{f}_k(x) + \sum_{p=1}^{k} \widetilde{a}_{p,k+1}x_px_{k+1} + \widetilde{a}_{k+1,k+1}x_{k+1}^2 + \Psi(x_1,\dots,x_{k+1})\\ &= \widetilde{f}_k(x) + \widetilde{a}_{k+1,k+1}x_{k+1}^2 + \Psi(x_1,\dots,x_{k+1})\end{aligned}$$

with $\widetilde{f}_k(x) := \dfrac{1}{2}\displaystyle\sum_{j=1}^{k}\sum_{i=1}^{k} \widetilde{a}_{ij}x_ix_j$. As $\widetilde{f}_k(x) = f_k^1(x)$, $\widetilde{a}_{k+1,k+1} \ge 0$, and $\Psi(x_1,\dots,x_{k+1})$ is an affine function, we see that $\widetilde{f}(x)$ is a convex function.

Next, suppose that $P \ne \emptyset$. By the assumption (3.2), we have $a_{k+1,k+1} \ne 0$ and $a_{pp} \ne 0$ for every $p \in P$. Then, choose $\widetilde{a}_{ij} = a_{ij}^1$ for all i, j from the set $\{1,\dots,k\}$ with $j \ne i$, $\widetilde{a}_{pp} = a_{pp}^1$ for all $p \notin P$, $\widetilde{a}_{pp} = |a_{pp}^1| + a_{p,k+1}^2$ for all $p \in P$, $\widetilde{a}_{p,k+1} = \widetilde{a}_{k+1,p} = 0$ for all $p \in \{1,\dots,k\}\setminus P$,

$$\widetilde{a}_{p,k+1} = \widetilde{a}_{k+1,p} = a_{p,k+1} \quad \forall p \in P,$$

and $\widetilde{a}_{k+1,k+1} = |a_{k+1,k+1}| + \dfrac{1}{4}\zeta$, where ζ denotes the number of the elements of P. Clearly, $\widetilde{a}_{k+1,k+1} \ne 0$ and $\widetilde{a}_{pp} \ne 0$ for every $p \in P$. Let $g_p(x_p,x_{k+1}) := a_{p,k+1}^2x_p^2 + a_{p,k+1}x_px_{k+1} + \dfrac{1}{4}x_{k+1}^2$

for every $p \in P$. Since

$$g_p(x_p, x_{k+1}) = \left(a_{p,k+1}x_p + \frac{1}{2}x_{k+1}\right)^2 \geq 0$$

for all $(x_p, x_{k+1}) \in \mathbb{R}^2$, $g_p(x_p, x_{k+1})$ is a convex function for every $p \in P$; see [15, Proposition 3.71]. It follows that

$$\widetilde{f}(x) := \frac{1}{2}\sum_{j=1}^{k+1}\sum_{i=1}^{k+1} \widetilde{a}_{ij}x_ix_j + \Psi(x_1, \ldots, x_{k+1})$$

is a convex function. Indeed, by the above choice of $\widetilde{a}_{ij}$ for i, j from the index set $\{1, \ldots, k+1\}$, one has

$$\begin{aligned} \widetilde{f}(x) &= f_k^1(x) + \sum_{p\in P} g_p(x_p, x_{k+1}) + |a_{k+1,k+1}|x_{k+1}^2 \\ &\quad + \sum_{p\in P} \left(|a_{pp}^1| - a_{pp}^1\right)x_p^2 + \Psi(x_1, \ldots, x_{k+1}) \end{aligned} \tag{3.5}$$

with $f_k^1(x) := \frac{1}{2}\sum_{j=1}^{k}\sum_{i=1}^{k} a_{ij}^1 x_i x_j$. Since the function $\widetilde{f}(x)$ in (3.5) is convex, the structural convexity of $f(x)$ is proved. □

From Definition 2.1 it follows that the function $f(x)$ in (3.1) has the structural convexity if and only if the quadratic function

$$g(x) := \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_ix_j = \langle Ax, x\rangle \tag{3.6}$$

has that property. Hence, the set of quadratic functions $g(x)$ of the form (3.6) having the structural convexity corresponds to a certain subset of matrices A of the space $\mathbb{R}_S^{n\times n}$ of symmetric $(n\times n)$–matrices, which is denoted by Σ. We now present some interesting properties of Σ.

