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SOLVABILITY AND NUMERICAL APPROXIMATION OF A DYNAMIC FRICTIONLESS THERMO-VISCOELASTIC CONTACT PROBLEM

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Abstract. In this paper, we investigate a dynamic contact problem involving a thermo-viscoelastic body in interaction with a thermally conductive foundation. The contact is modeled by using the normal compliance condition, which allows for a realistic description of the contact mechanics. We rigorously establish existence and uniqueness results for a weak solution of the problem by applying the Banach fixed-point theorem. Furthermore, we introduce a fully discrete approximation based on the finite element method combined with a backward Euler time-stepping scheme. Detailed error estimates are then derived, leading to a linear convergence rate under suitable additional regularity assumptions on the solution and the problem data.

Keywords. Error estimate; Finite element method; Thermo-viscoelastic material; Weak solvability.
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1. Introduction

The study of contact problems for thermo-viscoelastic materials is an important topic in applied mathematics and engineering. These problems describe how deformable bodies touch or press each other when temperature and mechanical effects happen together. These situations give mathematical models using partial differential equations with nonlinear boundary conditions. Thermal effects, viscoelastic behavior, and contact conditions make these problems hard to study, especially to prove that solutions exist and are unique. Recently, some works used variational and hemivariational methods and fractional viscoelastic models to study these problems. Examples are the analysis of time-fractional thermo-viscoelastic contact problems by Ouaanabi et al. [24] and the dynamic study of thermo-viscoelastic rods with damping effects [8], where weak solutions are found by using modern mathematical tools. It is important to understand how thermo-viscoelastic materials behave under contact. These materials are used in many applications such as sensors, actuators, energy harvesting devices, and smart structures. They can combine thermal, electrical, and mechanical effects, which helps engineers

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improve performance and efficiency. The mathematical study of contact problems with mechanical and thermal effects is hot. In [11], Bouallala et al. studied a dynamic contact problem with friction in thermo-viscoelasticity and proved existence and uniqueness using variational methods. In [9], Bouallala studied more advanced thermo-viscoelastic contact problems, including frictionless contact problems with time-fractional derivatives. In [23], Ouaanabi et al. investigated the contact problems with history-dependent operators and fractional effects. Indeed, they provided both theory and numerical simulations. Early results for dynamic frictional contact problems in thermo-viscoelastic materials were done in [14]. Later, the numerical study of a dynamic viscoelastic contact problem with a nonmonotone friction law, with or without thermal effect, was done in [7, 26, 27]. A general class of dynamic frictional contact problems using hemivariational inequalities was studied in [20, 21, 22], where nonsmooth analysis and fixed-point techniques were used. Recent studies also include the piezoelectric effect, showing the interactions between mechanical, thermal, and electrical fields. In [4], Alaoui et al. studied a dynamic Coulomb friction contact problem for thermo-piezoelectric materials and proved existence results. The work in [3, 5] studied the frictionless contact problem with normal compliance in thermo-piezoelectricity using a rigorous variational formulation. In [2], a quasistatic thermo-electro-viscoelastic contact problem was studied by using variational and hemivariational inequalities. The results presented in [23, 28, 29] focused on analysis and numerical approximation of hemivariational inequalities for frictional thermo-electro-viscoelastic contact problems with damage, showing that the schemes converge. The works in [10, 16, 25] studied unilateral contact problems with Coulomb friction in thermoelectro-viscoelasticity, with theory and possible numerical applications. These works show the growing complexity of coupled piezoelectric contact models and motivate more research on variational formulation, numerical approximation, and error analysis. In [12], a variational method was used to study such dynamic contact problems, with numerical simulations showing that the methods work well. In [1], optimal control for thermo-viscoelastic systems with contact and friction constraints was studied, giving theory for existence and system optimization. These works show the importance of combining mathematical analysis and numerical techniques for thermo-viscoelastic contact problems and motivate more research on variational formulation, numerical methods, and control.

In this work, we introduce a novel dynamic model describing the frictionless contact between a deformable body and a thermally conductive foundation, where the material behavior is governed by a thermo-viscoelastic constitutive law. The contact interaction is modeled by using a normal compliance condition, complemented by regularized thermal boundary conditions. The main novelty of the proposed approach lies in the strong coupling between mechanical and thermal effects in the dynamic regime, as expressed in equation (2.4), where the rate of temperature evolution explicitly depends on the mechanical deformation through the term $R(\dot{u}(t))$. This model improves previous ones by including the interaction between displacement and temperature, which is important for correctly describing the behavior of thermo-viscoelastic materials under dynamic loading. From a mathematical point of view, proving that this coupled system has a unique solution is a challenging task. We first rewrite the problem in a variational form, which gives a system with a hyperbolic equation for displacement and a parabolic equation for temperature. Then, we prove the existence and uniqueness of a weak solution. After that, we introduce a numerical method that gives a fully discrete approximation and estimates the errors. The main difficulty comes from the coupling between displacement and temperature, which requires careful estimates using fixed-point arguments. The estimation of errors for the numerical solution is also delicate because it involves both mechanical and thermal parts. These challenges are increased by nonlinear and time-dependent contact conditions, especially at the boundary Γ_3, where normal compliance and thermal flux act together.

The paper is organized as follows. Section 2 presents the mechanical problem and the governing equations. In Section 3, we specify the assumptions on the data and derive the variational formulation of the model. Section 4 is devoted to proving the existence and uniqueness of a solution using the Banach fixed-point theorem. Finally, in Section 5, we introduce a fully discrete numerical scheme along with error estimates for the approximations obtained.

