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AN ACCURATE AND STABLE NUMERICAL SCHEME FOR FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. This paper investigates a novel compact finite difference scheme for numerical solutions of a class of nonlinear Υ-Caputo time-fractional integro-differential equations. The fractional derivative and integral are discretized in time by using the $L1$ approximation, while the nonlinear convection term is approximated by a specially designed nonlinear compact difference operator. A rigorous theoretical analysis establishes the unconditional stability and convergence of the proposed method. Numerical experiments are presented to support the theoretical results and demonstrate the accuracy and computational efficiency of the scheme across various test scenarios.

Keywords. Compact finite difference scheme; Fractional calculus; Integro-differential equations; Nonlinear time-fractional equation; Stability analysis.
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1. Introduction

The modeling of complex physical phenomena involving memory effects, anomalous diffusion, and nonlocal interactions has long challenged the boundaries of classical integer-order calculus. Fractional partial differential equations (FPDEs) encode hereditary dynamics and long-range spatial dependencies directly into the convolution structure of their operators, a capacity that integer-order models, constrained by locality, cannot replicate by design. The results reveal an operator-theoretic reach that cuts across disciplinary boundaries: fractional operators quantify memory-driven attenuation in signal processing [24], encode non-integer damping in control loops [28], reproduce anomalous transport in fluid dynamics [5], capture subdiffusive drug-release profiles in biomedical models [18], and resolve creep and relaxation spectra in viscoelastic solids [4]. In each setting, the nonlocal character of fractional operators encodes physical information that local derivatives discard, and it is precisely this structural fidelity that drives the sustained growth of the field.

Analytical intractability, however, follows almost immediately from the nonlinear structure of these equations. Closed-form resolution holds only in narrowly constrained, largely academic cases, and our observations indicate that this gap forces the construction of numerical schemes that operate directly at the discrete level, each designed to exploit a specific structural

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feature of the problem. Among the strategies that have gained prominence in this endeavor, finite difference methods [13] discretize the problem on a grid by replacing differential operators with difference quotients, finite element methods [21] construct piecewise polynomial approximations over a mesh that can conform to complex geometries, spectral methods [14] expand the solution in global orthogonal bases such as Legendre polynomials to attain exponential convergence for smooth solutions, and integral transform methods [23] recast the equation in a transformed domain where quadrature and fast transform techniques may be efficiently applied. A particularly active line of research concerns the treatment of singular kernels arising in time-fractional integro-differential equations, where standard uniform discretizations suffer from order reduction near the initial time; in this direction, our recent contribution [6] developed accurate non-uniform grid schemes specifically designed to overcome this singularity barrier, establishing sharp error estimates. That singularity-resolution strategy now confronts a qualitatively harder adversary: the Υ-Caputo fractional derivative, which simultaneously couples a nonlinear convection term, a weakly singular integral kernel with power-law memory, and a viscous diffusion term into a single governing equation, and it is precisely this configuration that the present work targets.

A widely studied example of nonlinear FPDEs is the generalized fractional Burgers' equation, which arises in the modeling of turbulence, nonlinear diffusion, and wave propagation phenomena. The classical form of the Burgers' equation is given by

$$\frac{\partial\vartheta}{\partial s}(z,s)+\vartheta\frac{\partial\vartheta}{\partial z}(z,s)=c\frac{\partial^2\vartheta}{\partial z^2}(z,s),$$

where $\vartheta(z,s)$ represents the velocity field, $c>0$ is the viscosity coefficient, $s\geq 0$ denotes time, and $z\in[0,1]$ is the spatial domain. The term $\vartheta\frac{\partial\vartheta}{\partial z}$ represents the nonlinear convection, while $c\frac{\partial^2\vartheta}{\partial z^2}$ accounts for viscous diffusion.

In this context, Sanz-Serna [31] studied the following equation

$$\frac{\partial\vartheta}{\partial s}(z,s)+\vartheta\frac{\partial\vartheta}{\partial z}(z,s)=\int_0^s(s-\mathfrak{t})^{-1/2}\frac{\partial^2\vartheta}{\partial z^2}(z,\mathfrak{t})\,d\mathfrak{t},$$

where the integral term models viscoelastic forces similar to those in non-Newtonian fluids [27]. This equation combines the effects of viscoelasticity with nonlinear convection, providing a simplified framework for investigating complex scenarios involving the interplay of viscous forces and nonlinearity. Further generalizations involve fractional integral terms with arbitrary order $\beta\in(0,1)$

$$\begin{aligned}\frac{\partial\vartheta}{\partial s}(z,s)+\vartheta\frac{\partial\vartheta}{\partial z}(z,s)&=\frac{1}{\Gamma(\beta)}\int_0^s(s-\mathfrak{t})^{\beta-1}\frac{\partial^2\vartheta}{\partial z^2}(z,\mathfrak{t})\,d\mathfrak{t}+g(z,s),\\ \frac{\partial\vartheta}{\partial s}(z,s)-\frac{\partial\vartheta}{\partial z}(z,s)&=\frac{1}{\Gamma(\beta)}\int_0^s(s-\mathfrak{t})^{\beta-1}\frac{\partial^2\vartheta}{\partial z^2}(z,\mathfrak{t})\,d\mathfrak{t}+f(z,s,\vartheta(z,s)),\end{aligned}$$

where $g(z,s)$ is a smooth function and f is Lipschitz continuous. These equations have been studied by [25, 38, 39], among others.

To capture the dynamics of subdiffusion and anomalous diffusion processes more accurately, researchers have incorporated fractional derivatives into these models. [17, 35] investigated fractional partial integro-differential equations (FPIDEs) of the form

$$ {}^{C}\mathscr{D}^{\alpha}_{0^+}\vartheta(z,s)+\vartheta\frac{\partial\vartheta}{\partial z}(z,s)=\int_0^s(s-\mathfrak{t})^{\beta-1}\frac{\partial^2\vartheta}{\partial z^2}(z,\mathfrak{t})\,d\mathfrak{t}+g(z,s), $$

$$ {}^{C}\mathscr{D}^{\alpha}_{0^+}\vartheta(z,s)-\frac{\partial\vartheta}{\partial z}(z,s)=\frac{1}{\Gamma(\beta)}\int_0^s(s-\mathfrak{t})^{\beta-1}\frac{\partial^2\vartheta}{\partial z^2}(z,\mathfrak{t})\,d\mathfrak{t}+f(z,s,\vartheta(z,s)), $$

where ${}^{C}\mathscr{D}^{\alpha}_{0^+}$ denotes the Caputo fractional derivative of order $\alpha\in(0,1)$. These FPIDEs effectively model phenomena with memory and hereditary properties, such as heat conduction in materials with memory [4]. Classical fractional operators encode memory through a pure power-law kernel whose algebraic decay imposes no finite horizon on past influence, a structural feature that, in many physical and biological systems, overcounts the persistence of historical effects. The Υ-Caputo fractional calculus addresses this limitation not by exponential damping, but by replacing the standard time variable with a kernel-generating function Υ, which reshapes the power-law memory profile to match the intrinsic time scale of the underlying dynamics, a flexibility that has proved consistent with observed phenomena in geophysics [5], statistical physics [4], and financial modeling [24]. Fractional calculus with respect to another function takes the classical framework a decisive step further. The operator is not merely a formal extension. It fuses the kernel-generating function Υ directly into the convolution structure of the Caputo derivative, rendering the intrinsic time scale of memory tunable through the single independent choice of Υ, without any additional parameters. Almeida and colleagues [2] were among the first to investigate this operator analytically, and Fahad et al. [16] subsequently demonstrated that selecting Υ appropriately collapses the Υ-Caputo calculus onto the Hadamard-type calculus, revealing a structural kinship between two operator families that had previously appeared unrelated. A systematic treatment of composition rules, integration-by-parts identities, and existence-uniqueness theory for the associated differential equations was assembled in [3], providing the analytical scaffolding on which subsequent numerical work could stand. In [1], Almeida et al. examined equations driven by arbitrary kernels, mapping the landscape of well-posedness across a wide class of kernel choices. In [29], Saifullah et al. constructed an integral transform tailored to this fractional framework, converting operator equations into algebraic ones in the transformed domain; and variational and control-theoretic formulations followed in [20, 26, 34], where optimality conditions and Euler–Lagrange equations were extracted directly from functionals involving the Υ-Caputo operator.

Our own contributions approached the Υ-Caputo operator from the numerical side. In [7], the authors designed a finite difference scheme that discretizes a linear Υ-Caputo diffusion equation and established its stability through Von Neumann analysis. That scheme was subsequently extended in [8] to the nonlinear integro-differential setting with weakly singular kernels, where a high-order compact discretization achieves fourth-order spatial accuracy while preserving unconditional stability through discrete energy analysis. Our observations indicate that this operator sustains high-order numerical treatment without sacrificing stability, a finding that directly motivates the present work. On the analytical side, coincidence degree theory was applied in [10] to establish existence and uniqueness for higher-order nonlinear fractional equations, and [9] delivered accurate non-uniform grid schemes that overcome the order-reduction barrier imposed by initial-time singularities in time-fractional integro-differential equations. Related contributions involving Υ-Caputo operators were developed in [11, 12]. Taken together, these results reveal a coherent numerical framework for fractional operators, one that, to the best of our knowledge, has not yet been extended to nonlinear partial integro-differential equations governed by the Υ-Caputo derivative, which is precisely the gap the present paper closes.

In this study, we focus on solving the following nonlinear Υ-Caputo fractional partial integro-differential equation

$$\begin{aligned} &{}^{C}\mathscr{D}_{a^+}^{\alpha;\Upsilon}\vartheta(z,s) + \vartheta\frac{\partial\vartheta}{\partial z}(z,s) \\ &= \int_a^s \big(\Upsilon(s)-\Upsilon(\mathfrak{t})\big)^{1-\beta}\Upsilon'(\mathfrak{t})\frac{\partial^2\vartheta}{\partial z^2}(z,\mathfrak{t})\,d\mathfrak{t} + \mu\frac{\partial^2\vartheta}{\partial z^2}(z,s) + g(z,s), \end{aligned} \tag{1.1}$$

under the following initial and boundary conditions respectively

$$\vartheta(z,a) = \varphi(z), \quad z\in[0,\, L], \qquad \text{and} \qquad \vartheta(0,s) = \vartheta(L,s) = 0, \quad s\in[a,\, T], \tag{1.2}$$

where $(\alpha,\beta)\in(0,1)^2$, ${}^{C}\mathscr{D}_{a^+}^{\alpha;\Upsilon}$ denotes the Υ-Caputo fractional derivative of order α, $\mu>0$ is the viscosity coefficient, $a\geq 0$ is the initial time, Υ is a given monotone increasing function, φ is a specified initial condition, and g is a sufficiently smooth, known source term.

Equation (1.1) generalizes the standard FPIDEs studied in [17, 35] by replacing the classical Caputo derivative and convolution integral with their Υ-counterparts, and by supplementing the integro-differential structure with the viscous diffusion term $\mu\,\partial^2\vartheta/\partial z^2$, which accounts for the interplay between memory-driven transport and classical viscous dissipation. The power-law kernel $\big(\Upsilon(s)-\Upsilon(\mathfrak{t})\big)^{1-\beta}$ encodes the influence of past states through an algebraic decay whose rate is shaped entirely by the fractional order β, without imposing any artificial truncation of memory. The kernel-generating function Υ introduces an independent degree of freedom: by calibrating Υ, the discretization adapts to non-uniform temporal scales that arise naturally in subdiffusion phenomena [19], a flexibility that uniform-grid formulations cannot replicate. In general, analytical solutions to FPIDEs pose substantial difficulties due to the nonlocal nature of fractional derivatives, the singular kernels within integral terms, and the intrinsic nonlinearities present in these models. Such complexity highlights the importance of developing efficient and accurate numerical methods.

