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ON APPROXIMATE PARETO SOLUTIONS FOR NONSMOOTH VECTOR OPTIMIZATION PROBLEMS WITH UNCERTAIN DATA ON OBJECTIVE FUNCTION AND CONSTRAINTS

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Abstract. In this paper, we investigate optimality conditions and duality for approximate Pareto solutions of nonsmooth/nonconvex vector optimization problems involving intersection of closed sets with uncertain data on objective function and constraints.
Keywords. Approximate Pareto solution; Constraint qualification; Mordukhovich/limiting subdifferential; Robust optimization; Vector optimization problem.
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1. Introduction

Vector optimization problems play a crucial role in many scientific and practical fields, as multiple objectives often need to be achieved simultaneously in real-world problems. This enables decision-makers to select solutions that best align with their specific priorities. The applications of vector optimization problems are vast, ranging from economics and engineering to management and artificial intelligence. Therefore, the research and development of methods for solving such problems are increasingly necessary and of great practical significance. Optimization problem with uncertain data studies models in which input parameters are not precisely known or may vary over time. Rather than assuming fixed data, this approach accounts for uncertainties such as measurement errors, market volatility, or random noise. This allows the model to better reflect reality and produce more stable solutions. Two common approaches are robust optimization and stochastic optimization, each addressing uncertainty in different ways. As a result, the resulting decisions are more reliable in complex and unpredictable environments. Robust optimization plays a crucial role in the study of optimization problems with uncertain data, as it seeks solutions that remain feasible under various parameter variations. This approach does not rely on a specific probability distribution but instead focuses on worst-case scenarios to ensure the stability and reliability of the solutions. Therefore, robust optimization is one of the most effective tools to investigate optimization problems with uncertain data. In the last decades, robust optimality conditions and robust duality theorems for optimization problems with data uncertainty were studied by many researchers; see, e.g., [4, 5, 8, 9, 11, 16, 18, 25, 26, 27, 28, 29] and the cited references therein.

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In this paper, we consider the vector optimization problem involving intersection of closed sets with uncertain data:

$$\text{(UP)}\quad \begin{array}{l} \min\left(f_1(x,u_1),\dots,f_m(x,u_m)\right),\\ \text{s.t. } x\in S:=\left\{x\in \bigcap_{j=1}^{q} C_j \mid g_i(x,v_i)\le 0, i=1,\dots,p\right\}, \end{array}$$

where x is a decision variable, $u_k, k=1,\dots,m$ and $v_i, i=1,\dots,p$ are uncertain parameters, which reside in the uncertainty sets U_k and V_i, respectively, $U_k\subset\mathbb{R}^m, k=1,\dots,m$ and $V_i\subset\mathbb{R}^l, i=1,\dots,p$ are nonempty compact sets and $C_j\subset\mathbb{R}^n, j=1,\dots,q$ are nonempty closed subsets and $f_k:\mathbb{R}^n\times U_k\to\mathbb{R}, k=1,\dots,m$, $g_i:\mathbb{R}^n\times V_i\to\mathbb{R}, i=1,\dots,p$ are functions.

To treat the uncertainty problem (UP), we associate with it the following robust counterpart:

$$\text{(RP)}\quad \begin{array}{l} \min\left(\max_{u_1\in U_1} f_1(x,u_1),\dots,\max_{u_m\in U_m} f_m(x,u_m)\right),\\ \text{s.t. } x\in S:=\left\{x\in \bigcap_{j=1}^{q} C_j \mid g_i(x,v_i)\le 0, \forall v_i\in V_i, i=1,\dots,p\right\}, \end{array}$$

where the uncertain objective functions and uncertain constraint functions are enforced for all possible realizations within the corresponding uncertainty sets. In what follows, we use sometimes the notation $F:=(F_1,\dots,F_m)$ and $G:=(G_1,\dots,G_p)$, where $F_k(x):=\max_{u_k\in U_k} f_k(x,u_k)\in\mathbb{R}, k=1,\dots,m$ and $G_i:=\max_{v_i\in V_i} g_i(x,v_i)\in\mathbb{R}, i=1,\dots,p$ for $x\in\mathbb{R}^n$.

Definition 1.1. Let $\varepsilon:=(\varepsilon_1,\dots,\varepsilon_m)\in\mathbb{R}^m_+$. A vector $\bar{x}\in S$ is said to be

(i) a robust ε-quasi Pareto solution of problem (UP), written by $\bar{x}\in\varepsilon$-quasi-$\mathscr{S}$(RP), if $\bar{x}$ is an ε-quasi Pareto solution of problem (RP), i.e., for any $x\in S$,

$$F(x)-F(\bar{x})+\varepsilon\|x-\bar{x}\|\notin-\mathbb{R}^m_+\backslash\{0\}.$$

(ii) a robust ε-quasi weakly Pareto solution of problem (UP), written by $\bar{x}\in\varepsilon$-quasi-$\mathscr{S}^w$(RP), if $\bar{x}$ is an $\varepsilon-$quasi weakly Pareto solution of problem (RP), i.e., for any $x\in S$,

$$F(x)-F(\bar{x})+\varepsilon\|x-\bar{x}\|\notin-\text{int}\mathbb{R}^m_+.$$

On the other hand, finding approximate solutions is always easier than finding exact solutions in the complex optimization problems. Therefore, the study of approximate solutions is significant from both the theoretical aspect and computational applications. Among many interesting research for approximate efficient solutions, optimality conditions and duality for approximate efficient solutions of optimization problems were considered numerously by researchers. Optimality conditions and duality for approximate solutions of optimization problems with finite constraints were considered in [3, 6, 7, 17, 23] and the references therein. Optimality conditions and duality for approximate solutions of optimization problems with infinite constraints were given in [12, 19, 20, 21, 24] and the references therein. Up to our knowledge, only few papers studied optimality conditions/duality theorems for approximate solutions of nonsmooth/nonconvex vector optimization problems with uncertain data on objective function and constraints. Besides, to the best of our knowledge, there have not been any works on optimality conditions/duality theorems for ε-quasi Pareto solutions of nonsmooth/nonconvex vector optimization problems involving intersection of closed sets with uncertain data on objective function and constraints. Optimality conditions for an optimization problem of type (UP) with

$m = 1$ without uncertain data on objective function and constraints were obtained in [1, 10]. Optimality conditions/duality theorems for robust Pareto efficient solutions of problem (UP) (where $m = 1$) were investigated in [9]. Optimality conditions/duality theorems/saddle point theorems for ε-quasi Pareto solutions of vector optimization problems involving intersection of closed sets without uncertain data on objective function and constraints were obtained in [22]. Moreover, many other common classes of programming problems with specific types of constraints [2, 13] can be transformed into models involving intersection of closed sets. On the other hand, studying vector optimization problems involving intersection of closed sets is necessary and meaningful because a mathematical optimization model related to customer satisfaction in the automotive industry whose mathematical model structure is suitable for the optimization problems involving intersection of closed sets (see [10]).

In this paper, we modify the method from [9, 22] to obtain some new results on optimality conditions/duality theorems for ε-quasi Pareto solutions of nonsmooth/nonconvex vector optimization problems involving intersection of closed sets with uncertain data on objective function and constraints. The plan of the paper is organized as follows. Section 1, and Section 2 present introduction, notations and preliminaries. In Section 3, we investigate necessary and sufficient conditions for robust ε-quasi Pareto solutions of problem (UP). In Section 4, we study robust duality results between problem (UP) and its dual one in the sense of Mond-Weir type. Finally, conclusions are given in Section 5.

2. Preliminaries

Some basic concepts and tools from variational analysis [14, 15], which are needed in the sequel, are recalled in this section. Throughout the paper, all spaces under consideration are assumed to be Euclidean spaces $\mathbb{R}^n$ with $n \in \mathbb{N} := \{1, 2, \ldots\}$. The inner product and the norm in $\mathbb{R}^n$ are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. The symbol $\mathbb{B}$ stands for the closed unit ball in $\mathbb{R}^n$. The symbol $B(x, r)$ stands for the open ball centered at $x \in \mathbb{R}^n$ with the radius $r > 0$. Let Θ be a nonempty subset of $\mathbb{R}^n$, the interior, the convex hull and boundary of Θ are denoted respectively by $\mathrm{int}\Theta, \mathrm{co}\Theta$ and $\mathrm{bd}\Theta$. The notation $x \xrightarrow{\Theta} \bar{x}$ means that $x \to \bar{x}$ with $x \in \Theta$. The polar cone of Θ is $\Theta^\circ := \{y \in \mathbb{R}^n \mid \langle y, x \rangle \leq 0, \forall x \in \Theta\}$. Besides, the nonnegative (resp., nonpositive) orthant cone of Euclidean space $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \mid x_i \geq 0, i = 1, \ldots, n\}$ (resp., $\mathbb{R}^n_-$) for $n \in \mathbb{N} := \{1, 2, \ldots\}$, while $\mathrm{int}\mathbb{R}^n_+$ is used to indicate the topological interior of $\mathbb{R}^n_+$.

Let $G : \Theta \subset \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a multivalued function/set-valued map. G is closed at $\bar{x} \in \Theta$ if, for any sequence $\{x_k\} \subset \Theta, x_k \to \bar{x}$ and any sequence $\{y_k\} \subset \mathbb{R}^m, y_k \in G(x_k), y_k \to \bar{y}$ as $k \to +\infty$, $\bar{y} \in G(\bar{x})$. Given a set-valued map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, the sequential Painlevé-Kuratowski upper/outer limit of G at $\bar{x} \in \mathrm{dom}G$ is given by

$$\operatorname*{Lim\,sup}_{x \xrightarrow{\Theta} \bar{x}} G(x) := \{y \in \mathbb{R}^n \mid \exists \text{ sequence } x_k \xrightarrow{\Theta} \bar{x} \text{ and } y_k \to y \text{ with } y_k \in G(x_k), \forall k \in \mathbb{N}\},$$

where $\mathrm{dom}G := \{x \in \mathbb{R}^n \mid G(x) \neq \emptyset\}$.

The Fréchet/regular normal cone to Θ at $\bar{x} \in \Theta$ is defined by

$$\widehat{N}(\bar{x}; \Theta) := \left\{ y \in \mathbb{R}^n \mid \limsup_{x \xrightarrow{\Theta} \bar{x}} \frac{\langle y, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq 0 \right\}.$$

If $\bar{x} \in \mathbb{R}^n \backslash \Theta$, we put $\widehat{N}(\bar{x};\Theta) := \emptyset$.

The Mordukhovich/limiting normal cone $N(\bar{x};\Theta)$ to Θ at $\bar{x} \in \Theta$ is obtained from Fréchet/regular normal cones by taking the sequential Painlevé Kuratowski upper limit as

$$N(\bar{x};\Theta) := \operatorname*{Lim\,sup}_{x \xrightarrow{\Theta} \bar{x}} \widehat{N}(x;\Theta).$$

If $\bar{x} \in \mathbb{R}^n \backslash \Theta$, we put $N(\bar{x};\Theta) := \emptyset$.

Set Θ is convex if $\lambda a + (1-\lambda)b \in \Theta$ for all $a, b \in \Theta$ and $0 \le \lambda \le 1$. If set Θ is convex, then

$$\widehat{N}(\bar{x};\Theta) = N(\bar{x};\Theta) = \{y \in \mathbb{R}^n \mid \langle y, x - \bar{x}\rangle \le 0, \forall x \in \Theta\}.$$

Let $\psi : \mathbb{R}^n \to \bar{\mathbb{R}} := [-\infty, +\infty]$ be an extended real-valued function. The domain and epigraph of ψ are given by

$$\operatorname{dom}\psi := \{x \in \mathbb{R}^n \mid |\psi(x)| < +\infty\},$$
$$\operatorname{epi}\psi := \{(x,\mu) \in \mathbb{R}^n \times \mathbb{R} \mid \mu \ge \psi(x)\}.$$

Let $\psi : \mathbb{R}^n \to \bar{\mathbb{R}}$ be defined on $\mathbb{R}^n$. Then ψ is convex if

$$\psi(\lambda x + (1-\lambda)y) \le \lambda\psi(x) + (1-\lambda)\psi(y), \text{ for all } x, y \in \mathbb{R}^n \text{ and } 0 < \lambda < 1.$$

If the inequality is strict for $x \ne y$, then ψ is strictly convex.

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If the inequality is strict for $x \ne y$, then ψ is strictly concave.

