J. Appl. Numer. Optim. 8 (2026), No. 3, pp. 465-473
Available online at http://jano.biemdas.com
https://doi.org/10.23952/jano.8.2026.3.08

APPROXIMATE SOLUTIONS IN LINEAR FRACTIONAL VECTOR OPTIMIZATION

NGUYEN THI THU HUONG

Center for Applied Mathematics and Information Technology,
Institute of Information and Communication Technology, Le Quy Don Technical University,
236 Hoang Quoc Viet, Nghia Do, Hanoi, Vietnam

Abstract. This paper studies approximate solutions of a linear fractional vector optimization problem without requiring boundedness of the constraint set. We establish necessary and sufficient conditions for approximating weakly efficient points of such a problem via some properties of the objective function and a technical lemma related to the intersection of the topological closure of the cone generated by a subset of the Euclidean space and the interior of the negative orthant. We obtain necessary conditions and sufficient conditions for approximate efficient solutions of our problem. Applications of these results to linear vector optimization are also considered.
Keywords. Approximate solutions; Efficient solution; Fractional vector optimization; Weak efficiency.
2020 Mathematics Subject Classification. 90C29, 90C32.

1. Introduction

Linear fractional vector optimization problems (LFVOPs) are specific nonconvex vector optimization problems, which have been studied intensively due to their noteworthy properties and theoretical importance; see, e.g., [1, 2, 4, 5, 6, 8, 9, 10, 12, 13, 14, 16, 28, 31] and the references therein.

Connections of LFVOPs with monotone affine vector variational inequalities were firstly recognized by Yen and Phuong [29]. Topological properties of the solution sets of LFVOPs and monotone affine vector variational inequalities were studied by Choo and Atkins [5, 6], Benoist [1, 2], Huy and Yen [14], Hoa et al. [8, 9, 10], Huong et al. [12, 13] and other authors. By a fundamental theorem on stability of monotone affine variational inequalities, Yen and Yao [31] derived several results on solution stability and topological properties of the solution sets of LFVOPs. In [30], Yen and Yang initiated a study on infinite-dimensional LFVOPs via affine variational inequalities on normed spaces. Numerical methods for solving LFVOPs can be found in Malivert [22] and Steuer [24]. Interested readers are referred to the survey paper [28] for more results on linear fractional vector optimization problems.

The notions of efficient solutions and weakly efficient solutions are crucial for analyzing vector optimization problems (see, e.g., [20, 24]). But, sometimes the corresponding solution sets are empty. In that case, one may wish to find approximate solutions, which satisfy some

E-mail address: nguyenhuong2308.mta@gmail.com
Received 3 February 2026; Accepted 28 June 2026; Published online 1 August 2026.

requirements of the decision maker. In addition, note that some algorithms such as iterative algorithms and search algorithms often provide approximate solutions. Therefore, considering approximate solutions and studying their necessary and sufficient conditions is an important question from both theoretical and practical points of view. Based on the Kutateladze's concept [15] of approximate points, Loridan [19] introduced the notion of ε-efficient solutions to vector optimization and obtained some similar results as in [15]. Then, several authors (see, e.g., [7, 11, 17, 19, 25, 27] and the references therein) established further results in this direction.

In [17], Li and Wang proposed the concept of ε-proper efficiency in vector optimization and obtained several necessary and sufficient conditions for ε-proper efficiency via scalarization and an alternative theorem. Afterwards, Liu [18] obtained some scalarization results for the kind of approximate properly efficient solutions for general vector optimization problems. Recently, for a linear fractional vector optimization problem with a bounded constraint set, Tuyen [26, Theorem 3.2] demonstrated that there exists no difference between the ε-properly efficient solution set and the ε-efficient solution one. Later, in [11, Theorem 3.3], we presented necessary and sufficient conditions for a ε-proper efficient solution of a general vector optimization problem via Benson's approach [3]. Moreover, we also prove [11, Theorem 5.4] that, for any linear vector optimization problem with a pointed polyhedral convex cone K, either the e-properly efficient solution set is empty or it coincides with the e-efficient solution set, where e is any nonzero vector taken from the closed pointed ordering cone. It is well known that for a general vector optimization problem, the ε-properly efficient solution set is a subset of the ε-efficient solution set and the ε-efficient solution set is a subset of the ε-weakly efficient solution set.