This paper proposes a fourth-order compact finite difference scheme for Equation (1.1). The $L1$ approximation discretizes both the Υ-Caputo derivative and the weakly singular integral term, directly targeting the temporal nonlocality that standard difference quotients cannot handle. A nonlinear compact finite difference operator approximates the convection term $\vartheta\,\partial_z\vartheta$, maintaining fourth-order spatial accuracy in the presence of nonlinearity. Unconditional stability is established through discrete energy analysis and mathematical induction, not asserted, but proved. The scheme converges at order $(2-\alpha)$ in time and order four in space, and our results indicate that this accuracy profile is achieved with a stencil compact enough to reduce grid-point count and computational cost relative to existing approaches, without sacrificing stability over long simulation intervals. Numerical experiments validate each theoretical rate and quantify the gain in efficiency over competing schemes.

The paper proceeds as follows. Section 2 provides essential preliminary definitions and lemmas regarding Υ-Caputo fractional derivatives. In Section 3, we detail the construction of the fourth-order compact finite difference scheme applied to Equations (1.1)–(1.2), and rigorously demonstrate its stability and convergence properties. Numerical results in Section 4 illustrate the effectiveness and superiority of our proposed scheme. Finally, concluding remarks and potential directions for future research are presented in Section 5.

2. Preliminaries

In this section, we recall some definitions and properties of fractional integrals and derivatives with respect to another function Υ, which is useful for discussing the considered problem. In addition, we review some technical Lemmas.

Definition 2.1. [2] Let ϑ be a real-valued function that is piecewise continuous on $(0,T)$ and belongs to $\mathrm{L}([0,T])$. For $\alpha \in [0,\infty)$, the Riemann-Liouville fractional integral of order α is defined as

$$\mathscr{I}_{0^+}^{\alpha}\vartheta(s) = \frac{1}{\Gamma(\alpha)}\int_0^s (s-\mathfrak{t})^{\alpha-1}\vartheta(\mathfrak{t})d\mathfrak{t}. \tag{2.1}$$

A generalization of this integral, incorporating a positive, monotonic, and increasing function Υ such that $\Upsilon' \neq 0$ on $[a,T]$, leads to the Υ-Riemann-Liouville fractional integral

$$\mathscr{I}_{a^+}^{\alpha;\Upsilon}\vartheta(s) = \frac{1}{\Gamma(\alpha)}\int_a^s \Upsilon'(\mathfrak{t})\big(\Upsilon(s)-\Upsilon(\mathfrak{t})\big)^{\alpha-1}\vartheta(\mathfrak{t})d\mathfrak{t}. \tag{2.2}$$

Definition 2.2. [2] For $n-1<\alpha\leq n$ with $n\in\mathbb{N}$, the Caputo fractional derivative of order α is given by

$${}^{C}\mathscr{D}_{a^+}^{\alpha}\vartheta(s) = \frac{1}{\Gamma(n-\alpha)}\int_0^s (s-\mathfrak{t})^{n-\alpha-1}\frac{d^n}{d\mathfrak{t}^n}\vartheta(\mathfrak{t})d\mathfrak{t}. \tag{2.3}$$

If the derivative is formulated concerning a function Υ that is continuously differentiable and non-decreasing on $[a,T]$, the Υ-Caputo fractional derivative is defined as

$${}^{C}\mathscr{D}_{a^+}^{\alpha;\Upsilon}\vartheta(s) = \mathscr{I}_{a^+}^{n-\alpha;\Upsilon}\left(\frac{1}{\Upsilon'(s)}\frac{d}{ds}\right)^n\vartheta(s). \tag{2.4}$$

Remark 2.1. If $\Upsilon(s)=s$, then the Υ-fractional operators (2.2) and (2.4) become the classical fractional operators (2.1) and (2.3), respectively.

Lemma 2.1. [2, 3] *Let $\alpha\in(n-1,n]$ with $n\in\mathbb{N}^*$. Assume that the function $\Upsilon\in C^n([a,T])$ is non-decreasing and satisfies $\Upsilon'(s)\neq 0$ for all $s\in[a,T]$. Then, for any function $\vartheta\in C^n([a,T])$, the following properties hold:*

(1) The composition of the Υ-Riemann-Liouville fractional integral and the corresponding Υ-Caputo derivative yields the original function ${}^{C}\mathscr{D}_{a^+}^{\alpha;\Upsilon}\left(\mathscr{I}_{a^+}^{\alpha;\Upsilon}\vartheta\right)(s)=\vartheta(s)$.

(2) Conversely, applying the Υ-Riemann-Liouville integral to the Υ-Caputo derivative results in

$$\mathscr{I}_{a^+}^{\alpha;\Upsilon}\,{}^{C}\mathscr{D}_{a^+}^{\alpha;\Upsilon}\vartheta(s) = \vartheta(s) - \sum_{k=0}^{n-1}\frac{(\Upsilon(s)-\Upsilon(a))^k}{\Gamma(k+1)}\left[\mathscr{D}_{\Upsilon}^{[k]}\vartheta(s)\right]_{s=a},$$

where the differential operator $\mathscr{D}_{\Upsilon}^{[k]}$ is defined as

$$\mathscr{D}_{\Upsilon}^{[k]}\vartheta(s) = \left(\frac{1}{\Upsilon'(s)}\frac{d}{ds}\right)^k\vartheta(s).$$

(3) *For any $\alpha_1, \alpha_2 > 0$ and $\vartheta \in C([a,T])$, the Υ-Riemann-Liouville fractional integrals satisfies the semigroup property*

$$\mathscr{I}_{a^+}^{\alpha_1;\Upsilon}\mathscr{I}_{a^+}^{\alpha_2;\Upsilon}\vartheta(s) = \mathscr{I}_{a^+}^{\alpha_1+\alpha_2;\Upsilon}\vartheta(s), \quad \forall s \in [a,T].$$

The compact difference operator is a powerful mathematical tool known for its high accuracy and computational efficiency. This approach aims to develop high-precision numerical schemes by leveraging compact finite difference operators. This methodology enables the construction of efficient and stable discretization techniques that require fewer grid points while maintaining superior accuracy. To validate the effectiveness of the proposed numerical scheme, it will be applied to solve FPDEs, demonstrating its robustness, accuracy, and practical applicability in complex mathematical modeling.

To tackle the problem defined by equations (1.1) – (1.2), we discretize the domain $[0,L] \times [0,T] = \{(z,s) \mid 0 \le z \le L,\ 0 \le s \le T\}$ into grids described by the set of nodes $\{(z_i, s_n)\}$, in which $z_i := ih, i = 0,1,\cdots,M$ and $s_n = n\tau, n = 0,1,\cdots,N$, where $h := \frac{L}{M}$ and $\tau := \frac{T}{N}$ represent the spatial and temporal step sizes, respectively, and M and N are two given positive integers. The space domain $[0,L]$ is covered by the uniform mesh $\omega_h := \{z_i \mid 0 \le i \le M\}$ and the time domain $[0,T]$ is covered by the uniform mesh $\omega_\tau := \{s_n \mid 0 \le n \le N\}$. The full spatiotemporal grid is then given by $\omega_{h\tau} := \omega_h \times \omega_\tau$. For any grid function $v := \{v_i^n \mid 0 \le i \le M,\ 0 \le n \le N\}$ defined on $\omega_{h\tau}$, we introduce the following notations

$$v_i^{n-\frac{1}{2}} := \tfrac{1}{2}(v_i^n + v_i^{n-1}), \quad \delta_s v_i^{n-\frac{1}{2}} := \tfrac{1}{\tau}(v_i^n - v_i^{n-1}), \quad \delta_z v_{i-\frac{1}{2}}^n := \tfrac{1}{h}(v_i^n - v_{i-1}^n),$$
$$\triangle_z v_i^n := \tfrac{1}{2h}(v_{i+1}^n - v_{i-1}^n), \qquad \delta_z^2 v_i^n := \tfrac{1}{h}(\delta_z v_{i+\frac{1}{2}}^n - \delta_z v_{i-\frac{1}{2}}^n).$$

Let $\Omega_h := \{v \mid v = (v_0,\ v_1,\ \cdots, v_M)\}$ and $\mathring{\Omega}_h := \{v \mid v \in \Omega_h,\ v_0 = v_M = 0\}$ denote the spaces of grid functions on ω_h. For any u and v in the set $\mathring{\Omega}_h$, we introduce the following inner products and norms

$$(u,v) := h\sum_{i=1}^{M-1} u_i v_i, \qquad \langle \delta_z u,\ \delta_z v\rangle := h\sum_{i=1}^{M} (\delta_z u_{i-\frac{1}{2}})(\delta_z v_{i-\frac{1}{2}}),$$
$$\|v\| := \sqrt{(v,v)}, \qquad |v|_1 := \sqrt{\langle v,v\rangle}, \qquad \|v\|_\infty := \max_{0\le i\le M}|v_i|.$$

To discretize the nonlinear term $\vartheta\frac{\partial\vartheta}{\partial z}(z,s)$, we use the non-commutative bilinear function ξ defined by

$$\xi(\vartheta_i, v_i) := \frac{1}{3}\Big[\vartheta_i \triangle_z v_i + \triangle_z(\vartheta_i v_i)\Big], \quad 1 \le i \le M-1.$$

In what follows, we give some useful lemmas.

Lemma 2.2. [32] *Let $\vartheta(s) \in C^2[0,s_n]$. Then there exists a constant $C_1 > 0$ such that*

$$\left|\frac{1}{\Gamma(1-\alpha)}\int_0^{s_n}\frac{\vartheta'(\mathrm{t})}{(s_n-\mathrm{t})^\alpha}d\mathrm{t} - \frac{\tau^{-1}}{\Gamma(1-\alpha)}\mathscr{D}_s^{(\alpha)}\vartheta(s_n)\right| \le C_1\tau^{2-\alpha} \quad for \quad 0<\alpha<1,$$

$$\mathscr{D}_s^{(\alpha)}\vartheta(s_n) \ := a_0\vartheta(s_n) - \sum_{i=1}^{n-1}(a_{n-i-1} - a_{n-i})\vartheta(s_i) - a_{n-1}\vartheta(s_0),$$

where $a_i := \int_{s_i}^{s_{i+1}} \mathrm{t}^{-\alpha}d\mathrm{t} = \frac{\tau^{1-\alpha}}{1-\alpha}\left[(i+1)^{1-\alpha} - i^{1-\alpha}\right], i \ge 0$.

Lemma 2.3. [15, 17] *Let $\vartheta(s) \in C^2[0,T]$. Then there exist a positive constant C_2 depending only on β such that*

$$\left| \int_0^{s_n} (s_n - \mathfrak{t})^{\beta-1} \vartheta(\mathfrak{t}) d\mathfrak{t} - \sum_{j=0}^{n} b_{n-j} \vartheta(s_{n-j}) \right| \leq C_2 \max_{0\leq s\leq T} |\vartheta''(s)| s_n^{\beta} \tau^2,$$

where $b_{n-j} = \frac{\tau^\beta}{\beta(\beta+1)} \begin{cases} (n-1)^{\beta+1} - (n-\beta-1)n^\beta, & j = n \\ (j+1)^{\beta+1} + (j-1)^{\beta+1} - 2(j)^{\beta+1}, & j \in [1, n-1] \\ 1, & j = 0. \end{cases}$

Lemma 2.4. [8] *For any β and b_k $(0 \leq k \leq N)$ defined in Lemma 2.3 with $k = n - j$, it holds*

(1) $0 < \dfrac{(N-1)(T-\tau)^\beta + (\beta+1-N)T^\beta}{\beta(\beta+1)} \leq b_0 < \dfrac{\tau^\beta}{\beta+1}$, $1 \leq n \leq N$,

(2) $b_k > 0$, $0 \leq k \leq n$, $1 \leq n \leq N$,

(3) $\dfrac{\tau^\beta}{\beta+1} \leq \displaystyle\sum_{k=1}^{n-1} b_k \leq \dfrac{T^\beta}{\beta}$, $2 \leq n \leq N$.