Let $\psi : \mathbb{R}^n \to \bar{\mathbb{R}}$ be finite at $\bar{x} \in \operatorname{dom}\psi$, then the Fréchet subdifferential of ψ at $\bar{x}$ is defined by

$$\widehat{\partial}\psi(\bar{x}) := \left\{y \in \mathbb{R}^n \mid (y,-1) \in \widehat{N}\left((\bar{x},\psi(\bar{x}));\operatorname{epi}\psi\right)\right\}.$$

If $|\psi(\bar{x})| = +\infty$, then one puts $\widehat{\partial}\psi(\bar{x}) := \emptyset$. The Mordukhovich/limiting subdifferential of f at $\bar{x}$ is defined by

$$\partial\psi(\bar{x}) := \left\{y \in \mathbb{R}^n \mid (y,-1) \in N\left((\bar{x},\psi(\bar{x}));\operatorname{epi}\psi\right)\right\}.$$

If $|\psi(\bar{x})| = +\infty$, then one puts $\partial\psi(\bar{x}) := \emptyset$. If ψ is a convex function with $\bar{x} \in \operatorname{dom}\psi$, then

$$\begin{aligned} \widehat{\partial}\psi(\bar{x}) = \partial\psi(\bar{x}) &= \left\{y \in \mathbb{R}^n \mid (y,-1) \in N\left((\bar{x},\psi(\bar{x}));\operatorname{epi}\psi\right)\right\} \\ &= \left\{y \in \mathbb{R}^n \mid \psi(x) - \psi(\bar{x}) \ge \langle y, x - \bar{x}\rangle, \forall x \in \mathbb{R}^n\right\}. \end{aligned}$$

Let $\psi : \mathbb{R}^n \to \bar{\mathbb{R}}$ be finite at $\bar{x} \in \operatorname{dom}\psi$, then ψ is lower semi-continuous at $\bar{x}$ if $\liminf_{x \to \bar{x}} \psi(x) \ge \psi(\bar{x})$. Given $\Theta \subset \mathbb{R}^n$, consider the indicator function $\delta(\cdot;\Theta)$ defined by

$$\delta(x;\Theta) := \begin{cases} 0, & \text{if } x \in \Theta, \\ +\infty, & \text{otherwise.} \end{cases}$$

Furthermore, we have a relation between the Mordukhovich/limiting normal cone and the Mordukhovich/limiting subdifferential of the indicator function as follows

$$N(\bar{x};\Theta) = \partial\delta(\bar{x};\Theta), \forall \bar{x} \in \Theta.$$

We say that ψ is locally Lipschitz at $\bar{x} \in \mathbb{R}^n$ if there exist a positive constant $L > 0$ and a neighborhood U of $\bar{x}$, such that, for all $x_1, x_2 \in U$, $|\psi(x_1) - \psi(x_2)| \le L\|\psi_1 - x_2\|$. For a function ψ is locally Lipschitz at $\bar{x}$ with $L > 0$, it implies that (see [14, Corollary 1.81]), for all $x^* \in \partial\psi(\bar{x})$, $\|x^*\| \le L$.

Lemma 2.1. (See [14, Proposition 1.114]) *Let $\psi : \mathbb{R}^n \to \bar{\mathbb{R}}$ be finite at $\bar{x} \in \mathbb{R}^n$. If $\bar{x}$ is a local minimizer of ψ, then $0 \in \partial\psi(\bar{x})$.*

Lemma 2.2. (See [14, Therorem 3.36]) *Suppose that $\psi_k : \mathbb{R}^n \to \bar{\mathbb{R}}, k = 1,\ldots,m$ (with $m \geq 2$) are lower semi-continuous around $\bar{x} \in \mathbb{R}^n$, and let all but one of these functions be Lipschitz continuous around $\bar{x}$. Then $\partial(\psi_1 + \cdots + \psi_m)(\bar{x}) \subset \partial\psi_1(\bar{x}) + \cdots + \partial\psi_m(\bar{x})$.*

Lemma 2.3. (See [14, Therorem 3.36 and Therorem 3.46]) *Let the functions $\psi_k : \mathbb{R}^n \to \bar{\mathbb{R}}$, $k = 1,\ldots,m$ (with $m \geq 2$) be locally Lipschitz around $\bar{x} \in \mathbb{R}^n$ and denote the maximum function by $\phi(x) := \max\{\psi_k(x) \mid k = 1,\ldots,m\}$ for any $x \in \mathbb{R}^n$. Then*

$$\partial\phi(\bar{x}) \subset \bigcup\Big\{ \sum_{k=1}^{m} \alpha_k \partial\psi_k(\bar{x}) \mid (\alpha_1,\ldots,\alpha_m) \in \mathbb{R}^m_+, \sum_{k=1}^{m} \alpha_m = 1, \\ \alpha_k[\psi_k(\bar{x}) - \phi(\bar{x})] = 0, k = 1,\ldots,m \Big\}.$$

3. Robust Approximate Optimality Conditions

In this section, we establish optimality conditions for ε-quasi (weakly) Pareto solutions of for nonsmooth vector optimization problems involving intersection of closed sets under uncertain data. In what follows, we assume that the functions $f_1,\ldots,f_m$ together with the constraint functions $g_1,\ldots,g_p$ of problem (UP) satisfy the following assumptions:

(A1) Given a fixed $\bar{x} \in \mathbb{R}^n$ there exist $\delta_k > 0, k = 1,\ldots,m$ and $\eta_i > 0, i = 1,\ldots,p$ such that the functions $u_k \in U_k \mapsto f_k(x,u_k) \in \mathbb{R}$ and $v_i \in V_i \mapsto g_i(x,v_i) \in \mathbb{R}$ are upper semicontinuous for each $x \in B(\bar{x},\delta_k)$ and $x \in B(\bar{x},\eta_i)$, respectively, and the functions $f_k(\cdot,u_k), u_k \in U_k$ and $g_i(\cdot,v_i), v_i \in V_i$ are uniformly Lipschitz of given ranks $L_k > 0, k = 1,\ldots,m$ and $L_i > 0, i = 1,\ldots,p$ on $B(\bar{x},\delta_k)$ and $B(\bar{x},\eta_i)$, respectively, i.e.,

$$|f_k(x_1,u_k) - f_k(x_2,u_k)| \leq L_k\|x_1 - x_2\|, \forall x_1,x_2 \in B(\bar{x},\delta_k), \forall u_k \in U_k, k = 1,\ldots,m.$$

$$|g_i(x_1,v_i) - g_i(x_2,v_i)| \leq L_i\|x_1 - x_2\|, \forall x_1,x_2 \in B(\bar{x},\eta_i), \forall v_i \in V_i, i = 1,\ldots,p.$$

(A2) For the above $\bar{x} \in \mathbb{R}^n$, the multivalued functions $(x,u_k) \in B(\bar{x},\delta_k) \times U_k \rightrightarrows \partial_x f_k(x,u_k) \subset \mathbb{R}^n, k = 1,\ldots,m$ are closed at $(\bar{x},\bar{u}_k)$ for each $\bar{u}_k \in U_k(\bar{x})$ and the multivalued functions $(x,v_i) \in B(\bar{x},\eta_i) \times V_i \rightrightarrows \partial_x g_i(x,v_i) \subset \mathbb{R}^n, i = 1,\ldots,p$ are closed at $(\bar{x},\bar{v}_i)$ for each $\bar{v}_i \in V_i(\bar{x})$, where the symbol ∂_x stands for the Mordukhovich/limiting subdifferential operation with respect to the first variable x and

$$\begin{aligned} U_k(\bar{x}) &:= \{u_k \in U_k \mid f_k(\bar{x},u_k) = F_k(\bar{x})\}, \\ V_i(\bar{x}) &:= \{v_i \in V_i \mid g_i(\bar{x},v_i) = G_i(\bar{x})\} \end{aligned} \tag{3.1}$$

with

$$\begin{aligned} F_k(\bar{x}) &:= \max_{u_k \in U_k} f_k(\bar{x},u_k), k = 1,\ldots,m, \\ G_i(\bar{x}) &:= \max_{v_i \in V_i} g_i(\bar{x},v_i), i = 1,\ldots,p. \end{aligned} \tag{3.2}$$

Definition 3.1. (See [9]) Let $\bar{x} \in S$. We say that the following constraint qualification (CQ) is satisfied at $\bar{x}$ if there does not exist $(\mu_1,\ldots,\mu_p) \in \Lambda(\bar{x})$ such that

$$0 \in \sum_{i=1}^{p} \mu_i \mathrm{co}\{\partial_x g_i(\bar{x},v_i) \mid v_i \in V_i(\bar{x})\} + \sum_{j=1}^{q} N(\bar{x};C_j),$$

where $V_i(\bar{x})$ is defined by (3.1) and

$$\Lambda(\bar{x}) := \left\{ (\lambda_1, \ldots, \lambda_p) \in \mathbb{R}^p_+ \mid \sum_{i=1}^{p} \lambda_i = 1, \max_{v_i \in V_i} \lambda_i g_i(\bar{x}, v_i) = 0, i = 1, \ldots, p \right\}.$$

Now, we propose a necessary optimality condition for a robust ε-quasi weakly Pareto solution of problem (UP) under the condition (CQ).

Theorem 3.1. *Let $\varepsilon := (\varepsilon_1, \ldots, \varepsilon_m) \in \mathbb{R}^m_+ \backslash \{0\}, \bar{x} \in \varepsilon-quasi-\mathscr{S}^w$(RP) and the assumptions (A1) and (A2) hold at $\bar{x}$. Suppose that the equation $v_1 + \cdots + v_q = 0$, where $v_j \in N(\bar{x}; C_j), j = 1, \ldots, q$, has only the trivial solution $v_j = 0, j = 1, \ldots, q$. Then, there exist $\alpha_k \geq 0, k = 1, \ldots, m$, and $\mu_i \geq 0, i = 1, \ldots, p$, with $\sum_{k=1}^{m} \alpha_k + \sum_{i=1}^{p} \mu_i = 1$ such that*

$$\begin{aligned} 0 \in & \sum_{k=1}^{m} \alpha_k \mathrm{co} \left\{ \partial_x f_k(\bar{x}, u_k) \mid u_k \in U_k(\bar{x}) \right\} + \sum_{i=1}^{p} \mu_i \mathrm{co} \left\{ \partial_x g_i(\bar{x}, v_i) \mid v_i \in V_i(\bar{x}) \right\} \\ & + \sum_{k=1}^{m} \alpha_k \varepsilon_k \mathbb{B} + \sum_{j=1}^{q} N(\bar{x}; C_j) \end{aligned} \tag{3.3}$$

and

$$\mu_i \max_{v_i \in V_i} g_i(\bar{x}, v_i) = 0, i = 1, \ldots, p. \tag{3.4}$$

If condition (CQ) *holds at $\bar{x}$, then $\alpha_k, k = 1, \ldots, m$ in the relation of (3.3) can be chosen not all equal zero.*

Proof. Since $\bar{x} \in \varepsilon-$quasi$-\mathscr{S}^w$(RP), one has $\bar{x} \in S$ and

$$F(x) - F(\bar{x}) + \varepsilon \|x - \bar{x}\| \notin -\mathrm{int}\mathbb{R}^m, \forall x \in S, \tag{3.5}$$

where $F := (F_1, \ldots, F_m)$ and $F_k(x) := \max_{u_k \in U_k} f_k(x, u_k) \in \mathbb{R}, k = 1, \ldots, m$. Putting

$$\Phi(x) := \max_{1 \leq k \leq m, 1 \leq i \leq p} \left\{ F_k(x) - F_k(\bar{x}) + \varepsilon_k \|x - \bar{x}\|, G_i(x) \right\}, x \in \mathbb{R}^n,$$

where $F_k, k = 1, \ldots, m$ and $G_i, i = 1, \ldots, p$ are defined by (3.2).

We prove that

$$\Phi(x) \geq 0 = \Phi(\bar{x}), \forall x \in \bigcap_{j=1}^{q} C_j. \tag{3.6}$$

It is easy to see that the equality in (3.6) holds due to $\bar{x} \in S$. Indeed, from $\bar{x} \in S$, one sees that $G_i(\bar{x}) \leq 0, i = 1, \ldots, p$. Therefore, one has

$$\Phi(\bar{x}) = \max_{1 \leq k \leq m, 1 \leq i \leq p} \{F_k(\bar{x}) - F_k(\bar{x}) + \varepsilon_k \|\bar{x} - \bar{x}\|, G_i(x)\} = \max_{1 \leq i \leq p} \{0, G_i(\bar{x})\} = 0.$$

If $x \in S$, then $\Phi(x) \geq 0$. Otherwise, $\Phi(x) < 0$ implies that

$$\max_{1 \leq k \leq m, 1 \leq i \leq p} \{F_k(x) - F_k(\bar{x}) + \varepsilon_k \|x - \bar{x}\|, G_i(x)\} < 0,$$

which implies $F_k(x) - F_k(\bar{x}) + \varepsilon_k \|x - \bar{x}\| < 0, k = 1, \ldots, m$, which contradicts (3.5). If $x \in \bigcap_{j=1}^{q} C_j \backslash S$, then there exists $i_0 \in \{1, \ldots, p\}$ such that $G_{i_0}(x) > 0$, which deduces that $\Phi(x) > 0$. Therefore, (3.6) holds. Moreover, by (3.6), we see that $\bar{x}$ is a minimizer of $\min_{x \in \bigcap_{j=1}^{q} C_j} \Phi(x)$.