2. Physical Setting and Classical Formulation

In this section, we give a classic formulation of a dynamic frictionless contact problem for a thermo-viscoelastic body with a thermally conductive rigid foundation. We consider a thermo-viscoelastic body which initially occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ with a smooth boundary $\Gamma = \partial\Omega$. The body is acted upon by body forces of density f_0 and volume heat source term q_{th} on Ω. It is also subject to mechanical and thermal constraints at its boundary. To formulate these constraints, we divide Γ into three measurable and disjoint parts Γ_1, Γ_2, and Γ_3 such that $|\Gamma_1| > 0$. We assume that the body is clamped on Γ_1 and the temperature is zero on $\Gamma_1 \cup \Gamma_2$. We also assume that surface tractions of density f_2 act on Γ_2. Over the contact surface Γ_3, the body may come in frictionless contact with a conductive obstacle, the so called foundation, which its temperature are assumed to be maintained at θ_F. Let $T > 0$. We denote by $[0,T]$ the time interval of interest and by $x \in \Omega \cup \Gamma$ and $t \in [0,T]$ the spatial and time variables, respectively. Sometimes, we omit the explicit dependence of various functions on x. Throughout this paper, the indices i, j, k, l run from 1 to d. The summation convention over repeated indices is used. We denote the space of second-order symmetric tensors on $\mathbb{R}^d$ by $\mathbb{S}^d$. In addition, we define the inner product and its associated norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ by

$$u \cdot v = u_i v_i, \quad \|v\| = \sqrt{v \cdot v}, \qquad \forall u, v \in \mathbb{R}^d,$$

$$\sigma \cdot \tau = \sigma_{ij}\tau_{ij}, \quad \|\tau\| = \sqrt{\tau \cdot \tau}, \qquad \forall \sigma, \tau \in \mathbb{S}^d.$$

We denote by ν the unit outward normal on boundary Γ and we adopt the usual notation for normal and tangential components of vectors and tensors

$$u = u_\nu \nu + u_\tau, \quad u_\nu = u \cdot \nu, \qquad \forall u \in \mathbb{R}^d,$$

$$\sigma\nu = \sigma_\nu \nu + \sigma_\tau, \qquad \sigma_\nu = (\sigma\nu) \cdot \nu, \qquad \forall \sigma \in \mathbb{S}^d.$$

Next, we use the standard notation for the Lebesgue and the Sobolev spaces associated with Ω and Γ. For a real Banach space $(\mathscr{B}, \|\cdot\|_{\mathscr{B}})$, we denote by $\mathscr{B}^*$ the dual space of $\mathscr{B}$ and we use the notation $(\cdot,\cdot)_{\mathscr{B}^* \times \mathscr{B}}$ to represent the duality pairing between $\mathscr{B}^*$ and $\mathscr{B}$. We use the usual notation for the spaces $L^2(0,T;\mathscr{B})$ and $W^{1,2}(0,T;\mathscr{B})$.

Finally, to present our mechanical problem, we denote

- $u : \Omega \times (0,T) \to \mathbb{R}^d$ the displacement field,
- $\theta : \Omega \times (0,T) \to \mathbb{R}$ the temperature field,
- $\sigma : \Omega \times (0,T) \to \mathbb{S}^d$ the stress tensor,

- $q : \Omega \times (0,T) \to \mathbb{R}^d$ the heat flux vector.

Moreover, let $\varepsilon(u) = (u_{i,j} + u_{j,i})/2$ denote the linearized strain tensor, where the index that follows a comma indicates a partial derivative with respect to the corresponding component of the spatial variable.

Now, the classical model for the dynamic frictionaless contact problem over the finite time interval $(0,T)$ in thermo-viscoelasticity can be formulated as follows.

Problem 2.1. *Find a displacement field $u : \Omega \times (0,T) \to \mathbb{R}^d$ and a temperature field $\theta : \Omega \times (0,T) \to \mathbb{R}$ such that for all $t \in (0,T)$*

$$\sigma(t) = \mathscr{A}\varepsilon(\dot{u}(t)) + \mathscr{F}\varepsilon(u(t)) - \theta(t)\mathscr{M} \quad \text{in} \quad \Omega, \tag{2.1}$$

$$q(t) = -\mathscr{K}\nabla\theta(t) \quad \text{in} \quad \Omega, \tag{2.2}$$

$$\rho\ddot{u}(t) - \mathrm{Div}\sigma(t) = f_0(t) \quad \text{in} \quad \Omega, \tag{2.3}$$

$$\dot{\theta}(t) + \mathrm{div}q(t) = R(\dot{u}(t)) + q_{th}(t) \quad \text{in} \quad \Omega, \tag{2.4}$$

$$u(t) = 0 \quad \text{on} \quad \Gamma_1, \tag{2.5}$$

$$\sigma(t)\nu = f_2(t) \quad \text{on} \quad \Gamma_2, \tag{2.6}$$

$$\theta(t) = 0 \quad \text{on} \quad \Gamma_1 \cup \Gamma_2, \tag{2.7}$$

$$-\sigma_\nu(t) = p(u_\nu(t) - g), \quad \text{on} \quad \Gamma_3, \tag{2.8}$$

$$\sigma_\tau(t) = 0 \quad \text{on} \quad \Gamma_3, \tag{2.9}$$

$$q(t)\cdot\nu = K_c(\theta(t) - \theta_F) \quad \text{on} \quad \Gamma_3, \tag{2.10}$$

$$u(0) = u_0, \qquad \dot{u}(0) = v_0, \qquad \theta(0) = \theta_0 \quad \text{in} \quad \Omega. \tag{2.11}$$

Next, we provide an explanation of the equations and boundary conditions (2.1) to (2.11). Equations (2.1)-(2.2) represent the constitutive law for thermo-viscoelastic materials, where $\mathscr{A} = (\mathscr{A}_{ijkl})$ denotes the viscosity tensor, $\mathscr{F} = (\mathscr{F}_{ijkl})$ denotes the elasticity tensor, $\mathscr{M} = (\mathscr{M}_{ij})$ denotes the thermal expansion tensor and $\mathscr{K} = (\mathscr{K}_{ij})$ denotes the thermal conductivity tensor. Equations (2.3)-(2.4) represent the equilibrium equations for the stress and heat flux fields respectively, in which ρ denotes the density of mass. For simplicity, in what follows, we set $\rho \equiv 1$. Div and div denote the divergence operators for tensor and vector fields, respectively, that is, $\mathrm{Div}\sigma = (\sigma_{ij,j})$ and $\mathrm{div}D = D_{i,i}$. The function R describes the influence of the velocity field on the temperature. Conditions (2.5)-(2.6) and (2.7) represent the mechanical and thermal boundary conditions, respectively. Note that their physical interpretation was mentioned in the second paragraph of this section. (2.8) and (2.9) represent the frictionless contact condition with normal compliance, where g denotes the gap function that characterizes the separation between the body and the foundation at the contact surface Γ_3, p denotes the normal compliance function. Equation (2.10) describes the temperature boundary condition on Γ_3 in which positive constant K_c is the coefficient of heat exchange between the foundation and the body. Finally, (2.11) represents the initial conditions of the problem, where u_0, v_0 and θ_0 are given functions that denote the initial displacement, initial velocity and initial temperature, respectively.