Lemma 2.5. [22] *For $s_n = T > 0$ and for a nonnegative set $(\lambda_\ell)_{\ell=0}^{n-1}$, there exists a constant λ, independent of s, satisfying $\lambda \geq \sum_{\ell=0}^{n-1} \lambda_\ell$. Suppose that $\{\vartheta^n | 0 \leq n \leq N\}$ satisfies*

$$\mathfrak{D}_s^\alpha (\vartheta^n)^2 \leq \sum_{\ell=1}^{n} \lambda_{n-1} (\vartheta^\ell)^2 + \vartheta^n (\xi^n + \eta^n).$$

Let $\{\xi^n, \eta^n \mid 0 \leq n \leq N\}$ represent non-negative sequences. If $\tau_{n-1} \leq \tau_n$ for $2 \leq n \leq N$, and $\tau_N \leq \sqrt[\alpha]{\frac{1}{2\Gamma(2-\alpha)\lambda}}$, then

$$v^n \leq 2E_\alpha(2\lambda s_n^\alpha) \Big(\vartheta^0 + \max_{1\leq k\leq n} \sum_{l=1}^{k} P_{k-l}^{(k)} \xi^l + \omega_{1+\alpha}(s_n) \max_{1\leq k\leq n} \eta^k \Big), \quad 0 \leq n \leq N,$$

where $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k\alpha)}$ is Mittag-Leffler function and the discrete convolution kernel $P_{s-k}^{(s)}$ is defined by

$$P_{s-k}^{(s)} = \frac{1}{a_0^{(k)}} \begin{cases} 1, & k = s, \\ \displaystyle\sum_{j=k+1}^{s} (a_{j-k-1}^{(j)} - a_{j-k}^{(j)}) P_{s-j}^{(s)}, & 1 \leq k \leq s-1. \end{cases}$$

Lemma 2.6. [36] *Let $v(z) \in \mathscr{C}^5([z_{i-1}, z_{i+1}])$ and $w(z) := v''(z)$. Then*

$$v(z_i)v'(z_i) = \xi(v_i, v_i) - \frac{h^2}{2} \xi(w_i, v_i) + O(h^4).$$

Lemma 2.7. [33] *Let ϑ in Ω_h and v in $\mathring{\Omega}_h$. Then $(\xi(\vartheta, v), v) = 0$.*

Lemma 2.8. [36] *For any $v, \vartheta \in \mathring{\Omega}_h$ and $R \in \Omega_h$ satisfying $v_i = \delta_z^2 \vartheta_i - \frac{h^2}{12} \delta_z^2 v_i + R_i$, $1 \leq i \leq M-1$, it holds*

$$(v, \vartheta) = -|\vartheta|_1^2 - \frac{h^2}{12} \|v\|^2 + \frac{h^4}{144} |v|_1^2 + \frac{h^2}{12} (R, v) + (R, \vartheta),$$

$$(v, \vartheta) \leq -|\vartheta|_1^2 - \frac{h^2}{18} \|v\|^2 + \frac{h^2}{12} (R, v) + (R, \vartheta).$$

Lemma 2.9. [30] *For* $\varepsilon > 0$ *and any* $\vartheta \in \Omega_h$, *it holds that* $\|\vartheta\|_\infty^2 \le \varepsilon|\vartheta|_1^2 + (\frac{1}{\varepsilon} + \frac{1}{L})\|\vartheta\|^2$.

Lemma 2.10. [30] *For any grid function* $\vartheta \in \mathring{\Omega}_h$, *the following inequalities hold*

$$\|\vartheta\|_\infty \le \frac{\sqrt{L}}{2}|\vartheta|_1,\ \|\vartheta\| \le \frac{L}{\sqrt{6}}|\vartheta|_1,\ \|\triangle_z\vartheta\| \le |\vartheta|_1,\ |\vartheta|_1 \le \frac{2}{h}\|\vartheta\|.$$

3. Derivation of Compact Difference Scheme

Next, we proceed with the derivation of the compact difference scheme for the problem defined by equations (1.1) – (1.2). Let $w := \frac{\partial^2\vartheta}{\partial z^2}$. Then

$$\begin{cases} {}^C\mathscr{D}_{a^+}^{\alpha;\Upsilon}\vartheta(z,s) + \vartheta\dfrac{\partial\vartheta}{\partial z}(z,s) = \Gamma(\beta)\mathscr{I}_{a^+}^{\beta;\Upsilon}w(z,s) + \mu w(z,s) + g(z,s), & (z,s) \in (0,L)\times(a,T], \\ w(z,s) = \frac{\partial^2\vartheta}{\partial z^2}(z,s), & (z,s) \in (0,L)\times(a,T], \\ \vartheta(z,a) = \varphi(z), & z \in [0,L], \\ \vartheta(0,s) = \vartheta(L,s) = w(0,s) = w(L,s) = 0, & s \in [a,T]. \end{cases} \tag{3.1}$$

Assume that $\vartheta(z,s) \in \mathrm{C}_{z,s}^{6,2}([0,L]\times[0,T])$ and define the mesh function

$$V := \{V(z_i,\ s_n) \mid 0 \le i \le M,\ 0 \le n \le N\}$$

and

$$W := \{W(z_i,\ s_n) \mid 0 \le i \le M,\ 0 \le n \le N\}$$

on $\omega_{h\tau}$, where $V(z_i,\ s_n) := \vartheta(z_i,\ s_n)$ and $W(z_i,\ s_n) := w(z_i,\ s_n)$. Consider (3.1) at the point (z_i, s_n). To approximate the Υ-Caputo derivative, we begin by rewriting the integral term in Definition 2.1 through the change of variable

$$\phi(u) = V\left(z_i, \Upsilon^{-1}(u)\right), \qquad u = \Upsilon(s). \tag{3.2}$$

Under this transformation, the kernel $(\Upsilon(s_n) - \Upsilon(\mathrm{t}))^{-\alpha}$ becomes $(u_n - \xi)^{-\alpha}$, and the derivative transforms according to

$$\frac{1}{\Gamma(1-\alpha)}\int_a^{s_n}(\Upsilon(s_n) - \Upsilon(\mathrm{t}))^{-\alpha}\frac{d}{d\mathrm{t}}V(z_i,\mathrm{t})\,d\mathrm{t} = \frac{1}{\Gamma(1-\alpha)}\int_{u_0}^{u_n}(u_n - \xi)^{-\alpha}\phi'(\xi)\,d\xi.$$

This representation has exactly the structure required in Lemma 2.2. Applying the lemma to ϕ on a uniform mesh in the transformed variable u yields

$$\begin{aligned} {}^C\mathscr{D}_{a^+}^{\alpha;\Upsilon}V(z_i,s_n) =& \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[V(z_i,s_n) - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})V(z_i,\tilde{s}_k) - a_{n-1}V(z_i,s_0)\right] + O(\tau^{(2-\alpha)}), \\ :=& \mathfrak{D}_s^\alpha V_i^n + O(\tau^{(2-\alpha)}), \end{aligned} \tag{3.3}$$

where $a_k = (k+1)^{1-\alpha} - k^{1-\alpha}$ for $0 \le k \le n$ and $\tilde{s}_k = \Upsilon^{-1}(u_k)$.

Secondly, concerning the integral term in (3.1), the Υ-RL fractional integral

$$\mathscr{I}_{a^+}^{\beta;\Upsilon}W(z_i,s_n) = \frac{1}{\Gamma(\beta)}\int_a^{s_n}\big(\Upsilon(s_n) - \Upsilon(\mathrm{t})\big)^{\beta-1}\Upsilon'(\mathrm{t})\,W(z_i,\mathrm{t})\,d\mathrm{t}.$$

Setting $u = \Upsilon(t)$, this reduces to the standard convolution form $\frac{1}{\Gamma(\beta)}\int_{u_0}^{u_n}(u_n - \xi)^{\beta-1}W(z_i,\xi)\,d\xi$ (see [7] for details). A piecewise linear interpolation of W on the mesh $\{s_k\}$ combined with

Lemma 2.3 then yields

$$\begin{aligned}\mathscr{I}_{a^+}^{\beta;\Upsilon}W(z_i,s_n) =&\frac{\tau^\beta}{\Gamma(\beta+2)}\sum_{k=0}^{n} b_k W(z_i,\tilde{s}_k)+O(\tau^2),\\ :=&\mathfrak{I}_s^\beta W_i^n+O(\tau^2),\end{aligned} \tag{3.4}$$

where $b_k=\begin{cases}(n-1)^{\beta+1}-(n-\beta-1)n^\beta, & k=0\\ (n-k-1)^{\beta+1}+(n-k+1)^{\beta+1}-2(n-k)^{\beta+1}, & k\in[1,n-1]\\ 1, & k=n.\end{cases}$ Additionally, for the nonlinear term in (3.1), applying Lemma 2.6, we obtain

$$V(z_i,s_n)\frac{\partial V}{\partial z}(z_i,s_n)=\xi(V(z_i,s_n),V(z_i,s_n))-\frac{h^2}{2}\xi(W(z_i,s_n),V(z_i,s_n))+O(h^4). \tag{3.5}$$

The substitution of Eqs. (3.3), (3.4), and (3.5) into Eq. (3.1) produces for $0\le n\le N$

$$\begin{cases}\mathfrak{D}_s^\alpha V_i^n+\xi(V_i^n,V_i^n)-\dfrac{h^2}{2}\xi(W_i^n,V_i^n)=\Gamma(\beta)\mathfrak{I}_s^\beta W_i^n+\mu W_i^n+g_i^n+R_i^n, & 1\le i\le M-1,\\ W_i^n=\delta_z^2V_i^n-\frac{h^2}{12}\delta_z^2W_i^n+Q_i^n, & 1\le i\le M-1,\\ V_i^0=\varphi(z_i), & 0\le i\le M,\\ V_0^n=V_M^n=W_0^n=W_M^n=0.\end{cases} \tag{3.6}$$

The following estimate for truncation errors with positive constants c_1 and c_2

$$\left|R_i^n\right|\le c_1(\tau^{2-\alpha}+h^4)\qquad \textit{and}\qquad \left|Q_i^n\right|\le c_2h^4.$$

Omitting the truncation errors R_i^n and Q_i^n in (3.6), replacing the functions V_i^n, W_i^n with their numerical approximation ϑ_i^n, w_i^n, and using the respective methods, we formulate the compact difference scheme for $0\le n\le N$ as follows

$$\begin{cases}\mathfrak{D}_s^\alpha \vartheta_i^n+\xi(\vartheta_i^n,\vartheta_i^n)-\dfrac{h^2}{2}\xi(w_i^n,\vartheta_i^n)=\Gamma(\beta)\mathfrak{I}_s^\beta w_i^n+\mu w_i^n+g_i^n, & 1\le i\le M-1,\\ w_i^n=\delta_z^2\vartheta_i^n-\frac{h^2}{12}\delta_z^2w_i^n, & 1\le i\le M-1,\\ \vartheta_i^0=\varphi(z_i), & 0\le i\le M,\\ \vartheta_0^n=\vartheta_M^n=w_0^n=w_M^n=0.\end{cases} \tag{3.7}$$

Denote $e_i^n=V_i^n-\vartheta_i^n,\ f_i^n=W_i^n-w_i^n,\ 0\le i\le M,\ 0\le n\le N, c_3=\max\{|\vartheta(z,s)|,\ |\vartheta_z(z,s)|,\ |\vartheta_{zz}(z,s)|,\ |\vartheta_{zzz}(z,s)|\}$ and $c_4=c_3+\|\varphi\|$.

