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GENERALIZED HUKUHARA-PSEUDOCONVEX AND QUASICONVEX INTERVAL-VALUED FUNCTIONS AND THEIR APPLICATIONS IN OPTIMIZATION PROBLEMS WITH gH-CLARKE DERIVATIVE

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Abstract. In this paper, we study the notions of *gH-pseudoconvex* and *quasiconvex* interval-valued functions (IVFs) and their applications in optimization problems with gH-Clarke derivative. It is proved that a convex IVF is gH-pseudoconvex and a gH-pseudoconvex IVF is quasiconvex for gH-Fréchet differentiable IVFs. With the help of the gH-pseudoconvex, quasiconvex, and gH-Lipschitz continuous IVFs, we present some results on characterizing efficient solutions of an optimization problem with upper gH-Clarke and gH-Fréchet differentiable IVFs on a star-shaped constraint set. We also present some illustrative examples to support our main results.
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1. Introduction

Interval optimization problems (IOPs) has been a topic of intense research due to the inherent presence of imprecise and uncertain events in various real-world situations. In this paper, we attempt to define a generalized convexities for interval-valued functions (IVFs) and use them to characterize the efficient solutions to IOPs by using the concepts of gH-Clarke derivative [10]. The topics of this study are pseudoconvexity and quasiconvexity for IVFs, and the applications of gH-Clarke derivative in IOPs. In the following, we present the literature on convexity of IVFs and IOPs; see [10, 15] and the references therein. Moore [24] is the prime mover of interval analysis. However, there are few limitations of Moore's interval arithmetic (see [15] for details), such as, the additive inverse of a degenerate intervals, i.e., an intervals whose upper and lower limits are equal, exist only. For the same reason, many conventional properties for real numbers are not true for compact intervals, for instance, for two compact intervals $\mathbf{A}$ and $\mathbf{C}$, $(\mathbf{A} \ominus_{gH} \mathbf{C}) \oplus \mathbf{C} \neq \mathbf{A}$ (see [15, Remark 2.3.1]. To develop a theoretical framework of the calculus of IVFs and interval analysis, Hukuhara [18] introduced a new rule for the difference of compact intervals, known as Hukuhara difference (H-difference) of intervals. Although H-difference provides additive inverse of compact intervals, this difference of a compact interval $\mathbf{B}$

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from a compact interval **A** can be calculated only when the width of **A** is greater than or equal to that of **B** [8]. To overcome this difficulty, the nonstandard difference of intervals was introduced by Markov [21] and named as generalized Hukuhara difference (gH-difference) of intervals by Stefanini [27]. The gH-difference provides an additive inverse of any compact interval and is applicable for all pairs of compact intervals. To characterize the efficient solutions of an IOPs, convexity plays an essential role. In order to derive the KKT optimality condition of IOP, Wu [28] introduced the concept of convex IVF and it has been studied in several works. Zhang et al. [29] derived the concepts of preinvex and invex IVF and explained several properties related to these concepts. Chalco-Cano et al. [9] introduced the concepts of convexity, preinvexity and prequasi-invexity for fuzzy-valued functions. Zhang et al. [30] defined the pseudoconvex IVF with the help of the lower and upper real-valued function of IVF. The concept of h-convex to present new Jensen and Hermite-Hadamard type inequalities for IVF is developed by Zhao et al. [31]. The concept of p-convex was introduced by Abdeljawad et al. [1] to establish some integral inequalities of the Hermite-Hadamard type and Hermite-Hadamard-Fejér type as well as some product inequalities via the Katugampola fractional integral operator.

In the study of IOPs, in addition to interval arithmetic, appropriate ordering of intervals and the calculus of IVFs play key roles. Unlike the real numbers, intervals are not linearly ordered. Thus, the development of optimization theory with interval-valued function is not a trivial extension of the conventional optimization theory. IOPs have been analyzed with respect to a partial ordering [19]. Some researchers [6, 14] used ordering relations of intervals based on the parametric comparison of intervals. In [12], an ordering relation of intervals was defined by a bijective map from the set of intervals to the Euclidean plane $\mathbb{R}^2$. However, these ordering relations [6, 12, 14] of intervals can be derived from the relations described in [19]. Sengupta et al. [26] proposed an acceptability function for intervals, just like a fuzzy membership function. Recently, Ghosh et al. [16] investigated variable ordering relations for intervals and used them in IOPs. Despite of many attempts to develop a calculus of IVFs, the existing ideas in [5, 11, 15, 17] with help of generalized derivative (gH-directional derivative, gH-Gâteaux derivative, gH-Fréchet derivative, gH-Hadamard derivative, gH-subderivative, gH-Dini Hadamard ε-subderivative), and convexity of IVFs, are not adequate to find solutions to smooth as well as nonsmooth IOPs. Bhurjee and Panda [6] proposed some optimality conditions and duality results for nonsmooth convex IOPs by converting them to a real-valued problem through a parametric representation of IVFs. One needs the explicit expression of the IVF for the parametric representation, which is often practically difficult.

Motivation and contribution: Convexity plays an important role in optimization theory because a local optimum becomes a global one and a necessary optimality condition becomes a sufficient condition. However, practically, not all optimization problems can be formulated as convex problems. Thus, various generalizations of convexity [2, 7] have been developed, which enjoys the local-global property of optima and necessary-sufficient property of optimality conditions. From the literature on the analysis of IVFs, one see that the study of traditional generalized convexity (pseudoconvex and quasiconvex) for IVFs have not been developed so far. However, the basic properties of generalized convexity might be beneficial to solve the equilibrium problems (see [3, 4, 22]) and capture the optimal solutions of IOPs with nonsmooth IVFs.

In this paper, we define gH-pseudoconvex and quasiconvex IVFs. Further, with the help of the proposed concepts of gH-pseudoconvexity and quasiconvexity, we prove that every gH-Fréchet differential convex IVF is gH-pseudoconvex IVF, and a gH-Fréchet differentiable gH-pseudoconvex IVF is quasiconvex. Besides, with the help of upper gH-Clarke derivative [10] and gH-pseudoconvex IVF, we provide a necessary and sufficient conditions for characterizing the efficient solutions of IOPs.

The rest of the article is demonstrated as follows. The next section covers some basic terminologies and notions of convex analysis and interval analysis, followed by the convexity and calculus of IVFs. In addition, some properties of intervals, the gH-directional and gH-Fréchet derivatives of an IVF are discussed in Section 2. The concepts of gH-pseudoconvex and quasiconvex IVFs are illustrated in Section 3 and it is proved that a convex IVF is gH-pseudoconvex and a gH-pseudoconvex IVF is quasiconvex for gH-Fréchet differentiable IVFs. In Section 4, we prove that the upper gH-Clarke derivative at an efficient solution of IOP does not dominate zero. By using gH-pseudoconvexity, we also provide a sufficient condition for an efficient solution for IOP with IVFs. Finally, Section 5 concludes and draws future scopes of this study.

2. Preliminaries and Terminology

This section is devoted to some basic notions on intervals, the convexity and calculus of IVFs. Throughout the paper, we use the following notations.

- $\mathscr{X}$ denotes a real normed linear space with the norm $\|\cdot\|$
- $\mathscr{B}(\bar{x},\delta)$ denotes the open ball centered at $\bar{x}\in\mathscr{X}$ with radius δ
- $\text{cl}(\mathscr{S})$ denotes the closure of the nonempty set $\mathscr{S}$
- $\text{int}(\mathscr{S})$ denotes the set of interior points of the set $\mathscr{S}$
- $\text{cone}(\mathscr{S})$ denotes the cone generated by $\mathscr{S}$
- $\mathscr{T}(\mathscr{S},\bar{x})$ denotes the tangent cone to $\mathscr{S}$ at $\bar{x}$
- $\mathbb{R}$ denotes the set of real numbers
- $\mathbb{R}_+$ denotes the set of nonnegative real numbers.

A nonempty subset $\mathscr{C}$ of $\mathscr{X}$ is called *cone* [25] if $x\in\mathscr{C},\lambda\geq 0 \implies \lambda x\in\mathscr{C}$. Let $\mathscr{S}$ be a nonempty subset of $\mathscr{X}$. $\text{cone}(\mathscr{S})=\{\lambda s:\lambda\geq 0 \text{ and } s\in\mathscr{S}\}$ is called the *cone generated by* $\mathscr{S}$ [25]. A vector $h\in\mathscr{X}$ is called a *tangent vector* [20] to $\mathscr{S}$ at $\bar{x}\in\text{cl}(\mathscr{S})$ if there exist two sequences $\{x_n\}$ in $\mathscr{S}$ and $\{\lambda_n\}$ in positive real numbers with $\bar{x}=\lim_{n\to+\infty}x_n$ and $h=\lim_{n\to+\infty}\lambda_n(x_n-\bar{x})$. The set of all tangent vectors to $\mathscr{S}$ at $\bar{x}$ is called a *tangent cone* [20] to $\mathscr{S}$ at $\bar{x}$ and is denoted by $\mathscr{T}(\mathscr{S},\bar{x})$. Further, $\mathscr{S}$ of $\mathscr{X}$ is said to be *star-shaped* [20] with respect to some $\bar{x}\in\mathscr{S}$ if, for all $x\in\mathscr{S}$, $\lambda x+(1-\lambda)\bar{x}\in\mathscr{S}$ for all $\lambda\in[0,1]$.

Throughout the article, Moore's interval arithmetic [24] followed by the concepts of gH-difference of two intervals and ordering of intervals [19] is used. We denote the set of closed and bounded intervals by $I(\mathbb{R})$ and the elements of $I(\mathbb{R})$ by bold capital letters: $\mathbf{A},\mathbf{B},\mathbf{C},\ldots$. We represent an element $\mathbf{A}$ of $I(\mathbb{R})$ in its interval form with the help of the corresponding small letter in the following way:

$$\mathbf{A}=[\underline{a},\overline{a}],\text{ where } \underline{a} \text{ and } \overline{a} \text{ are real numbers such that } \underline{a}\leq\overline{a}.$$

It is noteworthy that any singleton $\{p\}$ of $\mathbb{R}$ or, a real number can be represented by an interval $\mathbf{P}=[\underline{p},\ \overline{p}]$, where $\underline{p}=p=\overline{p}$. In particular, $\mathbf{0}=\{0\}=[0,0]$ and $\mathbf{1}=\{1\}=[1,1]$.