Remark 2.1. Problem 2.1 describes the dynamic behavior of a thermo-viscoelastic body subjected to external forces, thermal effects, and unilateral frictionless contact on a portion of the boundary. The coupling between the mechanical and thermal equations arises through the dependence of the stress on the temperature field and through the presence of a heat source related to the rate of deformation. The mixed boundary conditions, combining clamped boundary parts, prescribed surface forces, normal contact with a foundation, and thermal exchanges, represent realistic physical situations encountered in engineering applications. From a mathematical point of view, the coupled structure of the system and the nonlinearity induced by the contact law make the analysis challenging and motivate the use of variational methods to establish existence and uniqueness results.

3. Variational Formulation and Main Results

In this section, we derive a weak formulation of Problem 2.1 and we present our main existence and uniqueness result. In order to achieve these, we need some notations. Let us denote by H, $\mathbf{H^1}$ and $\mathscr{D}$ the following spaces:

$$H=[L^2(\Omega)]^d=\left\{v=(v_i)\mid v_i\in L^2(\Omega)\right\},$$
$$\mathbf{H^1}=[H^1(\Omega)]^d=\left\{v=(v_i)\mid v_i\in H^1(\Omega)\right\},$$
$$\mathscr{D}=\{\sigma=(\sigma_{ij});\sigma_{ij}=\sigma_{ji}\in L^2(\Omega)\}.$$

The spaces H, $\mathbf{H^1}$ and $\mathscr{D}$ are real Hilbert spaces endowed with the following inner products:

$$(u,v)_H=\int_\Omega u_iv_idx,\quad \forall u,v\in H,$$
$$(\sigma,\tau)_{\mathscr{D}}=\int_\Omega \sigma_{ij}\tau_{ij}dx,\quad \forall\sigma,\tau\in\mathscr{D},$$
$$(u,v)_{\mathbf{H^1}}=(u,v)_H+(\varepsilon(u),\varepsilon(v))_{\mathscr{D}},\quad \forall u,v\in\mathbf{H^1},$$

and let $\|\cdot\|_H$, $\|\cdot\|_{\mathscr{D}}$ and $\|\cdot\|_{\mathbf{H^1}}$ be their associated norms, respectively.

For the displacement field, keeping in mind condition (2.5), we introduce $V=\{v\in\mathbf{H^1}\mid v=0\quad\text{on }\Gamma_1\}$, the closed subspace of $\mathbf{H^1}$, endowed with the following inner product

$$(u,v)_V=(\varepsilon(u),\varepsilon(v))_{\mathscr{D}},\quad \forall u,v\in V,$$

and its associated norm

$$\|v\|_V=\|\varepsilon(v)\|_{\mathscr{D}}. \tag{3.1}$$

Since $|\Gamma_1|>0$, the following Korn's inequality holds: There exists $c_k>0$ depending only on Ω and Γ_1 such that

$$\|\varepsilon(v)\|_{\mathscr{D}}\geq c_k\|v\|_{\mathbf{H^1}},\quad \forall v\in V. \tag{3.2}$$

A proof of Korn's inequality can be found in, e.g., [17, P. 16–17]. It follows from (3.1) and (3.2) that $\|\cdot\|_V$ is equivalent on V to the usual norm $\|\cdot\|_{\mathbf{H^1}}$. Thus $(V,\|\cdot\|_V)$ is a real Hilbert space.

On another hand, for the temperature field, keeping in mind (2.7) we introduce the closed functions subspace of $H^1(\Omega)$

$$Q=\{\eta\in H^1(\Omega)\mid \eta=0\quad\text{on}\quad\Gamma_1\cup\Gamma_2\},$$

endowed with the following inner product:

$$(\theta,\eta)_Q=(\nabla\theta,\nabla\eta)_H,\quad \forall\theta,\eta\in Q, \tag{3.3}$$

and the associated norm

$$\|\eta\|_Q = \|\nabla\eta\|_H. \tag{3.4}$$

Since $|\Gamma_1| > 0$, Friedrichs-Poincaré inequality holds. Thus there exists a constant $c_p > 0$ such that

$$\|\nabla\eta\|_H \geq c_p \|\eta\|_{H^1(\Omega)}, \quad \forall\eta \in Q. \tag{3.5}$$

It follows from (3.4)-(3.5) that $\|\cdot\|_Q$ is equivalent on Q to the usual norm $\|\cdot\|_{H^1(\Omega)}$ and then $(Q, \|\cdot\|_Q)$ is a real Hilbert space.

For simplicity, for an element ω in $\mathbf{H^1}$ or $H^1(\Omega)$, we still denote by ω its trace $\gamma(\omega)$ on Γ. By trace theorem, there exists a constant $c_0 > 0$ and $c_1 > 0$ such that

$$\|v\|_{[L^2(\Gamma_3)]^d} \leq c_0 \|v\|_V, \quad \forall v \in V,$$

$$\|\eta\|_{L^2(\Gamma_3)} \leq c_1 \|\eta\|_Q, \quad \forall\eta \in Q.$$

Finally, we recall the Gelfand triples $V \subset H \subset V^*$ and $Q \subset L^2(\Omega) \subset Q^*$, Let us denote

$$(u,v)_{V^*\times V} = (u,v)_H, \quad \forall u \in H, v \in V,$$

$$(\theta,\eta)_{Q^*\times Q} = (\theta,\eta)_{L^2(\Omega)}, \quad \forall\theta \in L^2(\Omega), \eta \in Q.$$

To study Problem 2.1, we need the following assumptions on its data.

