J. Appl. Numer. Optim. 1 (2019), No. 2, pp. 157-165
Available online at http://jano.biemdas.com
https://doi.org/10.23952/jano.1.2019.2.06

THREE CONVERGENCE RESULTS FOR INEXACT ORBITS OF NONEXPANSIVE MAPPINGS

ALEXANDER J. ZASLAVSKI

Department of Mathematics, The Technion – Israel Institute of Technology, 32000 Haifa, Isreal

Abstract. In this paper we study the convergence of inexact iterates of nonexpansive mappings which take a nonempty, closed subset of a complete metric space into the space, under the presence of summable errors, and generalize the known results in the literature for nonexpansive self-mappings of the complete metric space.

Keywords. Banach space; Complete metric space; Inexact iteration; Nonexpansive mapping.

2010 Mathematics Subject Classification. 47H09, 47H10, 54E50.

1. Introduction

In the past fifty-five years, there has been a lot of activities regarding the fixed point theory of nonexpansive (that is, 1-Lipschitz) mappings; see, for example, [3, 5, 11, 13, 14, 17, 18, 19, 21, 22, 23, 24, 27, 28] and the references cited therein. These activities stem from Banach's classical theorem [1] concerning the existence of a unique fixed point for a strict contraction. It also covers the convergence of (inexact) iterates of a nonexpansive mapping to one of its fixed points. Since that seminal result, many developments have taken place in this field including, in particular, studies of feasibility and common fixed point problems, which find important applications in engineering and medical sciences [8, 12, 15, 25, 26, 27, 28].

In [5] it was shown that if any exact orbit of a nonexpansive mapping converges to its fixed point, then this convergence property also holds for its inexact orbits with summable errors. This result was obtained for a self-mapping of a complete metric space X. In the present paper, we generalize this results for nonexpansive mappings which take a nonempty, closed subset of the complete metric space X into X.

2. Preliminaries

Let (X,ρ) be a complete metric space. For each $x\in X$ and each nonempty set $B\subset X$, put

$$\rho(x,B)=\inf\{\rho(x,y):\ y\in B\}.$$

For each $x\in X$ and each $r>0$, set

$$B(x,r)=\{y\in X:\ \rho(x,y)\le r\}.$$

For each mapping $A:X\to X$, let $A^0x=x$ for all $x\in X$.

In [5] it was studied the influence of errors on the convergence of orbits of nonexpansive mappings in metric spaces and it was obtained the following result (see also Theorem 2.72 of [24]).

Theorem 2.1. *Let $A:X\to X$ satisfy $\rho(Ax,Ay)\le\rho(x,y)$, for all $x,y\in X$. Let $F(A)$ be the set of all fixed points of A and let, for each $x\in X$, $\{A^nx\}_{n=1}^{\infty}$ be a sequence converging in (X,ρ). Assume that $\{x_n\}_{n=0}^{\infty}\subset X$, $\{r_n\}_{n=0}^{\infty}\subset(0,\infty)$ satisfies $\sum_{n=0}^{\infty}r_n<\infty$ and*

$$\rho(x_{n+1},Ax_n)\le r_n,\ n=0,1,\dots.$$

E-mail address: ajzasl@technion.ac.il.
Received May 24, 2019; Accepted July 30, 2019.

Then the sequence $\{x_n\}_{n=1}^{\infty}$ *converges to a fixed point of* A *in* (X,ρ).

In this paper, we generalize this result for nonexpansive mappings which take a nonempty, closed subset of the complete metric space X into X. More precisely, in Section 4, we prove the following result.

Theorem 2.2. *Let* K *be a nonempty closed subset of* X. *Let* $A : K \to X$ *satisfy*

$$\rho(Ax,Ay) \le \rho(x,y) \text{ for all } x,y \in K, \tag{2.1}$$

and let $F(A)$ *be the set of all fixed points of* A. *For each* $x \in X$, *if* $\{A^n x\}_{n=1}^{\infty}$ *is well-defined, then it converges in* (X,ρ). *Let*

$$\{x_n\}_{n=0}^{\infty} \subset K, \tag{2.2}$$

$\tilde{r} > 0$, *and* $\{r_n\}_{n=0}^{\infty} \subset (0,\infty)$ *satisfy*

$$\sum_{n=0}^{\infty} r_n < \infty, \tag{2.3}$$

$$\rho(x_{n+1},Ax_n) \le r_n,\ n = 0,1,\dots. \tag{2.4}$$

Let

$$B(x_n,\tilde{r}) \subset K \text{ for all sufficiently large natural numbers } n. \tag{2.5}$$

Then the sequence $\{x_n\}_{n=1}^{\infty}$ *converges to a fixed point of* A *in* (X,ρ).