It is easy to deduce that $\bar{x}$ is a minimizer of the following unconstrained scalar optimization problem

$$\min_{x\in\mathbb{R}^n}\left\{\Phi(x)+\delta\left(x;\bigcap_{j=1}^{q}C_j\right)\right\}. \tag{3.7}$$

Applying Lemma 2.1 to problem (3.7), one obtains

$$0\in\partial\left(\Phi+\delta\left(x;\bigcap_{j=1}^{q}C_j\right)\right)(\bar{x}).$$

Since Φ is Lipschitz continuous around $\bar{x}$, $C_1,...,C_q$ are closed sets, and $\delta(\cdot;C)$ is lower semicontinuous around $\bar{x}$, we deduce from $\partial\delta\left(x;\bigcap_{j=1}^{q}C_j\right)=N\left(\bar{x};\bigcap_{j=1}^{q}C_j\right)$ and Lemma 2.2 that

$$0\in\partial\Phi(\bar{x})+N\left(\bar{x};\bigcap_{j=1}^{q}C_j\right). \tag{3.8}$$

Besides, since $v_1+\cdots+v_q=0$, where $v_j\in N(\bar{x};C_j), j=1,\ldots,q$, has only one solution $v_j=0, j=1,\ldots,q$, we apply the formula of normal cones to finite set intersections (See [15, Corollary 2.17]) to arrive at

$$N\left(\bar{x};\bigcap_{j=1}^{q}C_j\right)\subset N(\bar{x};C_1)+\cdots+N\left(\bar{x};C_q\right). \tag{3.9}$$

Note that $\partial\left(\|\cdot-\bar{x}\|\right)(\bar{x})=\mathbb{B}$. According to Lemma 2.3 and Lemma 2.2, we have

$$\begin{aligned}\partial\Phi(\bar{x}) &=\partial\left(\max_{1\le k\le m,1\le i\le p}\{F_k(\cdot)-F_k(\bar{x})+\varepsilon_k\|\cdot-\bar{x}\|,G_i(x)\}\right)(\bar{x})\\ &\subset\left\{\sum_{k=1}^{m}\alpha_k\left(\partial F_k(\bar{x})+\varepsilon_k\mathbb{B}\right)+\sum_{i=1}^{p}\mu_i\partial G_i(\bar{x})\mid(\alpha_1,\ldots,\alpha_m)\in\mathbb{R}^m_+,\right.\\ &\left.(\mu_1,\ldots,\mu_p)\in\mathbb{R}^p_+,\sum_{k=1}^{m}\alpha_k+\sum_{i=1}^{p}\mu_i=1,\mu_iG_i(\bar{x})=0,i=1,\ldots,p\right\}.\end{aligned} \tag{3.10}$$

Following the proof of [4, Theorem 3.3], we see from assumptions (A1) and (A2) that

$$\begin{aligned}\partial F_k(\bar{x})=\partial\left(\max_{u_k\in U_k}f_k(\bar{x},u_k)\right)\subset\mathrm{co}\left\{\partial_x f_k(\bar{x},u_k)\mid u_k\in U_k(\bar{x})\right\},k=1,\ldots,m,\\ \partial G_i(\bar{x})=\partial\left(\max_{v_i\in V_i}g_i(\bar{x},v_i)\right)\subset\mathrm{co}\left\{\partial_x g_i(\bar{x},v_i)\mid v_i\in V_i(\bar{x})\right\},i=1,\ldots,p,\end{aligned} \tag{3.11}$$

where $U_k(\bar{x})$ and $V_i(\bar{x})$ are given as in (3.1). It yields from (3.8), (3.9), (3.10) and (3.11) that there exist $\alpha_k\ge 0,k=1,\ldots,m$, and $\mu_i\ge 0,i=1,\ldots,p$, with $\sum_{k=1}^{m}\alpha_k+\sum_{i=1}^{p}\mu_i=1$ such that

$$\begin{aligned}0\in&\sum_{k=1}^{m}\alpha_k\mathrm{co}\left\{\partial_x f_k(\bar{x},u_k)\mid u_k\in U_k(\bar{x})\right\}+\sum_{i=1}^{p}\mu_i\mathrm{co}\left\{\partial_x g_i(\bar{x},v_i)\mid v_i\in V_i(\bar{x})\right\}\\ &+\sum_{k=1}^{m}\alpha_k\varepsilon_k\mathbb{B}+\sum_{j=1}^{q}N(\bar{x};C_j)\end{aligned}$$

and $\mu_i\max_{v_i\in V_i}g_i(\bar{x},v_i)=0,i=1,\ldots,p$. So, (3.3) and (3.4) have been justified.

Now, let condition (CQ) be satisfied at $\bar{x}$. We obtain from (3.3) that $\sum_{k=1}^{m} \alpha_k \neq 0$. Otherwise, we assume on the contrary that $\alpha_k = 0$ for all $k = 1,\ldots,m$. Then there exist $\mu_i \geq 0, i = 1,\ldots,p$, with $\sum_{i=1}^{p} \mu_i = 1$ such that

$$0 \in \sum_{i=1}^{p} \mu_i \mathrm{co}\left\{\partial_x g_i(\bar{x}, v_i) \mid v_i \in V_i(\bar{x})\right\} + \sum_{j=1}^{q} N(\bar{x}; C_j),$$

which contradicts the fulfilment of the condition (CQ) at $\bar{x}$. The proof is complete. □

Remark 3.1. In Theorem 3.1, if $q = 1$ and $f_k, k = 1,\ldots,m, g_i, i = 1,\ldots,p$ do not have uncertain data, then we obtain [3, Theorem 3.7]. Besides, if $\varepsilon_k = 0, k = 1,\ldots,m$ and $f_k, k = 1,\ldots,m$ do not have uncertain data, then we have [4, Theorem 3.3]. On the other hand, if $m = 1$ and $\varepsilon_k = 0, k = 1,\ldots,m$, then we have [9, Theorem 3.1]. Furthermore, if $f_k, k = 1,\ldots,m, g_i, i = 1,\ldots,p$ do not have uncertain data, then we obtain [22, Theorem 3.3].

Now, we provide a type necessary optimality condition for problem (UP) under the smoothness of related functions and the convexity of uncertainty sets. In this case, both the hypotheses (A1) and (A2) are automatically satisfied. In what follows, we use $\nabla_x h_k(\bar{x}, \bar{u}_k), k = 1,\ldots,m$ to denote the derivative of the differentiable functions $h_k, k = 1,\ldots,m$ with respect to the first variable at the given points $(\bar{x}, \bar{u}_k), k = 1,\ldots,m$.

Corollary 3.1. *Let $\varepsilon := (\varepsilon_1,\ldots,\varepsilon_m) \in \mathbb{R}^m_+ \backslash \{0\}, \bar{x} \in \varepsilon-$quasi$-\mathscr{S}^w$(RP), where $U_k, k = 1,\ldots,m$ and $V_i, i = 1, p$ are convex sets. Suppose that there exist $\delta_k > 0, k = 1,\ldots,m$ and $\eta_i > 0, i = 1,\ldots,p$ such that the functions $f_k(x,\cdot)$ and $g_i(x,\cdot)$ are concave functions on U_k and V_i, for each $x \in B(\bar{x}, \delta_k)$ and $x \in B(\bar{x}, \eta_i)$, respectively. Assume that $f_k, k = 1,\ldots,m$ and $g_i, i = 1,\ldots,p$ are strictly differentiable with respect to the first variable on $B(\bar{x}, \delta_k) \times U_k$ and $B(\bar{x}, \eta_i) \times V_i$, respectively. Assume further that maps $(x, u_k) \mapsto \nabla_x f_k(x, u_k)$ and $(x, v_i) \mapsto \nabla_x g_i(x, v_i)$ are continuous on $B(\bar{x}, \delta_k) \times U_k$ and $B(\bar{x}, \eta_i) \times V_i$, respectively. If the system $v_1 + \cdots + v_q = 0$, where $v_j \in N(x; C_j), j = 1,\ldots,q$, has only one solution $v_j = 0, j = 1,\ldots,q$, then there exist $\bar{u}_k \in U_k(\bar{x}), k = 1,\ldots,m, \bar{v}_i \in V(\bar{x}), i = 1,\ldots,p$ and $(\alpha_1,\ldots,\alpha_m) \in \mathbb{R}^m_+, (\mu_1,\ldots,\mu_p) \in \mathbb{R}^p_+$ with $\sum_{k=1}^{m} \alpha_k + \sum_{i=1}^{p} \mu_i = 1$ such that*

$$0 \in \sum_{k=1}^{m} \alpha_k \nabla_x f_k(\bar{x}, \bar{u}_k) + \sum_{i=1}^{p} \mu_i \nabla_x g_i(\bar{x}, \bar{v}_i) + \sum_{k=1}^{m} \alpha_k \varepsilon_k \mathbb{B} + \sum_{j=1}^{q} N(\bar{x}; C_j) \tag{3.12}$$

and

$$\mu_i g_i(\bar{x}, \bar{v}_i) = 0, i = 1,\ldots,p. \tag{3.13}$$

If condition (CQ) *holds at $\bar{x}$, then $\alpha_k, k = 1,\ldots,m$ in the relation of (3.12) can be chosen not all equal zero.*

Proof. It is easy to see that hypothesis (A2) is automatically satisfied at $\bar{x}$ for our setting as the maps $(x, u_k) \mapsto \nabla_x f_k(x, u_k), k = 1,\ldots,m$ and $(x, v_i) \mapsto \nabla_x g_i(x, v_i), i = 1,\ldots,p$ are continuous on $B(\bar{x}, \delta_k) \times U_k$ and $B(\bar{x}, \eta_i) \times V_i$, respectively. To verify hypothesis (A1), we only justify for the function $f_k, k = 1,\ldots,m$ as the similarities go for the functions $g_i, i = 1,\ldots,p$. To see this, we first claim that, for each $\bar{\varepsilon} > 0$, there exists $\gamma > 0$ satisfying

$$\|\nabla_x f_k(x, u_k) - \nabla_x f_k(y, u_k)\| \leq \bar{\varepsilon}, \forall x, y \in B(\bar{x}, \gamma), \forall u_k \in U_k, k = 1,\ldots,m, \tag{3.14}$$

where $B(\bar{x},\gamma)$ can be chosen such that $B(\bar{x},\gamma) \subset B(\bar{x},\delta_k), k = 1,\ldots,m$ and it is a convex set. To prove (3.14), we suppose on the contrary that there exist $\bar{\varepsilon}_0 > 0$ and a sequence $\{(x_n, y_n, u_{k_n})\}$ in $B(\bar{x},\delta_{k_n}) \times B(\bar{x},\delta_{k_n}) \times U_{k_n}$ such that $(x_n, y_n) \to (\bar{x},\bar{x})$ as $n \to +\infty$ and

$$\|\nabla_x f_{k_n}(x_n, u_{k_n}) - \nabla_x f_{k_n}(y_n, u_{k_n})\| > \bar{\varepsilon}_0, \forall n \in \mathbb{N}. \tag{3.15}$$

In view of the compactness of $U_k, k = 1,\ldots,m$, passing to a subsequence if necessary, we may assume that $\{k_n\}$ converges to some $\bar{k}, \bar{k} = 1,\ldots,m$ and $\{u_{k_n}\}$ converges to some $\bar{u}_{\bar{k}} \in U_{\bar{k}}$. Besides, due to the continuity of $(x,u_k) \mapsto \nabla_x f_k(x,u_k)$ on $B(\bar{x},\delta_k) \times U_k$, it follows that

$$\nabla_x f_{k_n}(x_n, u_{k_n}) \to \nabla_x f_{\bar{k}}(\bar{x},\bar{u}_{\bar{k}}) \text{ and } \nabla_x f_{k_n}(y_n, u_{k_n}) \to \nabla_x f_{\bar{k}}(\bar{x},\bar{u}_{\bar{k}}) \text{ as } n \to +\infty,$$

which contradicts (3.15). Let $\bar{\varepsilon} > 0$ be such that (3.14) holds for a given $\gamma > 0$. For any $x, y \in B(\bar{x},\gamma)$ with $x \neq y$ and for any $u_k \in U_k, k = 1,\ldots,m$. Using the mean value theorem, we find $z \in (x,y) \subset B(\bar{x},\gamma)$ satisfying the condition

$$f_k(x,u_k) - f_k(y,u_k) = \langle \nabla_x f_k(z,u_k), x - y\rangle, k = 1,\ldots,m,$$

where $(x,y) := \text{co}\{x,y\} \backslash \{x,y\}$. This, together with (3.14) and the Cauchy–Schwarz inequality, gives us

$$\begin{aligned}
&|\langle \nabla_x f_k(y,u_k), x-y\rangle - f_k(x,u_k) + f_k(y,u_k)| \\
&= |\langle \nabla_x f_k(y,u_k) - \nabla_x f_k(z,u_k), x-y\rangle| \\
&\leq \|\nabla_x f_k(y,u_k) - \nabla_x f_k(z,u_k)\| \cdot \|x-y\| \\
&\leq \bar{\varepsilon}\|x-y\|.
\end{aligned} \tag{3.16}$$

From the Cauchy–Schwarz inequality, one has

$$\begin{aligned}
&|\langle \nabla_x f_k(y,u_k), x-y\rangle - f_k(x,u_k) + f_k(y,u_k)| \\
&= |f_k(x,u_k) - f(y,u_k) - \langle \nabla_x f_k(y,u_k), x-y\rangle| \\
&\geq |f_k(x,u_k) - f_k(y,u_k)| - |\langle \nabla_x f_k(y,u_k), x-y\rangle| \\
&\geq |f_k(x,u_k) - f_k(y,u_k)| - \|\nabla_x f_k(y,u_k)\| \cdot \|x-y\|
\end{aligned} \tag{3.17}$$

Besides, by (3.14), one has

$$\|\nabla_x f_k(y,u_k)\| - \|\nabla_x f_k(x,u_k)\| \leq \|\nabla_x f_k(x,u_k) - \nabla_x f_k(y,u_k)\| \leq \bar{\varepsilon},$$

which implies that $\|\nabla_x f_k(y,u_k)\| \leq \|\nabla_x f_k(\bar{x},u_k)\| + \bar{\varepsilon}$. Combining (3.16) and (3.17) gives us

$$\begin{aligned}
|f_k(x,u_k) - f_k(y,u_k)| \quad &\leq \|\nabla_x f_k(y,u_k)\| \cdot \|x-y\| + \bar{\varepsilon}\|x-y\| \\
&\leq (\|\nabla_x f_k(\bar{x},u_k)\| + 2\bar{\varepsilon}) \cdot \|x-y\|.
\end{aligned} \tag{3.18}$$

We conclude that (3.18) holds for every $x, y \in B(\bar{x},\gamma)$ and $u_k \in U_k, k = 1,\ldots,m$ as the case of $x = y$ also holds trivially. By the continuity of functions $u_k \mapsto \|\nabla_x f_k(\bar{x},u_k)\|$ on the compact sets $U_k, k = 1,\ldots,m$, it admits the maximum value over $U_k, k = 1,\ldots,m$, and so we can take $L_k \geq \max\limits_{u_k \in U_k} \left\{ \|\nabla_x f_k(\bar{x},u_k)\| + 2\bar{\varepsilon} \right\}, k = 1,\ldots,m$. Consequently, hypothesis (A1) is satisfied.