In this paper, we establish necessary and sufficient conditions for approximating weakly efficient points of a linear fractional vector optimization problem without requiring the constraint set's boundedness. As a consequence, we obtain necessary conditions and sufficient conditions for approximate efficient solutions to the considered problem. These results can be applied to linear vector optimization problems. Next, Section 2 recalls some definitions and auxiliary results. Section 3 presents the main results, while Section 4, the last section, describes some applications of these results to linear vector optimization.

2. Preliminaries

The scalar product and the norm in the Euclidean space $\mathbb{R}^p$ are denoted, respectively, by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$. Vectors in $\mathbb{R}^p$ are represented by columns of real numbers. If A is a matrix, then A^T denotes the transposed matrix of A. Thus, one has $\langle x,y\rangle = x^Ty$ for any $x,y\in\mathbb{R}^p$. For $x=(x_1,\dots,x_p)$ and $y=(y_1,\dots,y_p)$ from $\mathbb{R}^p$, one writes $x\leq y$ (resp., $x<y$) whenever $x_i\leq y_i$ (resp., $x_i<y_i$) for all $i=1,\dots,p$. The nonnegative orthant in $\mathbb{R}^p$ and the set of positive integers are denoted respectively by $\mathbb{R}^p_+$ and $\mathbb{N}$.

A nonempty set $K\subset\mathbb{R}^m$ is called a *cone* if $tv\in K$ for all $v\in K$ and $t\geq 0$. One says that K is *pointed* if $K\cap(-K)=\{0\}$. The smallest cone containing a nonempty set $D\subset\mathbb{R}^m$, i.e., the cone generated by D, is denoted by $\operatorname{cone}D$. The topological closure of D is denoted by $\overline{D}$ and $\overline{\operatorname{cone}}\,D := \overline{\operatorname{cone}D}$. A nonzero vector $v\in\mathbb{R}^n$ (see [23, p. 61]) is said to be a *direction of recession* of a nonempty convex set $D\subset\mathbb{R}^n$ if $x+tv\in D$ for every $t\geq 0$ and every $x\in D$. The set composed by $0\in\mathbb{R}^n$ and all the directions $v\in\mathbb{R}^n\setminus\{0\}$ satisfying the last condition is called the *recession cone* of D and denoted by 0^+D. If D is closed and convex, then

$$0^+D=\{v\in\mathbb{R}^n\ :\ \exists x\in\Omega \text{ s.t. } x+tv\in D \text{ for all } t>0\}.$$

Consider *linear fractional functions* $f_i : \mathbb{R}^n \to \mathbb{R},\ i = 1,\ldots,m$, of the form

$$f_i(x) = \frac{a_i^T x + \alpha_i}{b_i^T x + \beta_i},$$

where $a_i \in \mathbb{R}^n, b_i \in \mathbb{R}^n, \alpha_i \in \mathbb{R}$, and $\beta_i \in \mathbb{R}$. Let K be a *polyhedral convex set*, i.e., there exist $p \in \mathbb{N}$, a matrix $C \in \mathbb{R}^{p\times n}$, and a vector $d \in \mathbb{R}^p$ such that $K = \{x \in \mathbb{R}^n : Cx \le d\}$. Our standing condition is that $b_i^T x + \beta_i > 0$ for all $i \in I$ and $x \in K$, where $I := \{1,\ldots,m\}$. Put $f(x) = (f_1(x),\ldots,f_m(x))$ and let

$$\Omega = \{x \in \mathbb{R}^n : b_i^T x + \beta_i > 0,\ \forall i \in I\}.$$

Clearly, Ω is open and convex, $K \subset \Omega$, and f is continuously differentiable on Ω. The *linear fractional vector optimization problem* (LFVOP) given by f, K, and the ordering cone $\mathbb{R}^m_+$, is formally written as

$$\text{(VP)} \qquad \text{Minimize } f(x) \ \text{ subject to } \ x \in K.$$

Definition 2.1. A point $x \in K$ is said to be an *efficient solution* (or a *Pareto solution*) of (VP) if $\big(f(K) - f(x)\big) \cap \big(-\mathbb{R}^m_+ \setminus \{0\}\big) = \emptyset$. One calls $x \in K$ a *weakly efficient solution* (or a *weak Pareto solution*) of (VP) if $\big(f(K) - f(x)\big) \cap \big(-\operatorname{int}\mathbb{R}^m_+\big) = \emptyset$, where $\operatorname{int}\mathbb{R}^m_+ = \{\xi = (\xi_1,\ldots,\xi_m) \in \mathbb{R}^m : \xi_i > 0 \text{ for } i = 1,\ldots,m\}$ denotes the interior of $\mathbb{R}^m_+$.