Theorem 2.1 finds interesting applications and is an important ingredient in superiorization and perturbation resilience of algorithms; see [2, 4, 6, 7, 9, 10, 16, 20] and the references mentioned therein. The superiorization methodology works by taking an iterative algorithm, investigating its perturbation resilience, and then using proactively such perturbations in order to ”force” the perturbed algorithm to do in addition to its original task something useful. This methodology can be explained by the following result on convergence of inexact iterates.

Assume that $(X,\|\cdot\|)$ is a Banach space, $\rho(x,y) = \|x-y\|$ for all $x,y \in X$, for each $x \in X$, the sequence $\{A^n x\}_{n=1}^{\infty}$ converges in the norm topology. Assume that $x_0 \in X$, $\{\beta_k\}_{k=0}^{\infty}$ is a sequence of positive numbers satisfying

$$\sum_{k=0}^{\infty} \beta_k < \infty, \tag{2.6}$$

$\{v_k\}_{k=0}^{\infty} \subset X$ is a norm bounded sequence and, for any integer $k \ge 0$,

$$x_{k+1} = A(x_k + \beta_k v_k). \tag{2.7}$$

Then it follows from Theorem 2.1 that the sequence $\{x_k\}_{k=0}^{\infty}$ converges in the norm topology of X and its limit is a fixed point of A. In this case, mapping A is called the bounded perturbations resilient (see [6] and Definition 10 of [9]). In other words, if exact iterates of a nonexpansive mapping converge, then its inexact iterates with bounded summable perturbations converge too.

Now assume that $x_0 \in X$ and the sequence $\{\beta_k\}_{k=0}^{\infty}$ satisfying (2.6) are given and we need to find an approximate fixed point of A. In order to meet this goal, we construct a sequence $\{x_k\}_{k=1}^{\infty}$ defined by (2.7). Under an appropriate choice of the bounded sequence $\{v_k\}_{k=0}^{\infty}$, the sequence $\{x_k\}_{k=1}^{\infty}$ possesses some useful property. For example, the sequence $\{f(x_k)\}_{k=1}^{\infty}$ can be decreasing, where f is a given function.

3. Proof of Theorem 2.2

Set $A^0 x = x$ for all $x \in K$. It is sufficient to show that the sequence $\{x_n\}_{n=0}^{\infty}$ converges in (X,ρ).

Let $\varepsilon > 0$. Since the metric space (X,ρ) is complete, it is sufficient to show that there exists a natural number k such that, for each pair of integers $i,j \ge k$,

$$\rho(x_i,x_j) \le \varepsilon.$$

We may assume that

$$\varepsilon \in (0,\tilde{r}/4). \tag{3.1}$$

In view of (2.5), there exists a natural number k_1 such that

$$B(x_n,\tilde{r}) \subset K \text{ for all natural numbers } n \ge k_1. \tag{3.2}$$

By (2.3), there exists an integer $k_0 > k_1$ such that

$$\sum_{i=k_0}^{\infty} r_i < \varepsilon/4. \tag{3.3}$$

By induction, we show that

$$A^n x_{k_0} \in K$$

is well defined for all natural numbers n and that

$$\rho(A^n x_{k_0}, x_{n+k_0}) < \varepsilon.$$

for all integers $n \geq 0$. Assume that $n \geq 0$ is an integer,

$$A^i x_{k_0} \in K,\ i = 0, \dots, n \tag{3.4}$$

are well defined and that

$$\rho(A^n x_{k_0}, x_{n+k_0}) \leq \sum_{i=k_0}^{n+k_0} r_i - r_{n+k_0}. \tag{3.5}$$