In this setting, we can verify that (see, e.g., similar arguments as in [5, Corollary 3.4]), the sets $\left\{ \nabla_x f_k(\bar{x},u_k) \mid u_k \in U_k(\bar{x}) \right\}, k = 1,\ldots,m$ and $\left\{ \nabla_x g_i(\bar{x},v_i) \mid v_i \in V_i(\bar{x}) \right\}, i = 1,\ldots,p$ are convex. Applying Theorem 3.1, we find $(\alpha_1,\ldots,\alpha_m) \in \mathbb{R}^m_+$ and $(\mu_1,\ldots,\mu_p) \in \mathbb{R}^p_+$ with such that $\sum_{k=1}^m \alpha_k + \sum_{i=1}^p \mu_i = 1$ such that

$$0 \in \sum_{k=1}^m \alpha_k \left\{ \nabla_x f_k(\bar{x}, u_k) \mid u_k \in U_k(\bar{x}) \right\} + \sum_{i=1}^p \mu_i \left\{ \nabla_x g_i(\bar{x}, v_i) \mid v_i \in V_i(\bar{x}) \right\}$$
$$+ \sum_{k=1}^m \alpha_k \varepsilon_k \mathbb{B} + \sum_{j=1}^q N(\bar{x}; C_j), \mu_i \max_{v_i \in V_i} g_i(\bar{x}, v_i) = 0, i = 1, \ldots, p,$$

which imply the assertions in (3.12) and (3.13).

Now, let (CQ) be satisfied at $\bar{x}$. We obtain from (3.12) that $\sum_{k=1}^m \alpha_k \neq 0$. Otherwise, we assume on the contrary that $\alpha_k = 0$ for all $k = 1, \ldots, m$. Then there exist $\mu_i \geq 0, i = 1, \ldots, p$, with $\sum_{i=1}^p \mu_i = 1$ such that

$$0 \in \sum_{i=1}^p \mu_i \left\{ \nabla_x g_i(\bar{x}, \bar{v}_i) \mid \bar{v}_i \in V_i(\bar{x}) \right\} + \sum_{j=1}^q N(\bar{x}; C_j),$$

which contradicts the fulfilment of the condition (CQ) at $\bar{x}$. The proof is complete. □

Remark 3.2. In Corollary 3.1, if $\varepsilon_k = 0, k = 1, \ldots, m$ and $f_k, k = 1, \ldots, m$ do not have uncertain data, we obtain [4, Corollary 3.4]. Besides, if $\varepsilon_k = 0, k = 1, \ldots, m$ and $m = 1$, we obtain [9, Corollary 3.1].

Now, we give an example to illustrate the necessary conditions obtained in Theorem 3.1.

Example 3.1. Let $f_k : \mathbb{R}^2 \times U_k \to \mathbb{R}, k = 1, 2$, be defined by

$$f_1(x, u_1) = |x_1| + u_1 x_2^2, x = (x_1, x_2) \in \mathbb{R}^2, u_1 \in U_1 = [0, 1],$$
$$f_2(x, u_2) = x_1^2 + |x_2| + u_2, x = (x_1, x_2) \in \mathbb{R}^2, u_2 \in U_2 = [-1, 0],$$

and let $g_i : \mathbb{R}^2 \times V_i \to \mathbb{R}, i = 1, 2$, be given by

$$g_1(x, v_1) = -x_1^2 - |x_2| + v_1, x = (x_1, x_2) \in \mathbb{R}^2, v_1 \in V_1 = [-1, 0],$$
$$g_2(x, v_2) = -x_1^2 - |x_2| - v_2, x = (x_1, x_2) \in \mathbb{R}^2, v_2 \in V_2 = [0, 1].$$

We consider problem (UP) with $m = 2, p = 2, q = 1$ and $C = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq |x_1|\}$. We have $S = \{x \in C \mid g_i(x, v_i) \leq 0, \forall v_i \in V_i, i = 1, 2\}$. By simple computation, one has $S = C$ and $F_1(x) = |x_1| + x_2^2, F_2(x) = x_1^2 + |x_2|, G_i(x) = -x_1^2 - |x_2|, i = 1, 2$. Now, take $\bar{x} = (0, 0) \in S$ and $\varepsilon_1 = \varepsilon_2 = \frac{1}{2}$. Then, it is easy to show that $\bar{x} \in \varepsilon-$quasi$-\mathscr{S}^w$(RP). Indeed, we have

$$F_1(x) + \varepsilon_1 ||x - \bar{x}|| = |x_1| + x_2^2 + \frac{1}{2}\sqrt{x_1^2 + x_2^2} \geq 0 = F_1(\bar{x}), \forall x \in S,$$
$$F_2(x) + \varepsilon_1 ||x - \bar{x}|| = x_1^2 + |x_2| + \frac{1}{2}\sqrt{x_1^2 + x_2^2} \geq 0 = F_2(\bar{x}), \forall x \in S.$$

Besides, we have $U_1(\bar{x}) = \{1\}, U_2(\bar{x}) = \{0\}, V_i(\bar{x}) = \{0\}, i = 1, 2, N(\bar{x}; C) = \{(x^*, y^*) \in \mathbb{R}^2 \mid |x^*| \leq -y^*\}$ and

$$\left\{ \partial_x f_1(\bar{x}, u_1) \mid u_1 \in U_1(\bar{x}) \right\} = [-1, 1] \times \{0\},$$
$$\left\{ \partial_x f_2(\bar{x}, u_2) \mid u_2 \in U_2(\bar{x}) \right\} = \{0\} \times [-1, 1],$$
$$\left\{ \partial_x g_i(\bar{x}, v_i) \mid v_i \in V_i(\bar{x}) \right\} = \{(0, -1), (0, 1)\}, i = 1, 2.$$

On the other hand, taking $\alpha_1 = \alpha_2 = \dfrac{1}{4}, \mu_1 = \mu_2 = \dfrac{1}{4}, \mathbb{B} = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}$, we see that $\sum_{k=1}^m \alpha_k + \sum_{i=1}^p \mu_i = 1$ and

$$(0,0) \in \sum_{k=1}^m \alpha_k \mathrm{co}\Big\{\partial_x f_k(\bar{x}, u_k) \mid u_k \in U_k(\bar{x})\Big\} + \sum_{i=1}^p \mu_i \mathrm{co}\Big\{\partial_x g_i(\bar{x}, v_i) \mid v_i \in V_i(\bar{x})\Big\}$$
$$+ \sum_{k=1}^m \alpha_k \varepsilon_k \mathbb{B} + \sum_{j=1}^q N(\bar{x}; C_j)$$

and $\mu_i \max_{v_i \in V_i} g_i(\bar{x}, v_i) = 0, i = 1, 2$. Thus, the necessary optimality conditions in Theorem 3.1 hold for problem (UP).

Now, we introduce a concept of the robust ε-quasi approximate (KKT) condition for problem (UP).

Definition 3.2. Let $\varepsilon := (\varepsilon_1, \ldots, \varepsilon_m) \in \mathbb{R}^m_+$. A point $\bar{x} \in S$ is said to satisfy the robust ε-quasi approximate (KKT) condition with respect to problem (UP) if there exist $\alpha_k \geq 0, k = 1, \ldots, m$, and $\mu_i \geq 0, i = 1, \ldots, p$ such that $\alpha_k, k = 1, \ldots, m$ can be chosen not all equal zero and

$$0 \in \sum_{k=1}^m \alpha_k \mathrm{co}\Big\{\partial_x f_k(\bar{x}, u_k) \mid u_k \in U_k(\bar{x})\Big\} + \sum_{i=1}^p \mu_i \mathrm{co}\Big\{\partial_x g_i(\bar{x}, v_i) \mid v_i \in V_i(\bar{x})\Big\}$$
$$+ \sum_{k=1}^m \alpha_k \varepsilon_k \mathbb{B} + \sum_{j=1}^q N(\bar{x}; C_j)$$

and $\mu_i \max_{v_i \in V_i} g_i(\bar{x}, v_i) = 0, i = 1, \ldots, p$.

The following simple example proves that, in general, a feasible point of problem (UP) satifying the robust ε-quasi approximate (KKT) condition is not necessarily to be a robust ε-quasi weakly Pareto solution of problem (UP).

Example 3.2. Let $f_k : \mathbb{R} \times U_k \to \mathbb{R}, k = 1, 2$, be defined by

$$f_1(x, u_1) = -u_1|x| + x^5, x \in \mathbb{R}, u_1 \in U_1 = [1, 2],$$
$$f_2(x, u_2) = -|x| + x^5 - u_2, x \in \mathbb{R}, u_2 \in U_2 = [0, 1],$$

and let $g_i : \mathbb{R} \times V_i \to \mathbb{R}, i = 1, 2$, be given by

$$g_1(x, v_1) = v_1 x^2 - x - 2, x \in \mathbb{R}, v_1 \in V_1 = [0, 1],$$
$$g_2(x, v_2) = x^2 - x - 2 + v_2, x \in \mathbb{R}, v_2 \in V_2 = [-1, 0].$$

We consider problem (UP) with $m = 2, p = 2, q = 1$ and $C = \mathbb{R}$. We have $S = \{x \in C \mid g_i(x, v_i) \leq 0, \forall v_i \in V_i, i = 1, 2\} = [-1, 2]$ and $F_1(x) = -|x| + x^5, F_2(x) = -|x| + x^5, G_1(x) = x^2 - x - 2, G_2(x) = x^2 - x - 2$. Take $\bar{x} = 0 \in S$. By direct computation, one has $N(\bar{x}; C) = \{0\}, U_1(\bar{x}) = \{1\}, U_2(\bar{x}) = \{0\}, V_1(\bar{x}) = \{1\}, V_2(\bar{x}) = \{0\}$,

$$\Big\{\partial_x f_1(\bar{x}, u_1) \mid u_1 \in U_1(\bar{x})\Big\} = \{-1, 1\}, \Big\{\partial_x f_2(\bar{x}, u_2) \mid u_2 \in U_2(\bar{x})\Big\} = \{-1, 1\},$$

and

$$\Big\{\partial_x g_i(\bar{x}, v_i) \mid v_i \in V_i(\bar{x})\Big\} = \{-1\}, i = 1, 2.$$

On the other hand, taking $\alpha_1 = \alpha_2 = \frac{1}{4}$, $\mu_1 = \mu_2 = \frac{1}{4}, \varepsilon_1 = \varepsilon_2 = \frac{1}{2}, \mathbb{B} = [-1,1]$, we see that $\sum_{k=1}^m \alpha_k + \sum_{i=1}^p \mu_i = 1$ and

$$0 \in \sum_{k=1}^m \alpha_k \text{co}\Big\{\partial_x f_k(\bar{x}, u_k) \mid u_k \in U_k(\bar{x})\Big\} + \sum_{i=1}^p \mu_i \text{co}\Big\{\partial_x g_i(\bar{x}, v_i) \mid v_i \in V_i(\bar{x})\Big\}$$
$$+ \sum_{k=1}^m \alpha_k \varepsilon_k \mathbb{B} + \sum_{j=1}^q N(\bar{x}; C_j)$$

and $\mu_i \max_{v_i \in V_i} g_i(\bar{x}, v_i) = 0, i = 1,2$. Thus, the robust ε-quasi approximate (KKT) condition is satisfied at $\bar{x} = 0$. However, $\bar{x} = 0$ is not a robust ε-quasi weakly Pareto solution of problem (UP). To see this, we can choose $x = -1 \in S$, it is easy to see that

$$F_1(x) + \varepsilon_1 ||x - \bar{x}|| = -|x| + x_5 + \varepsilon_1 |x| = -1 - 1 + \frac{1}{2} = -\frac{3}{2} < 0 = F_1(\bar{x}),$$

$$F_2(x) + \varepsilon_2 ||x - \bar{x}|| = -|x| + x_5 + \varepsilon_2 |x| = -1 - 1 + \frac{1}{2} = -\frac{3}{2} < 0 = F_2(\bar{x}).$$

Recall here the notations $F := (F_1, \ldots, F_m)$ and $G := (G_1, \ldots, G_p)$, where, $k = 1, \ldots, m$, $F_k(x) := \max_{u_k \in U_k} f_k(x, u_k) \in \mathbb{R}$ and, $i = 1, \ldots, p$, $G_i(x) := \max_{v_i \in V_i} g_i(x, v_i) \in \mathbb{R}$ for $x \in \mathbb{R}^n$. Motivated by the definition of generalized convexity due to [9], we introduce a concept of generalized convexity as follows.

Definition 3.3. (i) We say that (F, G) is generalized convex at $\bar{x} \in \bigcap_{j=1}^q C_j$ if, for any $x \in \bigcap_{j=1}^q C_j$, there exists $w \in \Big(\sum_{j=1}^q N(\bar{x}; C_j)\Big)^\circ$ such that

$$f_k(x, u_k) - f_k(\bar{x}, u_k) \geq \langle x_k^*, w \rangle, \forall x_k^* \in \partial_x f_k(\bar{x}, u_k), \forall u_k \in U_k(\bar{x}), k = 1, \ldots, m,$$
$$g_i(x, v_i) - g_i(\bar{x}, v_i) \geq \langle x_i^*, w \rangle, \forall x_i^* \in \partial_x g_i(\bar{x}, v_i), \forall v_i \in V_i(\bar{x}), i = 1, \ldots, p,$$

and $\langle b^*, w \rangle \leq ||x - \bar{x}||, \forall b^* \in \mathbb{B}$, where $U_k(\bar{x}), k = 1, \ldots, m, V_i(\bar{x}), i = 1, \ldots, p$ are defined as in (3.1).

(ii) We say that (F, G) is strictly generalized convex at $\bar{x} \in \bigcap_{j=1}^q C_j$, if for any $x \in \bigcap_{j=1}^q C_j \backslash \{\bar{x}\}$, there exists $w \in \Big(\sum_{j=1}^q N(\bar{x}; C_j)\Big)^\circ$ such that

$$f_k(x, u_k) - f_k(\bar{x}, u_k) > \langle x_k^*, w \rangle, \forall x_k^* \in \partial_x f_k(\bar{x}, u_k), \forall u_k \in U_k(\bar{x}), k = 1, \ldots, m,$$
$$g_i(x, v_i) - g_i(\bar{x}, v_i) \geq \langle x_i^*, w \rangle, \forall x_i^* \in \partial_x g_i(\bar{x}, v_i), \forall v_i \in V_i(\bar{x}), i = 1, \ldots, p,$$

and $\langle b^*, w \rangle \leq ||x - \bar{x}||, \forall b^* \in \mathbb{B}$.