Theorem 3.2. *The set $\Sigma \subset \mathbb{R}_S^{n\times n}$ is a cone containing 0, which has nonempty interior. Moreover, if $n = 1$ then $\Sigma = \mathbb{R}_S^{1\times 1} = \mathbb{R}$. If $n \geq 2$, then the interior $\operatorname{int}\Sigma$ of Σ consists of those matrices $A = (a_{ij}) \in \mathbb{R}_S^{n\times n}$ with the diagonals not having any zero element, i.e., $a_{ii} \neq 0$ for all $i \in I$.*

Proof. The fact that the set $\Sigma \subset \mathbb{R}_S^{n\times n}$ is a cone containing 0 is immediate from Definition 2.1. For $n = 1$, by Theorem 3.1 one has $\Sigma = \mathbb{R}_S^{1\times 1} = \mathbb{R}$. In particular, Σ has nonempty interior. Now, suppose that $n \geq 2$.

Let $A = (a_{ij}) \in \mathbb{R}_S^{n\times n}$ be a matrix whose diagonal consists of nonzero elements. By Theorem 3.1, $A \in \Sigma$. Define $\varepsilon = \left(\max_{i\in I}\{|a_{ii}|\}\right)^{-1}$. Then, for any $B = (b_{ij}) \in \mathbb{R}_S^{n\times n}$ with $\|B - A\| < \varepsilon$, where the norm of a matrix $\|C\|$ is defined as the maximum of the absolute values of the elements of C, we have $|b_{ii} - a_{ii}| < \varepsilon$ for all $i \in I$. Consequently, $b_{ii} \neq 0$ for all $i \in I$. So, B belongs to Σ by Theorem 3.1. We have thus proved that $A \in \operatorname{int}\Sigma$.

If $A = (a_{ij}) \in \mathbb{R}_S^{n\times n}$ is an element of Σ and there exists $i_0 \in \{1, \ldots, n\}$ such that $a_{i_0i_0} = 0$. Given any $\varepsilon > 0$, we select an index $j_0 \in I \setminus \{i_0\}$. Since $A \in \Sigma$ and $a_{i_0i_0} = 0$, by Theorem 3.1 we can assert that $a_{i_0j_0} = a_{j_0i_0} = 0$. Let us define a matrix $B = (b_{ij}) \in \mathbb{R}_S^{n\times n}$ by setting $b_{ij} = a_{ij}$ for all $(i, j) \notin \{(i_0, j_0), (j_0, i_0)\}$, and $b_{i_0j_0} = b_{j_0i_0} = \frac{1}{2}\varepsilon$. Note that $\|B - A\| < \varepsilon$. If $B \in \Sigma$, then by Theorem 3.1 and the property $b_{i_0j_0} \neq 0$ we get $b_{i_0i_0} \neq 0$. But this is impossible because

$b_{i_0i_0} = a_{i_0i_0} = 0$. Thus, in any neighborhood of A there exists a matrix B which does not belong to Σ. Therefore, $A \notin \mathrm{int}\Sigma$.

Summing up all the above, we conclude that $\mathrm{int}\Sigma$ consists of the matrices $A = (a_{ij}) \in \mathbb{R}_S^{n\times n}$ with $a_{ii} \neq 0$ for all $i \in I$. □

If every symmetric $(n \times n)$–matrix $A = (a_{i,j})$ is identified with the vector $(a_{i,j})_{i\leq j}$ of the space $\mathbb{R}^{n(n+1)/2}$, then the space $\mathbb{R}_S^{n\times n}$ can be equipped with the product measure of the product of $n(n+1)/2$ copies of the real line $\mathbb{R}$, which has the Lebesgue measure. In what follows, the Lebesgue measure in $\mathbb{R}_S^{n\times n}$ is understood in that sense. The next corollary is immediate from Theorem 3.2.

Corollary 3.1. *The set Σ is dense in $\mathbb{R}_S^{n\times n}$. Moreover, Σ is a set of full Lebesgue measure in $\mathbb{R}_S^{n\times n}$.*

4. Quadratic Functions Have no Ravines

In this section, we will answer the question: *Can a linear-quadratic function have a ravine along a given linear subspace, or not?*

The following result complements Theorems 2.1 and 2.2. Here, linear-quadratic functions are considered and no convexity assumption is needed. Even if the function in question is convex, it is clear that the assertion of this theorem does not follow from the results recalled in Theorems 2.1 and 2.2.

Theorem 4.1. *Linear-quadratic functions cannot have a ravine along any linear subspace.*

Proof. Let $f(x) = \dfrac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_ix_j + \sum_{i=1}^{n} b_ix_i + \gamma$, where $a_{ij} = a_{ji}$ for all $i, j \in I$, be an arbitrary linear-quadratic function of n variables. Let $L \subset \mathbb{R}^n$ be a linear subspace with $L \neq \mathbb{R}^n$. Given an arbitrary sequence $\{x^k\}$ in $\mathbb{R}^n$, we will show that it cannot be an $L-$ravine sequence for f.