(It is clear that our assumption holds for $n = 0$). In view of (3.2) and the inequality $k_0 > k_1$, we have

$$B(x_{n+k_0}, \tilde{r}) \subset K.$$

By (3.4), we have

$$A^n x_{k_0} \in K \tag{3.6}$$

and $A^{n+1} x_{k_0} \in X$ is well defined. It follows from (2.1), (2.2), (2.4), (3.5) and (3.6) that

$$\begin{aligned}
\rho(A^{n+1} x_{k_0}, x_{n+1+k_0}) &\leq \rho(A^{n+1} x_{k_0}, A x_{n+k_0}) + \rho(A x_{n+k_0}, x_{n+k_0+1}) \\
&\leq \rho(A^n x_{k_0}, x_{n+k_0}) + r_{n+k_0} \\
&\leq \sum_{i=k_0}^{n+k_0} r_i \\
&= \sum_{i=k_0}^{n+k_0+1} r_i - r_{n+k_0+1}.
\end{aligned} \tag{3.7}$$

It follows from (3.1), (3.2), (3.3) and (3.7) that

$$\rho(A^{n+1} x_{k_0}, x_{n+1+k_0}) < \varepsilon < \tilde{r}/4,$$

$$A^{n+1} x_{k_0} \in K$$

and the assumption made for n also holds for $n+1$. Therefore by induction, we show that

$$A^i x_{k_0} \in K \text{ for all integers } i \geq 0$$

and that (3.5) holds for all integers $n \geq 0$. By our assumptions, there exists

$$z \in F(T)$$

such that

$$z = \lim_{n \to \infty} A^n x_{k_0}. \tag{3.8}$$

It follows from (3.3) and (3.5) that, for all integers $n \geq 0$,

$$\rho(A^n x_{k_0}, x_{n+k_0}) \leq \sum_{i=k_0}^{n+k_0} r_i \leq \sum_{i=k_0}^{\infty} r_i < \varepsilon/4. \tag{3.9}$$

In view of (3.8), there exists a natural number n_1 such that, for each integer $n \geq n_1$,

$$\rho(A^n x_{k_0}, z) \leq \varepsilon/4.$$

By the relation above and (3.9), we have, for each integer $n \geq n_1$,

$$\rho(z, x_{n+k_0}) \leq \rho(z, A^n x_{k_0}) + \rho(A^n x_{k_0}, x_{n+k_0}) \leq \varepsilon/4 + \varepsilon/4.$$

Thus, for all integers $n, m \geq n_1$,

$$\rho(x_{n+k_0}, x_{m+k_0}) \leq \rho(x_{n+k_0}, z) + \rho(z, x_{m+k_0}) \leq \varepsilon/2.$$

This completes the proof of Theorem 2.2.

4. Weak convergence of orbits

Let X be a nonempty closed subset of a Banach space $(E, \|\cdot\|)$ with a dual space $(E^*, \|\cdot\|_*)$ and let $A : X \to X$ satisfy

$$\|Ax - Ay\| \le \|x - y\| \text{ for each } x, y \in X.$$

As usual, we denote by A^0 the identity self-mapping of X. Consider the following assumptions.

(A1) For each $x \in X$, the sequence $\{A^n x\}_{n=1}^\infty$ converges weakly in X.

(A2) For each $x \in X$, the sequence $\{A^n x\}_{n=1}^\infty$ converges weakly in X to a fixed point of A.

The following result was obtained in [5] (see also Theorem 2.73 of [24]).

Theorem 4.1. *Assume that (A1) holds. Let $x_0 \in X$,*

$$\{r_n\}_{n=0}^\infty \subset (0, \infty), \ \sum_{n=0}^\infty r_n < \infty,$$

$$\{x_n\}_{n=0}^\infty \subset X, \ \|x_{n+1} - Ax_n\| \le r_n, \ n = 0, 1, \dots.$$

Then the sequence $\{x_n\}_{n=1}^\infty$ converges weakly in X. Moreover, if (A2) holds, then its limit is a fixed point of A.

In the present paper we generalize this result for nonexpansive mappings which take a nonempty, closed subset of the complete metric space X into X.

Let K be a nonempty closed subset of E and $A : K \to E$ satisfy

$$\|Ax - Ay\| \le \|x - y\| \text{ for each } x, y \in K.$$

Let $A^0 x = x$ for all $x \in K$.

We use following assumptions.

(A3) If $x \in K$ and the sequence $\{A^n x\}_{n=1}^\infty \subset K$ is well defined, then it converges weakly in E.

(A4) If $x \in K$ and the sequence $\{A^n x\}_{n=1}^\infty \subset K$ is well defined, then it converges weakly in E to a fixed point of A.