Remark 3.3. Note that, if $C_j, j = 1, \ldots, q$ are convex and $f_k, k = 1, \ldots, m, g_i, i = 1, \ldots, p$ are convex (resp., strictly convex), then (F, G) is generalized convex (resp., strictly generalized convex) at any $\bar{x} \in \bigcap_{j=1}^q C_j$ with $w := x - \bar{x}$ for each $x \in \bigcap_{j=1}^q C_j$. In addition, by minor modifications of [9, Example 3.3], we can prove that the class of generalized convex functions can contain some nonconvex functions.

Now, we give a sufficient condition for robust ε-quasi (weakly) Pareto solutions of problem (UP).

Theorem 3.2. *Let $\varepsilon := (\varepsilon_1, \ldots, \varepsilon_m) \in \mathbb{R}_+^m \backslash \{0\}$. Assume that $\bar{x} \in S$ satisfies the robust ε-quasi approximate* (KKT) *condition.*

(i) If (F,G) is generalized convex at $\bar{x}$, then $\bar{x} \in \varepsilon-$quasi$-\mathscr{S}^w$(RP).
(ii) If (F,G) is strictly generalized convex at $\bar{x}$, then $\bar{x} \in \varepsilon-$quasi$-\mathscr{S}$(RP).

Proof. Since $\bar{x} \in S$ satisfies the robust ε-quasi approximate (KKT) condition, we see that there exist $(\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m_+, (\mu_1, \ldots, \mu_p) \in \mathbb{R}^p_+$ with $\sum_{k=1}^m \alpha_k \neq 0$ and

$$
\begin{aligned}
x_k^* &\in \operatorname{co}\left\{ \partial_x f_k(\bar{x}, u_k) \mid u_k \in U_k(\bar{x}) \right\}, k = 1, \ldots, m, \\
x_i^* &\in \operatorname{co}\left\{ \partial_x g_i(\bar{x}, v_i) \mid v_i \in V_i(\bar{x}) \right\}, i = 1, \ldots, p,
\end{aligned} \tag{3.19}
$$

as well as $b^* \in \mathbb{B}$ such that

$$
-\left(\sum_{k=1}^m \alpha_k x_k^* + \sum_{i=1}^p \mu_i x_i^* + \sum_{k=1}^m \alpha_k \varepsilon_k b^* \right) \in \sum_{j=1}^q N\left(\bar{x}; C_j\right) \tag{3.20}
$$

and

$$
\mu_i \max_{v_i \in V_i} g_i(\bar{x}, v_i) = 0, i = 1, \ldots, p. \tag{3.21}
$$

It is clear from (3.19) that there exist $\alpha_{kl} \geq 0, x_{kl}^* \in \partial_x f_k(\bar{x}, u_{kl}), u_{kl} \in U_k(\bar{x}), l = 1, \ldots, l_k, l_k \in \mathbb{N}$ such that $\sum_{l=1}^{l_k} \alpha_{kl} = 1$, $\mu_{ir} \geq 0, x_{ir}^* \in \partial_x g_i(\bar{x}, v_{ir}), v_{ir} \in V_i(\bar{x}), r = 1, \ldots, r_i, r_i \in \mathbb{N}$ such that $\sum_{r=1}^{r_i} \mu_{ir} = 1$ and

$$
x_k^* = \sum_{l=1}^{l_k} \alpha_{kl} x_{kl}^*, k = 1, \ldots, m, \qquad x_i^* = \sum_{r=1}^{r_i} \mu_{ir} x_{ir}^*, i = 1, \ldots, p.
$$

Thus, we deduce from (3.20) that

$$
-\left(\sum_{k=1}^m \alpha_k \sum_{l=1}^{l_k} \alpha_{kl} x_{kl}^* + \sum_{i=1}^p \mu_i \sum_{r=1}^{r_i} \mu_{ir} x_{ir}^* + \sum_{k=1}^m \alpha_k \varepsilon_k b^* \right) \in \sum_{j=1}^q N\left(\bar{x}; C_j\right). \tag{3.22}
$$

We first prove (i). Suppose on contrary that $\bar{x} \notin \varepsilon-$quasi$-\mathscr{S}^w$(RP). It then follows that there exists $\hat{x} \in S$ such that

$$
F(\hat{x}) - F(\bar{x}) + \varepsilon ||\hat{x} - \bar{x}|| \in -\operatorname{int}\mathbb{R}^m_+, \tag{3.23}
$$

where $F_k(x) := \max_{u_k \in U_k} f_k(x, u_k), k = 1, \ldots, m$, for each $x \in \bigcap_{j=1}^q C_j$. By the generalized convexity of (F,G) at $\bar{x}$ and the definition of polar cone, we deduce from (3.22) that, for such $\hat{x}$ above, there exists $w \in \left(\sum_{j=1}^q N\left(\bar{x}; C_j\right) \right)^\circ$ such that

$$
\begin{aligned}
0 &\leq \sum_{k=1}^m \alpha_k \left(\sum_{l=1}^{l_k} \alpha_{kl} \langle x_{kl}^*, w \rangle \right) + \sum_{i=1}^p \mu_i \left(\sum_{r=1}^{r_i} \mu_{ir} \langle x_{ir}^*, w \rangle \right) + \sum_{k=1}^m \alpha_k \varepsilon_k \langle b^*, w \rangle \\
&\leq \sum_{k=1}^m \alpha_k \left(\sum_{l=1}^{l_k} \alpha_{kl} \left(f_k(\hat{x}, u_{kl}) - f_k(\bar{x}, u_{kl}) \right) \right) \\
&+ \sum_{i=1}^p \mu_i \left(\sum_{r=1}^{r_i} \mu_{ir} \left(g_i(\hat{x}, u_{ir}) - g_i(\bar{x}, u_{ir}) \right) \right) + \sum_{k=1}^m \alpha_k \varepsilon_k ||\hat{x} - \bar{x}||,
\end{aligned}
$$

or equivalently,

$$\begin{aligned}&\sum_{k=1}^{m}\alpha_k\left(\sum_{l=1}^{l_k}\alpha_{kl}f_k(\bar{x},u_{kl})\right)+\sum_{i=1}^{p}\mu_i\left(\sum_{r=1}^{r_i}\mu_{ir}g_i(\bar{x},u_{ir})\right)\\&\leq\sum_{k=1}^{m}\alpha_k\left(\sum_{l=1}^{l_k}\alpha_{kl}f_k(\hat{x},u_{kl})\right)+\sum_{i=1}^{p}\mu_i\left(\sum_{r=1}^{r_i}\mu_{ir}g_i(\hat{x},u_{ir})\right)+\sum_{k=1}^{m}\alpha_k\varepsilon_k||\hat{x}-\bar{x}||.\end{aligned}\tag{3.24}$$

It is easy to imply that from the definition of max functions $F_k(x), k=1,\dots,m$ and $G_i(x), i=1,\dots,p$, one has $f_k(\hat{x},u_{kl})\leq F_k(\hat{x}), f_k(\bar{x},u_{kl})=F_k(\bar{x}), l=1,\dots,l_k, l_k\in\mathbb{N}, u_{kl}\in U_k(\bar{x})$ and $g_i(\hat{x},v_{ir})\leq G_i(\hat{x})$, $g_i(\bar{x},v_{ir})=G_i(\bar{x})$, $r=1,\dots,r_i, r_i\in\mathbb{N}, v_{ir}\in V_i(\bar{x})$. Hence, we deduce from (3.24) that

$$\begin{aligned}&\sum_{k=1}^{m}\alpha_k\left(\sum_{l=1}^{l_k}\alpha_{kl}F_k(\bar{x})\right)+\sum_{i=1}^{p}\mu_i\left(\sum_{r=1}^{r_i}\mu_{ir}G_i(\bar{x})\right)\leq\sum_{k=1}^{m}\alpha_k\left(\sum_{l=1}^{l_k}\alpha_{kl}F_k(\hat{x})\right)\\&+\sum_{i=1}^{p}\mu_i\left(\sum_{r=1}^{r_i}\mu_{ir}G_i(\hat{x})\right)+\sum_{k=1}^{m}\alpha_k\varepsilon_k||\hat{x}-\bar{x}||.\end{aligned}$$

or equivalently,

$$\begin{aligned}&\sum_{k=1}^{m}\sum_{l=1}^{l_k}\alpha_k\alpha_{kl}F_k(\bar{x})+\sum_{i=1}^{p}\sum_{r=1}^{r_i}\mu_i\mu_{ir}G_i(\bar{x})\leq\sum_{k=1}^{m}\sum_{l=1}^{l_k}\alpha_k\alpha_{kl}F_k(\hat{x})\\&+\sum_{i=1}^{p}\sum_{r=1}^{r_i}\mu_i\mu_{ir}G_i(\hat{x})+\sum_{k=1}^{m}\alpha_k\varepsilon_k||\hat{x}-\bar{x}||.\end{aligned}\tag{3.25}$$

Note that, for any $\hat{x}\in S$ and by (3.21), $\mu_iG_i(\hat{x})\leq 0, i=1,\dots,p$ and $\mu_iG_i(\bar{x})=0, i=1,\dots,p$. Thus, it follows from (3.25) that

$$\sum_{k=1}^{m}\sum_{l=1}^{l_k}\alpha_k\alpha_{kl}F_k(\bar{x})\leq\sum_{k=1}^{m}\sum_{l=1}^{l_k}\alpha_k\alpha_{kl}F_k(\hat{x})+\sum_{k=1}^{m}\alpha_k\varepsilon_k||\hat{x}-\bar{x}||,$$

or equivalently, $\sum_{k=1}^{m}\alpha_kF_k(\bar{x})\leq\sum_{k=1}^{m}\alpha_kF_k(\hat{x})+\sum_{k=1}^{m}\alpha_k\varepsilon_k||\hat{x}-\bar{x}||$. This, along with the fact that $\alpha_k\geq 0, k=1,\dots,m$ with $\sum_{k=1}^{m}\alpha_k\neq 0$, it follows that there exists $k_0\in\{1,\dots,m\}$ such that $F_{k_0}(\bar{x})\leq F_{k_0}(\hat{x})+\varepsilon_{k_0}||\hat{x}-\bar{x}||$, which contradicts (3.23). The proof of (i) is complete.

We now prove (ii). Suppose on contrary that $\bar{x}\notin\varepsilon$-quasi-$\mathscr{S}$(RP). It then follows that there exists $\tilde{x}\in S$ such that

$$F(\hat{x})-F(\bar{x})+\varepsilon||\hat{x}-\bar{x}||\in-\mathbb{R}^m_+\backslash\{0\},\tag{3.26}$$

where $F_k(x):=\max_{u_k\in U_k}f_k(x,u_k), k=1,\dots,m$, for each $x\in\bigcap_{j=1}^{q}C_j$. By the strictly generalized convexity of (F,G) at $\bar{x}$, we deduce from (3.22) that, for such $\tilde{x}$ above, there exists $w\in\left(\sum_{j=1}^{q}N\left(\bar{x};C_j\right)\right)^{\circ}$ such that

$$\begin{aligned}0&\leq\sum_{k=1}^{m}\alpha_k\left(\sum_{l=1}^{l_k}\alpha_{kl}\left\langle x^*_{kl},w\right\rangle\right)+\sum_{i=1}^{p}\mu_i\left(\sum_{r=1}^{r_i}\mu_{ir}\left\langle x^*_{ir},w\right\rangle\right)+\sum_{k=1}^{m}\alpha_k\varepsilon_k\left\langle b^*,w\right\rangle\\&<\sum_{k=1}^{m}\alpha_k\left(\sum_{l=1}^{l_k}\alpha_{kl}\left(f_k(\hat{x},u_{kl})-f_k(\bar{x},u_{kl})\right)\right)\\&+\sum_{i=1}^{p}\mu_i\left(\sum_{r=1}^{r_i}\mu_{ir}\left(g_i(\hat{x},u_{ir})-g_i(\bar{x},u_{ir})\right)\right)+\sum_{k=1}^{m}\alpha_k\varepsilon_k||\hat{x}-\bar{x}||,\end{aligned}$$

By a similar procedure as in the proof of (i), one has

$$\sum_{k=1}^{m} \alpha_k F_k(\bar{x}) < \sum_{k=1}^{m} \alpha_k F_k(\hat{x}) + \sum_{k=1}^{m} \alpha_k \varepsilon_k ||\hat{x} - \bar{x}||,$$

This, along with the fact that $\alpha_k \geq 0, k = 1, \ldots, m$ with $\sum_{k=1}^{m} \alpha_k \neq 0$, it follows that there exists $k_0 \in \{1, \ldots, m\}$ such that $F_{k_0}(\bar{x}) < F_{k_0}(\hat{x}) + \varepsilon_{k_0} ||\hat{x} - \bar{x}||$, which contradicts (3.26). The proof of (ii) is complete. $\square$

Remark 3.4. In Theorem 3.2, if $q = 1$ and $f_k, k = 1, \ldots, m, g_i, i = 1, \ldots, p$ do not contain uncertain data, we obtain [3, Theorem 3.13]. Besides, if $\varepsilon_k = 0, k = 1, \ldots, m$ and $f_k, k = 1, \ldots, m$ do not contain uncertain data, we have [4, Theorem 3.11]. On the other hand, if $m = 1$ and $\varepsilon_k = 0, k = 1, \ldots, m$, we obtain [9, Theorem 3.2]. Furthermore, if $f_k, k = 1, \ldots, m, g_i, i = 1, \ldots, p$ do not have uncertain data, we get [22, Theorem 3.9].