$(\mathbf{H}_{\mathscr{A}})$ The viscosity tensor $\mathscr{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

(1) $\mathscr{A}(\cdot,\zeta)$ is measurable on Ω for all $\zeta \in \mathbb{S}^d$.
(2) There exists $m_{\mathscr{A}} > 0$ such that $(\mathscr{A}(x,\zeta_1) - \mathscr{A}(x,\zeta_2))(\zeta_1 - \zeta_2) \geq m_{\mathscr{A}} \|\zeta_1 - \zeta_2\|^2$ for all $\zeta_1, \zeta_2 \in \mathbb{S}^d$, a.e. $x \in \Omega$.
(3) There exists $L_{\mathscr{A}} > 0$ such that $\|\mathscr{A}(x,\zeta_1) - \mathscr{A}(x,\zeta_2)\| \leq L_{\mathscr{A}} \|\zeta_1 - \zeta_2\|$ for all $\zeta_1, \zeta_2 \in \mathbb{S}^d$, a.e. $x \in \Omega$.
(4) $\mathscr{A}(x,0) = 0$ for a.e. $x \in \Omega$.

$(\mathbf{H}_{\mathscr{F}})$ The elasticity tensor $\mathscr{F} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

(1) $\mathscr{F}(\cdot,\zeta)$ is measurable on Ω for all $\zeta \in \mathbb{S}^d$.
(2) There exists $m_{\mathscr{F}} > 0$ such that $(\mathscr{F}(x,\zeta_1) - \mathscr{F}(x,\zeta_2))(\zeta_1 - \zeta_2) \geq m_{\mathscr{F}} \|\zeta_1 - \zeta_2\|^2$ for all $\zeta_1, \zeta_2 \in \mathbb{S}^d$, a.e. $x \in \Omega$.
(3) There exists $L_{\mathscr{F}} > 0$ such that $\|\mathscr{F}(x,\zeta_1) - \mathscr{F}(x,\zeta_2)\| \leq L_{\mathscr{F}} \|\zeta_1 - \zeta_2\|$ for all $\zeta_1, \zeta_2 \in \mathbb{S}^d$, a.e. $x \in \Omega$.
(4) $\mathscr{F}(x,0) = 0$ for a.e. $x \in \Omega$.

$(\mathbf{H}_{\mathscr{K}})$ The thermal conductivity tensor $\mathscr{K} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

(1) $\mathscr{K}(\cdot,\zeta)$ is measurable on Ω for all $\zeta \in \mathbb{R}^d$.
(2) There exists $m_{\mathscr{K}} > 0$ such that $(\mathscr{K}(x,\zeta_1) - \mathscr{K}(x,\zeta_2))(\zeta_1 - \zeta_2) \geq m_{\mathscr{K}} \|\zeta_1 - \zeta_2\|^2$ for all $\zeta_1, \zeta_2 \in \mathbb{R}^d$, a.e. $x \in \Omega$.
(3) There exists $L_{\mathscr{K}} > 0$ such that $\|\mathscr{K}(x,\zeta_1) - \mathscr{K}(x,\zeta_2)\| \leq L_{\mathscr{K}} \|\zeta_1 - \zeta_2\|$ for all $\zeta_1, \zeta_2 \in \mathbb{R}^d$ for all $\zeta \in \mathbb{R}^d$ and a.e. $x \in \Omega$.
(4) $\mathscr{K}(x,0) = 0$ for a.e. $x \in \Omega$.

$(\mathbf{H}_{\mathscr{M}})$ The thermal expansion tensor $\mathscr{M} : \Omega \times \mathbb{R} \to \mathbb{S}^d$ satisfies

(1) $\mathscr{M}(\cdot,\zeta)$ is measurable on Ω for all $\zeta \in \mathbb{R}$.
(2) There exists $L_{\mathscr{M}} > 0$ such that $\|\mathscr{M}(x,\zeta_1) - \mathscr{M}(x,\zeta_2)\| \leq L_{\mathscr{M}} \|\zeta_1 - \zeta_2\|$ for all $\zeta_1, \zeta_2 \in \mathbb{R}$ and a.e. $x \in \Omega$.
(3) $\mathscr{M}(x,0) = 0$ for a.e. $x \in \Omega$.

$(\mathbf{H_R})$ The function $R:\mathbb{R}^d\to\mathbb{R}$ satisfies
(1) $R(\zeta)\in L^2(\Omega)$ for all $\zeta\in\mathbb{S}^d$.
(2) There exists $L_R>0$ such that $\|R(\zeta_1)-R(\zeta_2)\|_{L^2(\Omega)}\le L_R\|\zeta_1-\zeta_2\|_V$ for all $\zeta_1,\zeta_2\in\mathbb{S}^d$.
(3) $R(0)=0$.

$(\mathbf{H_p})$ The normal compliance function $p:\Gamma_3\times\mathbb{R}\to\mathbb{R}_+$ satisfy
(1) $p(\cdot,r)$ is continuous for all $r\in\mathbb{R}$.
(2) $p(x,r)=0$ for all $r<0$ and a.e. $x\in\Gamma_3$.
(3) There exists $M_p>0$ such that $p(x,r)\le M_p$ for all $r\in\mathbb{R}$ and a.e. $x\in\Gamma_3$.
(4) There exists $L_p>0$ such that $\|p(x,r_1)-p(x,r_2)\|\le L_p\|r_1-r_2\|$ for all $r_1,r_2\in\mathbb{R}$ and a.e. $x\in\Gamma_3$.