Let

$$F(A) = \{x \in K : \ Ax = x\}.$$

Clearly, $F(A)$ is a closed subset of E.

We prove the following result.

Theorem 4.2. *Let (A3) hold, $\{x_n\}_{n=0}^\infty \subset K$, $\tilde{r} > 0$, $\{r_n\}_{n=0}^\infty \subset (0, \infty)$ satisfy*

$$\sum_{n=0}^\infty r_n < \infty, \tag{4.1}$$

$$\|x_{n+1} - Ax_n\| \le r_n, \ n = 0, 1, \dots \tag{4.2}$$

and

$$B(x_n, \tilde{r}) \subset K \text{ for all sufficiently large natural numbers } n. \tag{4.3}$$

Then the sequence $\{x_n\}_{n=1}^\infty$ converges weakly in E and if K is weakly closed, then the limit belongs to K. Moreover, if (A4) holds, then the limit is a fixed point of A.

Proof. In view of (4.3), there exists a natural number n_0 such that

$$B(x_n, \tilde{r}) \subset K \text{ for all natural numbers } n \ge n_0. \tag{4.4}$$

By (4.1), there exists an integer

$$n_1 > n_0 \tag{4.5}$$

such that

$$\sum_{i=n_1}^\infty r_i < \tilde{r}/4. \tag{4.6}$$

Let

$$p \ge n_1 \tag{4.7}$$

be an integer. By induction, we show that $A^i x_p \in K$ is well defined for all integers $i \geq 0$. Assume that $n \geq 0$ is an integer,

$$A^i x_p \in K,\ i = 0, \dots, n \tag{4.8}$$

are well defined and that

$$\|A^n x_p - x_{n+p}\| \leq \sum_{i=p}^{n+p} r_i - r_{n+p}. \tag{4.9}$$

(It is clear that our assumption holds for $n = 0$). In view of (4.8), we find that $A^{n+1}x_p \in E$ is well defined. It follows from (4.4), (4.5) and (4.7) that

$$B(x_{n+p+1}, \tilde{r}) \subset K. \tag{4.10}$$

By (4.2), (4.9) and (4.10), we have

$$\begin{aligned} \|A^{n+1}x_p - x_{n+1+p}\| &\leq \|A^{n+1}x_p - Ax_{n+p}\| + \|Ax_{n+p} - x_{n+p+1}\| \\ &\leq \|A^n x_p - x_{n+p}\| + r_{n+p} \\ &\leq \sum_{i=p}^{n+p} r_i. \end{aligned} \tag{4.11}$$

It follows from (4.6), (4.7) and (4.11) that

$$\|A^{n+1}x_p - x_{n+1+p}\| < \tilde{r}/4. \tag{4.12}$$

In view of (4.4), (4.5), (4.7) and (4.12), we have

$$A^{n+1}x_p \in K. \tag{4.13}$$

By (4.11) and (4.13), we have that the assumption made for n also holds for $n+1$. Therefore by induction, we obtain that (4.8) and (4.9) hold for all integers $n \geq 0$. In view of (A3), the sequence $\{A^n x_p\}_{n=0}^{\infty}$ converges weakly to $y_p \in E$. If the set K is weakly closed, then

$$y_p \in K \tag{4.14}$$

and if (A4) holds, then

$$Ay_p = y_p. \tag{4.15}$$

Fix an integer $q \geq 1$. By (4.9), we have

$$\|A^q x_p - x_{q+p}\| \leq \sum_{j=p}^{\infty} r_j. \tag{4.16}$$

It follows that, for each integer $i \geq 0$,

$$\|A^{q+i}x_p - A^i x_{q+p}\| \leq \|A^q x_p - x_{q+p}\| \leq \sum_{j=p}^{\infty} r_j. \tag{4.17}$$

The equations

$$y_p = \lim_{n\to\infty} A^n x_p \text{ in the weak topology,} \tag{4.18}$$

and

$$y_{q+p} = \lim_{n\to\infty} A^n x_{q+p} \text{ in the weak topology} \tag{4.19}$$

hold. By (4.17), (4.18) and ((4.19)), for each $f \in E^*$ satisfying $\|f\|_* \leq 1$, we have

$$\begin{aligned} |f(y_q) - f(y_{q+p})| &= \lim_{n\to\infty} |f(A^{n+q}x_p) - f(A^n x_{q+p})| \\ &\leq \limsup_{n\to\infty} \|f\| \|A^{n+q}x_p - A^n x_{q+p}\| \\ &\leq \sum_{j=p}^{\infty} r_j. \end{aligned}$$