The efficient solution set (resp., the weakly efficient solution set) of (VP) are denoted, respectively, by E and E^w.

According to [5, 22] (see also [16, Theorem 8.1]), *for any $x \in K$, one has $x \in E$ (resp., $x \in E^w$) if and only if there exists a multiplier $\xi = (\xi_1,\ldots,\xi_m) \in \operatorname{int}\mathbb{R}^m_+$ (resp., $\xi = (\xi_1,\ldots,\xi_m) \in \mathbb{R}^m_+ \setminus \{0\}$) such that*

$$\Big\langle \sum_{i=1}^m \xi_i \left[(b_i^T x + \beta_i)a_i - (a_i^T x + \alpha_i)b_i\right], y - x \Big\rangle \ge 0, \quad \forall y \in K.$$

Let $\varepsilon = (\varepsilon_1,...,\varepsilon_m)$ be a vector in $\mathbb{R}^m_+$. Specializing the concept of ε-efficiency of general vector optimization problems in [17, 18, 19] to (VP), we have the following definition.

Definition 2.2. A point $x \in X$ is said to be an *ε-efficient solution* of (VP) if there exists no $y \in K$ such that $f(y) \le f(x) - \varepsilon$ and $f(y) \ne f(x) - \varepsilon$.

Slightly weakening the requirement of ε-efficiency, we get the following notion of ε-weak efficiency.

Definition 2.3. A point $x \in K$ is said to be an *ε-weakly efficient solution* of (VP) if there exists no $y \in K$ such that $f(y) < f(x) - \varepsilon$.

The ε-efficient solution set (resp., the ε-weakly efficient solution set) of (VP) are denoted, respectively, by E_ε and E^w_ε.

Remark 2.1. Let $\bar{x} \in K$. Then, $\bar{x}$ is an ε-efficient solution of (VP) if and only if $\big[f(K) - (f(\bar{x}) - \varepsilon)\big] \cap \big(-\mathbb{R}^m_+ \setminus \{0\}\big) = \emptyset$. Similarly, $\bar{x}$ is an ε-weakly efficient solution of (VP) if and only if $\big[f(K) - (f(\bar{x}) - \varepsilon)\big] \cap \big(-\operatorname{int}\mathbb{R}^m_+\big) = \emptyset$. Clearly, $E_\varepsilon \subset E^w_\varepsilon$.

When $\varepsilon = 0$, the notion of ε-efficient solution (resp., the notion of ε-weakly efficient solution) reduces to the notion of efficient solution, respectively, the notion of weakly efficient solution, i.e., $E_0 = E$ and $E_0^w = E^w$.

In the sequel, to establish verifiable necessary sufficient conditions for a point $\bar{x} \in K$ to belong to E_ε^w, we need the following two lemmas. The first one is a fact related to the intersection of the topological closure of the cone generated by a subset of $\mathbb{R}^m$ and the interior of the negative orthant.

Lemma 2.1. *Let Ω be a nonempty subset of $\mathbb{R}^m$. Then the following two properties are equivalent:*

: (i) $\Omega \cap \left(-\operatorname{int}\mathbb{R}^m_+\right) = \emptyset$;
: (ii) $\overline{\operatorname{cone}}\,\Omega \cap \left(-\operatorname{int}\mathbb{R}^m_+\right) = \emptyset$.

Proof. Since $\Omega \subset \operatorname{cone}\Omega \subset \overline{\operatorname{cone}}\,\Omega$ by definition, (ii) implies (i). To prove the reverse implication, we can argue by contradiction. Suppose that (i) holds, but (ii) is invalid. Then, there exists a vector $\bar{v} = (\bar{v}_1, ..., \bar{v}_m) \in -\operatorname{int}\mathbb{R}^m_+$ such that $\bar{v} = \lim\limits_{k\to\infty} v^k$ with $v^k = t_k x^k$, for $x^k \in \Omega$ and $t_k > 0$ for all $k \in \mathbb{N}$. As $\lim\limits_{k\to\infty} v_i^k = \bar{v}_i < 0$ for every $i \in I$, there exists $\bar{k} \in \mathbb{N}$ such that $v_i^k < 0$ for all $k > \bar{k}$ and $i \in I$. So,

$$x^k \in \Omega \cap \left(-\operatorname{int}\mathbb{R}^m_+\right) \quad \forall k \geq \bar{k},$$