$(\mathbf{H_D})$ The forces, tractions, heat source strength, temperature of the foundation, gap function, and initial conditions are assumed to possess the following regularity.

$$f_0\in L^2(0,T;H),\qquad f_2\in L^2(0,T;[L^2(\Gamma_2)]^d),\qquad q_{th}\in L^2(0,T;L^2(\Omega)),$$
$$\theta_F\in L^2(\Gamma_3),\qquad g\in L^2(\Gamma_3),\qquad u_0\in V,\qquad v_0\in H,\qquad \theta_0\in L^2(\Omega).$$

Next, we define the elements $f(t)\in V^*$ and $q_c(t)\in Q^*$ by

$$(f(t),w)_{V^*\times V}=(f_0(t),w)_H+(f_2(t),w)_{[L^2(\Gamma_2)]^d},$$
$$(q_c(t),\eta)_{Q^*\times Q}=(q_{th}(t),\eta)_{L^2(\Omega)}+(k\theta_F,\eta)_{L^2(\Gamma_3)},$$

for all $w\in V$ and $\eta\in Q$. We also define the mapping $j:V\times V\to\mathbb{R}$ by

$$j(u,w)=\int_{\Gamma_3}p(u_\nu-g)w_\nu da.$$

By applying Green's formula, we derive the following variational formulation of Problem 2.1, expressed in terms of the displacement and temperature fields.

Problem 3.1. *Find a displacement field $u:(0,T)\to V$ and a temperature field $\theta:(0,T)\to Q$ such that for a.e. $t\in(0,T)$ and all $(w,\eta)\in V\times Q$*

$$\begin{aligned}(\ddot u(t),w)_{V^*\times V}&+(\mathscr{A}\varepsilon(\dot u(t)),\varepsilon(w))_{\mathscr{D}}+(\mathscr{F}\varepsilon(u(t)),\varepsilon(w))_{\mathscr{D}}\\&-(\theta(t)\mathscr{M},\varepsilon(w))_{\mathscr{D}}+j(u(t),w)=(f(t),w)_{V^*\times V},\end{aligned}\tag{3.6}$$

$$\begin{aligned}(\dot\theta(t),\eta)_{Q^*\times Q}&+(\mathscr{K}\nabla\theta(t),\nabla\eta)_H-(R(\dot u(t)),\eta)_{L^2(\Omega)}\\&+K_c(\theta(t),\eta)_{L^2(\Gamma_3)}=(q_c(t),\eta)_{Q^*\times Q},\end{aligned}\tag{3.7}$$

$$u(0)=u_0,\qquad \dot u(0)=v_0,\qquad \theta(0)=\theta_0.$$

We now present our primary existence and uniqueness result, which will be proven in the next section, as follows:

Theorem 3.1. *Assume that $(\mathbf{H}_{\mathscr{A}})$, $(\mathbf{H}_{\mathscr{F}})$, $(\mathbf{H}_{\mathscr{K}})$, $(\mathbf{H}_{\mathscr{M}})$, $(\mathbf{H}_{\mathscr{P}})$, $(\mathbf{H_R})$, $(\mathbf{H_p})$ and $(\mathbf{H_D})$ hold, and the following smallness assumption*

$$\min\{m_{\mathscr{A}},\ m_{\mathscr{K}}-K_cc_1^2\}>\frac{L_{\mathscr{M}}+L_R}{2c_p},\tag{3.8}$$

is satisfied. Then Problem 3.1 has a unique solution (u,θ) which satisfies the following regularity conditions

$$u \in C^1(0,T;H) \cap W^{1,2}(0,T;V), \quad \ddot{u} \in L^2(0,T;V^*),$$
$$\theta \in C(0,T;H) \cap L^2(0,T;Q), \quad \dot{\theta} \in L^2(0,T;Q^*).$$

It is important to note that, once the displacement field u and the temperature field θ are determined by solving **Problem** 3.1, the stress σ and heat flux q can then be calculated using the thermo-viscoelastic constitutive laws (2.1)-(2.2).

4. Proof of **Theorem 3.1**

The proof of **Theorem 3.1** will be conducted in three steps, utilizing arguments based on fixed point theory.

Step 1: Let $\zeta \in L^2(0,T;V^*)$. We introduce the following intermediate problem in terms of the displacement and temperature fields.

Problem 4.1. *Find a displacement field $u_\zeta : (0,T) \to V$ and a temperature field $\theta_\zeta : (0,T) \to Q$ such that for a.e. $t \in (0,T)$ and all $(w,\eta) \in V \times Q$*

$$\begin{aligned}&(\ddot{u}_\zeta(t),w)_{V^*\times V} + (\mathscr{A}\varepsilon(\dot{u}_\zeta(t)),\varepsilon(w))_{\mathscr{D}} - (\theta_\zeta(t)\mathscr{M},\varepsilon(w))_{\mathscr{D}}\\ &\quad + (\zeta(t),w)_{V^*\times V} = (f(t),w)_{V^*\times V},\end{aligned}$$

$$\begin{aligned}&(\dot{\theta}_\zeta(t),\eta)_{Q^*\times Q} + (\mathscr{K}\nabla\theta_\zeta(t),\nabla\eta)_H - (R(\varepsilon(\dot{u}_\zeta(t))),\eta)_{L^2(\Omega)}\\ &\quad + K_c(\theta_\zeta(t),\eta)_{L^2(\Gamma_3)} = (q_c(t),\eta)_{Q^*\times Q},\end{aligned}$$

$$u_\zeta(0) = u_0, \qquad \dot{u}_\zeta(0) = v_0, \qquad \theta_\zeta(0) = \theta_0.$$

The result below concerns the unique solvability and regularity of solution to **Problem 4.1**.