Consider two intervals $\mathbf{A} = [\underline{a}, \overline{a}]$ and $\mathbf{B} = \left[\underline{b}, \overline{b}\right]$. The *addition* of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \oplus \mathbf{B}$, is defined by $\mathbf{A} \oplus \mathbf{B} = \left[\, \underline{a} + \underline{b},\ \overline{a} + \overline{b} \,\right]$. The *subtraction* of $\mathbf{B}$ from $\mathbf{A}$, denoted by $\mathbf{A} \ominus \mathbf{B}$, is defined by $\mathbf{A} \ominus \mathbf{B} = \left[\, \underline{a} - \overline{b},\ \overline{a} - \underline{b} \,\right]$. The *multiplication* of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \odot \mathbf{B}$, is defined by

$$\mathbf{A} \odot \mathbf{B} = \left[\min\left\{\underline{a}\,\underline{b},\ \underline{a}\overline{b},\ \overline{a}\underline{b},\ \overline{a}\overline{b}\right\},\ \max\left\{\underline{a}\,\underline{b},\ \underline{a}\overline{b},\ \overline{a}\underline{b},\ \overline{a}\overline{b}\right\}\right].$$

The multiplication of $\mathbf{A}$ by a real constant λ, denoted by $\lambda \odot \mathbf{A}$ or $\mathbf{A} \odot \lambda$, is defined by

$$\lambda \odot \mathbf{A} = \begin{cases} [\lambda \underline{a},\ \lambda \overline{a}] & \text{if } \lambda \geq 0 \\ [\lambda \overline{a},\ \lambda \underline{a}] & \text{if } \lambda < 0. \end{cases}$$

Notice that the definition of $\lambda \odot \mathbf{A}$ follows from the fact $\lambda = [\lambda, \lambda]$ and the definition of multiplication $\mathbf{A} \odot \mathbf{B}$. Let $0 \notin \mathbf{B}$. The *division* of $\mathbf{A}$ by $\mathbf{B}$, denoted by $\mathbf{A} \oslash \mathbf{B}$, is defined by

$$\mathbf{A} \oslash \mathbf{B} = \left[\min\left\{\underline{a}/\underline{b},\ \underline{a}/\overline{b},\ \overline{a}/\underline{b},\ \overline{a}/\overline{b}\right\},\ \max\left\{\underline{a}/\underline{b},\ \underline{a}/\overline{b},\ \overline{a}/\underline{b},\ \overline{a}/\overline{b}\right\}\right].$$

Since $\mathbf{A} \ominus \mathbf{A} \neq \mathbf{0}$ for any nondegenerate interval $\mathbf{A}$, we use the following concept of difference of intervals in this article.

Definition 2.1. (*gH-difference of intervals* [27]). Let $\mathbf{A} = [\underline{a}, \overline{a}]$ and $\mathbf{B} = [\underline{b}, \overline{b}]$ be two elements of $I(\mathbb{R})$. The *gH-difference* between $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \ominus_{gH} \mathbf{B}$, be defined by the interval $\mathbf{C}$ such that $\mathbf{A} = \mathbf{B} \oplus \mathbf{C}$ or $\mathbf{B} = \mathbf{A} \ominus \mathbf{C}$. It is to be noted that, for $\mathbf{A} = [\underline{a},\ \overline{a}]$ and $\mathbf{B} = \left[\underline{b},\ \overline{b}\right]$,

$$\mathbf{A} \ominus_{gH} \mathbf{B} = \left[\min\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\},\ \max\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}\right] \text{ and } \mathbf{A} \ominus_{gH} \mathbf{A} = \mathbf{0}.$$

In the following, we provide a dominance relation on intervals that is used throughout the paper. We remark that *domination* in the following definition is based on a *minimization* type optimization problems: the *smaller value* the *better*.

Definition 2.2. (*Dominance of intervals* [28]). Let $\mathbf{A} = [\underline{a}, \overline{a}]$ and $\mathbf{B} = [\underline{b}, \overline{b}]$ be two intervals in $I(\mathbb{R})$.

(i) $\mathbf{B}$ is said to be *dominated by* $\mathbf{A}$ if $\underline{a} \leq \underline{b}$ and $\overline{a} \leq \overline{b}$. We write $\mathbf{A} \preceq \mathbf{B}$;
(ii) $\mathbf{B}$ is said to be *strictly dominated by* $\mathbf{A}$ if either '$\underline{a} \leq \underline{b}$ and $\overline{a} < \overline{b}$' or '$\underline{a} < \underline{b}$ and $\overline{a} \leq \overline{b}$'. We write $\mathbf{A} \prec \mathbf{B}$;
(iii) if $\mathbf{B}$ is not dominated by $\mathbf{A}$, we write $\mathbf{A} \npreceq \mathbf{B}$; if $\mathbf{B}$ is not strictly dominated by $\mathbf{A}$, we write $\mathbf{A} \nprec \mathbf{B}$;
(iv) if either $\mathbf{A}$ is dominated by $\mathbf{B}$ or $\mathbf{B}$ is dominated by $\mathbf{A}$, then $\mathbf{A}$ and $\mathbf{B}$ are *comparable*,
(v) if $\mathbf{A} \npreceq \mathbf{B}$ and $\mathbf{B} \npreceq \mathbf{A}$, then *none of* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *dominates the other*, or $\mathbf{A}$ and $\mathbf{B}$ are *not comparable*.

If $\mathbf{B}$ is strictly dominated by $\mathbf{A}$, then $\mathbf{B}$ is dominated by $\mathbf{A}$. Moreover, if $\mathbf{B}$ is not dominated by $\mathbf{A}$, then $\mathbf{B}$ is not strictly dominated by $\mathbf{A}$.

Definition 2.3. (*Norm on* $I(\mathbb{R})$ [23]). Let $\mathbf{A} = [\underline{a}, \overline{a}]$. A function $\|\cdot\|_{I(\mathbb{R})} : I(\mathbb{R}) \to \mathbb{R}_+$, defined by $\|\mathbf{A}\|_{I(\mathbb{R})} = \|[\underline{a}, \overline{a}]\|_{I(\mathbb{R})} = \max\{|\underline{a}|, |\overline{a}|\}$, is called a *norm* on $I(\mathbb{R})$.

Lemma 2.1. *For all* $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D} \in I(\mathbb{R})$,

(i) $\boldsymbol{B} \nprec \boldsymbol{A} \ominus_{gH} (\boldsymbol{A} \ominus_{gH} \boldsymbol{B})$,
(ii) if $\boldsymbol{B} \nprec \boldsymbol{A} \ominus_{gH} \boldsymbol{C}$ *and* $\boldsymbol{0} \prec \boldsymbol{A}$, *then* $\boldsymbol{B} \nprec (-1) \odot \boldsymbol{C}$,
(iii) $\boldsymbol{A} \nprec \boldsymbol{0}$ *and* $\boldsymbol{A} \preceq \boldsymbol{B} \implies \boldsymbol{B} \nprec \boldsymbol{0}$,
(iv) $\boldsymbol{A} \ominus_{gH} \boldsymbol{B} \nprec \boldsymbol{0}$ *and* $\boldsymbol{C} \preceq \boldsymbol{B} \implies \boldsymbol{A} \ominus_{gH} \boldsymbol{C} \nprec \boldsymbol{0}$,

(v) if $\mathbf{C} \preceq \mathbf{B}$, then $\mathbf{A} \ominus_{gH} \mathbf{B} \preceq \mathbf{A} \ominus_{gH} \mathbf{C}$.

Proof. See Appendix 6. □

In the following, a few properties of convexity and calculus of IVFs are defined. In the rest of the paper, by the interval-valued function, we refer to the following definition.

A function $\mathbf{F}$ from a nonempty subset $\mathscr{S}$ of $\mathscr{X}$ to $I(\mathbb{R})$ is known as an IVF. For each argument point $x \in \mathscr{S}$, $\mathbf{F}$ can be presented by intervals

$$\mathbf{F}(x) = \left[\underline{f}(x),\ \overline{f}(x)\right],$$

where $\underline{f}$ and $\overline{f}$ are real-valued functions on $\mathscr{S}$ such that $\underline{f}(x) \leq \overline{f}(x)$ for all $x \in \mathscr{S}$.

If $\mathscr{S}$ is convex, then the IVF $\mathbf{F}$ is said to be convex [28] on $\mathscr{S}$ if, for any $x_1, x_2 \in \mathscr{S}$,

$$\mathbf{F}(\lambda_1 x_1 + \lambda_2 x_2) \preceq \lambda_1 \odot \mathbf{F}(x_1) \oplus \lambda_2 \odot \mathbf{F}(x_2) \text{ for all } \lambda_1, \lambda_2 \in [0,1] \text{ with } \lambda_1 + \lambda_2 = 1.$$

The IVF $\mathbf{F}$ is said to be *gH-continuous* [13] at an interior point $\bar{x} \in \mathscr{S}$ if

$$\lim_{\|d\| \to 0} (\mathbf{F}(\bar{x}+d) \ominus_{gH} \mathbf{F}(\bar{x})) = \mathbf{0}.$$

If $\mathbf{F}$ is gH-continuous at each x in $\mathscr{S}$, then $\mathbf{F}$ is said to be gH-continuous on $\mathscr{S}$. The IVF $\mathbf{F}$ is said to be *gH-Lipschitz continuous* [15] at $\bar{x} \in \mathscr{S}$ if there exist constants $K' > 0$ and $\delta > 0$ such that

$$\|\mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{F}(y)\|_{I(\mathbb{R})} \leq K' \|\bar{x} - y\| \text{ for all } y \in \mathscr{S} \cap \mathscr{B}(\bar{x}, \delta).$$

The constant K' is called a Lipschitz constant of $\mathbf{F}$ at $\bar{x}$. If there exists a $K > 0$ such that

$$\|\mathbf{F}(x) \ominus_{gH} \mathbf{F}(y)\|_{I(\mathbb{R})} \leq K \|x - y\| \text{ for all } x, y \in \mathscr{S},$$

then the IVF $\mathbf{F}$ is said to be gH-Lipschitz continuous on $\mathscr{S}$ and the constant K is said to be a Lipschitz constant of $\mathbf{F}$ on $\mathscr{S}$.

Lemma 2.2. [28] *$\mathbf{F}$ is a convex IVF on a convex set $\mathscr{S} \subseteq \mathscr{X}$ if and only if $\underline{f}$ and $\overline{f}$ are convex on $\mathscr{S}$.*

Definition 2.4. (*Efficient point* [15]). Let $\mathscr{S}$ be a nonempty subset of $\mathscr{X}$ and $\mathbf{F} : \mathscr{X} \to I(\mathbb{R})$ be an IVF. A point $\bar{x} \in \mathscr{S}$ is said to be an *efficient point* of the IOP $\min_{x \in \mathscr{S}} \mathbf{F}(x)$ if $\mathbf{F}(x) \nprec \mathbf{F}(\bar{x})$ for all $x \in \mathscr{S}$.

Notice that each efficient point $\bar{x}$ is not strictly dominated by any other point x of $\mathscr{S}$. It may, however, be dominated by some other point x' of $\mathscr{S}$ satisfying $\mathbf{F}(\bar{x}) = \mathbf{F}(x')$.