a contradiction to (i). □

Lemma 2.2 (See, e.g., [16, Lemma 8.1] and [22])**.** *Let $\varphi(x) = \dfrac{a^T x + \alpha}{b^T x + \beta}$ be a linear fractional function defined by $a, b \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$. Suppose that $b^T x + \beta \neq 0$ for every $x \in K_0$, where $K_0 \subset \mathbb{R}^n$ is an arbitrary convex set. Then,*

$$\varphi(y) - \varphi(x) = \frac{b^T x + \beta}{b^T y + \beta} \langle \nabla\varphi(x), y - x\rangle,$$

for any $x, y \in K_0$, where $\nabla\varphi(x)$ denotes the Fréchet derivative of φ at x.

3. Main Results

First, necessary and sufficient conditions for a feasible point to be an ε-weakly efficient solution of (VP), where $\varepsilon \in \mathbb{R}^m_+$, are provided by the following theorem.

Theorem 3.1. *Let $\varepsilon \in \mathbb{R}^m_+$ and $\bar{x} \in K$. Then, $\bar{x}$ is an ε-weakly efficient solution of* (VP) *if and only if there exists a vector $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+ \setminus \{0\}$ such that*

$$\sum_{i=1}^{m} \lambda_i [(b_i^T \bar{x} + \beta_i)\langle \nabla f_i(\bar{x}), y - \bar{x}\rangle + \varepsilon_i (b_i^T y + \beta_i)] \geq 0 \tag{3.1}$$

for all $y \in K$.

Proof. By definition, $\bar{x}$ is an ε-weakly efficient solution of (VP) if and only if the following system has no $y \in K$ such that

$$f_i(y) < f_i(\bar{x}) - \varepsilon_i, \;\; \forall i \in I. \tag{3.2}$$

For every $i \in I$, by Lemma 2.2, one has

$$f_i(y) - f_i(\bar{x}) = \frac{b_i^T \bar{x} + \beta_i}{b_i^T y + \beta_i} \langle \nabla f_i(\bar{x}), y - \bar{x} \rangle.$$

So, the system (3.2) of inequalities can be rewritten as follows:

$$\frac{b_i^T \bar{x} + \beta_i}{b_i^T y + \beta_i} \langle \nabla f_i(\bar{x}), y - \bar{x} \rangle + \varepsilon_i < 0, \ \ \forall i \in I.$$

Since $b_i^T y + \beta_i > 0$ for all $i \in I$ and $y \in K$, the latter is equivalent to the condition

$$(b_i^T \bar{x} + \beta_i) \langle \nabla f_i(\bar{x}), y - \bar{x} \rangle + \varepsilon_i (b_i^T y + \beta_i) < 0, \ \ \forall i \in I.$$

Let

$$A := \begin{pmatrix} (b_1^T \bar{x} + \beta_1) \nabla f_1(\bar{x})^T + \varepsilon_1 b_1^T \\ \vdots \\ (b_m^T \bar{x} + \beta_m) \nabla f_m(\bar{x})^T + \varepsilon_m b_m^T \end{pmatrix}, \ \ b := \begin{pmatrix} -(b_1^T \bar{x} + \beta_1) \langle \nabla f_1(\bar{x}), \bar{x} \rangle + \varepsilon_1 \beta_1 \\ \vdots \\ -(b_m^T \bar{x} + \beta_m) \langle \nabla f_m(\bar{x}), \bar{x} \rangle + \varepsilon_m \beta_m \end{pmatrix}.$$

Then, $\bar{x}$ is an ε-weakly efficient solution of (VP) if and only if *there is no* $y \in K$ *such that* $Ay + b \in (-\operatorname{int} \mathbb{R}^m_+)$. The last condition means that

$$D \cap (-\operatorname{int} \mathbb{R}^m_+) = \emptyset, \tag{3.3}$$

where $D := \{Ay + b \,:\, y \in K\}$. Put $P = \operatorname{cone} D$ and observe that (3.3) can be rewritten as $P \cap (-\operatorname{int} \mathbb{R}^m_+) = \emptyset$. So, by Lemma 2.1, (3.3) is equivalent to the condition

$$\overline{P} \cap (-\operatorname{int} \mathbb{R}^m_+) = \emptyset. \tag{3.4}$$