Lemma 4.1. *Let the assumptions of Theorem 3.1 hold. Then Problem 4.1 has a unique solution (u_ζ,θ_ζ) which satisfies $u_\zeta \in C^1(0,T;H) \cap W^{1,2}(0,T;V)$ and $\theta_\zeta \in C(0,T;H) \cap L^2(0,T;Q)$ with $\ddot{u}_\zeta \in L^2(0,T;V^*)$ and $\dot{\theta}_\zeta \in L^2(0,T;Q^*)$. Moreover, for $\zeta_1,\zeta_2 \in L^2(0,T;V^*)$, let us denote by $(u_{\zeta_1},\theta_{\zeta_1})$ and $(u_{\zeta_2},\theta_{\zeta_2})$ the solutions of Problem 4.1 corresponding to ζ_1 and ζ_2, respectively. Then there exists a constant $c>0$ such that, for all $t \in (0,T)$,*

$$\left\|u_{\zeta_1}(t) - u_{\zeta_2}(t)\right\|_V^2 \le c\int_0^t \|\zeta_1(s) - \zeta_2(s)\|_{V^*}^2\, ds. \tag{4.1}$$

Proof. To solve Problem 4.1, we introduce the velocity field v_ζ, which is related to the displacement field u_ζ by the following relationship:

$$v_\zeta(t) = \dot{u}_\zeta(t) \quad \text{and} \quad u_\zeta(t) = u_0 + \int_0^t v_\zeta(s)ds.$$

Then, we reformulate Problem 4.1 in terms of the velocity and temperature fields as follows.

Problem 4.2. *Find a velocity field* $v_\zeta : (0,T) \to V$ *and a temperature field* $\theta_\zeta : (0,T) \to Q$ *such that for a.e.* $t \in (0,T)$ *and all* $(w,\eta) \in V \times Q$

$$(\dot{v}_\zeta(t), w)_{V^*\times V} + (\mathscr{A}\varepsilon(v_\zeta(t)), \varepsilon(w))_{\mathscr{D}} - (\theta_\zeta(t)\mathscr{M}, \varepsilon(w))_{\mathscr{D}} = (f_\zeta(t), w)_{V^*\times V}, \tag{4.2}$$

$$\begin{aligned}(\dot{\theta}_\zeta(t), \eta)_{Q^*\times Q} + (\mathscr{K}\nabla\theta_\zeta(t), \nabla\eta)_H - (R(v_\zeta(t)), \eta)_{L^2(\Omega)}& \\ + K_c(\theta_\zeta(t), \eta)_{L^2(\Gamma_3)} = (q_c(t), \eta)_{Q^*\times Q},&\end{aligned} \tag{4.3}$$

$$u_\zeta(0) = u_0, \qquad \dot{u}_\zeta(0) = v_0, \qquad \theta_\zeta(0) = \theta_0,$$

where

$$(f_\zeta(t), w)_{V^*\times V} = (f(t) - \zeta(t), w)_{V^*\times V}, \quad \forall w \in V, t \in (0,T).$$

In solving **Problem 4.2**, we utilize [30, Theorem 2.29] within the following functional framework: $\mathscr{V} = V \times Q$ and $\mathscr{H} = H \times L^2(\Omega)$ equipped with the following inner products:

$$\begin{aligned}(X,Y)_{\mathscr{V}} &= (v,w)_V + (\theta,\eta)_Q, \quad \text{for all } X = (v,\theta),\ Y = (w,\eta) \in \mathscr{V},\\ (X,Y)_{\mathscr{H}} &= (v,w)_H + (\theta,\eta)_{L^2(\Omega)}, \quad \text{for all } X = (v,\theta),\ Y = (w,\eta) \in \mathscr{H}.\end{aligned}$$

Let $\|\cdot\|_{\mathscr{V}}$ and $\|\cdot\|_{\mathscr{H}}$ be their associated norms, respectively. We introduce the operator $A : \mathscr{V} \to \mathscr{V}^*$ defined by

$$\begin{aligned}(AX,Y)_{\mathscr{V}^*\times\mathscr{V}} = (\mathscr{A}\varepsilon(v), \varepsilon(w))_{\mathscr{D}} + (\mathscr{K}\nabla\theta, \nabla\eta)_H - (\theta\mathscr{M}, \varepsilon(w))_{\mathscr{D}}&\\ - (R(v), \eta)_{L^2(\Omega)} + K_c(\theta_\zeta, \eta)_{L^2(\Gamma_3)},&\end{aligned}$$

for all $X = (v,\theta),\ Y = (w,\eta) \in \mathscr{V}$. By using $(\mathbf{H}_{\mathscr{A}})$, $(\mathbf{H}_{\mathscr{K}})$, $(\mathbf{H}_{\mathscr{M}})$ and $(\mathbf{H_R})$, it is easy to see that A is continuous, an then it is a forteriori hemicontinuous operator. By using again $(\mathbf{H}_{\mathscr{A}})$, $(\mathbf{H}_{\mathscr{K}})$, $(\mathbf{H}_{\mathscr{M}})$, and $(\mathbf{H_R})$, we obtain that

$$\begin{aligned}\|(AX,Y)_{\mathscr{V}^*\times\mathscr{V}}\| \le{}& L_{\mathscr{A}}\|v\|_V\|w\|_V + L_{\mathscr{K}}\|\theta\|_Q\|\eta\|_Q + L_{\mathscr{M}}\|\theta\|_{L^2(\Omega)}\|w\|_V\\ &+ L_R\|v\|_V\|\eta\|_{L^2(\Omega)} + K_c c_1^2\|\theta\|_Q\|\eta\|_Q,\end{aligned} \tag{4.4}$$

For all $X = (v,\theta)$ and $Y = (w,\eta)$ in $\mathscr{V}$, we can easily deduce from (3.5) that

$$\|\eta\|_{L^2(\Omega)} \le \frac{1}{c_p}\|\eta\|_Q, \quad \forall \eta \in Q. \tag{4.5}$$