Definition 2.5. (*gH-directional derivative* [15]). Let $\mathbf{F}$ be an IVF on a nonempty subset $\mathscr{S}$ of $\mathscr{X}$. Let $\bar{x} \in \mathscr{S}$ and $h \in \mathscr{X}$. If $\lim_{\lambda \to 0+} \frac{1}{\lambda} \odot \left(\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x})\right)$ exists, then the limit is said to be *gH-directional derivative* of $\mathbf{F}$ at $\bar{x}$ in the direction h, and it is denoted by $\mathbf{F}_{\mathscr{D}}(\bar{x})(h)$.

Definition 2.6. (*gH-Fréchet derivative* [15]). Let $\mathscr{S}$ be a nonempty open subset of $\mathscr{X}$ and $\mathbf{F} : \mathscr{S} \to I(\mathbb{R})$ be an IVF. For an $\bar{x} \in \mathscr{S}$, if there exists a gH-continuous linear IVF $\mathbf{G} : \mathscr{X} \to I(\mathbb{R})$ with

$$\lim_{\|h\| \to 0} \frac{\|\mathbf{F}(\bar{x}+h) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{G}(h)\|_{I(\mathbb{R})}}{\|h\|} = 0,$$

then $\mathbf{G}$ is said to be a *gH-Fréchet derivative* of $\mathbf{F}$ at $\bar{x}$, and we write $\mathbf{G} = \mathbf{F}_{\mathscr{F}}(\bar{x})$.

Definition 2.7. (*Supremum and limit superior of an IVF*[10]). Let $\mathscr{S}$ be a nonempty subset of $\mathscr{X}$ and $\mathbf{F} : \mathscr{S} \to I(\mathbb{R})$ be an IVF. Then, the *supremum* of $\mathbf{F}$ over $\mathscr{S}$ is defined by

$$\sup_{\mathscr{S}} \mathbf{F} = \left[\sup_{\mathscr{S}} \underline{f},\ \sup_{\mathscr{S}} \overline{f}\right],$$

where $\sup_{\mathscr{S}} \underline{f} = \sup\left\{\underline{f}(x) : x \in \mathscr{S}\right\}$ and $\sup_{\mathscr{S}} \overline{f} = \sup\left\{\overline{f}(x) : x \in \mathscr{S}\right\}$.

Definition 2.8. (*Upper gH-Clarke derivative* [10]). Let $\mathbf{F}$ be an IVF on a nonempty subset $\mathscr{S}$ of $\mathscr{X}$. For $\bar{x} \in \mathscr{S}$ and $h \in \mathscr{X}$, if the limit superior

$$\limsup_{\substack{x \to \bar{x} \\ \lambda \to 0+}} \frac{1}{\lambda} \odot \left(\mathbf{F}(x+\lambda h) \ominus_{gH} \mathbf{F}(x)\right) = \lim_{\delta \to 0} \left(\sup_{x \in \overline{\mathscr{B}}(\bar{x},\delta) \cap \mathscr{S},\ \lambda \in (0,\delta)} \frac{1}{\lambda} \odot \left(\mathbf{F}(x+\lambda h) \ominus_{gH} \mathbf{F}(x)\right) \right)$$

exists finitely, then the limit superior value is called *upper gH-Clarke derivative* of $\mathbf{F}$ at $\bar{x}$ in the direction h, and it is denoted by $\mathbf{F}_{\mathscr{C}}(\bar{x})(h)$. If this limit superior exists for all $h \in \mathscr{X}$, then $\mathbf{F}$ is said to be *upper gH-Clarke differentiable* at $\bar{x}$.

3. gH-Pseudoconvex and Quasiconvex Interval-valued Functions

In this section, we study the concepts of gH-pseudoconvex and quasiconvex IVFs. In the case of gH-pseudoconvex IVF, a necessary condition for the existence of an efficient solution becomes a sufficient condition. Further, we show that the class of gH-pseudoconvex IVFs includes the class of all differentiable convex IVFs and is included in the class of all differentiable quasiconvex IVFs.

Definition 3.1. (gH*-pseudoconvex IVF*). Let $\mathscr{S}$ be a nonempty convex subset of $\mathscr{X}$ and $\mathbf{F} : \mathscr{S} \to I(\mathbb{R})$ be an IVF which has gH-directional derivative $\mathbf{F}_{\mathscr{D}}(\bar{x})(y-\bar{x})$ at some $\bar{x} \in \mathscr{S}$ in every direction $y-\bar{x}$, $y \in \mathscr{S}$. Then, $\mathbf{F}$ is called gH*-pseudoconvex* at $\bar{x}$ if, for all $y \in \mathscr{S}$,

$$\mathbf{F}_{\mathscr{D}}(\bar{x})(y-\bar{x}) \nprec \mathbf{0} \implies \mathbf{F}(y) \ominus_{gH} \mathbf{F}(\bar{x}) \nprec \mathbf{0}.$$

Remark 3.1. Definition 3.1 does not assume that the intervals in the range set $\mathbf{F}(\mathscr{S})$ of $\mathbf{F}$ are comparable. This makes the definition nonrestrictive. Furthermore, in the degenerate case, i.e., in the case of $\underline{f} = \overline{f}$ on $\mathscr{S}$, the definition reduces to $\mathbf{F}_{\mathscr{D}}(\bar{x})(y-\bar{x}) \geq 0 \implies \mathbf{F}(y) \geq \mathbf{F}(\bar{x})$, which is the conventional pseudoconvexity. Thus, Definition 3.1 is a true and nonrestrictive generalization of the conventional pseudoconvexity.

Remark 3.2. Notice that one may define gH-pseudoconvexity of an IVF $\mathbf{F} : \mathscr{S} \to I(\mathbb{R})$ at $\bar{x} \in \mathscr{S}$ by

$$\mathbf{F}_{\mathscr{D}}(\bar{x})(y-\bar{x}) \succeq \mathbf{0} \implies \mathbf{F}(y) \ominus_{gH} \mathbf{F}(\bar{x}) \succeq \mathbf{0} \text{ for all } y \in \mathscr{S}. \tag{3.1}$$

$$\text{i.e.,} \quad \mathbf{F}_{\mathscr{D}}(\bar{x})(y-\bar{x}) \succeq \mathbf{0} \implies \mathbf{F}(y) \succeq \mathbf{F}(\bar{x}) \text{ for all } y \in \mathscr{S}.$$

However, this definition is very restrictive as it is applicable only for an IVF $\mathbf{F} : \mathscr{S} \to I(\mathbb{R})$ for which the intervals in the range set $\mathbf{F}(\mathscr{S})$ are comparable. Thus, Definition 3.1 of gH-pseudoconvex IVF is more general than that in the sense of (3.1), and the results that are derived using Definition 3.1 are true for gH-pseudoconvex IVFs in the sense of (3.1).

Theorem 3.1. *Let $\mathscr{S}$ be a nonempty convex subset of $\mathscr{X}$ and $\mathbf{F} : \mathscr{S} \to I(\mathbb{R})$ be a convex IVF on $\mathscr{S}$ which has gH-directional derivative at some $\bar{x} \in \mathscr{S}$ in every direction $y-\bar{x}, y \in \mathscr{S}$. Then, $\mathbf{F}$ is gH-pseudoconvex at $\bar{x}$.*

Proof. Since $\mathbf{F}$ is a convex IVF on $\mathscr{S}$, for every $\bar{x}, y \in \mathscr{S}$ and $\lambda, \lambda' \in [0,1]$ with $\lambda + \lambda' = 1$, we have $\mathbf{F}(\bar{x} + \lambda(y - \bar{x})) = \mathbf{F}(\lambda y + \lambda' \bar{x}) \preceq \lambda \odot \mathbf{F}(y) \oplus \lambda' \odot \mathbf{F}(\bar{x})$. Thus

$$\begin{aligned}
\mathbf{F}(\bar{x} + \lambda(y - \bar{x})) \ominus_{gH} \mathbf{F}(\bar{x}) &\preceq (\lambda \odot \mathbf{F}(y) \oplus \lambda' \odot \mathbf{F}(\bar{x})) \ominus_{gH} \mathbf{F}(\bar{x}) \\
&= \big[\min\{\lambda \underline{f}(y) + \lambda' \underline{f}(\bar{x}) - \underline{f}(\bar{x}), \lambda \overline{f}(y) + \lambda' \overline{f}(\bar{x}) - \overline{f}(\bar{x})\}, \\
&\quad \max\{\lambda \underline{f}(y) + \lambda' \underline{f}(\bar{x}) - \underline{f}(\bar{x}), \lambda \overline{f}(y) + \lambda' \overline{f}(\bar{x}) - \overline{f}(\bar{x})\}\big] \\
&= \lambda \odot (\mathbf{F}(y) \ominus_{gH} \mathbf{F}(\bar{x})),
\end{aligned}$$

which implies

$$\mathbf{F}_{\mathscr{D}}(\bar{x})(y - \bar{x}) \preceq \mathbf{F}(y) \ominus_{gH} \mathbf{F}(\bar{x}) \text{ for all } y \in \mathscr{S}. \tag{3.2}$$

From the inequality (iii) of Lemma 2.1 and (3.2), we obtain

$$\mathbf{F}_{\mathscr{D}}(\bar{x})(y - \bar{x}) \nprec \mathbf{0} \implies \mathbf{F}(y) \ominus_{gH} \mathbf{F}(\bar{x}) \nprec \mathbf{0} \text{ for all } y \in \mathscr{S}.$$

Hence, $\mathbf{F}$ is gH-pseudoconvex at $\bar{x} \in \mathscr{S}$. $\square$

In the following example, we show an IVF which is gH-pseudoconvex but not convex.