Now, we present an example to show the importance of the generalized convexity of (F, G) in Theorem 3.2.

Example 3.3. Let $f_k : \mathbb{R} \times U_k \to \mathbb{R}, k = 1, 2$, be defined by

$$f_1(x, u_1) = \begin{cases} x^2 \sin \dfrac{1}{x} - u_1, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases} , x \in \mathbb{R}, u_1 \in U_1 = [0, 1],$$

$$f_2(x, u_2) = \begin{cases} x^2 \sin \dfrac{1}{x} + u_2, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases} , x \in \mathbb{R}, u_2 \in U_2 = [-1, 0],$$

and let $g_i : \mathbb{R} \times V_i \to \mathbb{R}, i = 1, 2$, be given by

$$g_1(x, v_1) = x^2 - x + v_1, x = (x_1, x_2) \in \mathbb{R}^2, v_1 \in V_1 = [-1, 0],$$
$$g_2(x, v_2) = x^2 - x - v_2, x = (x_1, x_2) \in \mathbb{R}^2, v_2 \in V_2 = [0, 1].$$

We consider problem (UP) with $m = 2, p = 2, q = 1$ and $C = \mathbb{R}$. We have $S = \{x \in C \mid g_i(x, v_i) \leq 0, \forall v_i \in V_i, i = 1, 2\} = [0, 1]$ and $F_1(x) = x^2 \sin \dfrac{1}{x}, F_2(x) = x^2 \sin \dfrac{1}{x}, G_1(x) = x^2 - x, G_2(x) = x^2 - x$. Take $\bar{x} = 0 \in S$. By direct computation, one has $N(\bar{x}; C) = \{0\}, U_1(\bar{x}) = \{0\}, U_1(\bar{x}) = \{0\}, V_1(\bar{x}) = \{0\}, V_2(\bar{x}) = \{0\}$ and

$$\left\{ \partial_x f_1(\bar{x}, u_1) \mid u_1 \in U_1(\bar{x}) \right\} = [-1, 1], \left\{ \partial_x f_2(\bar{x}, u_2) \mid u_2 \in U_2(\bar{x}) \right\} = [-1, 1],$$

$$\left\{ \partial_x g_i(\bar{x}, v_i) \mid v_i \in V_i(\bar{x}) \right\} = \{-1\}, i = 1, 2.$$

On the other hand, taking $\alpha_1 = \alpha_2 = \dfrac{1}{4}$, $\mu_1 = \mu_2 = \dfrac{1}{4}, \varepsilon_1 = \varepsilon_2 = \dfrac{1}{3\pi}, \mathbb{B} = [-1, 1]$, we see that $\sum_{k=1}^{m} \alpha_k + \sum_{i=1}^{p} \mu_i = 1$ and

$$0 \in \sum_{k=1}^{m} \alpha_k \mathrm{co} \left\{ \partial_x f_k(\bar{x}, u_k) \mid u_k \in U_k(\bar{x}) \right\} + \sum_{i=1}^{p} \mu_i \mathrm{co} \left\{ \partial_x g_i(\bar{x}, v_i) \mid v_i \in V_i(\bar{x}) \right\}$$
$$+ \sum_{k=1}^{m} \alpha_k \varepsilon_k \mathbb{B} + \sum_{j=1}^{q} N(\bar{x}; C_j)$$

and $\mu_i \max_{v_i \in V_i} g_i(\bar{x}, v_i) = 0, i = 1,2$. Thus, the robust ε-quasi approximate (KKT) condition is satisfied at $\bar{x} = 0$. However, $\bar{x} = 0$ is not a robust ε-quasi weakly Pareto solution of problem (UP). To see this, we can choose $x = \frac{2}{3\pi} \in S$, it is easy to see that

$$F_1(x) + \varepsilon_1 ||x - \bar{x}|| = -\frac{4}{9\pi^2} + \frac{2}{9\pi^2} = -\frac{2}{9\pi^2} < 0 = F_1(\bar{x}),$$
$$F_2(x) + \varepsilon_2 ||x - \bar{x}|| = -\frac{4}{9\pi^2} + \frac{2}{9\pi^2} = -\frac{2}{9\pi^2} < 0 = F_2(\bar{x}),$$

The reason is that (F,G) is not generalized convex at $\bar{x} = 0$. Indeed, take $x = -\frac{2}{5\pi} \in C, x_k^* = 0 \in \left\{ \partial_x f_k(\bar{x}, u_k) \mid u_k \in U_k(\bar{x}) \right\} = [-1,1], k = 1,2$. Then, it is easy to see that $N(\bar{x};C)^\circ = \mathbb{R}$ and

$$f_k(x,u_k) - f_k(\bar{x},u_k) = -\frac{4}{25\pi^2} < 0 = \left\langle x_k^*, w \right\rangle, \forall w \in N(\bar{x};C)^\circ = \mathbb{R},$$
$$\forall u_k \in U_k(\bar{x}), k = 1,2.$$

4. Robust Approximate Duality Theorems

In this section, we consider dual problem of robust problem (UP) in a Mond–Weir type. For convenience's sake, we recall here the notation $F := (F_1, \ldots, F_m)$ and $G := (G_1, \ldots, G_p)$, with

$$F_k(y) := \max_{u_k \in U_k} f_k(y,u_k) \in \mathbb{R}, k = 1, \ldots, m,$$
$$G_i(y) := \max_{v_i \in V_i} g_i(y,v_i) \in \mathbb{R}, i = 1, \ldots, p,$$

and

$$U_k(y) := \{u_k \in U_k \mid f_k(y,u_k) = F_k(y)\}, k = 1, \ldots, m,$$
$$V_i(y) := \{v_i \in V_i \mid g_i(y,v_i) = G_i(y)\}, i = 1, \ldots, p,$$

where $y \in \mathbb{R}^n$. Denote $\alpha := (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m_+ \backslash \{0\}$ with $\sum_{k=1}^m \alpha_k = 1$, $\mu := (\mu_1, \ldots, \mu_p) \in \mathbb{R}^p_+$, $C_j \subset \mathbb{R}^n, j = 1, \ldots, q$ are nonempty closed subsets. We consider Mond–Weir type uncertain dual problem (UMWD) with respect to problem (UP) as follows:

$$\text{(UMWD)} \qquad \max \left\{ f_1(y,u_1), \ldots, f_m(y,u_m) \mid (y, \alpha, \mu) \in S_{\text{MWD}} \right\}.$$

For $\varepsilon := (\varepsilon_1, \ldots, \varepsilon_m) \in \mathbb{R}^m_+$, the feasible set of problem (UMWD) is defined by

$$\begin{aligned} S_{\text{MWD}} := \Big\{ & (y, \alpha, \mu) \in \bigcap_{j=1}^q C_j \times \left(\mathbb{R}^m_+ \backslash \{0\}\right) \times \mathbb{R}^p_+ \mid \\ & 0 \in \sum_{k=1}^m \alpha_k \text{co} \Big\{ \partial_x f_k(y,u_k) \mid u_k \in U_k(y) \Big\} + \sum_{i=1}^p \mu_i \text{co} \Big\{ \partial_x g_i(y,v_i) \mid v_i \in V_i(y) \Big\} \\ & + \sum_{k=1}^m \alpha_k \varepsilon_k \mathbb{B} + \sum_{j=1}^q N(y;C_j), \sum_{i=1}^p \mu_i G_i(y) \geq 0, \sum_{k=1}^m \alpha_k = 1 \Big\}. \end{aligned} \tag{4.1}$$

The robust counterpart of problem (UMWD) can be captured by the following problem:

$$\text{(RMWD)} \qquad \max \left\{ \max_{u_1 \in U_1} f_1(y,u_1), \ldots, \max_{u_m \in U_m} f_m(y,u_m) \mid (y, \alpha, \mu) \in S_{\text{MWD}} \right\},$$

where S_{MWD} is given as in (4.1).

In what follows, we use the following notation for convenience:

$$u \prec v \Leftrightarrow u - v \in -\mathrm{int}\mathbb{R}^m_+, \ \ u \nprec v \text{ is the negation of } u \prec v,$$

$$u \preceq v \Leftrightarrow u - v \in -\mathbb{R}^m_+ \backslash \{0\}, \ \ u \npreceq v \text{ is the negation of } u \preceq v.$$

Now, we introduce definition of ε-quasi (weakly) Pareto solutions for problem (UMWD).

Definition 4.1. Let $\varepsilon := (\varepsilon_1, \ldots, \varepsilon_m) \in \mathbb{R}^m_+$. A vector $(\bar{y}, \bar{\alpha}, \bar{\mu}) \in S_{\text{MWD}}$ is said to be

(i) a robust ε-quasi Pareto solution of problem (UMWD), written by $(\bar{y}, \bar{\alpha}, \bar{\mu}) \in \varepsilon$-quasi-$\mathscr{S}$(RMWD), if $(\bar{y}, \bar{\alpha}, \bar{\mu})$ is an ε-quasi Pareto solution of problem (RMWD), i.e., for any $(y, \alpha, \mu) \in S_{\text{MWD}}$, $F(y) - F(\bar{y}) - \varepsilon ||\bar{y} - y|| \notin \mathbb{R}^m_+ \backslash \{0\}$.

(ii) a robust ε-quasi weakly Pareto solution of problem (UMWD), written by $(\bar{y}, \bar{\alpha}, \bar{\mu}) \in \varepsilon$-quasi-$\mathscr{S}^w$(RMWD), if $(\bar{y}, \bar{\alpha}, \bar{\mu})$ is an ε-quasi weakly Pareto solution of problem (RMWD), i.e., for any $(y, \alpha, \mu) \in S_{\text{MWD}}$, $F(y) - F(\bar{y}) - \varepsilon ||\bar{y} - y|| \notin \mathrm{int}\mathbb{R}^m_+$.

Theorem 4.1. (Robust ε-quasi weak duality) *Let $\varepsilon := (\varepsilon_1, \ldots, \varepsilon_m) \in \mathbb{R}^m_+$. Suppose that $x \in S$ and $(y, \alpha, \mu) \in S_{\text{MWD}}$.*

(i) *If (F, G) is generalized convex at y, then $F(x) \nprec F(y) - \varepsilon ||x - y||$.*

(ii) *If (F, G) is strictly generalized convex at y, then $F(x) \npreceq F(y) - \varepsilon ||x - y||$.*

Proof. Since $(y, \alpha, \mu) \in S_{\text{MWD}}$, we see that there exist $(\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m_+ \backslash \{0\}$ with $\sum_{k=1}^m \alpha_k = 1, \mu := (\mu_1, \ldots, \mu_p) \in \mathbb{R}^p_+$ and

$$x_k^* \in \mathrm{co}\left\{\partial_x f_k(y, u_k) \mid u_k \in U_k(y)\right\}, k = 1, \ldots, m, \tag{4.2}$$

$$x_i^* \in \mathrm{co}\left\{\partial_x g_i(y, v_i) \mid v_i \in V_i(y)\right\}, i = 1, \ldots, p, \tag{4.3}$$

as well as $b^* \in \mathbb{B}$ such that

$$-\left(\sum_{k=1}^m \alpha_k x_k^* + \sum_{i=1}^p \mu_i x_i^* + \sum_{k=1}^m \alpha_k \varepsilon_k b^*\right) \in \sum_{j=1}^q N\left(y; C_j\right) \tag{4.4}$$

and

$$\sum_{i=1}^p \mu_i G_i(y) \geq 0. \tag{4.5}$$

We first prove (i). Let $x \in S$. we suppose on contrary that $F(x) \prec F(y) - \varepsilon ||x - y||$. It follows that

$$F(x) - F(y) + \varepsilon ||x - y|| \in -\mathrm{int}\mathbb{R}^m_+. \tag{4.6}$$

By the generalized convexity of (F, G) at y, for such $x \in \bigcap_{j=1}^q C_j$, there exists $w \in \left(\sum_{j=1}^q N\left(y; C_j\right)\right)^\circ$ such that

$$f_k(x, u_k) - f_k(y, u_k) \geq \langle u^*, w \rangle, \forall u^* \in \partial_x f_k(y, u_k), \forall u_k \in U_k(y), k = 1, \ldots, m, \tag{4.7}$$

$$g_i(x, v_i) - g_i(y, v_i) \geq \langle v^*, w \rangle, \forall v^* \in \partial_x g_i(y, v_i), \forall v_i \in V_i(y), i = 1, \ldots, p \tag{4.8}$$

and

$$\langle b^*, w \rangle \leq ||x - y||, \forall b^* \in \mathbb{B}. \tag{4.9}$$

It is clear from (4.2) that there exist $\alpha_{kl} \geq 0, x^*_{kl} \in \partial_x f_k(y,u_{kl}), u_{kl} \in U_k(y), l = 1,\ldots,l_k, l_k \in \mathbb{N}$ such that $\sum_{l=1}^{l_k} \alpha_{kl} = 1$ and $x^*_k = \sum_{l=1}^{l_k} \alpha_{kl} x^*_{kl}, k = 1,\ldots,m$. Thus, from (4.7), one has

$$\langle x^*_k, w\rangle = \sum_{l=1}^{l_k} \alpha_{kl} \langle x^*_{kl}, w\rangle \leq \sum_{l=1}^{l_k} \alpha_{kl} \left(f_k(x,u_{kl}) - f_k(y,u_{kl})\right).$$

From the definition of max functions $F_k(x), k = 1,\ldots,m$, one has $f_k(x,u_{kl}) \leq F_k(x), f_k(y,u_{kl}) = F_k(y), l = 1,\ldots,l_k, l_k \in \mathbb{N}, u_{kl} \in U_k(\bar{x})$ and

$$\langle x^*_k, w\rangle \leq \sum_{l=1}^{l_k} \alpha_{kl} \left(F_k(x) - F_k(y)\right) = F_k(x) - F_k(y). \tag{4.10}$$