Theorem 3.1 (Convergence)**.** *Suppose that* $\{\vartheta(z,s),w(z,s)\}$ *satisfies problem* (3.1) *and* $\{\vartheta_i^n,w_i^n\mid 0\le i\le M,\ 0\le n\le N\}$ *are the solutions of scheme* (3.7)*. Then* $|e^n|_1\le c_7(\tau^{2-\alpha}+h^4), 0\le n\le N$, *where*

$$c_5=\frac{56}{16\rho-9\sigma}c_3^2(1+\frac{L^2}{6})+\frac{56}{16\rho-9\sigma}\Big(\frac{12544c_4^2}{(16\rho-9\sigma)^2}+\frac{1}{L}\Big)+\frac{28}{16\rho-9\sigma}\Big[\frac{\tilde{h}^2}{6}\Big(2c_3+\frac{c_3L}{\sqrt{6}}\Big)\Big]^2,$$

$$c_6=\max_{0\le n\le N}\Big(\sigma Lc_2^2+\frac{(16\rho-9\sigma)Lc_2^2}{448}+\frac{28Lc_1^2}{16\rho-9\sigma}+\frac{28\sigma^2Lc_2^2}{16\rho-9\sigma}+2\mu Lc_2^2\Big),$$

$$c_7=\sqrt{\frac{2s_n^\alpha}{\Gamma(1+\alpha)}c_6E_\alpha(2c_5s_n^\alpha)}$$

and the constants ρ and σ are now given by

$$\rho=\frac{\tau^{-\beta}}{\beta(\beta+1)}\min\left\{\frac{(N-1)(T-\tau)^{\beta}+(\beta+1-N)T^{\beta}}{\beta(\beta+1)};\frac{\tau^{\beta}}{\beta+1};1\right\}>0,$$

$$\sigma=\frac{\tau^{-\beta}}{\beta(\beta+1)}\max\left\{\frac{\tau^{\beta}}{\beta+1};\frac{T^{\beta}}{\beta};1\right\}.$$

Proof. By subtracting (3.6) from (3.7), we have the system of error equations for $0\le n\le N$

$$\begin{cases}\mathfrak{D}_s^{\alpha}e_i^n+\left[\xi(V_i^n,V_i^n)-\xi(\vartheta_i^n,\vartheta_i^n)\right]-\frac{h^2}{2}\left[\xi(W_i^n,V_i^n)-\xi(W_i^n,\vartheta_i^n)\right] & \\ -\Gamma(\beta)\mathfrak{I}_s^{\beta}f_i^n-\mu f_i^n=R_i^n, & 1\le i\le M-1,\\ f_i^n=\delta_z^2e_i^n-\frac{h^2}{12}\delta_z^2f_i^n+Q_i^n, & 1\le i\le M-1,\\ e_i^0=0, & 0\le i\le M,\\ e_0^n=e_M^n=f_0^n=f_M^n=0. & \end{cases}\tag{3.8}$$

By taking the inner product of the first equation in (3.8) with $-\delta_z^2e^n$ and substituting the second equation in (3.8) into the obtained expression, we arrive at the following equation for $0\le n\le N$

$$\begin{aligned}&\left(\mathfrak{D}_s^{\alpha}e^n,-\delta_z^2e^n\right)+\left(\Gamma(\beta)\mathfrak{I}_s^{\beta}\delta_z^2e^n,\delta_z^2e^n\right)+\mu\left(\delta_z^2e^n,\delta_z^2e^n\right)=\frac{\mu h^2}{12}\left(\delta_z^2f^n,\delta_z^2e^n\right)\\&\frac{h^2}{12}\left(\Gamma(\beta)\mathfrak{I}_s^{\beta}\delta_z^2f^n,\delta_z^2e^n\right)+\left(\xi(V^n,e^n)+\xi(e^n,V^n),\delta_z^2e^n\right)-\left(\xi(e^n,e^n),\delta_z^2e^n\right)\\&-\frac{h^2}{2}\left(\xi(W^n,V^n)-\xi(w^n,\vartheta^n),\delta_z^2e^n\right)-(R^n,\delta_z^2e^n)-(\mathfrak{I}_s^{\beta}Q^n,\delta_z^2e^n).\end{aligned}\tag{3.9}$$

The estimation for the first term on the left-hand side of (3.9) is given as follows

$$\begin{aligned}(\mathfrak{D}_s^{\alpha}e^n,-\delta_z^2e^n)=&-\frac{h\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{i=1}^{M-1}\left[e_i^n-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})e_i^n\right]\delta_z^2e_i^n\\=&\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[|e^n|_1^2-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\langle\delta_ze^n,\delta_ze^n\rangle\right]\\\ge&\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[|e^n|_1^2-\frac{1}{2}\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\langle\delta_ze^n,\delta_ze^n\rangle\right.\\&\left.-\frac{1}{2}\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\langle\delta_ze^n,\delta_ze^n\rangle\right]\\\ge&\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[|e^n|_1^2-\frac{1}{2}|e^n|_1^2-\frac{1}{2}\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})|e^n|_1^2\right]\\=&\frac{1}{2}\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[|e^n|_1^2-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})|e^n|_1^2\right]\\=&\frac{1}{2}\mathfrak{D}_s^{\alpha}|e^n|_1^2,\qquad 0\le n\le N.\end{aligned}\tag{3.10}$$

By using Lemma 2.4, we estimate the second term on the left-hand side of (3.9) by

$$\begin{aligned}\left(\Gamma(\beta)\mathfrak{I}_s^\beta\delta_z^2 e^n,\delta_z^2 e^n\right) =&\frac{h\tau^{-\beta}}{\beta(\beta+1)}\sum_{i=1}^{M-1}\Big(\sum_{k=0}^{n} b_k\delta_z^2 e_i^n\Big)\delta_z^2 e_i^n\\ =&\frac{\tau^{-\beta}}{\beta(\beta+1)}\Big(\sum_{k=0}^{n} b_k\left(\delta_z^2 e^n,\delta_z^2 e^n\right)\Big)\\ \geq&\frac{\tau^{-\beta}}{\beta(\beta+1)}\sum_{k=0}^{n} b_k\|\delta_z^2 e^n\|^2\geq\rho\|\delta_z^2 e^n\|^2,\qquad 0\leq n\leq N,\end{aligned}\tag{3.11}$$

where $\rho=\frac{\tau^{-\beta}}{\beta(\beta+1)}\min\left(\frac{(N-1)(T-\tau)^\beta+(\beta+1-N)T^\beta}{\beta(\beta+1)};\frac{\tau^\beta}{\beta+1};1\right)>0$. Since $\mu>0$, the additional local diffusion term contributes positively

$$\mu\left(\delta_z^2 e^n,\delta_z^2 e^n\right)=\mu\|\delta_z^2 e^n\|^2\geq 0,\qquad 0\leq n\leq N.\tag{3.12}$$

By taking the inner product of the second equation of (3.8) with f^n and using the inequalities from Lemma 2.10, we obtain $\|f^n\|\leq\frac{3}{2}(\|\delta_z^2 e^n\|+\|Q^n\|)$. Concerning the first term arising from the viscous diffusion on the right-hand side of (3.9), we have

$$\begin{aligned}\frac{\mu h^2}{12}\left(\delta_z^2 f^n,\delta_z^2 e^n\right)&\leq\frac{\mu h^2}{12}\|\delta_z^2 f^n\|\|\delta_z^2 e^n\|\\ &\leq\frac{\mu}{3}\|f^n\|\|\delta_z^2 e^n\|\\ &\leq\frac{\mu}{2}\left(\|\delta_z^2 e^n\|+\|Q^n\|\right)\|\delta_z^2 e^n\|\\ &\leq\frac{\mu}{2}\|\delta_z^2 e^n\|^2+\frac{\mu}{16}\|\delta_z^2 e^n\|^2+2\mu Lc_2^2h^8,\qquad 0\leq n\leq N.\end{aligned}\tag{3.13}$$

The second term on the right-hand side of (3.9), we have

$$\begin{aligned}\frac{h^2}{12}\left(\Gamma(\beta)\mathfrak{I}_s^\beta\delta_z^2 f^n,\delta_z^2 e^n\right)=&\frac{h^3\tau^{-\beta}}{12\beta(\beta+1)}\sum_{i=1}^{M-1}\Big(\sum_{k=0}^{n} b_k\delta_z^2 f_i^n\Big)\delta_z^2 e_i^n\\ \leq&\frac{h^2\tau^{-\beta}}{12\beta(\beta+1)}\sum_{k=0}^{n} b_k\|\delta_z^2 f^n\|\|\delta_z^2 e^n\|\\ \leq&\frac{h^2\sigma}{12}\|\delta_z^2 f^n\|\|\delta_z^2 e^n\|\\ \leq&\frac{\sigma}{3}\|f^n\|\|\delta_z^2 e^n\|\\ \leq&\frac{\sigma}{2}\|\delta_z^2 e^n\|^2+\frac{\sigma}{16}\|\delta_z^2 e^n\|^2+\sigma Lc_2^2h^8,\qquad 0\leq n\leq N,\end{aligned}\tag{3.14}$$

where $\sigma=\frac{\tau^{-\beta}}{\beta(\beta+1)}\max\left(\frac{\tau^\beta}{\beta+1};\frac{T^\beta}{\beta};1\right)$. This estimation is achieved by applying the Cauchy-Schwarz inequality and Lemma 2.10. For $0\leq n\leq N$, the third term on the right-hand side of (3.9) can be estimated as follows

$$\begin{aligned}\big(\xi(V^n,e^n)+\xi(e^n,V^n),\delta_z^2e^n\big)&=\frac{h}{3}\sum_{i=1}^{M-1}\Big[V_i^n\triangle_z e_i^n+e_i^n\triangle_z V_i^n+2\triangle_z(Ve)_i^n\Big]\delta_z^2e_i^n\\&=\frac{h}{3}\sum_{i=1}^{M-1}\Big[V_i^n\triangle_z e_i^n+3e_i^n\triangle_z V_i^n+V_{i+1}^n\delta_z e_{i+\frac12}^n+V_{i-1}^n\delta_z e_{i-\frac12}^n\Big]\delta_z^2e_i^n\\&\le c_3\Big(|e^n|_1+\|e^n\|\Big)\|\delta_z^2e^n\|\le\frac{16\rho-9\sigma}{112}\|\delta_z^2e^n\|^2+\frac{56}{16\rho-9\sigma}c_3^2(1+\frac{L^2}{6})|e^n|_1^2.\end{aligned}\tag{3.15}$$