Let $P^* := \{z^* \in \mathbb{R}^m \,:\, \langle z^*, z \rangle \geq 0, \ \forall z \in P\}$. We claim that (3.4) holds if and only if

$$P^* \cap \big(\mathbb{R}^m_+ \setminus \{0\}\big) \neq \emptyset. \tag{3.5}$$

Indeed, arguing by contradiction, one can easily show that (3.5) implies (3.4). We now assume that (3.4) holds. Then, applying the separation theorem [23, Theorem 11.3] to the convex sets $\overline{P}$ and $(-\mathbb{R}^m_+)$ yields a vector $\xi \in \mathbb{R}^m \setminus \{0\}$ such that

$$\langle \xi, u \rangle \leq \langle \xi, v \rangle, \ \ \forall u \in -\mathbb{R}^m_+, \ \ \forall v \in \overline{P}. \tag{3.6}$$

Fixing any $v \in \overline{P}$, from (3.6) we can deduce that $\xi \in \mathbb{R}^m_+ \setminus \{0\}$. Moreover, substituting $u = 0$ to the inequality in (3.6) shows that $\xi \in P^*$. So, (3.5) holds.

Clearly, (3.5) means there exists a vector $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+ \setminus \{0\}$ such that $\langle \lambda, v \rangle \geq 0$ for all $v \in \bar{P}$. Since the latter can be rewritten as $\langle \lambda, Ay + b \rangle \geq 0$ for all $y \in K$ and, by the constructions of A and b,

$$\langle \lambda, Ay + b \rangle = \sum_{i=1}^m \lambda_i [(b_i^T \bar{x} + \beta_i) \langle \nabla f_i(\bar{x}), y - \bar{x} \rangle + \varepsilon_i (b_i^T y + \beta_i)],$$

we have thus proved that $\bar{x}$ is an ε-weakly efficient solution of (VP) if and only if there exists a vector $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+ \setminus \{0\}$ such that the inequality (3.1) is fulfilled for every $y \in K$.

The proof is complete. □

Then, for ε-efficient solutions of (VP), the following result holds.

Theorem 3.2. *Let $\varepsilon \in \mathbb{R}^m_+$ and $\bar{x} \in K$. If $\bar{x}$ is an ε-efficient solution of* (VP)*, then there exists a vector $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m_+ \setminus \{0\}$ such that* (3.1) *holds. Conversely, if* (3.1) *is fulfilled for some $\lambda = (\lambda_1, \dots, \lambda_m) \in \operatorname{int} \mathbb{R}^m_+$, then $\bar{x}$ is an ε-efficient solution of* (VP).

Proof. Since $E_\varepsilon \subset E^w_\varepsilon$, the first assertion follows from Theorem 3.1. To prove the second assertion, suppose to the contrary that there exists a vector $\lambda = (\lambda_1, \dots, \lambda_m)$ with $\lambda_i > 0$ for all $i \in I$ such that condition (3.1) is satisfied for all $y \in K$, but $\bar{x} \notin E_\varepsilon$. Then, by Remark 2.1, there exists a vector $v \in \big[f(K) - (f(\bar{x}) - \varepsilon)\big]$, where $v = (v_1, ..., v_m)$, $v_i \le 0$ for all $i \in I$, and one has $v_{i_0} < 0$ for some $i_0 \in I$. So, there exists $y \in K$ such that $v = f(y) - (f(\bar{x}) - \varepsilon)$ and

$$\begin{cases} f_i(y) - \big(f_i(\bar{x}) - \varepsilon_i\big) & \le 0 \ \ \forall i \in I \setminus \{i_0\}, \\ f_{i_0}(y) - \big(f_{i_0}(\bar{x}) - \varepsilon_{i_0}\big) & < 0. \end{cases} \tag{3.7}$$

By Lemma 2.2, for every $i \in I$, one has

$$f_i(y) - f_i(\bar{x}) = \frac{b_i^T \bar{x} + \beta_i}{b_i^T y + \beta_i} \langle \nabla f_i(\bar{x}), y - \bar{x} \rangle.$$

Hence, system (3.7) can be rewritten as

$$\begin{cases} \dfrac{b_i^T \bar{x} + \beta_i}{b_i^T y + \beta_i} \langle \nabla f_i(\bar{x}), y - \bar{x} \rangle + \varepsilon_i & \le 0 \ \ \forall i \in I \setminus \{i_0\}, \\ \dfrac{b_{i_0}^T \bar{x} + \beta_{i_0}}{b_{i_0}^T y + \beta_{i_0}} \langle \nabla f_{i_0}(\bar{x}), y - \bar{x} \rangle + \varepsilon_{i_0} & < 0. \end{cases}$$