Similarly, from (4.3) and (4.8), we can verify that, for each $i = 1,\ldots,p$, there exist $\mu_{ir} \geq 0, x^*_{ir} \in \partial_x g_i(y,v_{ir}), v_{ir} \in V_i(y), r = 1,\ldots,r_i, r_i \in \mathbb{N}$ such that $\sum_{r=1}^{r_i} \mu_{ir} = 1$ and

$$\langle x^*_i, w\rangle = \sum_{r=1}^{r_i} \mu_{ir} \langle x^*_{ir}, w\rangle \leq \sum_{r=1}^{r_i} \mu_{ir} \left(g_i(x,v_{ir}) - g_i(y,v_{ir})\right).$$

From the definition of max functions $G_i(x), i = 1,\ldots,p$, one has $g_i(x,v_{ir}) \leq G_i(x), g_i(y,v_{ir}) = G_i(y), r = 1,\ldots,r_i, r_i \in \mathbb{N}, v_{ir} \in V_i(\bar{x})$ and

$$\langle x^*_i, w\rangle \leq \sum_{r=1}^{r_i} \mu_{ir} \left(G_i(x) - G_i(y)\right) = G_i(x) - G_i(y). \tag{4.11}$$

Hence, we deduce from (4.10), (4.11), (4.4), the definition of polar cones, and (4.9) that

$$\begin{aligned} 0 &\leq \sum_{k=1}^{m} \alpha_k \langle x^*_k, w\rangle + \sum_{i=1}^{p} \mu_i \langle x^*_i, w\rangle + \sum_{k=1}^{m} \alpha_k \varepsilon_k \langle b^*, w\rangle \\ &\leq \sum_{k=1}^{m} \alpha_k \left[F_k(x) - F_k(y)\right] + \sum_{i=1}^{p} \mu_i \left[G_i(x) - G_i(y)\right] + \sum_{k=1}^{m} \alpha_k \varepsilon_k ||x-y||. \end{aligned}$$

or equivalently,

$$\sum_{k=1}^{m} \alpha_k F_k(y) + \sum_{i=1}^{p} \mu_i G_i(y) \leq \sum_{k=1}^{m} \alpha_k F_k(x) + \sum_{i=1}^{p} \mu_i G_i(x) + \sum_{k=1}^{m} \alpha_k \varepsilon_k ||x-y||. \tag{4.12}$$

By $x \in S$, it is obvious that $G_i(x) \leq 0, i = 1,\ldots,p$. Thus $\sum_{i=1}^{p} \mu_i G_i(x) \leq 0$. From (4.12), we can assert that

$$\sum_{k=1}^{m} \alpha_k F_k(y) + \sum_{i=1}^{p} \mu_i G_i(y) \leq \sum_{k=1}^{m} \alpha_k F_k(x) + \sum_{k=1}^{m} \alpha_k \varepsilon_k ||x-y||. \tag{4.13}$$

From (4.5) and (4.13), it is easy to follow that

$$\sum_{k=1}^{m} \alpha_k F_k(y) \leq \sum_{k=1}^{m} \alpha_k F_k(x) + \sum_{k=1}^{m} \alpha_k \varepsilon_k ||x-y||.$$

This, along with the fact that $\alpha := (\alpha_1,\ldots,\alpha_m) \in \mathbb{R}^m_+ \backslash \{0\}$ with $\sum_{k=1}^{m} \alpha_k = 1$, it follows that there exists $k_0 \in \{1,\ldots,m\}$ such that $F_{k_0}(y) \leq F_{k_0}(x) + \varepsilon_{k_0}||x-y||$. This implies that $F_{k_0}(x) - F_{k_0}(y) + \varepsilon_{k_0}||x-y|| \geq 0$, which contradicts (4.6). The proof of (i) is complete.

We now prove (ii). Let $x \in S$. We suppose on contrary that $F(x) \preceq F(y) - \varepsilon||x-y||$, which is equivalent to the following inequality

$$F(x) - F(y) + \varepsilon||x-y|| \in -\mathbb{R}^m_+ \backslash \{0\}. \tag{4.14}$$

By the strictly generalized convexity of (F,G) at y, for such $x \in \bigcap_{j=1}^{q} C_j$ above, there exists $w \in \left(sum_{j=1}^{q} N\left(y;C_j\right)\right)^{\circ}$ such that

$$f_k(x,u_k) - f_k(y,u_k) > \langle u^*, w\rangle, \forall u^* \in \partial_x f_k(y,u_k), \forall u_k \in U_k(y), k = 1,\dots,m,$$
$$g_i(x,v_i) - g_i(y,v_i) \geq \langle v^*, w\rangle, \forall v^* \in \partial_x g_i(y,v_i), \forall v_i \in V_i(y), i = 1,\dots,p$$

and $\langle b^*, w\rangle \leq ||x-y||, \forall b^* \in \mathbb{B}$. By a similar procedure as in the proof of (i), we arrive at

$$\begin{aligned} 0 &\leq \sum_{k=1}^{m} \alpha_k \langle x_k^*, w\rangle + \sum_{i=1}^{p} \mu_i \langle x_i^*, w\rangle + \sum_{k=1}^{m} \alpha_k \varepsilon_k \langle b^*, w\rangle \\ &< \sum_{k=1}^{m} \alpha_k \left(F_k(x) - F_k(y)\right) + \sum_{i=1}^{p} \mu_i \left(G_i(x) - G_i(y)\right) + \sum_{k=1}^{m} \alpha_k \varepsilon_k ||x-y||, \end{aligned}$$

or equivalently,

$$\sum_{k=1}^{m} \alpha_k F_k(y) + \sum_{i=1}^{p} \mu_i G_i(y) < \sum_{k=1}^{m} F_k(x) + \sum_{i=1}^{p} \mu_i G_i(x) + \sum_{k=1}^{m} \alpha_k \varepsilon_k ||x-y||. \tag{4.15}$$

By $x \in S$ it is obvious that $G_i(x) \leq 0, i = 1,\dots,p$. Thus $\sum_{i=1}^{p} \mu_i G_i(x) \leq 0$. From (4.15), we can assert that

$$\sum_{k=1}^{m} \alpha_k f_k(y) + \sum_{i=1}^{p} \mu_i g_i(y) < \sum_{k=1}^{m} \alpha_k f_k(x) + \sum_{k=1}^{m} \alpha_k \varepsilon_k ||x-y||. \tag{4.16}$$

From (4.5) and (4.16), it is easy to follow that

$$\sum_{k=1}^{m} \alpha_k F_k(y) < \sum_{k=1}^{m} \alpha_k F_k(x) + \sum_{k=1}^{m} \alpha_k \varepsilon_k ||x-y||.$$

This, along with the fact that $\alpha := (\alpha_1,\dots,\alpha_m) \in \mathbb{R}_+^m \backslash \{0\}$ with $\sum_{k=1}^{m} \alpha_k = 1$, it follows that there exists $k_0 \in \{1,\dots,m\}$ such that $F_{k_0}(y) < F_{k_0}(x) + \varepsilon_{k_0} ||x-y||$. This implies that $F_{k_0}(x) - F_{k_0}(y) + \varepsilon_{k_0} ||x-y|| > 0$, which contradicts (4.14). The proof of (ii) is complete. □

Remark 4.1. In Theorem 4.1, if $m = 1$ and $\varepsilon_k = 0, k = 1,\dots,m$, we obtain [9, Theorem 4.1]. Furthermore, if $f_k, k = 1,\dots,m, g_i, i = 1,\dots,p$ do not have uncertain data, we have [22, Theorem 4.6].

Theorem 4.2. (Robust ε-quasi strong duality) *Let $\varepsilon := (\varepsilon_1,\dots,\varepsilon_m) \in \mathbb{R}_+^m, \bar{x} \in \varepsilon-quasi-\mathcal{S}^w$*(RP) *and assumptions* (A1) *and* (A2) *hold. Suppose that the condition* (CQ) *is satisfied at $\bar{x}$ and that the equation $v_1 + \dots + v_q = 0$, where $v_j \in N(\bar{x};C_j), j = 1,\dots,q$, has only the trivial solution $v_j = 0, j = 1,\dots,q$. Then there exists $(\bar{\alpha},\bar{\mu}) \in \left(\mathbb{R}_+^m \backslash \{0\}\right) \times \mathbb{R}_+^p$ such that $(\bar{x},\bar{\alpha},\bar{\mu}) \in S_{\mathrm{MWD}}$.*

(i) *If (F,G) is generalized convex at any $y \in \bigcap_{j=1}^{q} C_j$, then $(\bar{y},\bar{\alpha},\bar{\mu}) \in \varepsilon$-quasi-$\mathcal{S}^w$*(RMWD).

(ii) *If (F,G) is strictly generalized convex at any $y \in \bigcap_{j=1}^{q} C_j$, then $(\bar{y},\bar{\alpha},\bar{\mu}) \in \varepsilon$-quasi-$\mathcal{S}$*(RMWD).

Proof. Let $\bar{x} \in \varepsilon$-quasi-$\mathcal{S}^w$(RP). It follows from Theorem 3.1 that there exist $\alpha := (\alpha_1,\dots,\alpha_m) \in \mathbb{R}_+^m$ with $\sum_{k=1}^{m} \alpha_k \neq 0, \mu := (\mu_1,\dots,\mu_p) \in \mathbb{R}_+^p$ such that

$$\begin{aligned} 0 \in &\sum_{k=1}^{m} \alpha_k \mathrm{co}\left\{ \partial_x f_k(\bar{x},\bar{u}_k) \mid \bar{u}_k \in U_k(\bar{x}) \right\} + \sum_{i=1}^{p} \mu_i \mathrm{co}\left\{ \partial_x g_i(\bar{x},\bar{v}_i) \mid \bar{v}_i \in V_i(\bar{x}) \right\} \\ &+ \sum_{k=1}^{m} \alpha_k \varepsilon_k \mathbb{B} + \sum_{j=1}^{q} N(\bar{x};C_j) \end{aligned} \tag{4.17}$$

and $\mu_i \max_{v_i\in V_i} g_i(\bar{x},\bar{v}_i)=0, i=1,\dots,p$. Putting

$$\bar{\alpha}_k := \frac{\alpha_k}{\sum_{k=1}^{m}\alpha_k}, k=1,\dots,m, \bar{\mu}_i := \frac{\mu_i}{\sum_{k=1}^{m}\alpha_k}, i=1,\dots,p,$$

one has $\bar{\alpha} := (\bar{\alpha}_1,\dots,\bar{\alpha}_m)\in\mathbb{R}^m_+\backslash\{0\}$ with $\sum_{k=1}^m \bar{\alpha}_k = 1$ and $\bar{\mu} := (\bar{\mu}_1,\dots,\bar{\mu}_p)\in\mathbb{R}^p_+$. Furthermore, the assertion in (4.17) is also valid when α_k's and μ_i's are replaced by $\bar{\alpha}_k$'s and $\bar{\mu}_i$'s, respectively. So, $(\bar{x},\bar{\alpha},\bar{\mu})\in S_{\text{MWD}}$.

(i) If (F,G) is generalized convex at any $y\in\bigcap_{j=1}^q C_j$, we find by invoking (i) of Theorem 4.1 that $F(\bar{x})\not\prec F(y)-\varepsilon||\bar{x}-y||$, for any $(y,\alpha,\mu)\in S_{\text{MWD}}$. This implies that $(\bar{y},\bar{\alpha},\bar{\mu})\in\varepsilon-$quasi$-\mathscr{S}^w$(RMWD). The proof of (i) is complete.

(ii) If (F,G) is strictly generalized convex at any $y\in\bigcap_{j=1}^q C_j$, we obtain by invoking (ii) of Theorem 4.1 that $F(\bar{x})\not\preceq F(y)-\varepsilon||\bar{x}-y||$, for any $(y,\alpha,\mu)\in S_{\text{MWD}}$. Therefore, $(\bar{y},\bar{\alpha},\bar{\mu})\in\varepsilon$-quasi-$\mathscr{S}$(RMWD). The proof of (ii) is complete. □

Remark 4.2. In Theorem 4.2, if $m=1$ and $\varepsilon_k=0, k=1,\dots,m$, we have [9, Theorem 4.2]. Furthermore, if $f_k, k=1,\dots,m, g_i, i=1,\dots,p$ do not have uncertain data, we see [22, Theorem 4.7].

Since $C_j, j=1,\dots,q$ are convex and $f_k, k=1,\dots,m, g_i, i=1,\dots,p$ are convex, then (F,G) is generalized convex at any $\bar{x}\in\bigcap_{j=1}^q C_j$ with $w:=x-\bar{x}$ for each $x\in\bigcap_{j=1}^q C_j$. Therefore, the next corollary is a direct consequence of Theorem 4.2.

Corollary 4.1. *Let* $\varepsilon:=(\varepsilon_1,\dots,\varepsilon_m)\in\mathbb{R}^m_+, \bar{x}\in\varepsilon-$*quasi*$-\mathscr{S}^w$(RP) *and assumptions* (A1) *and* (A2) *hold. Suppose that the condition* (CQ) *is satisfied at* $\bar{x}$ *and that the equation* $v_1+\cdots+v_q=0$, *where* $v_j\in N(\bar{x};C_j), j=1,\dots,q$, *has only the trivial solution* $v_j=0, j=1,\dots,q$. *Then there exists* $(\bar{\alpha},\bar{\mu})\in\left(\mathbb{R}^m_+\backslash\{0\}\right)\times\mathbb{R}^p_+$ *such that* $(\bar{x},\bar{\alpha},\bar{\mu})\in S_{\text{MWD}}$. *Let* $C_j, j=1,\dots,q$ *be convex. If* $f_k(\cdot,u_k), k=1,\dots,m$ *and* $g_i(\cdot,v_i), i=1,\dots,p$ *are convex at any* $y\in\bigcap_{j=1}^q C_j$, *then* $(\bar{y},\bar{\alpha},\bar{\mu})\in\varepsilon$-*quasi*-$\mathscr{S}^w$(RMWD).