By using Lemma 2.9 and noticing $\|e^n\|\le c_4$, the fourth term on the right-hand side of (3.9) yields the following

$$\begin{aligned}-(\xi(e^n,e^n),\delta_z^2e^n)&=-\frac{h}{3}\sum_{i=1}^{M-1}(e_{i+1}^n+e_i^n+e_{i-1}^n)\triangle_z e_i^n\delta_z^2e_i^n\\&\le\|\delta_z e^n\|_\infty\|e^n\|\|\delta_z^2e^n\|\le\frac{16\rho-9\sigma}{224}\|\delta_z^2e^n\|^2+\frac{56c_4^2}{16\rho-9\sigma}\|\delta_z^2e^n\|_\infty\\&\le\frac{16\rho-9\sigma}{224}\|\delta_z^2e^n\|^2+\frac{56c_4^2}{16\rho-9\sigma}\Big[\varepsilon\|\delta_z^2e^n\|^2+(\frac{1}{\varepsilon}+\frac{1}{L})|e^n|_1^2\Big]\\&\le\frac{16\rho-9\sigma}{112}\|\delta_z^2e^n\|^2+\frac{56}{16\rho-9\sigma}\Big(\frac{12544c_4^2}{(16\rho-9\sigma)^2}+\frac{1}{L}\Big)|e^n|_1^2,\end{aligned}\tag{3.16}$$

where $\varepsilon=\frac{(16\rho-9\sigma)^2}{12544c_4^2}$. Setting $\tilde h=\frac{16\rho-9\sigma}{56c_3+168\sqrt{L}c_4}$, when $h\le h_0$, we can estimate the fifth term on the right-hand side of (3.9) as follows

$$\begin{aligned}-\frac{h^2}{2}\big(\xi(W^n,V^n)-\xi(w^n,\vartheta^n),\delta_z^2e^n\big)&=-\frac{h^3}{2}\sum_{i=1}^{M-1}\Big(\xi(W_i^n,e_i^n)+\xi(f_i^n,V_i^n)-\xi(f_i^n,e_i^n)\Big)\delta_z^2e_i^n\\=-\frac{h^3}{6}\sum_{i=1}^{M-1}&\Big(2W_i^n\triangle_z e_i^n+\frac12e_{i+1}^n\delta_z W_{i+\frac12}^n+\frac12e_{i-1}^n\delta_z W_{i-\frac12}^n\Big)\delta_z^2e_i^n\\-\frac{h^3}{6}\sum_{i=1}^{M-1}&\Big(2f_i^n\triangle_z V_i^n+\frac12V_{i+1}^n\delta_z f_{i+\frac12}^n+\frac12V_{i-1}^n\delta_z f_{i-\frac12}^n\Big)\delta_z^2e_i^n\\+\frac{h^3}{6}\sum_{i=1}^{M-1}&\Big(2f_i^n\triangle_z e_i^n+\frac12e_{i+1}^n\delta_z f_{i+\frac12}^n+\frac12e_{i-1}^n\delta_z f_{i-\frac12}^n\Big)\delta_z^2e_i^n\\\le\frac{h^3c_3}{6}&\Big(2|e^n|_1+\|e^n\|+2\|f^n\|+|f^n|_1\Big)\|\delta_z^2e^n\|+\frac{h^3}{6}\Big(2\|f^n\|_\infty|e^n|_1+\|e^n\|_\infty|f^n|_1\Big)\|\delta_z^2e^n\|\\\le\frac{h^2}{6}&(2c_3+\frac{c_3L}{\sqrt6})|e^n|_1\|\delta_z^2e^n\|+\frac{h^2}{3}(c_3+\frac{c_3}{h})\|f^n\|\|\delta_z^2e^n\|+\sqrt{L}hc_4\|f^n\|\|\delta_z^2e^n\|\\\le\frac{h^2}{6}&\Big(2c_3+\frac{c_3L}{\sqrt6}\Big)|e^n|_1\|\delta_z^2e^n\|+\frac32\Big[\frac{h^2}{3}(c_3+\frac{c_3}{h})+\sqrt{L}hc_4\Big]\Big(\|\delta_z^2e^n\|^2+\|\delta_z^2e\|\|Q^j\|\Big)\\\le\frac{3(16\rho-9\sigma)}{112}&\|\delta_z^2e^n\|^2+\frac{28}{16\rho-9\sigma}\Big[\frac{\tilde h^2}{6}\Big(2c_3+\frac{c_3L}{\sqrt6}\Big)\Big]^2|e^n|_1^2+\frac{(16\rho-9\sigma)Lc_2^2}{448}h^8.\end{aligned}\tag{3.17}$$

Moreover, it is evident that the sixth and the last terms on the right-hand side of (3.9) provide the following results

$$-(P^n,\delta_z^2 e^n)\le\frac{16\rho-9\sigma}{112}\|\delta_z^2 e^n\|^2+\frac{28Lc_1^2}{16\rho-9\sigma}(\tau^{2-\alpha}+h^4)^2,\qquad 0\le n\le N. \tag{3.18}$$

$$-(\mathfrak{I}_s^{\beta}Q^n,\delta_z^2 e^n)\le\frac{16\rho-9\sigma}{112}\|\delta_z^2 e^n\|^2+\frac{28\sigma^2Lc_2^2}{16\rho-9\sigma}h^8,\qquad 0\le n\le N. \tag{3.19}$$

Substituting (3.10)-(3.19) into (3.9), we have

$$\begin{aligned}\mathfrak{D}_s^{\alpha}|e^n|_1^2\le&2\Bigg[\frac{56}{16\rho-9\sigma}c_3^2\Big(1+\frac{L^2}{6}\Big)+\frac{56}{16\rho-9\sigma}\Big(\frac{12544c_4^2}{(16\rho-9\sigma)^2}+\frac{1}{L}\Big)\\&+\frac{28}{16\rho-9\sigma}\Big[\frac{\tilde{h}^2}{6}\Big(2c_3+\frac{c_3L}{\sqrt{6}}\Big)\Big]^2\Bigg]|e^n|_1^2\\&+2\left[\sigma Lc_2^2+\frac{(16\rho-9\sigma)Lc_2^2}{448}+\frac{28Lc_1^2}{16\rho-9\sigma}+\frac{28\sigma^2Lc_2^2}{16\rho-9\sigma}+2\mu Lc_2^2\right](\tau^{2-\alpha}+h^4)^2\\\le&c_5|e^n|_1^2+2c_6(\tau^{2-\alpha}+h^4)^2.\end{aligned}$$

Using the discrete fractional Grönwall type inequality by Lemma 2.5 under the uniform grid, we have for $0\le n\le N$

$$|e^n|_1^2\le 2\Big[|e^0|_1^2+\frac{s_n^{\alpha}}{\Gamma(1+\alpha)}c_6(\tau^{2-\alpha}+h^4)^2\Big]E_{\alpha}(2c_5s_n^{\alpha})=c_7^2(\tau^{2-\alpha}+h^4)^2.$$

The proof is concluded. □

Remark 3.1. Under the conditions of Theorem 3.1, combining with Lemma 2.10, we can obtain

$$\|e^n\|\le\frac{c_7L}{\sqrt{6}}(\tau^{2-\alpha}+h^4)\,,\ \|e^n\|_\infty\le\frac{c_7\sqrt{L}}{2}(\tau^{2-\alpha}+h^4)\,,\ 0\le n\le N.$$

Next, we analyze the stability of the proposed fourth-order compact difference scheme. Now, we suppose some perturbation terms. Letting $\{\tilde{\vartheta}_i^n,\tilde{w}_i^n\mid 0\le i\le M,\ 0\le n\le N\}$, we get the following system of equations for $0\le n\le N$

$$\begin{cases}\mathfrak{D}_s^{\alpha}\tilde{\vartheta}_i^n+\xi(\tilde{\vartheta}_i^n,\tilde{\vartheta}_i^n)-\dfrac{h^2}{2}\xi(\tilde{w}_i^n,\tilde{\vartheta}_i^n)=\Gamma(\beta)\mathfrak{I}_s^{\beta}\tilde{w}_i^n+\mu\tilde{w}_i^n+s_i^n, & 1\le i\le M-1,\\ \tilde{w}_i^n=\delta_z^2\tilde{\vartheta}_i^n-\frac{h^2}{12}\delta_z^2\tilde{w}_i^n, & 1\le i\le M-1,\\ \tilde{\vartheta}_i^0=\varphi(z_i)+p(z_i), & 0\le i\le M,\\ \tilde{\vartheta}_0^n=\tilde{\vartheta}_M^n=\tilde{w}_0^n=\tilde{w}_M^n=0.\end{cases} \tag{3.20}$$

and denote

$$\zeta_i^n=\tilde{\vartheta}_i^n-\vartheta_i^n,\quad \eta_i^n=\tilde{w}_i^n-w_i^n,\quad 0\le i\le M,\,0\le n\le N,$$

$$|\tilde{\vartheta}^n|_1\le|V^n|_1+|e^n|_1\le c_3+c_4(\tau^{2-\alpha}+h^4)\le c_8,\quad \|\vartheta\|_\infty\le\frac{\sqrt{L}}{2}c_8.$$

Subtracting (3.20) from (3.6), we have

$$\begin{cases} \mathfrak{D}_s^\alpha \zeta_i^n + \Big[\xi(\tilde{\vartheta}_i^n, \tilde{\vartheta}_i^n) - \xi(\vartheta_i^n, \vartheta_i^n)\Big] - \dfrac{h^2}{2}\Big[\xi(\tilde{w}_i^n, \tilde{\vartheta}_i^n) & \\ -\xi(w_i^n, \vartheta_i^n)\Big] - \Gamma(\beta)\mathfrak{I}_s^\beta \eta_i^n - \mu\eta_i^n = s_i^n, & 1 \le i \le M-1,\ 0 \le n \le N, \\ \eta_i^n = \delta_z^2 \zeta_i^n - \frac{h^2}{12}\delta_z^2 \eta_i^n, & 1 \le i \le M-1,\ 0 \le n \le N, \\ \zeta_i^0 = p(z_i), & 0 \le i \le M, \\ \zeta_0^n = \zeta_M^n = \eta_0^n = \eta_M^n = 0, & 0 \le n \le N. \end{cases} \tag{3.21}$$

By setting $c_9 = 1 + \frac{9c_8^2 L}{4\nu}$, we thus obtain the following stability result

Theorem 3.2 (Stability)**.** *Let $\{\zeta_i^n, \eta_i^n\}$ be the solutions of* (3.21) *. It holds*

$$\|\zeta^n\|^2 \le 2\Big[\|p\|^2 + \frac{s_n^\alpha}{\Gamma(1+\alpha)} \max_{0\le k\le n} \|s^k\|^2 E_\alpha(2c_9 s_n^\alpha)\Big],\ 0 \le n \le N.$$

Proof. Taking an inner product of (3.21) with ζ^n, we have, for $0 \le n \le N$,

$$\begin{aligned} \big(\mathfrak{D}_s^\alpha \zeta^n, \zeta^n\big) - \big(\Gamma(\beta)\mathfrak{I}_s^\beta \eta^n, \zeta^n\big) - \big(\mu\eta^n, \zeta^n\big) &= \big(\xi(\tilde{\vartheta}_i^n, \tilde{\vartheta}_i^n) - \xi(\vartheta_i^n, \vartheta_i^n), -\zeta^n\big) \\ &\quad + \frac{h^2}{2}\big(\xi(\tilde{w}^n, \tilde{\vartheta}^n) - \xi(w^n, \vartheta^n), \zeta^n\big) + (s^n, \zeta^n). \end{aligned} \tag{3.22}$$

By calculating, we find that the first term on the left-hand side of (3.22) is given as follows

$$(\mathfrak{D}_s^\alpha \zeta^n, \zeta^n) \ge \frac{1}{2}\mathfrak{D}_s^\alpha \|\zeta^n\|^2, \qquad 0 \le n \le N. \tag{3.23}$$

Together with Lemma 2.8 and Lemma 2.4, computing the second term on the left-hand side of (3.22) yields that

$$\begin{aligned} -\big(\Gamma(\beta)\mathfrak{I}_s^\beta \eta^n, \zeta^n\big) &= -\frac{h\tau^{-\beta}}{\beta(\beta+1)} \sum_{i=1}^{M-1} \Big(\sum_{k=0}^n b_k \eta_i^n\Big)\zeta_i^n \\ &= -\frac{\tau^{-\beta}}{\beta(\beta+1)}\Big(\sum_{k=0}^n b_k \big(\eta^n, \zeta^n\big)\Big) \\ &\ge \frac{\tau^{-\beta}}{\beta(\beta+1)} \sum_{k=0}^n b_k \big(|\zeta^n|_1^2 + \frac{h^2}{18}\|\eta^n\|^2\big) \\ &\ge \nu\big(|\zeta^n|_1^2 + \frac{h^2}{18}\|\eta^n\|^2\big), \qquad 0 \le n \le N, \end{aligned} \tag{3.24}$$

where

$$\nu = \frac{\tau^{-\beta}}{\beta(\beta+1)} \min\Big(\frac{(N-1)(T-\tau)^\beta + (\beta+1-N)T^\beta}{\beta(\beta+1)}; \frac{\tau^\beta}{\beta+1}; 1\Big).$$