Example 3.1. Consider $\mathscr{X}$ as the Euclidean space $\mathbb{R}$, $\mathscr{S} = \mathscr{X}$ and the IVF $\mathbf{F} : \mathscr{S} \to I(\mathbb{R})$, which is defined by

$$\begin{aligned}
\mathbf{F}(x) &= x \odot [1,2] \oplus x^3 \odot [5,8] \\
&= \begin{cases} [x + 5x^3, 2x + 8x^3] & \text{for } x \geq 0, \\ [2x + 8x^3, x + 5x^3] & \text{for } x < 0. \end{cases}
\end{aligned}$$

Therefore,

$$\underline{f}(x) = \begin{cases} x + 5x^3 & \text{for } x \geq 0, \\ 2x + 8x^3 & \text{for } x < 0 \end{cases} \quad \text{and} \quad \overline{f}(x) = \begin{cases} 2x + 8x^3 & \text{for } x \geq 0, \\ x + 5x^3 & \text{for } x < 0. \end{cases}$$

The IVF $\mathbf{F}$ is depicted in Figure 1 by the shaded region. Note that, for any $y \in \mathscr{S}$ and $\bar{x} = 0$,

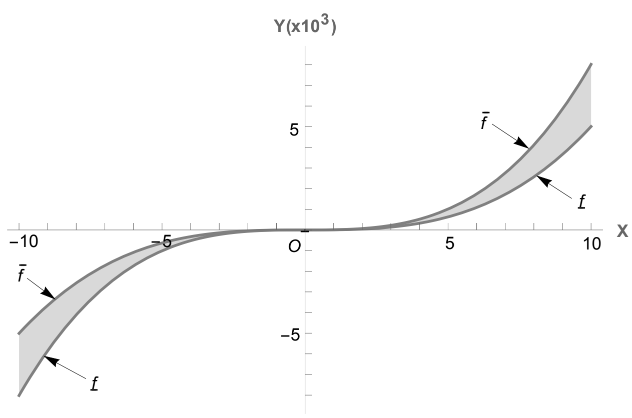

FIGURE 1. The IVF $\mathbf{F}$ of Example 3.1

$$\mathbf{F}_{\mathscr{D}}(\bar{x})(y - \bar{x}) = \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot \big(\lambda y \odot [1,2] \oplus \lambda^3 y^3 \odot [5,8]\big) = y \odot [1,2].$$

Again,

$$\mathbf{F}_{\mathscr{D}}(\bar{x})(y-\bar{x}) \not\prec 0 \implies y \odot [1,2] \not\prec 0 \implies y \geq 0.$$

Further, for $y \geq 0$, $\mathbf{F}(y) \ominus_{gH} \mathbf{F}(0) = y \odot [1,2] \oplus y^3 \odot [5,8] \not\prec \mathbf{0}$. Hence, $\mathbf{F}$ is gH-pseudoconvex at $\bar{x} = 0$. However, the functions $\underline{f}$ and $\overline{f}$ are not convex on $\mathscr{S}$ (see Figure 1). Therefore, the IVF $\mathbf{F}$ is not convex on $\mathscr{S}$.

Definition 3.2. (*Quasiconvex IVF*). Let $\mathscr{S}$ be a nonempty convex subset of $\mathscr{X}$. An IVF $\mathbf{F} : \mathscr{S} \to I(\mathbb{R})$ is said to be *quasiconvex* on $\mathscr{S}$ if, for all $x, y \in \mathscr{S}$,

$$\text{either } \mathbf{F}(x) \not\prec \mathbf{F}(\lambda x + \lambda' y) \text{ or } \mathbf{F}(y) \not\prec \mathbf{F}(\lambda x + \lambda' y), \text{ where } \lambda, \lambda' \in [0,1] \text{ with } \lambda + \lambda' = 1.$$

Remark 3.3. It is a convention to prefix gH in a nomenclature if we use gH-difference in its definition. As there is no role of gH-difference in Definition 3.2, we have not prefixed gH with *quasiconvex* and *convex*.

Remark 3.4. Definition 3.2 does not assume that $\mathbf{F}(x)$ and $\mathbf{F}(y)$ are comparable for all $x, y \in \mathscr{S}$. This makes the definition nonrestrictive. In addition, in the degenerate case, i.e., in the case of $\underline{f} = \overline{f}$ on $\mathscr{S}$, the definition reduces to the conventional quasiconvexity. Thus, Definition 3.2 is a true and nonrestrictive generalization of the conventional quasiconvexity. In the example of Remark 3.5, the IVF $\mathbf{F}$ is quasiconvex but there are a few points in the domain of $\mathbf{F}$ for which $\mathbf{F}(x)$ and $\mathbf{F}(y)$ are not comparable.

Using the definition of quasiconvex IVFs, we obtain the following relation between gH-pseudoconvex and quasiconvex IVFs.

Theorem 3.2. *Let $\mathscr{S}$ be a nonempty convex subset of $\mathscr{X}$ and $\mathbf{F}$ be an IVF defined on an open superset of $\mathscr{S}$. If $\mathbf{F}$ is gH-Fréchet differentiable and gH-pseudoconvex at every $\bar{x} \in \mathscr{S}$, then $\mathbf{F}$ is quasiconvex on $\mathscr{S}$.*

Proof. Suppose that $\mathbf{F}$ is not quasiconvex on $\mathscr{S}$. Then, there exists a $\hat{\lambda} \in [0,1]$ such that, for all $x, y \in \mathscr{S}$,

$$\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\hat{\lambda}x + (1-\hat{\lambda})y) \prec \mathbf{0} \text{ and } \mathbf{F}(y) \ominus_{gH} \mathbf{F}(\hat{\lambda}x + (1-\hat{\lambda})y) \prec \mathbf{0}. \tag{3.3}$$

Since $\mathbf{F}$ is gH-Fréchet differentiable on $\mathscr{S}$, $\mathbf{F}$ is gH-continuous on $\mathscr{S}$. Thus there exists a $\bar{\lambda} \in (0,1)$ with

$$\mathbf{F}(\lambda x + (1-\lambda)y) \preceq \mathbf{F}(\bar{\lambda}x + (1-\bar{\lambda})y) \text{ for all } \lambda \in (0,1). \tag{3.4}$$

As $\mathbf{F}$ is gH-Fréchet differentiable, $\mathbf{F}$ is also gH-directionally differentiable. With the help of [15, Theorem 3.1], we have

$$\mathbf{0} \not\prec \mathbf{F}_{\mathscr{D}}(\bar{x})(x-\bar{x}) \text{ and } \mathbf{0} \not\prec \mathbf{F}_{\mathscr{D}}(\bar{x})(y-\bar{x}), \text{ where } \bar{x} = \bar{\lambda}x + (1-\bar{\lambda})y. \tag{3.5}$$

Since

$$y - \bar{x} = y - \bar{\lambda}x - (1-\bar{\lambda})y = -\bar{\lambda}(x-y) \tag{3.6}$$

and

$$x - \bar{x} = x - \bar{\lambda}x - (1-\bar{\lambda})y = (1-\bar{\lambda})(x-y), \tag{3.7}$$

due to inequality (3.5) and equation (3.7), we obtain

$$\begin{aligned} & \mathbf{0} \nprec \mathbf{F}_{\mathscr{D}}(\bar{x})\,(x-y) \\ \Longrightarrow\quad & \mathbf{0} \nsucc \mathbf{F}_{\mathscr{D}}(\bar{x})\,\left(-\bar{\lambda}(x-y)\right) \\ \Longrightarrow\quad & \mathbf{0} \nsucc \mathbf{F}_{\mathscr{D}}(\bar{x})(y-\bar{x}) \text{ by equation (3.6).} \end{aligned} \tag{3.8}$$

From (3.5) and (3.8), we have

$$\text{either } \mathbf{F}_{\mathscr{D}}(\bar{x})(y-\bar{x}) = \mathbf{0} \text{ or `}\mathbf{F}_{\mathscr{D}}(\bar{x})(y-\bar{x}) \text{ and } \mathbf{0} \text{ are not comparable'.} \tag{3.9}$$

Since $\mathbf{F}$ is gH-pseudoconvex at $\bar{x} \in \mathscr{S}$, we have from (3.9) that

$$\mathbf{F}(y) \ominus_{gH} \mathbf{F}(\bar{x}) \nprec \mathbf{0} \text{ for all } y \in \mathscr{S}. \tag{3.10}$$

However, for any $y \in \mathscr{S}$,

$$\begin{aligned} \mathbf{F}(y) \ominus_{gH} \mathbf{F}(\bar{x}) &= \mathbf{F}(y) \ominus_{gH} \mathbf{F}(\bar{\lambda}x + (1-\bar{\lambda})y) \\ &\preceq \mathbf{F}(y) \ominus_{gH} \mathbf{F}(\hat{\lambda}x + (1-\hat{\lambda})y), \\ &\quad \text{by the inequality (v) of Lemma 2.1 and (3.4)} \\ &\prec \mathbf{0}, \text{ by (3.3).} \end{aligned} \tag{3.11}$$

Hence, (3.10) contradicts (3.11). Therefore, $\mathbf{F}$ is a quasiconvex IVF on $\mathscr{S}$. $\square$

In the following example, we show that there are some gH-Fréchet differentiable IVFs which are quasiconvex but not gH-pseudoconvex.

Example 3.2. Consider $\mathscr{X}$ as the Euclidean space $\mathbb{R}$, $\mathscr{S} = \mathbb{R}_+$ and the IVF $\mathbf{F} : \mathscr{S} \to I(\mathbb{R})$, which is defined by $\mathbf{F}(x) = x^3 \odot [3,\ 9]$. For any $\bar{x}$ in $\mathscr{S}$ and h in $\mathscr{X}$ such that $\bar{x} + \lambda h \in \mathscr{S}$ with $\lambda \geq 0$, we have

$$\begin{aligned} \mathbf{F}_{\mathscr{D}}(\bar{x})(h) &= \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x} + \lambda h) \ominus_{gH} \mathbf{F}(\bar{x})) \\ &= \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot [3(\bar{x}+\lambda h)^3 - 3\bar{x}^3,\ 9(\bar{x}+\lambda h)^3 - 9\bar{x}^3] \\ &= \bar{x}^2 h \odot [9,\ 27]. \end{aligned}$$

Therefore,

$$\begin{aligned} & \lim_{\|h\| \to 0} \frac{1}{\|h\|} \odot \left(\|\mathbf{F}(\bar{x}+h) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{F}_{\mathscr{D}}(\bar{x})(h)\|_{I(\mathbb{R})} \right) \\ = \ & \lim_{\|h\| \to 0} \frac{1}{\|h\|} \odot \left(\|[3(\bar{x}+h)^3 - 3\bar{x}^3,\ 9(\bar{x}+h)^3 - 9\bar{x}^3] \ominus_{gH} \bar{x}^2 h \odot [9,\ 27]\|_{I(\mathbb{R})} \right) \\ = \ & 0. \end{aligned}$$

Hence, $\mathbf{F}$ is gH-Fréchet differentiable at every $\bar{x} \in \mathscr{S}$ and every direction $h \in \mathscr{X}$. Also, for any $x, y \in \mathscr{S}$ and $\lambda, \lambda' \in [0,1]$ with $\lambda + \lambda' = 1$, we have

$$\begin{aligned} & \text{either } \mathbf{F}(x) \succeq \mathbf{F}(\lambda x + \lambda' y) \text{ or } \mathbf{F}(y) \succeq \mathbf{F}(\lambda x + \lambda' y) \\ \Longrightarrow\ & \text{either } \mathbf{F}(x) \nprec \mathbf{F}(\lambda x + \lambda' y) \text{ or } \mathbf{F}(y) \nprec \mathbf{F}(\lambda x + \lambda' y). \end{aligned}$$

Therefore, $\mathbf{F}$ is a quasiconvex IVF on $\mathscr{S}$. Note that

$$\mathbf{F}_{\mathscr{D}}(\bar{x})(y-\bar{x}) \nprec \mathbf{0} \implies \bar{x}^2 y \odot [9,27] \nprec \mathbf{0} \implies y \geq 0.$$

Further, for $\bar{x} = 2$ with $y = 1$,

$$\mathbf{F}(y) \ominus_{gH} \mathbf{F}(\bar{x}) = 1^3 \odot [3,9] \ominus_{gH} 2^3 \odot [3,9] = [-63,-21] \prec \mathbf{0}.$$

Since $\mathbf{F}_{\mathscr{D}}(\bar{x})(y-\bar{x}) \nprec 0 \not\Longrightarrow \mathbf{F}(y) \ominus_{gH} \mathbf{F}(\bar{x}) \nprec \mathbf{0}$ for $\bar{x} = 2$ with $y = 1$, $\mathbf{F}$ is not gH-pseudoconvex at $\bar{x} \in \mathscr{S}$.