Theorem 4.3. (Robust ε-quasi converse duality) *Let* $\varepsilon:=(\varepsilon_1,\dots,\varepsilon_m)\in\mathbb{R}^m_+$. *Suppose that* $(\bar{x},\bar{\alpha},\bar{\mu})\in S_{\text{MWD}}$.

(i) If $\bar{x}\in S$ *and* (F,G) *is generalized convex at* $\bar{x}$, *then* $\bar{x}\in\varepsilon$-*quasi*-$\mathscr{S}^w$(RP).

(ii) If $\bar{x}\in S$ *and* (F,G) *is strictly generalized convex at* $\bar{x}$, *then* $\bar{x}\in\varepsilon$-*quasi*-$\mathscr{S}$(RP).

Proof. Since $(\bar{x},\bar{\alpha},\bar{\mu})\in S_{\text{MWD}}$, we see tgat there exist $\bar{\alpha}:=(\bar{\alpha}_1,\dots,\bar{\alpha}_m)\in\mathbb{R}^m_+\backslash\{0\}$ with $\sum_{k=1}^m\bar{\alpha}_k=1, \bar{\mu}:=(\bar{\mu}_1,\dots,\bar{\mu}_p)\in\mathbb{R}^p_+$ and

$$x_k^*\in\operatorname{co}\left\{\partial_x f_k(\bar{x},u_k)\mid u_k\in U_k(\bar{x})\right\}, k=1,\dots,m, \tag{4.18}$$

$$x_i^*\in\operatorname{co}\left\{\partial_x g_i(\bar{x},v_i)\mid v_i\in V_i(\bar{x})\right\}, i=1,\dots,p, \tag{4.19}$$

as well as $b^*\in\mathbb{B}$ such that

$$-\left(\sum_{k=1}^m\bar{\alpha}_k x_k^*+\sum_{i=1}^p\bar{\mu}_i x_i^*+\sum_{k=1}^m\bar{\alpha}_k\varepsilon_k b^*\right)\in\sum_{j=1}^q N\left(\bar{x};C_j\right) \tag{4.20}$$

and

$$\sum_{i=1}^{p} \bar{\mu}_i G_i(\bar{x}) \geq 0. \tag{4.21}$$

We first prove (i). Suppose on contrary that $\bar{x} \notin \varepsilon-$quasi$-\mathscr{S}^w$(RP). It then follows that there exists $\tilde{x} \in S$ satisfying

$$F_k(\tilde{x}) < F_k(\bar{x}) - \varepsilon_k ||\tilde{x} - \bar{x}||, k = 1, \ldots, m. \tag{4.22}$$

By the generalized convexity of (F,G) at $\bar{x}$, for such $\tilde{x} \in \bigcap_{j=1}^{q} C_j$, there exists $w \in \left(\sum_{j=1}^{q} N(\bar{x}; C_j)\right)^{\circ}$ such that

$$f_k(\tilde{x}, u_k) - f_k(\bar{x}, u_k) \geq \langle u^*, w \rangle, \forall u^* \in \partial_x f_k(\bar{x}, u_k), \forall u_k \in U_k(\bar{x}), k = 1, \ldots, m, \tag{4.23}$$

$$g_i(\tilde{x}, v_i) - g_i(\bar{x}, v_i) \geq \langle v^*, w \rangle, \forall v^* \in \partial_x g_i(\bar{x}, v_i), \forall v_i \in V_i(\bar{x}), i = 1, \ldots, p, \tag{4.24}$$

and

$$\langle b^*, w \rangle \leq ||\tilde{x} - \bar{x}||, \forall b^* \in \mathbb{B}. \tag{4.25}$$

It is clear from (4.18) that there exist $\alpha_{kl} \geq 0, x_{kl}^* \in \partial_x f_k(\bar{x}, u_{kl}), u_{kl} \in U_k(\bar{x}), l = 1, \ldots, l_k, l_k \in \mathbb{N}$ such that $\sum_{l=1}^{l_k} \alpha_{kl} = 1$ and $x_k^* = \sum_{l=1}^{l_k} \alpha_{kl} x_{kl}^*, k = 1, \ldots, m$. Thus, from (4.23), one has

$$\langle x_k^*, w \rangle = \sum_{l=1}^{l_k} \alpha_{kl} \langle x_{kl}^*, w \rangle \leq \sum_{l=1}^{l_k} \alpha_{kl} \left(f_k(\tilde{x}, u_{kl}) - f_k(\bar{x}, u_{kl}) \right).$$

From the definition of max functions $F_k(\tilde{x}), k = 1, \ldots, m$, one has $f_k(\tilde{x}, u_{kl}) \leq F_k(\tilde{x}), f_k(\bar{x}, u_{kl}) = F_k(\bar{x}), l = 1, \ldots, l_k, l_k \in \mathbb{N}, u_{kl} \in U_k(\bar{x})$ and

$$\langle x_k^*, w \rangle \leq \sum_{l=1}^{l_k} \alpha_{kl} \left(F_k(\tilde{x}) - F_k(\bar{x}) \right) = F_k(\tilde{x}) - F_k(\bar{x}). \tag{4.26}$$

From (4.19) and (4.24), we can verify that, for each $i = 1, \ldots, p$, there exist $\mu_{ir} \geq 0, x_{ir}^* \in \partial_x g_i(\bar{x}, v_{ir}), v_{ir} \in V_i(\bar{x}), r = 1, \ldots, r_i, r_i \in \mathbb{N}$ such that $\sum_{r=1}^{r_i} \mu_{ir} = 1$ and

$$\langle x_i^*, w \rangle = \sum_{r=1}^{r_i} \mu_{ir} \langle x_{ir}^*, w \rangle \leq \sum_{r=1}^{r_i} \mu_{ir} \left(g_i(\tilde{x}, v_{ir}) - g_i(\bar{x}, v_{ir}) \right).$$

From the definition of max functions $G_i(\tilde{x}), i = 1, \ldots, p$, one has $g_i(\tilde{x}, v_{ir}) \leq G_i(\tilde{x}), g_i(\bar{x}, v_{ir}) = G_i(\bar{x}), r = 1, \ldots, r_i, r_i \in \mathbb{N}, v_{ir} \in V_i(\bar{x})$ and

$$\langle x_i^*, w \rangle \leq \sum_{r=1}^{r_i} \mu_{ir} \left(G_i(\tilde{x}) - G_i(\bar{x}) \right) = G_i(\tilde{x}) - G_i(\bar{x}). \tag{4.27}$$

From the definition of polar cones, (4.20), (4.25), (4.26), and (4.27), it is easy to deduce that

$$\begin{aligned} 0 &\leq \sum_{k=1}^{m} \bar{\alpha}_k \langle u_k^*, w \rangle + \sum_{i=1}^{p} \bar{\mu}_i \langle x_i^*, w \rangle + \sum_{k=1}^{m} \bar{\alpha}_k \varepsilon_k \langle b^*, w \rangle \\ &\leq \sum_{k=1}^{m} \bar{\alpha}_k \left(F_k(\tilde{x}) - F_k(\bar{x}) \right) + \sum_{i=1}^{p} \bar{\mu}_i \left(G_i(\tilde{x}) - G_i(\bar{x}) \right) + \sum_{k=1}^{m} \bar{\alpha}_k \varepsilon_k ||\tilde{x} - \bar{x}||. \end{aligned} \tag{4.28}$$

By $\tilde{x} \in S$, it is obvious that $G_i(\tilde{x}) \leq 0, i = 1,\ldots,p$. Thus $\sum_{i=1}^{p} \bar{\mu}_i G_i(\tilde{x}) \leq 0$. From (4.21) and (4.28), it is easy to see

$$\sum_{k=1}^{m} \bar{\alpha}_k F_k(\tilde{x}) + \sum_{k=1}^{m} \bar{\alpha}_k \varepsilon_k ||\tilde{x} - \bar{x}|| \geq \sum_{k=1}^{m} \bar{\alpha}_k F_k(\bar{x}).$$

This, along with the fact that $\bar{\alpha} := (\bar{\alpha}_1,\ldots,\bar{\alpha}_m) \in \mathbb{R}^m_+ \backslash \{0\}$ with $\sum_{k=1}^{m} \bar{\alpha}_k = 1$, it follows that there exists $k_0 \in \{1,\ldots,m\}$ such that $F_{k_0}(\tilde{x}) + \varepsilon_{k_0} ||\tilde{x} - \bar{x}|| \geq F_{k_0}(\bar{x})$, which contradicts (4.22). The proof of (i) is complete.

We now prove (ii). Suppose on contrary that $\bar{x} \notin \varepsilon$-quasi-$\mathscr{S}$(RP). It then follows that there exists $\hat{x} \in S$ satisfying

$$F_k(\hat{x}) + \varepsilon_k ||\hat{x} - x|| \leq F_k(\bar{x}), k = 1,\ldots,m, \tag{4.29}$$

where at least one inequality is strict. By the strictly generalized convexity of (F,G) at $\bar{x}$, for such $\hat{x} \in \bigcap_{j=1}^{q} C_j$, there exists $w \in \left(\sum_{j=1}^{q} N\left(\bar{x}; C_j\right) \right)^{\circ}$ such that

$$f_k(\hat{x},u_k) - f_k(\bar{x},u_k) > \langle u^*, w \rangle, \forall u^* \in \partial_x f_k(\bar{x},u_k), \forall u_k \in U_k(\bar{x}), k = 1,\ldots,m,$$

$$g_i(\hat{x},v_i) - g_i(\bar{x},v_i) \geq \langle v^*, w \rangle, \forall v^* \in \partial_x g_i(\bar{x},v_i), \forall v_i \in V_i(\bar{x}), i = 1,\ldots,p,$$

and $\langle b^*, w \rangle \leq ||\hat{x} - \bar{x}||, \forall b^* \in \mathbb{B}$. By a similar procedure as in the proof of (i), we arrive at

$$\begin{aligned} 0 &\leq \sum_{k=1}^{m} \bar{\alpha}_k \langle x_k^*, w \rangle + \sum_{i=1}^{p} \bar{\mu}_i \langle x_i^*, w \rangle + \sum_{k=1}^{m} \bar{\alpha}_k \varepsilon_k \langle b^*, w \rangle \\ &< \sum_{k=1}^{m} \bar{\alpha}_k \left[F_k(\hat{x}) - F_k(\bar{x}) \right] + \sum_{i=1}^{p} \bar{\mu}_i \left(G_i(\hat{x}) - G_i(\bar{x}) \right) + \sum_{k=1}^{m} \bar{\alpha}_k \varepsilon_k ||\hat{x} - \bar{x}||. \end{aligned} \tag{4.30}$$

By $\hat{x} \in S$ it is obvious that $G_i(\hat{x}) \leq 0, i = 1,\ldots,p$. Thus, one implies that

$$\sum_{i=1}^{p} \bar{\mu}_i G_i(\hat{x}) \leq 0. \tag{4.31}$$

From (4.21), (4.30) and (4.31), it is easy to follow that

$$\sum_{k=1}^{m} \bar{\alpha}_k F_k(\hat{x}) + \sum_{k=1}^{m} \bar{\alpha}_k \varepsilon_k ||\hat{x} - \bar{x}|| > \sum_{k=1}^{m} \bar{\alpha}_k F_k(\bar{x}).$$

This, along with the fact that $\bar{\alpha} := (\bar{\alpha}_1,\ldots,\bar{\alpha}_m) \in \mathbb{R}^m_+ \backslash \{0\}$ with $\sum_{k=1}^{m} \bar{\alpha}_k = 1$, it follows that there exists $k_0 \in \{1,\ldots,m\}$ such that $F_{k_0}(\hat{x}) + \varepsilon_{k_0} ||\hat{x} - \bar{x}|| > F_{k_0}(\bar{x})$, which contradicts (4.29). The proof of (ii) is complete. □

Remark 4.3. In Theorem 4.3, if $m = 1$ and $\varepsilon_k = 0, k = 1,\ldots,m$, we see n [9, Theorem 4.2]. Furthermore, if $f_k, k = 1,\ldots,m, g_i, i = 1,\ldots,p$ do not have uncertain data, we see [22, Theorem 4.8].

Since $C_j, j = 1,\ldots,q$ are convex and $f_k, k = 1,\ldots,m, g_i, i = 1,\ldots,p$ are convex, then (F,G) is generalized convex at any $\bar{x} \in \bigcap_{j=1}^{q} C_j$ with $w := x - \bar{x}$ for each $x \in \bigcap_{j=1}^{q} C_j$. Therefore, the next corollary is a direct consequence of Theorem 4.3.

Corollary 4.2. *Let $\varepsilon := (\varepsilon_1,\ldots,\varepsilon_m) \in \mathbb{R}^m_+$ and $C_j, j = 1,\ldots,q$ be convex. Let $(\bar{x},\bar{\alpha},\bar{\mu}) \in S_{\mathrm{MWD}}$. If $\bar{x} \in S, f_k(\cdot,u_k), k = 1,\ldots,m$ and $g_i(\cdot,v_i), i = 1,\ldots,p$ are convex at $\bar{x}$, then $\bar{x} \in \varepsilon$-quasi-$\mathscr{S}^w$(RP).*

5. Conclusion

In this paper, we first obtained necessary optimality conditions for ε-quasi Pareto solutions of problem (UP) in terms of the Mordukhovich/limiting subdifferentials. We next introduced the new definition of generalized (strictly) convex via the Mordukhovich/limiting subdifferentials of locally Lipschitz functions. We then established sufficient optimality conditions and duality theorems for ε-quasi Pareto solutions of the considered problem under the assumption of generalized (strictly) convexity. The obtained results in this paper are new and different from the results presented in [3, 4, 9, 22].

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