Since $\mu > 0$, by Lemma 2.8 and Lemma 2.4, the third term on the left-hand side of (3.22) satisfies

$$-\big(\mu\eta^n, \zeta^n\big) \ge \mu\nu\Big(|\zeta^n|_1^2 + \frac{h^2}{18}\|\eta^n\|^2\Big) \ge 0, \qquad 0 \le n \le N. \tag{3.25}$$

From Lemma 2.7, the first term on the right-hand side of (3.22) is given by

$$
\begin{aligned}
-\left(\xi\left(\tilde{\vartheta}^n,\tilde{\vartheta}^n\right)-\xi\left(\vartheta^n,\vartheta^n\right),\zeta^n\right) &= -\left(\xi\left(\zeta^n,\vartheta^n\right),\zeta^n\right)\\
&= -\frac{h}{3}\sum_{i=1}^{M-1}\left[\zeta_i^n\Delta_z\vartheta_i^n+\Delta_z\left(\zeta^n\vartheta^n\right)_i\right]\zeta_i^n\\
&= \frac{h}{3}\sum_{i=1}^{M-1}\left[\vartheta_i^n\Delta_z\left(\left(\zeta_i^n\right)^2\right)_i+\zeta_i^n\vartheta_i^n\Delta_z\zeta_i^n\right]\\
&\leqslant \|\vartheta^n\|_\infty|\zeta^n|_1\|\zeta^n\|\\
&\leqslant \frac{c_8\sqrt{L}}{2}|\zeta^n|_1\|\zeta^n\|,\qquad 0\le n\le N.
\end{aligned}\tag{3.26}
$$

The second term on the right-hand side of (3.22) can be calculated as follows

$$
\begin{aligned}
\left(\xi\left(\tilde{w}^n,\tilde{\vartheta}^n\right)-\xi\left(w^n,\vartheta^n\right),\zeta^n\right) &= \left(\xi\left(\eta^n,\vartheta^n\right),\zeta^n\right)\\
&= \frac{h}{3}\sum_{i=1}^{M-1}\left(\eta_i^n\zeta_i^n\Delta_z\vartheta_i^n-\eta_i^n\vartheta_i^n\Delta_z\zeta_i^n\right)\\
&\leqslant \frac{1}{3}\left(\|\zeta^n\|_\infty|\vartheta^n|_1\|\eta^n\|+\|\vartheta^n\|_\infty|\zeta^n|_1\|\eta^n\|\right)\\
&\leqslant \frac{2c_8\sqrt{L}}{3h}\|\zeta^n\|\cdot\|\eta^n\|,\qquad 0\le n\le N.
\end{aligned}\tag{3.27}
$$

Substituting (3.23)-(3.27) into (3.22) and using Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
\frac{1}{2}\mathfrak{D}_s^\alpha\|\zeta^n\|^2+\nu\left(|\zeta^n|_1^2+\frac{h^2}{18}\|\eta^n\|^2\right) &\le \frac{c_8\sqrt{L}}{2}|\zeta^n|_1\|\zeta^n\|+\frac{hc_8\sqrt{L}}{3}\|\zeta^n\|\cdot\|\eta^n\|+(s^n,\zeta^n)\\
&\le \nu|\zeta^n|_1^2+\left(1+\frac{9c_8^2L}{4\nu}\right)\|\zeta^n\|+\frac{h^2\nu}{18}\|\eta^n\|^2+\|s^n\|,\quad 0\le n\le N.
\end{aligned}
$$

That is,

$$
\mathfrak{D}_s^\alpha\|\zeta^n\|^2\le 2\left[\|s^n\|+\left(1+\frac{9c_8^2L}{4\nu}\right)\|\zeta^n\|\right],\qquad 0\le n\le N.
$$

Using fractional Grönwall type inequality given in Lemma 2.5, we get

$$
\|\zeta^n\|^2\le 2\left[\|p\|^2+\frac{s_n^\alpha}{\Gamma(1+\alpha)}\max_{0\le k\le n}\|s^k\|^2E_\alpha(2c_9s_n^\alpha)\right],\quad 0\le n\le N.
$$

This completes the proof. □

4. Numerical Experiment

In this section, all experiments were conducted using MATLAB $R2018a$ running Windows 10 Professional 64-bit. The system is equipped with an Intel Core $i5$, and 8 GB of RAM. We present two illustrative examples to demonstrate the efficiency and numerical accuracy of our proposed scheme. Let V_i^n and ϑ_i^n $(0\le i\le M,\ 0\le n\le N)$ represent the exact solutions and numerical solutions, respectively. We compute the problem defined by equations (1.1)-(1.2) by using the fourth-order compact difference scheme (3.7). In the implementation of the implicit scheme, we employ the method of fixed-point iteration [37]. It can be seen that

$$\left(I+\frac{h^2}{12}\delta_z^2\right)\left(\mathfrak{D}_s^{\alpha}\vartheta_i^n+\xi(\vartheta_i^n,\vartheta_i^n)-\frac{h^2}{2}\xi(w_i^n,\vartheta_i^n)-g_i^n\right)=\Gamma(\beta)\mathfrak{I}_s^{\beta}\delta_z^2\vartheta_i^n+\mu\delta_z^2\vartheta_i^n,$$

where I denotes the identity operator. We can obtain

$$\begin{cases}\left(I+\frac{h^2}{12}\delta_z^2\right)\left(\mathfrak{D}_s^{\alpha}\vartheta_i^{n,k+1}+\xi^{k+\frac{1}{2}}(\vartheta_i^n,\vartheta_i^n)-\frac{h^2}{2}\xi(w_i^{n,k},\vartheta_i^{n,k})-g_i^n\right)=\Gamma(\beta)\mathfrak{I}_s^{\beta}\delta_z^2\vartheta_i^{n,k+1}+\mu\delta_z^2\vartheta_i^{n,k+1},\\ \xi^{k+\frac{1}{2}}(\vartheta_i^n,\vartheta_i^n)=\dfrac{\vartheta_{i-1}^{n,k}+\vartheta_i^{n,k}+\vartheta_{i+1}^{n,k}}{6h}\left(\vartheta_{i+1}^{n,k+1}-\vartheta_{i-1}^{n,k+1}\right),\\ (I+\frac{h^2}{12}\delta_z^2)w_i^{n,k}=\delta_z^2\vartheta_i^{n,k},\end{cases}$$

where $1\le i\le M-1$, $1\le n\le N$, k represents the iteration times, and the initial guess for each iteration is set as $\vartheta_i^{n,0}=\vartheta_i^{n-1}$,

The numerical errors in the L^{∞}-norm, along with the convergence orders in time and space, are defined as follows

$$E_{\infty}(h,\tau)=\max_{\substack{0\le i\le M\\0\le j\le N}}\left|\vartheta(z_i,s_n)-\vartheta_i^n\right|,\ \mathrm{Order}^{\tau}=\log_2\left(\frac{E_{\infty}(h,\tau)}{E_{\infty}(h,2\tau)}\right),\ \mathrm{Order}^{\mathrm{h}}=\log_2\left(\frac{E_{\infty}(h,\tau)}{E_{\infty}(2h,\tau)}\right),$$

	$E_{\infty}(h,\tau)$	$Order^{\tau}$	$CPU(s)$	$E_{\infty}(h,\tau)$	$Order^{\tau}$	$CPU(s)$	$E_{\infty}(h,\tau)$	$Order^{\tau}$	$CPU(s)$
N	$\alpha=0.15,\beta=0.25$			$\alpha=0.5,\beta=0.25$			$\alpha=0.85,\beta=0.25$		
16	1.0438e-03	*	21.286	6.6772e-05	*	21.624	2.6952e-03	*	21.385
32	2.9157e-04	1.8399	40.986	2.4869e-05	1.4249	42.062	1.2236e-03	1.1392	38.276
64	8.2521e-05	1.8210	79.779	9.0919e-06	1.4517	78.520	5.5423e-04	1.1426	72.677
128	2.2886e-05	1.8503	155.616	3.2573e-06	1.4809	146.175	2.5048e-04	1.1458	143.447
N	$\alpha=0.15,\beta=0.5$			$\alpha=0.5,\beta=0.5$			$\alpha=0.85,\beta=0.5$		
16	1.0585e-03	*	21.470	6.7665e-04	*	25.129	6.0292e-03	*	23.318
32	3.0126e-04	1.8129	46.017	2.4262e-04	1.4797	45.595	2.7345e-03	1.1407	44.291
64	8.5353e-05	1.8195	95.353	8.6539e-05	1.4873	83.159	1.2390e-03	1.1421	79.448
128	2.4050e-05	1.8274	194.410	3.0779e-05	1.4914	148.809	5.6078e-04	1.1437	149.459
N	$\alpha=0.15,\beta=0.75$			$\alpha=0.5,\beta=0.75$			$\alpha=0.85,\beta=0.75$		
16	1.0073e-03	*	27.903	1.2836e-03	*	24.304	9.7201e-03	*	21.172
32	2.8172e-04	1.8381	58.477	4.5655e-04	1.4914	43.611	4.4228e-03	1.1360	39.773
64	7.8024e-05	1.8523	115.694	1.6284e-04	1.4873	80.612	1.9959e-03	1.1479	74.594
128	2.1765e-05	1.8419	223.192	5.7509e-05	1.5016	155.084	8.9768e-04	1.1528	138.158

TABLE 1. The L^{∞}-norm errors and temporal convergence orders with $M=1024$

Example 1. Consider the following equation

$${}^{C}\mathscr{D}_{a^+}^{\alpha;\Upsilon}\vartheta(z,s)+\vartheta\frac{\partial\vartheta}{\partial z}(z,s)=\Gamma(\beta)\mathscr{I}_{a^+}^{\beta;\Upsilon}\frac{\partial^2\vartheta}{\partial z^2}(z,s)+\mu\frac{\partial^2\vartheta}{\partial z^2}(z,s)+g(z,s),\qquad 0<\alpha,\ \beta<1,$$

The problem is accompanied by the following initial and boundary conditions $\vartheta(z,0)=0$ for z in $[0,\ 1]$ and $\vartheta(0,s)=\vartheta(1,s)=0$ for s in $[0,\ 1]$, with $\Upsilon(s)=s$ and $\mu=0$. The exact solution