Theorem 3.3. *Let $\mathscr{S}$ be a nonempty convex subset of $\mathscr{X}$ and $\mathbf{F} : \mathscr{S} \to I(\mathbb{R})$ be a convex IVF on $\mathscr{S}$ which is gH-Fréchet differentiable on $\mathscr{S}$. Then $\mathbf{F}$ is a quasiconvex IVF on $\mathscr{S}$.*

Proof. Since $\mathbf{F}$ is convex and gH-Freéchet differentiable, by Theorem 3.1, $\mathbf{F}$ is gH-pseudoconvex. Also, by Theorem 3.2, $\mathbf{F}$ is a quasiconvex IVF on $\mathscr{S}$. $\square$

Remark 3.5. The converse of Theorem 3.3 is not true. For example, we consider $\mathscr{X}$ as the Euclidean space $\mathbb{R}$, $\mathscr{S} = [0,2]$, and the IVF $\mathbf{F} : \mathscr{S} \to I(\mathbb{R})$ which is defined by

$$\begin{aligned} \mathbf{F}(x) &= \begin{cases} x \odot [1,2] \oplus [-1,0] & \text{for } 0 \le x \le 1, \\ [1,0] \oplus x \odot [-1,2] & \text{for } 1 \le x \le 2. \end{cases} \\ &= [-|x-1|, 2x]. \end{aligned}$$

The IVF is depicted in Figure 2 by gray shaded region.

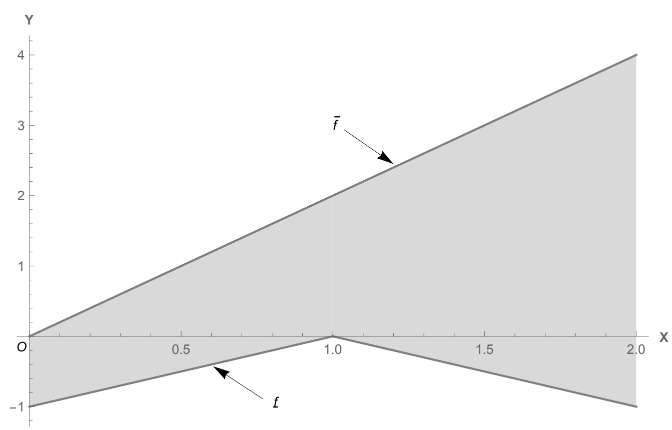

FIGURE 2. IVF $\mathbf{F}$ of Remark 3.5

From Figure 2, we obtain that, for any $\lambda \in (0,1)$,

$$\mathbf{F}(2) \nprec \mathbf{F}(2\lambda + 0(1-\lambda)) = \mathbf{F}(2\lambda).$$

Thus $\mathbf{F}$ is quasiconvex on $\mathscr{S}$. However, $\underline{f}(x) = -|x-1|$ is not convex on $\mathscr{S}$ (see Figure 2). Therefore, the IVF $\mathbf{F}$ is not convex on $\mathscr{S}$.

The results of Theorem 3.1, Theorem 3.2, and Theorem 3.3 can be summarized as follows.

If $\mathbf{F}$ is an IVF on a nonempty convex subset $\mathscr{S}$ of $\mathscr{X}$, which is gH-Fréchet differentiable at every $\bar{x} \in \mathscr{X}$, then

$$\begin{aligned} \mathbf{F} \text{ is convex IVF on } \mathscr{S} &\Longrightarrow \mathbf{F} \text{ is } gH\text{-pseudoconvex IVF at every } \bar{x} \in \mathscr{S} \\ &\Longrightarrow \mathbf{F} \text{ is quasiconvex IVF on } \mathscr{S}. \end{aligned}$$

4. Characterization of Efficient Solutions

In this section, we present several characterizations of efficient solutions for IOPs based on the properties of quasiconvex and gH-pseudoconvex IVFs, contingent cone, upper gH-Clarke and gH-Fréchet derivatives.

Theorem 4.1. *Let $\mathscr{S}$ be a nonempty convex subset of $\mathscr{X}$ and $\boldsymbol{F}:\mathscr{S}\to I(\mathbb{R})$ be an IVF. Consider the IOP:*

$$\min_{x\in\mathscr{S}} \boldsymbol{F}(x). \tag{4.1}$$

(i) If $\boldsymbol{F}$ is continuous and convex on $\mathscr{S}$, then, for every efficient point $\bar{x}\in\mathscr{S}$ of the IOP (4.1),

$$\boldsymbol{F}(\bar{x}+h) \nprec \boldsymbol{F}(\bar{x}) \text{ for all } h\in\mathscr{T}(\mathscr{S},\bar{x}). \tag{4.2}$$

(ii) If $\mathscr{S}$ is star-shaped with respect to some $\bar{x}\in\mathscr{S}$ and inequality (4.2) is true for $\bar{x}$, then $\bar{x}$ is an efficient point of the IOP (4.1).

Proof. (i) For $h=0_{\mathscr{X}}$, the result is trivial. We assume that inequality (4.2) does not hold at an efficient point $\bar{x}\in\mathscr{S}$ of the IOP (4.1). Then, there exists a vector $h\in\mathscr{T}(\mathscr{S},\bar{x})\setminus\{0_{\mathscr{X}}\}$ such that

$$\begin{aligned}
& \mathbf{F}(\bar{x}+h) \prec \mathbf{F}(\bar{x}) \\
\Longrightarrow\quad & \mathbf{F}(\bar{x})\ominus_{gH}\mathbf{F}(\bar{x}+h) \succ \mathbf{0} \\
\Longrightarrow\quad & \max\{\underline{g}(h),\overline{g}(h)\} > 0 \text{ and } \min\{\underline{g}(h),\overline{g}(h)\} \geq 0,
\end{aligned} \tag{4.3}$$

where $\underline{g}(h)=\underline{f}(\bar{x})-\underline{f}(\bar{x}+h)$ and $\overline{g}(h)=\overline{f}(\bar{x})-\overline{f}(x\bar{+}h)$. Since $\mathbf{F}$ is gH-continuous, $\overline{f}$ and $\underline{f}$ are continuous by [10, Lemma 2.4]. Therefore, $\overline{g}$ and $\underline{g}$ are continuous. By Archimedean property, there exists ε such that $\max\{\underline{g}(h),\overline{g}(h)\}>\varepsilon>0$, which together with (4.3) yields

$$\mathbf{F}(\bar{x})\ominus_{gH}\mathbf{F}(\bar{x}+h)\succ\mathbf{A}\succ\mathbf{0},\ \text{where } \mathbf{A}=[\varepsilon,0]. \tag{4.4}$$

As $h\in\mathscr{T}(\mathscr{S},\bar{x})\setminus\{0_{\mathscr{X}}\}$, there exist a sequence $\{x_n\}$ in $\mathscr{S}$ and a sequence $\{\lambda_n\}$ of positive real numbers such that $\bar{x}=\lim_{n\to+\infty}x_n$ and $h=\lim_{n\to+\infty}h_n$, where $h_n=\lambda_n(x_n-\bar{x})$ for all $n\in\mathbb{N}$. As $h\neq 0_{\mathscr{X}}$, we obtain $\lim_{n\to+\infty}\frac{1}{\lambda_n}=0$. Since $\lim_{n\to+\infty}\frac{1}{\lambda_n}=0$, without loss of generality, we assume $0<\frac{1}{\lambda_n}<1$ for all $n\in\mathbb{N}$. Since $\mathbf{F}$ is convex and gH-continuous on $\mathscr{S}$, for sufficiently large $n\in\mathbb{N}$, we have

$$\begin{aligned}
\mathbf{F}(x_n) &= \mathbf{F}\left(\tfrac{1}{\lambda_n}\bar{x}+x_n-\bar{x}+\bar{x}-\tfrac{1}{\lambda_n}\bar{x}\right) \\
&= \mathbf{F}\left(\tfrac{1}{\lambda_n}(\bar{x}+h_n)+\left(1-\tfrac{1}{\lambda_n}\right)\bar{x}\right) \\
&\preceq \tfrac{1}{\lambda_n}\odot\mathbf{F}(\bar{x}+h_n)\oplus\left(1-\tfrac{1}{\lambda_n}\right)\odot\mathbf{F}(\bar{x}) \\
&\preceq \tfrac{1}{\lambda_n}\odot(\mathbf{F}(\bar{x}+h_n)\oplus\mathbf{A})\oplus\left(1-\tfrac{1}{\lambda_n}\right)\odot\mathbf{F}(\bar{x}) \\
&\prec \tfrac{1}{\lambda_n}\odot\mathbf{F}(\bar{x})\oplus\left(1-\tfrac{1}{\lambda_n}\right)\odot\mathbf{F}(\bar{x}),\ \text{by (4.4)} \\
&\prec \mathbf{F}(\bar{x}),
\end{aligned}$$

which is a contradiction to $\bar{x}$, an efficient point of the IOP (4.1). Therefore, relation (4.2) is true.

(ii) Let $\mathscr{S}$ be star-shaped with respect to some $\bar{x} \in \mathscr{S}$. Then, it follows from [20, Theorem 4.8] that $\mathscr{S} \setminus \{\bar{x}\} \subset \mathscr{T}(\mathscr{S}, \bar{x})$. Hence, by (4.2), we have $\mathbf{F}(\bar{x}+h) \nprec \mathbf{F}(\bar{x})$ for all $h \in \mathscr{S} \setminus \{\bar{x}\}$, i.e., $\bar{x}$ is an efficient point of the IOP (4.1).