This implies that

$$\|y_p - y_{p+q}\| \leq \sum_{j=k}^{\infty} r_j. \tag{4.20}$$

Since the above inequality holds for each integer $p \geq n_1$ and each integer $q \geq 1$ and $\sum_{j=0}^{\infty} r_j < \infty$, we conclude that $\{y_k\}_{k=1}^{\infty}$ is a Cauchy sequence and there exists

$$y_* = \lim_{k\to\infty} y_k \tag{4.21}$$

in the norm topology of E. Clearly, if K is weakly closed, then $y_* \in K$ and if (A4) holds, then $Ay_* = y_*$. By (4.20) and (4.21), we have

$$\|y_p - y_*\| \le \sum_{j=p}^{\infty} r_j \tag{4.22}$$

for all integers $p \ge n_1$. In order to complete the proof, it is sufficient to show that $\lim_{k\to\infty} x_k = y_*$ in the weak topology. Let $f \in E^*$ be a continuous linear functional on E such that $\|f\|_* \le 1$ and let $\varepsilon > 0$ be given. It is sufficient to show that $|f(y_* - x_i)| \le \varepsilon$ for all large enough integers $i \ge 0$. By (4.1), there is an integer $p \ge n_1$ such that

$$\sum_{j=p}^{\infty} r_j < \varepsilon/4. \tag{4.23}$$

By (4.16) and (4.22), for each integer $n \ge 1$, we have

$$\begin{aligned} |f(y_* - x_{p+n})| &\le |f(y_* - y_p)| + |f(y_p - A^n x_p)| + |f(A^n x_p - x_{n+p})| \\ &\le \|y_* - y_p\| + |f(y_p - A^n x_p)| + \|A^n x_p - x_{n+p}\| \\ &\le \sum_{j=p}^{\infty} r_j + |f(y_p - A^n x_p)| + \sum_{j=p}^{\infty} r_j. \end{aligned} \tag{4.24}$$

Since $y_p = \lim_{n\to\infty} A^n x_p$ in the weak topology of E, there exits a natural number i_0 such that

$$|f(y_p - A^n x_p)| < \varepsilon/4 \text{ for all natural numbers } n \ge i_0. \tag{4.25}$$

By (4.23) and (4.25), we have, for each integer $n \ge i_0$,

$$|f(y_* - x_{p+n})| \le \varepsilon/4 + \varepsilon/4 + \varepsilon/4 = 3\varepsilon/4.$$

This completes the proof. □

5. AN EXTENSION OF THEOREM 2.2

Let K be a nonempty closed subset of a complete metric space (X,ρ). Let $F \subset K$ be a nonempty closed set in (X,ρ) and let $A_i : K \to X$, $i = 0,1,\ldots$ satisfy for all integers $i \ge 0$,

$$\rho(A_i x, A_i y) \le \rho(x,y) \text{ for all } x,y \in K, \tag{5.1}$$

$$A_i x = x \text{ for all } x \in F. \tag{5.2}$$

We use the following assumptions.

(A1) If $x \in K$, $p \ge 0$ is an integer and the sequence $\{\prod_{i=p}^{n} A_i x\}_{n=p}^{\infty} \subset K$ is well defined, then it converges in (X,ρ).

(A2) If $x \in K$, $p \ge 0$ is an integer and the sequence $\{\prod_{i=p}^{n} A_i x\}_{n=p}^{\infty} \subset K$ is well defined, then it converges in (X,ρ) and

$$\lim_{n\to\infty} \prod_{i=p}^{n} A_i x \in F.$$

We prove the following result.