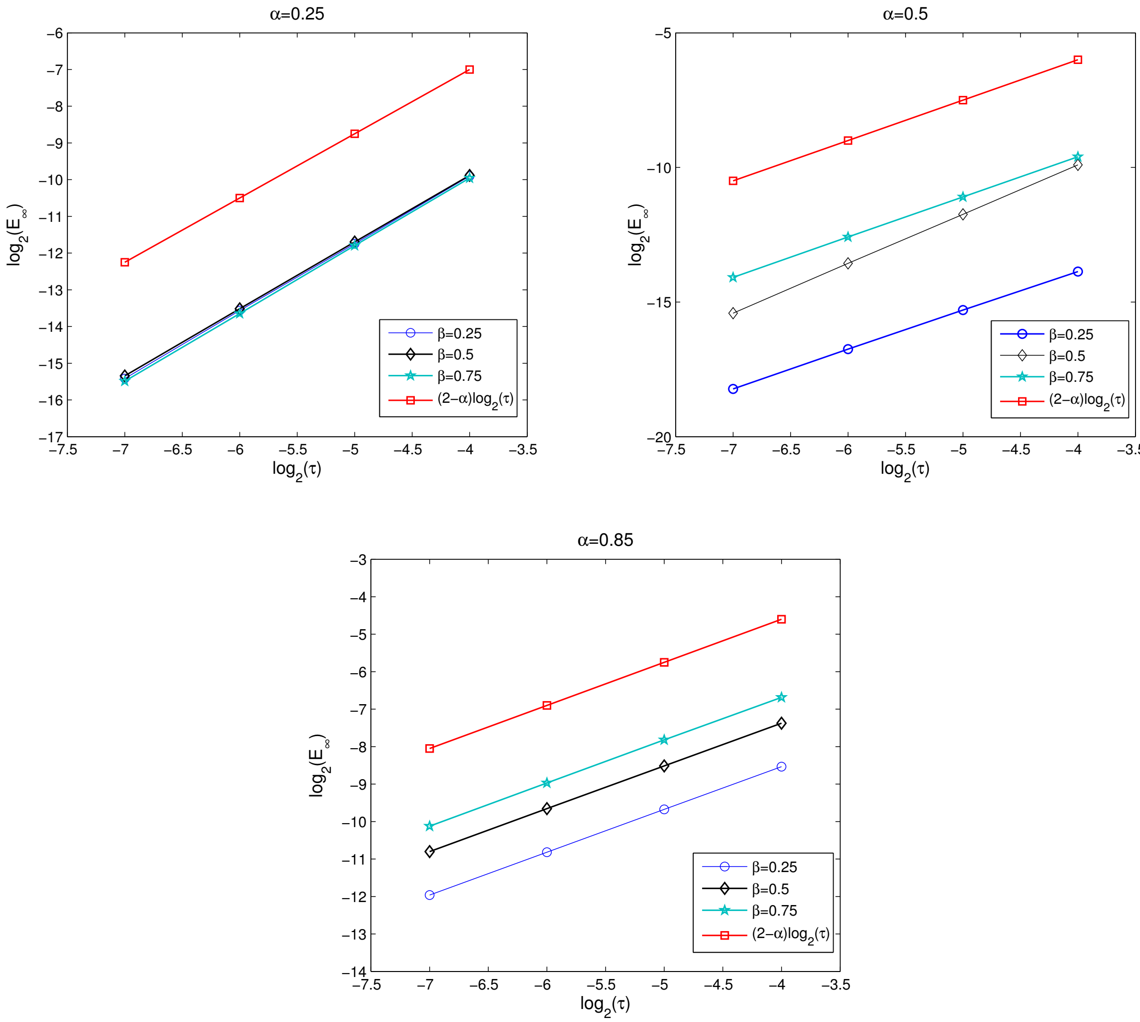


FIGURE 1. $log_2(E_\infty)$ as a function of $log_2(\tau)$ for $\alpha = 0.25, 0.5, 0.85$ and given β

to this problem is given by $\vartheta(z,s) = s^3 \sin(\pi z)$. Furthermore, we have the constructed source term denoted as g, which complies with the aforementioned conditions. It is expressed as

$$g(z,s) = \sin(\pi z)\Big(\frac{6s^{3-\alpha}}{\Gamma(4-\alpha)} + \frac{(6\pi^2\Gamma(\beta))s^{\beta+3}}{\Gamma(\beta+4)} + \pi s^6 \cos(\pi z)\Big).$$

Firstly, we investigate the convergence properties in both time and space for the fourth-order compact finite difference scheme (3.7). Using a fixed spatial grid size $h = 1/1024$, Figure 1 displays the discrete L^∞-norm error and demonstrates a temporal convergence order of $(2-\alpha)$. Similarly, Figure 2, obtained with a fixed temporal step size $\tau = 1/2000$, confirms the expected fourth-order spatial accuracy. Subsequently, we assess the accuracy of the proposed numerical method for various fractional orders α and β in the interval $(0,1)$. Tables 1 and 2 summarize these results. Table 1 highlights a convergence order of $(2-\alpha)$ in time, while Table 2 verifies the fourth-order convergence in space, in complete agreement with our theoretical analysis. Finally, Figure 3 provides visual representations of the numerical solution, exact solution, absolute error surface, and error contour plots corresponding to the specific parameter values

$\alpha = \beta = 0.5$, $h = 1/1024$, and $\tau = 1/256$. These graphical results further confirm the efficiency and accuracy of the numerical method proposed.

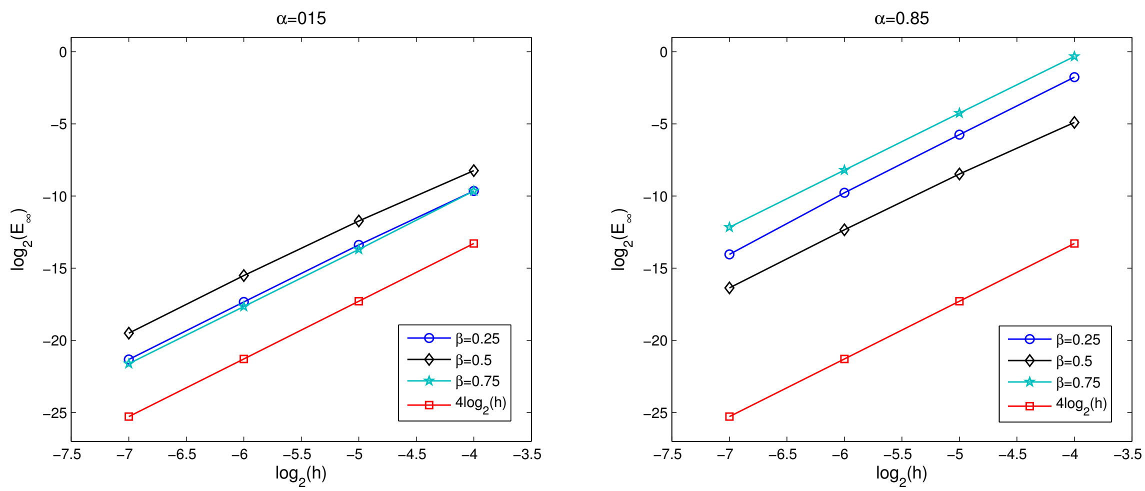


FIGURE 2. $log_2(E_\infty)$ as a function of $log_2(h)$ for $\alpha = 0.15, 0.85$ and given β

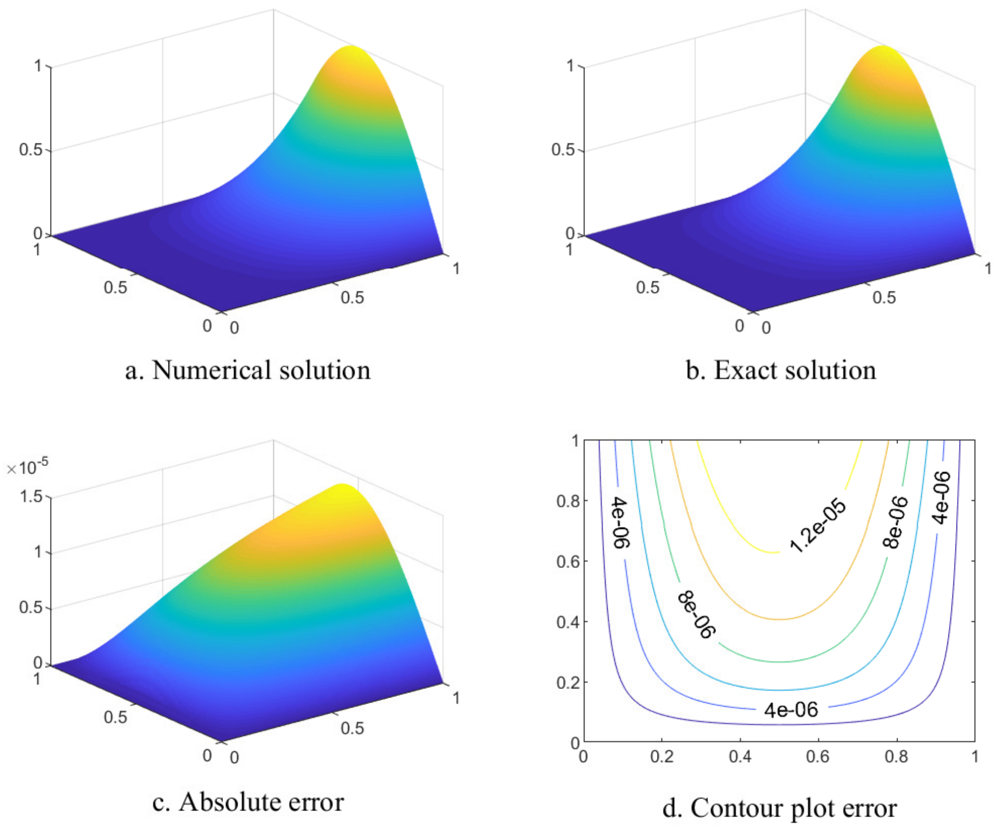
	$E_\infty(h,\tau)$	$Order^h$	$CPU(s)$	$E_\infty(h,\tau)$	$Order^h$	$CPU(s)$	$E_\infty(h,\tau)$	$Order^h$	$CPU(s)$
M	$\alpha = 0.15, \beta = 0.25$			$\alpha = 0.55, \beta = 0.25$			$\alpha = 0.85, \beta = 0.25$		
10	1.2440e-03	*	52.469	5.3702e-03	*	44.374	2.9507e-01	*	43.166
20	9.3135e-05	3.7395	53.644	3.9937e-04	3.7492	45.254	1.8631e-02	3.9853	44.091
40	6.0183e-06	3.9519	56.151	2.5356e-05	3.9773	49.828	1.1412e-03	4.0291	46.132
80	3.8121e-07	3.9807	61.258	1.5895e-06	3.9957	53.026	5.9096e-05	4.2713	50.735
M	$\alpha = 0.15, \beta = 0.5$			$\alpha = 0.55, \beta = 0.5$			$\alpha = 0.85, \beta = 0.5$		
10	3.3114e-03	*	72.821	1.2321e-02	*	47.298	3.3482e-02	*	44.187
20	2.9397e-04	3.4937	73.716	1.2583e-03	3.2916	48.369	2.8144e-03	3.5725	44.690
40	2.1349e-05	3.7834	76.122	9.1332e-05	3.7842	50.597	1.9287e-04	3.8671	46.477
80	1.3439e-06	3.9897	80.398	5.7759e-06	3.9830	55.316	1.1808e-05	4.0298	50.818
M	$\alpha = 0.15, \beta = 0.75$			$\alpha = 0.55, \beta = 0.75$			$\alpha = 0.85, \beta = 0.75$		
10	1.2440e-03	*	68.983	2.1629e-02	*	45.004	7.9768e-01	*	40.857
20	7.5198e-05	4.0481	71.604	1.4036e-03	3.9458	48.173	5.2337e-02	3.9299	42.159
40	4.8324e-06	3.9599	78.636	9.2218e-05	3.9279	50.794	3.3895e-03	3.9487	43.992
80	3.1088e-07	3.9583	85.269	5.7385e-06	4.0063	52.597	2.1777e-04	3.9602	48.960

TABLE 2. The L^∞-norm errors and spatial convergence orders with $N = 2000$

Example 2. Consider the following equation

$$^{C}\mathscr{D}_{a^+}^{\alpha;\Upsilon}\vartheta(z,s) + \vartheta\frac{\partial\vartheta}{\partial z}(z,s) = \Gamma(\beta)\mathscr{I}_{a^+}^{\beta;\Upsilon}\frac{\partial^2\vartheta}{\partial z^2}(z,s) + \mu\frac{\partial^2\vartheta}{\partial z^2}(z,s) + g(z,s),\ 0 < \alpha, \beta < 1.$$

The following initial and boundary conditions $\vartheta(z,0) = (1-z)^2 z^2$ for $z \in [0,\ 1]$ and $\vartheta(0,s) = \vartheta(1,s) = 0$ for $s \in [0,\ 1]$. The exact solution $\vartheta(z,s) = \left(s^{5/2}+1\right)z^2(1-z)^2$ with $\mu = 0$ and



a. Numerical solution

b. Exact solution

c. Absolute error

d. Contour plot error

FIGURE 3. Numerical solution, Exact solution, absolute error and contour plot error for example 1 with $\alpha = \beta = 0.5$, $M = 1024$ and $N = 256$.