□

Example 4.1. Consider $\mathscr{S} = [0, 10]$ and the IVF $\mathbf{F} : \mathscr{S} \to I(\mathbb{R})$, which is defined by $\mathbf{F}(x) = |x| \odot [6, 15]$. Hence, $\mathbf{F}$ is gH-continuous convex IVF as $\underline{f}$ and $\overline{f}$ are continuous convex real-valued functions. Also, we can easily check that $\bar{x} = 0$ is an efficient solution of IOP (4.1) and $\mathscr{T}(\mathscr{S}, \bar{x}) = [0, +\infty)$ with $\mathbf{F}(\bar{x}+h) \nprec \mathbf{F}(\bar{x})$ for all $h \in \mathscr{T}(\mathscr{S}, \bar{x})$. Since $\mathscr{S}$ is star-shaped with respect to efficient point $\bar{x}$ and $\mathscr{S} \subset \mathscr{T}(\mathscr{S}, \bar{x})$. Therefore the result (ii) of above theorem is also true.

In the next theorem, using the notion of upper gH-Clarke derivative, we present a necessary condition for efficient solutions.

Theorem 4.2. *Let $\mathscr{W}$ be a subset of $\mathscr{X}$ with nonempty interior. Let $\mathscr{S}$ be a nonempty subset of $\mathscr{W}$, and $\bar{x} \in \mathscr{S} \cap int(\mathscr{W})$ be an efficient point of the following IOP:*

$$\min_{x \in \mathscr{S}} \boldsymbol{F}(x). \tag{4.5}$$

If $\mathscr{S}$ is star-shaped with respect to $\bar{x}$ and the IVF $\boldsymbol{F}$ of the IOP (4.5) is gH-Lipschitz continuous at $\bar{x}$, then $\boldsymbol{F}_{\mathscr{C}}(\bar{x})(x-\bar{x}) \nprec \boldsymbol{0}$ for all $x \in \mathscr{S}$.

Proof. Since $\mathbf{F}$ is gH-Lipschitz continuous at $\bar{x}$, due to [10, Theorem 3.1], $\mathbf{F}$ is upper gH-Clarke differentiable at $\bar{x}$ and

$$\limsup_{\substack{x \to 0 \\ \lambda \to 0+}} \frac{1}{\lambda} \odot \left(\mathbf{F}(\bar{x}+\lambda(x-\bar{x})) \ominus_{gH} \mathbf{F}(\bar{x})\right) = \mathbf{F}_{\mathscr{C}}(\bar{x})(x-\bar{x}). \tag{4.6}$$

As $\bar{x}$ is an efficient point of $\mathbf{F}$ and $\mathscr{S}$ is star-shaped with respect to $\bar{x}$, for any $x \in \mathscr{S}$ and $\lambda > 0$ with $\bar{x}+\lambda(x-\bar{x}) \in \mathscr{S}$, we

$$\begin{aligned}
& \mathbf{F}(\bar{x}+\lambda(x-\bar{x}) \nprec \mathbf{F}(\bar{x}) \\
\Longrightarrow \quad & \mathbf{F}(\bar{x}+\lambda(x-\bar{x}) \ominus_{gH} \mathbf{F}(\bar{x}) \nprec \mathbf{0}, \text{ by Lemma 2.1 of [15]} \\
\Longrightarrow \quad & \frac{1}{\lambda} \odot \left(\mathbf{F}(\bar{x}+\lambda(x-\bar{x})) \ominus_{gH} \mathbf{F}(\bar{x})\right) \nprec \mathbf{0} \\
\Longrightarrow \quad & \limsup_{\substack{x \to 0 \\ \lambda \to 0+}} \frac{1}{\lambda} \odot \left(\mathbf{F}(\bar{x}+\lambda(x-\bar{x})) \ominus_{gH} \mathbf{F}(\bar{x})\right) \nprec \mathbf{0} \\
\Longrightarrow \quad & \mathbf{F}_{\mathscr{C}}(\bar{x})(x-\bar{x}) \nprec \mathbf{0} \text{ for all } x \in \mathscr{S} \text{ from (4.6).}
\end{aligned}$$

□

Remark 4.1. The converse of Theorem 4.2 is not true. For example, we consider $\mathscr{X}$ as the Euclidean space $\mathbb{R}$ and the IOP

$$\min_{x \in \mathscr{S} = (-\infty, 0]} \mathbf{F}(x), \text{ with } \mathbf{F}(x) = x^2 \odot [-5, -3] = \left[-5x^2, -3x^2\right]. \tag{4.7}$$

For any x in $\mathscr{S}$ and an arbitrary $h \in \mathscr{X}$ with $x+\lambda h$, we have

$$\lim_{\lambda \to 0+} \frac{1}{\lambda} \odot \left(\mathbf{F}(x+\lambda h) \ominus_{gH} \mathbf{F}(x)\right)$$

$$\begin{aligned} &= \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot \Big[\min\{5x^2 - 5(x+\lambda h)^2,\ 3x^2 - 3(x+\lambda h)^2\}, \\ &\qquad \max\{5x^2 - 5(x+\lambda h)^2,\ 3x^2 - 3(x+\lambda h)^2\}\Big] \\ &= [\min\{-10xh, -6xh\},\ \max\{-10xh, -6xh\}]. \end{aligned}$$

Then,

$$\lim_{\substack{x \to 0 \\ \lambda \to 0+}} \frac{1}{\lambda} \odot (\mathbf{F}(x+\lambda h) \ominus_{gH} \mathbf{F}(x)) = [\min\{0,0\}, \max\{0,0\}] = \mathbf{0}.$$

Consequently, considering $\bar{x} = 0$, we have $\mathbf{F}_{\mathscr{C}}(\bar{x})(h) = \mathbf{0}$ for all $h \in \mathscr{X}$. Therefore, $\mathbf{F}_{\mathscr{C}}(\bar{x})(h) \nprec \mathbf{0}$ for all $h \in \mathscr{X}$. Let $\mathscr{W} = (-\infty, 12]$. Thus $\bar{x} \in \mathscr{S} \cap \text{int}(\mathscr{W})$. Further, $\mathscr{S}$ is starshaped at $\bar{x}$. As $\underline{f}(x) = -5x^2$ and $\overline{f}(x) = -3x^2$ are Lipschitz continuous at $\bar{x}$, we find by [10, Lemma 2.4] that the IVF $\mathbf{F}$ is Lipschitz continuous at $\bar{x}$. However, $\bar{x}$ is not an efficient point of the IOP (4.7) as

$$\mathbf{F}(x) \prec \mathbf{0} = \mathbf{F}(\bar{x}) \text{ for all } x \in (-\infty, 0).$$

With the help of tangent cones, we derive a necessary optimality condition for IOPs whose objective functions are gH-Fréchet differentiable.

Theorem 4.3. *Let $\mathscr{S}$ be a nonempty subset of $\mathscr{X}$ and $\mathbf{F}$ be an IVF defined on an open superset of $\mathscr{S}$. If $\bar{x}$ is an efficient point of the IOP (4.1) with $\mathbf{F}(\bar{x}) \preceq \mathbf{F}(x)$ for all $x \in \mathscr{S}$ and $\mathbf{F}$ is gH-Fréchet differentiable at $\bar{x}$, then $\mathbf{F}_{\mathscr{F}}(\bar{x})(h) \nprec \mathbf{0}$ for all $h \in \mathscr{T}(\mathscr{S}, \bar{x})$.*

Proof. For $h = 0_{\mathscr{X}}$, we have $\mathbf{F}_{\mathscr{F}}(\bar{x})(h) = \mathbf{0}$, and hence the result is trivial. Let $\bar{x} \in \mathscr{S}$ be an efficient point of $\mathbf{F}$ on $\mathscr{S}$, and let h be an arbitrary element of $\mathscr{T}(\mathscr{S}, \bar{x}) \setminus \{0_{\mathscr{X}}\}$. By the definition of h, there exists a sequence $\{x_n\}$ of elements in $\mathscr{S}$ and a sequence $\{\lambda_n\}$ of positive real numbers such that $\bar{x} = \lim_{n \to +\infty} x_n$ and $h = \lim_{n \to +\infty} h_n$, where $h_n = \lambda_n(x_n - \bar{x})$ for all $n \in \mathbb{N}$. Since $\mathbf{F}$ is gH-Fréchet differentiable at $\bar{x}$,

$$\begin{aligned} \mathbf{F}_{\mathscr{F}}(\bar{x})(h) &= \mathbf{F}_{\mathscr{F}}(\bar{x}) \left(\lim_{n \to +\infty} \lambda_n (x_n - \bar{x}) \right) \\ &= \lim_{n \to +\infty} \lambda_n \odot \mathbf{F}_{\mathscr{F}}(\bar{x})(x_n - \bar{x}) \\ &\nprec \lim_{n \to +\infty} \lambda_n \odot \big[\big(\mathbf{F}(x_n) \ominus_{gH} \mathbf{F}(\bar{x})\big) \ominus_{gH} \big(\mathbf{F}(x_n) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{F}_{\mathscr{F}}(\bar{x})(x_n - \bar{x})\big)\big], \\ &\qquad \text{by inequality (i) of Lemma 2.1} \\ &\nprec (-1) \odot \lim_{n \to +\infty} \lambda_n \odot \big(\mathbf{F}(x_n) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{F}_{\mathscr{F}}(\bar{x})(x_n - \bar{x})\big), \text{ by (ii) of Lemma 2.1} \\ &= (-1) \odot \lim_{n \to +\infty} \left(\|h_n\| \odot \frac{\mathbf{F}(x_n) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{F}_{\mathscr{F}}(\bar{x})(x_n - \bar{x})}{\|x_n - \bar{x}\|} \right) \\ &= \mathbf{0}. \end{aligned}$$

Hence, $\mathbf{F}_{\mathscr{F}}(\bar{x})(h) \nprec \mathbf{0}$ for all $h \in \mathscr{T}(\mathscr{S}, \bar{x})$. □

Remark 4.2. In Theorem 4.3, instead of considering the fact that the IVF $\mathbf{F}$ is defined on an open superset of $\mathscr{S}$, we may consider that

(i) the IVF $\mathbf{F}$ is simply defined on $\mathscr{S}$, or
(ii) $\mathscr{S}$ is open and the IVF $\mathbf{F}$ is defined on $\mathscr{S}$.