Since $b_i^T y + \beta_i > 0$ for all $i \in I$, the last system is equivalent to

$$\begin{cases} (b_i^T \bar{x} + \beta_i) \langle \nabla f_i(\bar{x}), y - \bar{x} \rangle + \varepsilon_i (b_i^T y + \beta_i) & \le 0 \ \ \forall i \in I \setminus \{i_0\}, \\ (b_{i_0}^T \bar{x} + \beta_{i_0}) \langle \nabla f_{i_0}(\bar{x}), y - \bar{x} \rangle + \varepsilon_{i_0} (b_{i_0}^T y + \beta_{i_0}) & < 0. \end{cases}$$

As $\lambda_i > 0$ for all $i \in I$, one has

$$\begin{cases} \lambda_i \big[(b_i^T \bar{x} + \beta_i) \langle \nabla f_i(\bar{x}), y - \bar{x} \rangle + \varepsilon_i (b_i^T y + \beta_i)\big] & \le 0 \ \ \forall i \in I \setminus \{i_0\}, \\ \lambda_{i_0} \big[(b_{i_0}^T \bar{x} + \beta_{i_0}) \langle \nabla f_{i_0}(\bar{x}), y - \bar{x} \rangle + \varepsilon_{i_0} (b_{i_0}^T y + \beta_{i_0})\big] & < 0. \end{cases}$$

Therefore, summing up the last inequalities yields

$$\sum_{i=1}^m \lambda_i [(b_i^T \bar{x} + \beta_i) \langle \nabla f_i(\bar{x}), y - \bar{x} \rangle + \varepsilon_i (b_i^T y + \beta_i)] < 0.$$

This obviously contradicts the condition (3.1). The proof is complete. □

4. Applications to Linear Vector Optimization

Consider the problem (VP) as in Section 2. If $b_i = 0$ and $\beta_i = 1$ for all $i \in I$, then (VP) coincides with the classical *linear vector optimization problem* (LVOP) (see Luc [21] and the references therein).

From Theorem 3.1, we can easily obtain necessary and sufficient conditions for ε-weakly efficient solutions of a linear vector optimization problem as follows.

Theorem 4.1. *Let $\varepsilon \in \mathbb{R}^m_+$ and $\bar{x} \in K$. Then, $\bar{x}$ is an ε-weakly efficient solution of* (LVOP) *if and only if there exists a vector $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+ \setminus \{0\}$ such that*

$$\sum_{i=1}^{m} \lambda_i \big(\langle a_i, y - \bar{x} \rangle + \varepsilon_i \big) \geq 0 \tag{4.1}$$

for all $y \in K$.

On account of Theorem 3.2, we have the following result for ε-efficient solutions of (LVOP)

Theorem 4.2. *Let $\varepsilon \in \mathbb{R}^m_+$ and $\bar{x} \in K$. If $\bar{x}$ is an ε-efficient solution of* (LVOP)*, then there exists a vector $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+ \setminus \{0\}$ such that* (4.1) *holds. Conversely, if* (4.1) *is fulfilled for some $\lambda = (\lambda_1, \ldots, \lambda_m) \in \operatorname{int} \mathbb{R}^m_+$, then $\bar{x}$ is an ε-efficient solution of* (LVOP).

The linear vector optimization problem in the next example has no weakly efficient solution, while both approximate efficient solution set and approximate weakly efficient solution set are nonempty.

Example 4.1. Consider problem (VP) with $n = m = 2$,

$$K = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0\},$$

and $f(x) = (x_1, x_2)$. One has $E^w = E = \emptyset$. Let $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2_+$ be such that $\varepsilon_1 > 0$. Using Remark 2.1, we see that

$$E_\varepsilon = \{x = (x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 < \varepsilon_1\}$$

and

$$E^w_\varepsilon = \{x = (x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq \varepsilon_1\}.$$

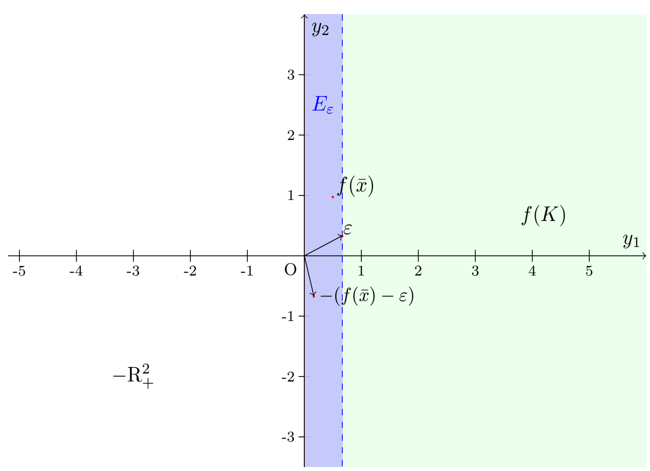

FIGURE 1. The set E_ε in Example 4.1.