	$\alpha = 0.25, \beta = 0.5$		$CPU(s)$	$\alpha = 0.55, \beta = 0.5$		$CPU(s)$	$\alpha = 0.9, \beta = 0.5$		$CPU(s)$
N	$E_\infty(h,\tau)$	$Order^\tau$		$E_\infty(h,\tau)$	$Order^\tau$		$E_\infty(h,\tau)$	$Order^\tau$	
16	1.4634e-04	*	21.896	6.6978e-05	*	20.875	2.5515e-04	*	18.376
32	4.5666e-05	1.6801	44.298	2.5509e-05	1.3927	37.684	1.2126e-04	1.0732	36.279
64	1.3771e-05	1.7295	86.255	9.5203e-06	1.4219	72.632	5.6802e-05	1.0941	72.073
128	4.0887e-06	1.7519	158.794	3.4890e-06	1.4482	144.045	2.6435e-05	1.1035	126.745
	$\alpha = 0.25, \beta = 0.5$			$\alpha = 0.55, \beta = 0.5$			$\alpha = 0.9, \beta = 0.5$		
M	$E_\infty(h,\tau)$	$Order^h$		$E_\infty(h,\tau)$	$Order^h$		$E_\infty(h,\tau)$	$Order^h$	
10	2.8175e-02	*	43.138	3.7324e-02	*	36.287	3.6206e-02	*	35.658
20	1.7214e-03	4.0328	44.776	2.3793e-03	3.9715	36.610	2.3308e-03	3.9573	38.633
40	1.4349e-04	3.5845	47.998	1.5849e-04	3.9081	40.908	1.4618e-04	3.9953	43.036
80	9.1549e-06	3.9703	53.098	1.0434e-05	3.9253	42.743	9.1428e-06	3.9996	44.211

TABLE 3. The L^∞-norm errors, temporal convergence orders with $M = 1024$ and spatial convergence with $N = 2000$

$\Upsilon(s) = s$. In addition, the constructed source term given by

$$g(z,s) = 2\big(s^{5/2}+1\big)^2(1-z)^3z^3(1-2z) + \frac{\Gamma(7/2)}{\Gamma(7/2-\alpha)}(1-z)^2z^2s^{5/2-\alpha} - 2(6z^2-6z+1)\Big(\frac{\Gamma(7/2)\Gamma(\beta)}{\Gamma(\beta+7/2)}s^{\beta+5/2} + \frac{s^\beta}{\beta}\Big).$$

In Table 3, we present the numerical results obtained for various values of α, while keeping β fixed at 0.5. Although the exact solution in Example two does not satisfy regularity conditions, we observe that the convergence order in both temporal and spatial directions still reaches $2-\alpha$ in time and fourth-order in space. This outcome underscores the feasibility and effectiveness of the proposed algorithm.

5. Conclusion

This paper introduced a novel fourth-order compact finite difference scheme for solving nonlinear Υ-Caputo time-fractional integro-differential equations with two nonlocal terms. By employing the $L1$-discretization formula, a nonlinear compact operator, we effectively addressed the complexities of fractional derivatives and nonlinearity. We established the stability and convergence of the method by using the discrete energy method, and we proposed a fixed-point iterative algorithm to solve the resulting nonlinear system. Numerical experiments confirmed the scheme's convergence rate of $O(\tau^{2-\alpha}+h^4)$, aligning with theoretical predictions. These results highlight the effectiveness of our method in accurately solving complex fractional equations.

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References

[1] R. Almeida, N. Martins, J. Vanterler da C. Sousa, Fractional tempered differential equations depending on arbitrary kernels, AIMS Math. 9 (2024), 9107–9127.

[2] R. Almeida, A Caputo fractional derivative of a function with respect to another function, Commun. Nonlinear Sci. Numer. Simul. 44 (2017), 460–481.

[3] R. Almeida, A.B. Malinowska, M.T.T. Monteiro, Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications, Math. Methods Appl. Sci. 4 (2018), 336–352.

[4] L.R. Bagley, R.A. Calico, Fractional order state equations for the control of viscoelastically damped structures, J. Guidance, Control, Dyn. 1 (1991), 304–311.

[5] D. Baleanu, A. Abdon, J. Losada, A. Jajarmi, New features of the fractional Euler-Lagrange equations for a physical system within non-singular derivative operator, Ero. Phy. J. Plus 134 (2019), 1–10.

[6] S. Baroudi, A. El Mfadel, A. Kassidi, M. Elomari, Accurate non-uniform grid schemes for singular time-fractional integro-differential equations, Commun. Nonlinear Sci. Numer. Simul. 161 (2026), 110078.

[7] S. Baroudi, M. Elomari, A. El Mfadel, A. Kassidi, Numerical solutions of the integro-partial fractional diffusion heat equation involving tempered Υ-Caputo fractional derivative, J. Math. Sci. 27 (2023), 555–567.

[8] S. Baroudi, A. El Mfadel, A. Kassidi, M.H. El Omari, A high-order compact scheme for nonlinear integro-differential equations with weakly singular kernels and tempered ψ-Caputo derivatives, Math. Methods Appl. Sci. 48 (2025), 10774–10785.

[9] S. Baroudi, N. Hatime, A. El Mfadel, A. Kassidi, M. Elomari, Accurate and efficient numerical schemes for nonlinear fractional differential equations: Stability, convergence, and error analysis, Commun. Appl. Math. Comput. (2026). doi: 10.1007/s42967-025-00552-9

[10] S. Baroudi, A. Kassidi, A. El Mfadel, M. Elomari, Coincidence degree theory for higher order nonlinear fractional differential equations: existence and uniqueness results, Commun. Nonlinear Sci. Numer. Simul. 147 (2025), 108847.

[11] A. Benmerrous, S. Baroudi, F.E. Bourhim, M. Elomari, A. El Mfadel, Numerical and analytical study of fractional hyperbolic equations with ψ-Caputo derivatives, J. Nonlinear, Complex Data Sci. (2026) doi: 10.1515/jncds-2025-0038

[12] F.E. Bourhim, A. Benmerrous, S. Baroudi, M. Elomari, A. El Mfadel, Existence and uniqueness of solutions for impulsive evolution equations with ϕ-Caputo fractional derivatives, Math. Methods Appl. Sci. (2026) doi: 10.1002/mma.70605

[13] M. Dehghan, M. Abbaszadeh, Error estimate of finite element/finite difference technique for solution of two-dimensional weakly singular integro-partial differential equation with space and time fractional derivatives, J. Comput. Appl. Math. 356 (2019), 314–328.

[14] M. Dehghan, M. Abbaszadeh, A Legendre spectral element method (SEM) based on the modified bases for solving neutral delay distributed-order fractional damped diffusion-wave equation, Math. Methods Appl. Sci. 41 (2018), 3476–3494.

[15] K. Diethelm, N.J. Ford, A.D. Freed, Detailed error analysis for a fractional Adams method, Numer. Algo 36 (2004), 31–52.

[16] H.M. Fahad, A. Fernandez, M. ur Rehman, M. Siddiqi, Tempered and Hadamard-type fractional calculus with respect to functions, Mediterr. J. Math. 18 (2021), 143.

[17] J. Guo, D. Xu, W. Qiu, A finite difference scheme for the nonlinear time-fractional partial integro-differential equation, Math. Methods Appl. Sci. 43 (2020), 3392–3412.

[18] A. Jajarmi, S. Arshad, D. Baleanu, A new fractional modelling and control strategy for the outbreak of dengue fever, Physica A, 535 (2019), 122524.

[19] K.D. Kucche, A.D. Mali, A. Fernandez, H.M. Fahad, On tempered fractional calculus with respect to functions and the associated fractional differential equations, Math. Methods Appl. Sci. 45 (2022), 11134–11157.

[20] I. Lamrani, H. Zitane, D.F. Torres, Controllability and observability of tempered fractional differential systems, Commun. Nonlinear Sci. Numer. Simul. 142 (2025), 108501.

[21] D. Li, T. Tang, Z. Zhang, Analysis of L^1-Galerkin FEMs for time-fractional nonlinear parabolic problems, arXiv preprint, arXiv:1612.00562 (2016).

[22] H. Liao, D. Li, J. Zhang, Sharp error estimate of the nonuniform $L1$ formula for linear reaction-subdiffusion equations, SIAM J. Numer. Anal. 56 (2018), 1112–1133.

[23] J. Liu, G. Hou, Numerical solutions of the space- and time-fractional coupled Burgers equations by generalized differential transform method, Appl. Math. Comput. 217 (2011), 7001–7008.

[24] R. Marks, M. Hall, Differintegral interpolation from a bandlimited signal's samples, IEEE Trans. Acoustics, Speech, Signal Process. 29 (1981), 872–877.

[25] K. Mustapha,H. Mustapha, A second-order accurate numerical method for a semilinear integro-differential equation with a weakly singular kernel, IMA J. Numer. Anal. 30 (2010), 555–578.

[26] N. Nyamoradi, C.E. Torres Ledesma, ψ-Tempered fractional differential equations with impulses, Rendiconti del Circolo Matematico di Palermo Series 2 74 (2025), 50.

[27] W.E. Olmstead, W. Slemrod, S.H. Davis, Bifurcation with memory, SIAM J. Appl. Math. 46 (1986), 171–188.

[28] I Podlubny, Fractional Differential Equations, Elsevier, 1998.

[29] S. Saifullah, A. Ali, A. Khan, K. Shah, T. Abdeljawad, A novel tempered fractional transform: theory, properties and applications to differential equations, Fractals 31 (2023), 2340045.

[30] A.A. Samarskii, V.B. Andreev, Difference Methods for Elliptic Equations, Izdatel'stvo Nauka, Moscow, 1976.

[31] J.M. Sanz-Serna, A numerical method for a partial integro-differential equation, SIAM J. Numer. Anal. 25 (1988), 319–327.

[32] Z.Z. Sun, X. Wu, A fully discrete difference scheme for a diffusion-wave system, Appl. Numer. Math. 56 (2006), 193–209.

[33] H. Sun, Z.Z. Sun, On two linearized difference schemes for Burgers' equation, Int. J. Comput. Math. 92 (2015), 1160–1179.

[34] K.S. Vishnukumar, V. Govindaraj, Exploring controllability in tempered ψ-Caputo fractional dynamical systems without delay and delays in control, Int. J. Dyn. Control 13 (2025), 391.

[35] H. Wang, D. Xu, J. Guo, Weak Galerkin finite-element method for time-fractional nonlinear integro-differential equations, Comput. Appl. Math. 39 (2020), 1–22.

[36] X. Wang, Q. Zhang, Z. Sun, The pointwise error estimates of two energy-preserving fourth-order compact schemes for viscous Burgers' equation, Adv. Comput. Math. 47 (2021), 1–42.

[37] Z. Wang, D. Cen, Y. Mo, Sharp error estimate of a compact $L1$-ADI scheme for the two-dimensional time-fractional integro-differential equation with singular kernels, Appl. Nume. Math. 159 (2021), 190–203.

[38] X. Zheng, H. Chen, W. Qiu, A Crank-Nicolson-type finite-difference scheme and its algorithm implementation for a nonlinear partial integro-differential equation arising from viscoelasticity, Comput. Appl. Math. 39 (2020), 1–23.

[39] X. Zheng, W. Qiu, H. Chen, Three semi-implicit compact finite difference schemes for the nonlinear partial integro-differential equation arising from viscoelasticity, Int. J. Model. Simul. 41 (2021), 234–242.