Then we deduce from (4.4) that

$$\|(AX,Y)_{\mathscr{V}^*\times\mathscr{V}}\| \le \left[\left(L_{\mathscr{A}} + \frac{L_R}{c_p}\right)\|v\|_V + \left(L_{\mathscr{K}} + K_c c_1^2 + \frac{L_{\mathscr{M}}}{c_p}\right)\|\theta\|_Q\right]\|Y\|_{\mathscr{V}},$$

for all $X = (v,\theta),\ Y = (w,\eta) \in \mathscr{V}$. This leads to

$$\|AX\|_{\mathscr{V}^*} \le \left(L_{\mathscr{A}} + L_{\mathscr{K}} + K_c c_1^2 + \frac{L_R + L_{\mathscr{M}}}{c_p}\right)\|X\|_{\mathscr{V}},$$

for all $X \in \mathscr{V}$. This leads to $\|AX\|_{\mathscr{V}^*} \le M_A\|X\|_{\mathscr{V}}, \quad \forall X \in \mathscr{V}$, where $M_A = L_{\mathscr{A}} + L_{\mathscr{K}} + kc_1^2 + \frac{L_R + L_{\mathscr{M}}}{c_p}$. Given the smallness assumption (3.8), we assert that A satisfies $\underline{(H_2)}$. To illustrate, let

$X=(v,\theta)\in\mathscr{V}$. By applying $(\mathbf{H}_{\mathscr{A}})$, $(\mathbf{H}_{\mathscr{K}})$, $(\mathbf{H}_{\mathscr{M}})$, and $(\mathbf{H}_{\mathbf{R}})$, we deduce that

$$\begin{aligned}(AX,X)_{\mathscr{V}^*\times\mathscr{V}} \geq & m_{\mathscr{A}}\|v\|_V^2+m_{\mathscr{K}}\|\theta\|_Q^2-L_{\mathscr{M}}\|\theta\|_{L^2(\Omega)}\|v\|_V \\ & -L_R\|v\|_V\|\theta\|_{L^2(\Omega)}-K_c c_1^2\|\theta\|_Q^2.\end{aligned}$$

Therefore, by (4.5) and $ab\leq\frac{1}{2}(a^2+b^2)$ for all $a,b\in\mathbb{R}_+$, we obtain

$$(AX,X)_{\mathscr{V}^*\times\mathscr{V}}\geq\left(m_{\mathscr{A}}-\frac{L_{\mathscr{M}}+L_R}{2c_p}\right)\|v\|_V^2+\left(m_{\mathscr{K}}-K_c c_1^2-\frac{L_{\mathscr{M}}+L_R}{2c_p}\right)\|\theta\|_Q^2.$$

Thus A satisfies $(AX,X)_{\mathscr{V}^*\times\mathscr{V}}\geq m_A\|X\|_{\mathscr{V}}^2$, for all $X\in\mathscr{V}$, where

$$m_A=\min\{m_{\mathscr{A}},\ m_{\mathscr{K}}-K_c c_1^2\}-\frac{L_{\mathscr{M}}+L_R}{2c_p}.$$

Next, we define the elements $X_0\in\mathscr{V}$ and $F_\zeta\in L^2(0,T;\mathscr{V}^*)$ by

$$X_0=(v_0,\theta_0),\ (F_\zeta(t),Y)_{\mathscr{V}^*\times\mathscr{V}}=(f_\zeta(t),w)_{V^*\times V}+(q_c(t),\eta)_{Q^*\times Q},$$

for all $Y=(w,\eta)\in\mathscr{V}$ and $t\in(0,T)$. We apply [30, Theorem 2.29] to deduce the following variational equality

$$\begin{gathered}(\dot{X}_\zeta(t),Y)_{\mathscr{V}^*\times\mathscr{V}}+(AX_\zeta(t),Y)_{\mathscr{V}^*\times\mathscr{V}}=(F_\zeta(t),Y)_{\mathscr{V}^*\times\mathscr{V}},\\ \forall Y\in\mathscr{V}\text{ and } a.e. t\in(0,T),\end{gathered}\tag{4.6}$$

(it can be reduced to $\dot{X}_\zeta(t)+AX_\zeta(t)=F_\zeta(t),\forall Y\in\mathscr{V}$ and $a.e.\ t\in(0,T)$) with the initial data

$$X_\zeta(0)=X_0.\tag{4.7}$$

has a unique solution $X_\zeta=(v_\zeta,\theta_\zeta)$ which satisfies $X_\zeta\in C(0,T;\mathscr{H})\cap L^2(0,T;\mathscr{V})$ and $\dot{X}_\zeta\in L^2(0,T;\mathscr{V}^*)$. Now, it remains to show that we recover from (4.6)-(4.7) the coupled system (4.2)-(4.3). By choosing $Y=(w,0)$ in (4.6), where w is an arbitrary element of V, we obtain that $v_\zeta(t)\in V$ is a solution to (4.2) for a.e. $t\in(0,T)$. Moreover, if we choose $Y=(0,\eta)$ in (4.6), where η is an arbitrary element of Q, we find that $\theta_\zeta(t)\in Q$ is the solution of (4.3) for a.e. $t\in(0,T)$. Therefore, we conclude that (4.6)-(4.7) is equivalent to system (4.2)-(4.3). It follows that the couple (v_ζ,θ_ζ) represents the unique solution to Problem 4.2.

Now, Let $u_\zeta:[0,T]\to V$ be the function defined by

$$u_\zeta(t)=u_0+\int_0^t v_\zeta(s)ds,\qquad\forall t\in[0,T].\tag{4.8}$$

Then, we find that (u_ζ,θ_ζ) is the unique solution of Problem 4.1 which satisfies

$$\begin{gathered}u_\zeta\in C^1(0,T;H)\cap W^{1,2}(0,T;V),\quad \ddot{u}_\zeta\in L^2(0,T;V^*),\\ \theta_\zeta\in C(0,T;L^2(\Omega))\cap L^2(0,T;Q),\quad \dot{\theta}_\zeta\in L^2(0,T;Q^*).\end{gathered}$$

We now proceed to demonstrate inequality (4.1). Consider two elements, denoted as ζ_1 and ζ_2, from $L^2(0,T;V^*)$. We apply variational equality (4.6) separately to ζ_1 and ζ_2, substituting $Y=X_{\zeta_1}(t)-X_{\zeta_2}(t)$ in both the first in the second equality. By subtracting these resulting equalities, we obtain the following equality for all $t\in(0,T)$

$$\begin{aligned}&(\dot{X}_{\zeta_1}(t)-\dot{X}_{\zeta_2}(t),X_{\zeta_1}(t)-X_{\zeta_2}(t))_{\mathscr{V}^*\times\mathscr{V}}+(A(X_{\zeta_1}(t)-X_{\zeta_2}(t)),X_{\zeta_1}(t)-X_{\zeta_2}(t))_{\mathscr{V}^*\times\mathscr{V}}\\ &=((\zeta_1(t),0)-(\zeta_2(t),0),X_{\zeta_1}(t)-X_{\zeta_2}(t))_{\mathscr{V}^*\times\mathscr{V}}.\end{aligned}$$