We can check that $\bar{x} = (0,0)$, $\varepsilon_1 > 0$ is an ε-weakly efficient solution by using Theorem 4.1. Indeed, choosing $\lambda = (\lambda_1, 0) \in \mathbb{R}^2_+ \setminus \{0\}$, $\lambda_1 > 0$, we see that condition (4.1) is satisfied. Now, we assume that there exist some $\lambda = (\lambda_1, \lambda_2) \in \operatorname{int} \mathbb{R}^2_+$ such that the condition (4.1) holds. This means that

$$\lambda_1 \big(\langle a_1, y - \bar{x} \rangle + \varepsilon_1 \big) + \lambda_2 \big(\langle a_2, y - \bar{x} \rangle + \varepsilon_2 \big) \geq 0$$

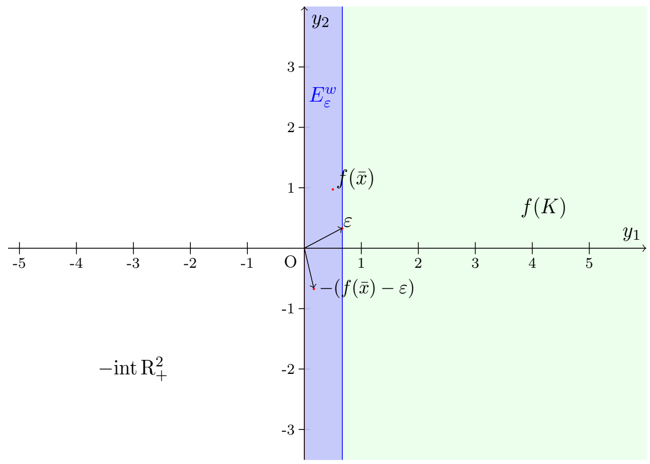

FIGURE 2. The set E_ε^w in Example 4.1.

for all $y \in K$. As $a_1 = (1.0)$ and $a_1 = (0.1)$, one has

$$\lambda_1(y_1+\varepsilon_1)+\lambda_2(y_2+\varepsilon_2) \geq 0 \tag{4.2}$$

for all $y \in K$. Since $\lambda = (\lambda_1, \lambda_2) \in \mathrm{int}\,\mathbb{R}^2_+$, we can rewrite (4.2) equivalently as

$$-y_2 \leq \frac{\lambda_1(y_1+\varepsilon_1)+\lambda_2\varepsilon_2}{\lambda_2} \tag{4.3}$$

for all $y \in K$. We observer that (4.3) does not hold for all $y = (y_1, y_2) \in K, y_2 < 0$ and for any $\lambda \in \mathrm{int}\,\mathbb{R}^2_+$. This means that there is no any $\lambda = (\lambda_1, \lambda_2) \in \mathrm{int}\,\mathbb{R}^2_+$ such that (4.2) is satisfied. But $(0,0)$ is sill an ε-efficient solution by Remark 2.1. Thus, Theorem 4.2 cannot be used to assure that $\bar{x} \in E_\varepsilon$.

Acknowledgements
The author would like to thank Professor Nguyen Dong Yen for helpful discussions on the subject and the anonymous referee for very helpful comments and suggestions. The author was also supported by Le Quy Don Technical University.

REFERENCES

[1] J. Benoist, Connectedness of the efficient set for strictly quasiconcave sets, J. Optim. Theory Appl. 96 (1998), 627–654.
[2] J. Benoist, Contractibility of the efficient set in strictly quasiconcave vector maximization, J. Optim. Theory Appl. 110 (2001), 325–336.
[3] H.B. Benson, An improved definition of proper efficiency for vector maximization with respect to cones, J. Math. Anal. Appl. 7 (1979), 232–241.
[4] E.U. Choo, Proper efficiency and the linear fractional vector maximum problem, Oper. Res. 32 (1984), 216–220.
[5] E.U. Choo, D.R. Atkins, Bicriteria linear fractional programming, J. Optim. Theory Appl. 36 (1982), 203–220.
[6] E.U. Choo, D.R. Atkins, Connectedness in multiple linear fractional programming, Manag. Sci. 29 (1983), 250–255.