As L is a proper linear subspace of $\mathbb{R}^n$, we can find a hyperplane H containing L. Suppose that $c = (c_1,\ldots,c_n) \neq 0$ is a normal vector to H. Let

$$\alpha(c,x^k) := \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(c_ix_j^k + c_jx_i^k).$$

Select any $\mu \in (0,\|c\|)$. Choose $\delta = \|c\| + \mu$, $\varepsilon = \|c\| - \mu$, and $y^k = x^k$.

First, consider the case where $\alpha(c,x^k) \nrightarrow +\infty$ as $k \to \infty$. In this case, for each k, we define $z^k = (z_1^k,\ldots,z_n^k)$, where $z_i^k := x_i^k + c_i$, $i = 1,\ldots,n$. Then we have $z^k = x^k + c$. Therefore,

$$\begin{cases} \|x^k - y^k\| = 0 < \delta, \quad y^k \in x^k + L, \\ \|x^k - z^k\| = \|c\| < \delta, \\ d(z^k, x^k + L) \geq d(z^k, x^k + H) = \|c\| > \varepsilon, \end{cases} \tag{4.1}$$

and

$$\begin{aligned} f(z^k) &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} z_i^k z_j^k + \sum_{i=1}^{n} b_i z_i^k + \gamma \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x_i^k + c_i)(x_j^k + c_j) + \sum_{i=1}^{n} b_i (x_i^k + c_i) + \gamma \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x_i^k x_j^k + c_i x_j^k + c_j x_i^k + c_i c_j) + \sum_{i=1}^{n} b_i x_i^k + \sum_{i=1}^{n} b_i c_i + \gamma \\ &= f(x^k) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(c_i x_j^k + c_j x_i^k) + f(c) - \gamma. \end{aligned}$$

By (4.1), the sequences $\{y^k\}$ and $\{z^k\}$ satisfy the conditions in (2.1). Since

$$\begin{aligned} f(z^k) - f(y^k) &= f(z^k) - f(x^k) \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(c_i x_j^k + c_j x_i^k) + f(c) - \gamma \\ &= \frac{1}{2}\alpha(c, x^k) + f(c) - \gamma, \end{aligned}$$

one sees that $f(z^k) - f(y^k) \nrightarrow +\infty$ as $k \to \infty$.

Next, consider the case where $\alpha(c, x^k) \to +\infty$ as $k \to \infty$. For each k, put $z^k = (z_1^k, \ldots, z_n^k)$, where $z_i^k = x_i^k - c_i$, $i = 1, \ldots, n$. Then $z^k = x^k - c$ and we also have (4.1), because $-c$ is a normal vector to H. Clearly,

$$\begin{aligned} f(z^k) &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} z_i^k z_j^k + \sum_{i=1}^{n} b_i z_i^k + \gamma \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x_i^k - c_i)(x_j^k - c_j) + \sum_{i=1}^{n} b_i (x_i^k - c_i) + \gamma \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x_i^k x_j^k - c_i x_j^k - c_j x_i^k + c_i c_j) + \sum_{i=1}^{n} b_i x_i^k - \sum_{i=1}^{n} b_i c_i + \gamma \\ &= f(x^k) - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(c_i x_j^k + c_j x_i^k) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} c_i c_j - \sum_{i=1}^{n} b_i c_i. \end{aligned}$$

The system (4.1) guarantees that $\{y^k\}$ and $\{z^k\}$ satisfy the conditions in (2.1). Since

$$\begin{aligned} f(z^k) - f(y^k) &= f(z^k) - f(x^k) \\ &= -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(c_i x_j^k + c_j x_i^k) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} c_i c_j - \sum_{i=1}^{n} b_i c_i \\ &= -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(c_i x_j^k + c_j x_i^k) + f(c) - 2\sum_{i=1}^{n} b_i c_i - \gamma \\ &= -\frac{1}{2}\alpha(c, x^k) + f(c) - 2\sum_{i=1}^{n} b_i c_i - \gamma, \end{aligned}$$

we have $\lim_{k\to\infty}\left[f(z^k)-f(y^k)\right]=-\infty$.

The above analysis shows that $\{x^k\}$ is not an L-ravine sequence for f. Since $\{x^k\}$ is given arbitrarily, this completes the proof. □

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