Theorem 5.1. *Let $\tilde{r} > 0$, $\{r_n\}_{n=0}^{\infty} \subset (0,\infty)$ satisfy*

$$\sum_{n=0}^{\infty} r_n < \infty, \tag{5.3}$$

$$\{x_n\}_{n=0}^{\infty} \subset K, \tag{5.4}$$

$$\rho(x_{n+1}, A_n x_n) \le r_n,\ n = 0,1,\ldots \tag{5.5}$$

and

$$B(x_n, \tilde{r}) \subset K \text{ for all sufficiently large natural numbers } n. \tag{5.6}$$

Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges in (X,ρ). Moreover, if (A2) holds, then

$$\lim_{n\to\infty} x_n \in F.$$

Proof. In view of (5.6), there exists a natural number n_0 such that

$$B(x_n, \tilde{r}) \subset K \text{ for all natural numbers } n \geq n_0. \tag{5.7}$$

Let

$$\varepsilon \in (0, \tilde{r}/2). \tag{5.8}$$

By (5.3), we have that there exists an integer

$$n_1 \geq n_0 \tag{5.9}$$

such that

$$\sum_{i=n_1}^{\infty} r_i < \varepsilon/4. \tag{5.10}$$

By induction, we show that, for all integer $n \geq n_1$,

$$\prod_{i=n_1}^{n} A_i x_{n_1} \in K \tag{5.11}$$

is well defined and

$$\rho(\prod_{i=n_1}^{n} A_i x_{n_1}, x_{n+1}) \leq \sum_{i=n_1}^{n} r_i. \tag{5.12}$$

It follows from (5.5), (5.7), (5.8) and (5.10) that

$$\rho(A_{n_1} x_{n_1}, x_{n_1+1}) \leq r_{n_1} < \tilde{r} \tag{5.13}$$

and

$$A_{n_1} x_{n_1} \in B(x_{n_1+1}, \tilde{r}) \subset K. \tag{5.14}$$

Assume that $n \geq n_1$ is an integer, $\prod_{i=n_1}^{n} A_i x_{n_1} \in K$ is well defined and (5.12) holds. (By (5.13) and (5.14), our assumption holds for $n = n_1$). It follows from (5.1), (5.5) and (5.12) that

$$\begin{aligned}
\rho(\prod_{i=n_1}^{n+1} A_i x_{n_1}, x_{n+2}) &\leq \rho(\prod_{i=n_1}^{n+1} A_i x_{n_1}, A_{n+1} x_{n+1}) + \rho(A_{n+1} x_{n+1}, x_{n+2}) \\
&\leq \rho(\prod_{i=n_1}^{n} A_i x_{n_1}, x_{n+1}) + r_{n+1} \leq \sum_{i=n_1}^{n+1} r_i.
\end{aligned} \tag{5.15}$$

By (5.8), (5.10) and (5.15), we have

$$\rho(\prod_{i=n_1}^{n+1} A_i x_{n_1}, x_{n+2}) \leq \tilde{r}. \tag{5.16}$$

In view of (5.7) and (5.16), we have

$$\prod_{i=n_1}^{n+1} A_i x_{n_1} \in B(x_{n+2}, \tilde{r}) \subset K.$$

It follows from the inclusion above and (5.15) that the assumption made for n also holds for $n+1$. Therefore by induction, we show that

$$\prod_{i=n_1}^{n} A_i x_{n_1} \in K \text{ for all integers } n \geq n_1$$

and that (5.12) holds for all integers $n \geq n_1$. From our assumptions, there exists

$$y = \lim_{n \to \infty} \prod_{i=n_1}^{n} A_i x_{n_1} \tag{5.17}$$

and if (A2) holds, then $y \in F$. In view of (5.17), there exists an integer $n_2 > n_1$ such that for all integers $n \geq n_1$,

$$\rho(y, \prod_{i=n_1}^{n} A_i x_{n_1}) < \varepsilon/4. \tag{5.18}$$

It follows from (5.10), (5.12) and (5.18) that

$$\rho(x_{n+p}, y) \leq \rho(x_{n+p}, \prod_{i=n_1}^{n} A_i x_{n_1}) + \rho(\prod_{i=n_1}^{n} A_i x_{n_1}, y) < \varepsilon/2. \tag{5.19}$$

Since ε is any number satisfying (5.8), we conclude that $\{x_i\}_{i=0}^{\infty}$ is a Cauchy sequence and there exists $\lim_{i\to\infty} x_i \in K$. By (5.19), we have $\rho(y, \lim_{i\to\infty} x_i) \leq \varepsilon/2$. If (A2) holds, the inequality above implies that $\rho(\lim_{i\to\infty} x_i, F) < \varepsilon$. Since ε is any number satisfying (5.8), we conclude that $\lim_{i\to\infty} x_i \in F$. This completes the proof. □

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