[7] T.D. Chuong, D.S. Kim, Approximate solutions of multiobjective optimization problems, Positivity 20 (2016), 187–207.
[8] T.N. Hoa, T.D. Phuong, N.D. Yen, Linear fractional vector optimization problems with many compo nents in the solution sets, J. Ind. Manag. Optim. 1 (2005), 477–486.
[9] T.N. Hoa, T.D. Phuong, N.D. Yen, On the parametric affine variational inequality approach to linear fractional vector optimization problems, Vietnam J. Math. 33 (2005), 477–489.
[10] T.N. Hoa, N.Q. Huy, T.D. Phuong, N.D. Yen, Unbounded components in the solution sets of strictly quasiconcave vector maximization problems, J. Glob. Optim. 37 (2007), 1–10.
[11] N.T.T. Huong, Approximate proper efficiency in vector optimization via Benson's approach, Vietnam J. Math. (2026). doi: 10.1007/s10013-026-00810-0
[12] N.T.T. Huong, J.C. Yao, N.D. Yen, Geoffrion's proper efficiency in linear fractional vector optimization with unbounded constraint sets, J. Global Optim. 78 (2020), 545–562.
[13] N.T.T. Huong, J.C. Yao, N.D. Yen, Connectedness structure of the solution sets of vector variational inequalities, Optimization 66 (2017), 889–901.
[14] N.Q. Huy, N.D. Yen, Remarks on a conjecture of J. Benoist, Nonlinear Anal. Forum 9 (2004), 109–117.
[15] S.S. Kutateladze, Convex e-Programming, Soviet Math. Doklady 20 (1979), 391–393.
[16] G.M. Lee, N.N. Tam, N.D. Yen, Quadratic Programming and Affine Variational Inequalities. A Qualitative Study, Springer-Verlag, New York, 2005.
[17] Z.Li, S. Wang, ε-approximate solutions in multiobjective optimization, Optimization 44 (1998), 161–174.
[18] J.C. Liu, ε-properly efficient solutions to nondifferentiable multiobjective programming problems, Appl. Math. Lett. 12 (1999), 109–113.
[19] P. Loridan, ε-solutions in vector minimization problems, J. Optim. Theory Appl. 43 (1984), 265–276.
[20] D.T. Luc, Theory of Vector Optimization, Springer, Berlin, 1989.
[21] D.T. Luc, Multiobjective Linear Programming: An Introduction, Springer, Berlin, 2016.
[22] C. Malivert, Multicriteria fractional programming. In: Sofonea, M., Corvellec, J.N. (eds.) Proceedings of the 2nd Catalan Days on Applied Mathematics. pp. 189–198, Presses Universitaires de Perpinan, 1995.
[23] R.T. Rockafellar, Convex Analysis. Princeton University Press, Princeton, New Jersey, 1970.
[24] R.E. Steuer, Multiple Criteria Optimization: Theory, computation and application, John Wiley & Sons, New York, 1986.
[25] B. Soleimani, C. Tammer, Concepts for approximate solutions of vector optimization problems with variable order structures, Vietnam J. Math. 42 (2014), 543–566.
[26] N.V. Tuyen, A note on approximate proper efficiency in linear fractional vector optimization, Optim. Lett. 16 (2022), 1835–1845.
[27] D.J. White, Epsilon efficiency, J. Optim. Theory Appl. 49 (1986), 319–337.
[28] N.D. Yen, Linear fractional and convex quadratic vector optimization problems, In: Ansari, Q.H., Yao, J.-C. (eds.) Recent Developments in Vector Optimization, 297–328, Springer, Berlin, 2012.
[29] N.D. Yen, T.D. Phuong, Connectedness and Stability of the Solution Sets in Linear Fractional Vector Optimization Problems, In: P. Giannessi, F. (eds) Vector Variational Inequalities and Vector Equilibria. Nonconvex Optimization and Its Applications, vol 38. Springer, Boston, 2020.
[30] N.D. Yen, X.Q. Yang, Affine variational inequalities on normed spaces, J. Optim. Theory Appl. 178 (2018), 36–55.
[31] N.D. Yen, J.C. Yao, Monotone affine vector variational inequalities, Optimization 60 (2011), 53–68.