However, this assumptions are quite restrictive. For instance, let $\mathscr{S} = [0,1] \cap \mathbb{Q}$, where $\mathbb{Q}$ is the set of rational number, $\mathscr{W} = (-2,+\infty)$ and $\mathscr{X}$ as the Euclidean space $\mathbb{R}$. Then, by [20, Theorem 4.8 and Theorem 4.9], $\mathscr{T}(\mathscr{S},\bar{x}) = [0,+\infty)$, where $\bar{x} = 0$. Let $\mathbf{F}$ be an IVF defined on open superset $\mathscr{W}$ of $\mathscr{S}$ by $\mathbf{F}(x) = x^2 \odot [1,2]$. Here, $\bar{x}$ is an efficient point of the IOP (4.1) with $\mathbf{F}(\bar{x}) \preceq \mathbf{F}(x)$ for all $x \in \mathscr{S}$ and $\mathbf{F}$ is gH-Fréchet differentiable at $\bar{x} = 0$ with $\mathbf{F}_{\mathscr{F}}(\bar{x})(h) = \mathbf{0} \nprec \mathbf{0}$ for all $h \in \mathscr{T}(\mathscr{S},\bar{x})$. Hence, the result of Theorem 4.3 is true. If we restrict $\mathbf{F}$ on $\mathscr{S}$ only, then, for any $h \in \mathscr{T}(\mathscr{S},\bar{x}) \subseteq \mathbb{R}$, $\bar{x}+\lambda_n h \notin \mathscr{S}$ for some sequence $\{\lambda_n\}$ of positive real numbers such that $\lambda_n \to 0$. Therefore, we cannot find gH-Fréchet derivative at $\bar{x} = 0 \in \mathscr{S}$ in any direction $h \in \mathscr{T}(\mathscr{S},\bar{x})$. Hence, the result of Theorem 4.3 is not applicable for this case. Further, $\mathscr{S} = [0,1] \cap \mathbb{Q}$ assures the restrictiveness of assumption (ii).

Remark 4.3. The condition '$\mathbf{F}_{\mathscr{F}}(\bar{x})(h) \nprec \mathbf{0}$ for all $h \in \mathscr{T}(\mathscr{S},\bar{x})$' in Theorem 4.3 is necessary for an efficient point but not sufficient. For instance, consider $\mathscr{S} = (-\infty,0]$ and $\mathscr{X}$ as the Euclidean space $\mathbb{R}$. Then, by [20, Theorem 4.8 and Theorem 4.9], $\mathscr{T}(\mathscr{S},\bar{x}) = (-\infty,0]$, where $\bar{x} = 0$. Let $\mathbf{F}$ be an IVF with IOP:

$$\min_{x\in\mathscr{S}=[0,+\infty)} \mathbf{F}(x) = x^2 \odot [-6,-2]. \tag{4.8}$$

Note that, for $\bar{x}$ in $\mathscr{S}$ and h in $\mathscr{T}(\mathscr{S},\bar{x})$ with $\bar{x}+\lambda h \in \mathscr{S}$, we have

$$\lim_{\lambda\to 0+} \frac{1}{\lambda} \odot (\mathbf{F}(\bar{x}+\lambda h) \ominus_{gH} \mathbf{F}(\bar{x})) = \lim_{\lambda\to 0+} \frac{1}{\lambda} \odot \left((\lambda h)^2 \odot [-6,-2]\right) = \mathbf{0}$$

and

$$\lim_{\|h\|\to 0} \frac{\|\mathbf{F}(\bar{x}+h) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{0}\|_{I(\mathbb{R})}}{\|h\|} = \lim_{\|h\|\to 0} \frac{\|h^2 \odot [-6,-2]\|_{I(\mathbb{R})}}{\|h\|} = 0.$$

Hence, $\mathbf{F}$ is gH-Fréchet differentiable at $\bar{x} \in \mathscr{S}$ with $\mathbf{F}_{\mathscr{F}}(\bar{x})(h) = \mathbf{0} \nprec \mathbf{0}$ for all $h \in \mathscr{T}(\mathscr{S},\bar{x})$. However, $\mathbf{F}(x) \preceq \mathbf{0} = \mathbf{F}(0)$ for all $x \in \mathscr{S}$. Therefore, $\bar{x} = 0$ is not an efficient solution of IOP (4.8).

Next, we look for some conditions under which Theorem 4.3 becomes a sufficient optimality condition. In the next theorem, we show that the result in Theorem 4.3 becomes a sufficient optimality condition for gH-pseudoconvex IVFs.

Theorem 4.4. *Let $\mathscr{S}$ be a nonempty subset of $\mathscr{X}$ and $\mathbf{F}$ be a gH-directionally differentiable IVF defined on an open superset of $\mathscr{S}$. Let $\mathscr{S}$ be star-shaped with respect to some $\bar{x} \in \mathscr{S}$, $\mathbf{F}_{\mathscr{D}}(\bar{x})(h)$ be gH-directional derivative at $\bar{x}$ in the direction of h, and $\mathbf{F}$ be gH-pseudoconvex at $\bar{x}$. Then,*

$$\mathbf{F}_{\mathscr{D}}(\bar{x})(h) \nprec \mathbf{0} \text{ for all } h \in \mathscr{T}(\mathscr{S},\bar{x}) \tag{4.9}$$

if and only if $\bar{x}$ is an efficient point of the IOP (4.1) on $\mathscr{S}$.

Proof. Let the IVF $\mathbf{F}$ satisfy condition (4.9) at some $\bar{x} \in \mathscr{S}$. Since $\mathscr{S}$ is star-shaped with respect to $\bar{x} \in \mathscr{S}$. Then, by [20, Theorem 4.8], we have $\mathscr{S} \setminus \{\bar{x}\} \subset \mathscr{T}(\mathscr{S},\bar{x})$. Thus $\mathbf{F}_{\mathscr{D}}(\bar{x})(x-\bar{x}) \nprec \mathbf{0}$ for all $x \in \mathscr{S}$. Since $\mathbf{F}$ is gH-pseudoconvex at $\bar{x}$,

$$\begin{aligned} & \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \nprec \mathbf{0} \text{ for all } x \in \mathscr{S} \\ \Longrightarrow\quad & \mathbf{F}(x) \nprec \mathbf{F}(\bar{x}) \text{ for all } x \in \mathscr{S}, \text{ by Lemma 2.1 of [15]).} \end{aligned}$$

Hence, $\bar{x}$ is an efficient point of the IOP (4.1). The converse part is followed by [15, Theorem 3.2]. ∎

Example 4.2. Consider $\mathscr{X} = \mathbb{R}$, $\mathscr{S} = [0,4]$, and the gH-pseudoconvex IVF $\mathbf{F} : \mathscr{S} \to I(\mathbb{R})$, which is defined by $\mathbf{F}(x) = [2x^2 - 8x + 8, 2x^2 + 10]$. Hence,

$$\mathscr{T}(\mathscr{S}, x) = \begin{cases} [0\,, +\infty) & \text{if } x = 0,\ 4, \\ \mathbb{R} & \text{if } x \in (0,4). \end{cases}$$

For $x \in \mathscr{S}$ and $h \in \mathscr{T}(\mathscr{S}, x)$, we have

$$\begin{aligned}
&\mathbf{F}_{\mathscr{D}}(x)(h) \\
&= \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\mathbf{F}(x + \lambda h) \ominus_{gH} \mathbf{F}(x)) \\
&= \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot \left(\left[2(x+\lambda h)^2 - 8(x+\lambda h) + 8, 2(x+\lambda h)^2 + 10\right] \ominus_{gH} \left[2x^2 - 8x + 8, 2x^2 + 10\right] \right) \\
&= \begin{cases} [4h(x-2),\ 4hx] & \text{if } h \geq 0, \\ [4hx,\ 4h(x-2)] & \text{otherwise.} \end{cases}
\end{aligned}$$

• *Case* 1. Let $x \in [0,2]$.
In this case, $4hx \geq 0$ for $h \geq 0$ and $4h(x-2) \geq 0$ for $h \leq 0$. Hence, $\mathbf{F}_{\mathscr{D}}(x)(h) \nprec \mathbf{0}$ for all $h \in \mathscr{T}(\mathscr{S}, x)$.

• *Case* 2. Let $x \in (2,4]$.
Then, for $h < 0$ we see that $\mathbf{F}_{\mathscr{D}}(x)(h) \prec \mathbf{0}$. Thus, the relation (4.9) of Theorem 4.4 holds only for $x \in [0,2]$. In Figure 3, the objective function $\mathbf{F}$ is depicted by the shaded region. From Figure 3, we also see that each $x \in [0,2]$ is an efficient point of the IOP (4.1).

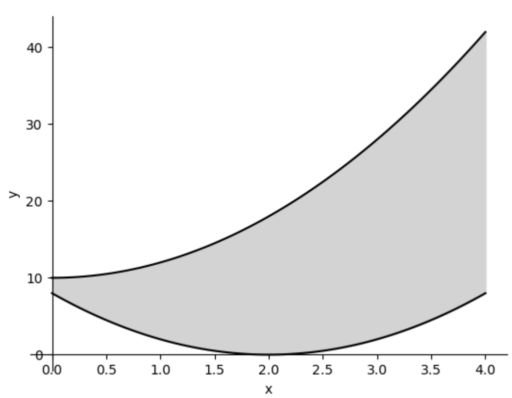

FIGURE 3. The objective function $\mathbf{F}(x)$ of Example 4.2

5. CONCLUSION AND FUTURE DIRECTIONS

In this paper, two concepts on IVFs were studied— gH-pseudoconvex IVF (Definition 3.1) and quasiconvex IVF (Definition 3.2). In the degenerate case, each of Definitions 3.1 and 3.2 reduces to the respective conventional definition for the real-valued functions. Note that, in most of the definitions, we used the $\nsucc$-inequality not $\preceq$-inequality. Use of the $\nsucc$-inequality makes the definitions more general than $\preceq$-inequality, as $\mathbf{A} \nsucc \mathbf{B}$ implies either $\mathbf{A} \preceq \mathbf{B}$ or '$\mathbf{A}$ and $\mathbf{B}$

are not comparable'. Note that a gH-directional differentiable convex IVF is gH-pseudoconvex (Theorem 3.1) and a gH-pseudoconvex gH-Fréchet differentiable IVF is quasiconvex (Theorem 3.2). It has demonstrated that if the objective function of an IOP is continuous and convex on the feasible set $\mathscr{S} \subseteq \mathscr{X}$ and $\bar{x} \in \mathscr{S}$ is an efficient point of that IOP, then the value of the objective function at $\bar{x}$ does not dominate its value on the contingent cone $\mathscr{T}(\mathscr{S}, \bar{x})$, and the converse of this result is true if $\mathscr{S}$ is star-shaped (Theorem 4.1). Further, if the objective function of an IOP is gH-Clarke differentiable at an efficient point, then, at that point, it was demonstrated that zero cannot be dominated by the upper gH-Clarke derivative of the objective function (Theorem 4.2), and if the objective function is gH-Fréchet differentiable at an efficient point, then gH-Fréchet derivative of the objective function on contingent cone cannot dominate zero (Theorem 4.3). Besides, we note that if the objective function of an IOP is gH-pseudoconvex at a feasible point $\bar{x} \in \mathscr{S}$, where $\mathscr{S}$ is star-shaped, and if the gH-directional derivative of the objective function at $\bar{x}$ does not dominate zero for all the directions on the contingent cone to $\mathscr{S}$, then $\bar{x}$ is an efficient point of that IOP (Theorem 4.4).

In this paper, we used quasiconvexity and gH-pseudoconvexity of IVFs to derive the optimality conditions (in the sense of minimum) with the help of upper gH-Clarke derivative. It is clear that one can define concavity and use lower gH-Clarke derivative to derive optimality conditions for maximization type IOPs. Furthermore, one may attempt to investigate constrained IOPs with nonsmooth interval-valued objective functions using upper and lower gH-Clarke derivatives. Also, we shall try to propose a numerical technique with the help of these derivatives to determine the complete set of efficient points of a nonsmooth IOP. Below, we propose a constraint IOP which we want to solve via this derivative.

Suppose a constrained IOP is given by

$$\min \mathbf{F}(x) \text{ subject to } x \in \mathscr{S}, \tag{5.1}$$

where $\mathscr{S} = \{x \in \mathscr{X} : \mathbf{G}_i(x) \preceq \mathbf{0},\ i = 1, 2, \ldots, m \text{ and } \mathbf{G}_i \text{ is an IVF}\}$ and $\mathbf{F} : \mathscr{S} \to I(\mathbb{R})$ is gH-Lipschitz continuous IVF on $\mathscr{S}$. We attempt to replace the above constrained IOP by the following unconstrained IOP

$$\min\ \left(\mathbf{F}(x) \oplus d_{\mathscr{S}}(x) \odot \mathbf{C}\right), \tag{5.2}$$

where $d_{\mathscr{S}}(x) = \min\{\|x - u\| : u \in \mathscr{S}\}$ and $\mathbf{C} \succeq \mathbf{0}$. As $d_{\mathscr{S}}(x) \odot \mathbf{C}$ is not gH-differentiable, we cannot solve the IOP (5.2) by any gradient based technique provided in the literature of IOP. However, due to its gH-Lipschitz continuity on $\mathscr{X}$ (See Appendix 7), the objective function of (5.2) is upper gH-Clarke differentiable for all $x \in \mathscr{X}$, and with the help of upper gH-Clarke derivative, we can characterize the efficient solutions of the constrained IOP (5.1).

6. Proof of Lemma 2.1

Proof. Let $\mathbf{A} = [\underline{a}, \overline{a}]$, $\mathbf{B} = [\underline{b}, \overline{b}]$ and $\mathbf{C} = [\underline{c}, \overline{c}]$.

(i) Suppose that inequality $\mathbf{B} \not\prec \mathbf{A} \ominus_{gH} (\mathbf{A} \ominus_{gH} \mathbf{B})$ is not true. Then,

$$\mathbf{B} \prec \mathbf{A} \ominus_{gH} (\mathbf{A} \ominus_{gH} \mathbf{B}). \tag{6.1}$$

Now, we have the following two cases.

- **Case** 1. If $\underline{a} - \underline{b} \leq \overline{a} - \overline{b}$, then $\mathbf{A} \ominus_{gH} \mathbf{B} = [\underline{a} - \underline{b}, \overline{a} - \overline{b}]$ and $\mathbf{A} \ominus_{gH} (\mathbf{A} \ominus_{gH} \mathbf{B}) = [\underline{b}, \overline{b}] = \mathbf{B}$, which contradicts (6.1).

- **Case** 2. If $\overline{a}-\overline{b}<\underline{a}-\underline{b}$, then $\mathbf{A}\ominus_{gH}\mathbf{B}=[\overline{a}-\overline{b},\underline{a}-\underline{b}]$;
  If $\mathbf{A}\ominus_{gH}(\mathbf{A}\ominus_{gH}\mathbf{B})=[\underline{a}-(\overline{a}-\overline{b}),\overline{a}-(\underline{a}-\underline{b})]$, by (6.1), we have
  $$\overline{b}\leq\overline{a}-(\underline{a}-\underline{b}) \implies \underline{a}-\underline{b}\leq\overline{a}-\overline{b},$$
  which contradicts $\overline{a}-\overline{b}<\underline{a}-\underline{b}$.
  If $\mathbf{A}\ominus_{gH}(\mathbf{A}\ominus_{gH}\mathbf{B})=[\overline{a}-(\underline{a}-\underline{b}),\underline{a}-(\overline{a}-\overline{b})]$, by (6.1), we obtain
  $$\overline{b}\leq\underline{a}-(\overline{a}-\overline{b}) \implies \overline{a}\leq\underline{a} \implies \overline{a}=\underline{a},$$
  and $\mathbf{A}\ominus_{gH}(\mathbf{A}\ominus_{gH}\mathbf{B})=\mathbf{B}$, which contradicts (6.1).

Hence, from **Case** 1 and **Case** 2, we obtain $\mathbf{B}\not\prec\mathbf{A}\ominus_{gH}(\mathbf{A}\ominus_{gH}\mathbf{B})$.

(ii) Since $\mathbf{0}\prec\mathbf{A} \implies 0\leq\underline{a}$ and $0<\overline{a}$, for any $\mathbf{C}\in I(\mathbb{R})$, we have
$$-\overline{c}\leq-\underline{c}\leq\underline{a}-\underline{c} \text{ and } -\overline{c}<\overline{a}-\overline{c}.$$
Therefore, we obtain
$$\begin{aligned}&[-\overline{c},-\underline{c}]\prec[\min\{\underline{a}-\underline{c},\overline{a}-\overline{c}\},\max\{\underline{a}-\underline{c},\overline{a}-\overline{c}\}]\\ \implies&(-1)\odot\mathbf{C}\prec\mathbf{A}\ominus_{gH}\mathbf{C}.\end{aligned}$$
Hence, for any $\mathbf{B}\in I(\mathbb{R})$, we have $\mathbf{B}\not\prec\mathbf{A}\ominus_{gH}\mathbf{C} \implies \mathbf{B}\not\prec(-1)\odot\mathbf{C}$.

(iii) If $\mathbf{A}\not\prec\mathbf{0}$, then $\overline{a}\geq 0$. Since $\mathbf{A}\preceq\mathbf{B}$, $\overline{b}\geq 0$, $\mathbf{B}\not\prec\mathbf{0}$.

(iv) According to the dominance of intervals, we have
$$\begin{aligned}&\mathbf{A}\ominus_{gH}\mathbf{B}\not\prec\mathbf{0}\\ \implies&\max\{\underline{a}-\underline{b},\overline{a}-\overline{b}\}\geq 0\\ \implies&\underline{a}-\underline{b}\geq 0 \text{ or}, \overline{a}-\overline{b}\geq 0.\end{aligned}\tag{6.2}$$
Since $\mathbf{C}\preceq\mathbf{B}$,
$$\underline{c}\leq\underline{b} \text{ and } \overline{c}\leq\overline{b} \implies \underline{b}-\underline{c}\geq 0 \text{ and } \overline{b}-\overline{c}\geq 0.\tag{6.3}$$
From (6.2) and (6.3), we have either $\underline{a}-\underline{c}\geq 0$ or $\overline{a}-\overline{c}\geq 0$. Therefore, $\mathbf{A}\ominus_{gH}\mathbf{C}\not\prec\mathbf{0}$.

(v) Since $\mathbf{C}\preceq\mathbf{B}$, we have
$$\begin{aligned}&\underline{c}\leq\underline{b} \text{ and } \overline{c}\leq\overline{b}\\ \implies&\underline{a}-\underline{c}\geq\underline{a}-\underline{b} \text{ and } \overline{a}-\overline{c}\geq\overline{a}-\overline{b}\\ \implies&[\min\{\underline{a}-\underline{c},\overline{a}-\overline{c}\},\max\{\underline{a}-\underline{c},\overline{a}-\overline{c}\}]\\ &\succeq[\min\{\underline{a}-\underline{b},\overline{a}-\overline{b}\},\max\{\underline{a}-\underline{b},\overline{a}-\overline{b}\}]\\ \implies&\mathbf{A}\ominus_{gH}\mathbf{B}\preceq\mathbf{A}\ominus_{gH}\mathbf{C}.\end{aligned}$$

□

7. LIPSCHITZ

Proof. Let $\mathbf{G}(x)=d_{\mathscr{S}}(x)\odot\mathbf{C}$. Then, for all $x,y\in\mathscr{X}$, we have
$$\begin{aligned}\|\mathbf{G}(x)\ominus_{gH}\mathbf{G}(y)\|_{I(\mathbb{R})}&=\left\|\left(\min_{u\in\mathscr{S}}\|x-u\|\right)\odot\mathbf{C}\ominus_{gH}\left(\min_{u\in\mathscr{S}}\|y-u\|\right)\odot\mathbf{C}\right\|_{I(\mathbb{R})}\\ &\leq\left\|(\|x-u\|)\odot\mathbf{C}\ominus_{gH}(\|y-u\|)\odot\mathbf{C}\right\|_{I(\mathbb{R})}\end{aligned}$$

$$
\begin{aligned}
&= \Big\|\Big[\min\{\|x-u\|\underline{c}-\|y-u\|\underline{c},\ \|x-u\|\overline{c}-\|y-u\|\overline{c}\},\\
&\qquad \max\{\|x-u\|\underline{c}-\|y-u\|\underline{c},\ \|x-u\|\overline{c}-\|y-u\|\overline{c}\}\Big]\Big\|_{I(\mathbb{R})}\\
&\leq \text{ either } \underline{c}\|x-y\| \text{ or } \overline{c}\|x-y\|.
\end{aligned}
$$

Hence, $\mathbf{G}$ is gH-Lipschitz continuous on $\mathscr{X}$. If $\mathbf{F}$ is gH-Lipschitz continuous on $\mathscr{S}$ and $\mathbf{H}(x) = \mathbf{F}(x) \oplus \mathbf{G}(x)$, then, for all $x, y \in \mathscr{S}$,

$$
\begin{aligned}
& \|\mathbf{H}(x) \ominus_{gH} \mathbf{H}(y)\|_{I(\mathbb{R})}\\
= \ & \big\|\big[\underline{f}(x)+\underline{g}(x),\ \overline{f}(x)+\overline{g}(x)\big] \ominus_{gH} \big[\underline{f}(y)+\underline{g}(y),\ \overline{f}(y)+\overline{g}(y)\big]\big\|_{I(\mathbb{R})}\\
\leq \ & \text{either } |\underline{f}(x)-\underline{f}(y)|+|\underline{g}(x)-\underline{g}(y)| \text{ or, } |\underline{f}(x)-\underline{f}(y)|+|\underline{g}(x)-\underline{g}(y)|.
\end{aligned}
$$

Since $\underline{f},\ \underline{g},\ \overline{f}$ and $\overline{g}$ are Lipschitz continuous on $\mathscr{S}$, $\mathbf{H}$ is a gH-Lipschitz continuous IVF on $\mathscr{S}$. □

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