By employing the Cauchy-Schwarz inequality, the strong monotonicity of A which previously established, we find that for all $t \in (0,T)$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|X_{\zeta_1}(t)-X_{\zeta_2}(t)\right\|^2_{\mathscr{H}}+m_A\left\|X_{\zeta_1}(t)-X_{\zeta_2}(t)\right\|^2_{\mathscr{V}}\\&\leq\|(\zeta_1(t),0)-(\zeta_2(t),0)\|_{\mathscr{V}^*}\left\|X_{\zeta_1}(t)-X_{\zeta_2}(t)\right\|_{\mathscr{V}}.\end{aligned}$$

By utilizing the following Young's inequality, $ab \leq \varepsilon a^2+\frac{1}{4\varepsilon}b^2$, for all $a,b \in \mathbb{R},\ \varepsilon>0$, we can easily find that, for all $t \in (0,T)$,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|X_{\zeta_1}(t)-X_{\zeta_2}(t)\right\|^2_{\mathscr{H}}+(m_A-\varepsilon)\left\|X_{\zeta_1}(t)-X_{\zeta_2}(t)\right\|^2_{\mathscr{V}}\\&\leq\frac{1}{4\varepsilon}\|(\zeta_1(t),0)-(\zeta_2(t),0)\|^2_{\mathscr{V}^*},\end{aligned}$$

where ε is a real positive parameter chosen to be small enough. We integrate the previous inequality from 0 to $t \in (0,T)$ to find that there exists a constant $c>0$ such that

$$\int_0^t\left\|X_{\zeta_1}(s)-X_{\zeta_2}(s)\right\|^2_{\mathscr{V}}ds\leq c\int_0^t\|(\zeta_1(s),0)-(\zeta_2(s),0)\|^2_{\mathscr{V}^*}ds.$$

By exploiting the definitions of the norms $\|\cdot\|_{\mathscr{V}}$ and $\|\cdot\|_{\mathscr{H}}$, we can easily find that for all $t \in (0,T)$,

$$\int_0^t\left\|v_{\zeta_1}(s)-v_{\zeta_2}(s)\right\|^2_V ds+\int_0^t\left\|\theta_{\zeta_1}(s)-\theta_{\zeta_2}(s)\right\|^2_Q ds\leq c\int_0^t\|\zeta_1(s)-\zeta_2(s)\|^2_{V^*}ds. \tag{4.9}$$

From (4.8), we have

$$\left\|u_{\zeta_1}(t)-u_{\zeta_2}(t)\right\|^2_V\leq T\int_0^t\left\|v_{\zeta_1}(s)-v_{\zeta_2}(s)\right\|^2_V ds,\quad \forall t\in(0,T). \tag{4.10}$$

This leads to conclude from (4.9) that estimate (4.1) holds. □

<u>Step 2</u>: For $\zeta \in L^2(0,T;V^*)$, we denote by (u_ζ,θ_ζ) the solution of Problem 4.1, and we consider the operator $\Lambda : L^2(0,T;V^*) \to L^2(0,T;V^*)$ defined by

$$(\Lambda\zeta(t),w)_{V^*\times V}=(\mathscr{F}\varepsilon(u_\zeta(t)),\varepsilon(w))_{\mathscr{Q}}+j(u_\zeta(t),w), \tag{4.11}$$

for all $w \in V$ and a.e. $t \in (0,T)$.

Lemma 4.2. *There exists a unique $\bar{\zeta} \in L^2(0,T;V^*)$ such that $\Lambda\bar{\zeta}=\bar{\zeta}$.*

Proof. Let $\zeta_1,\zeta_2 \in L^2(0,T;V^*)$. From (4.11), after some algebra, we find that there exists $c>0$ such that, for all $t\in(0,T)$, $\|\Lambda\zeta_1(t)-\Lambda\zeta_2(t)\|^2_{V^*}\leq c\left\|u_{\zeta_1}(t)-u_{\zeta_2}(t)\right\|^2_V$. We combine this inequality with (4.1) to conclude that there exists $c>0$ such that

$$\|\Lambda\zeta_1(t)-\Lambda\zeta_2(t)\|^2_{V^*}\leq c\int_0^t\|\zeta_1(s)-\zeta_2(s)\|^2_{V^*}ds,$$

for all $t\in(0,T)$. It follows from [19, Lemma 7] that Λ has a unique fixed point $\bar{\zeta}\in L^2(0,T;V^*)$. □

Finally, we turn to prove Theorem 3.1.

We denote by $(u_{\bar{\zeta}}, \theta_{\bar{\zeta}})$ the solution of Problem 4.1 corresponding to $\bar{\zeta}$ the unique fixed point of the operator Λ. Since $(\bar{\zeta}, w)_{V^*\times V} = (\mathscr{F}\varepsilon(u_{\bar{\zeta}}(t)), \varepsilon(w))_{\mathscr{D}} + j(u_{\bar{\zeta}}(t), w)$ for all $w \in V$, we conclude that $(u_{\bar{\zeta}}, \theta_{\bar{\zeta}})$ is a solution to Problem 3.1. The uniqueness part of the solution follows from uniqueness of the fixed point of operator Λ.

5. Numerical Approximation and Error Analysis

In this section, we present a fully discrete approximation of Problem 3.1 and derive an error estimate for the corresponding approximate solution.

Let $T^h = \{T_r\}_r$ be a finite element triangulation of $\overline{\Omega}$, compatible with the boundary partition, where h denotes the spatial discretization parameter. For each element T_r, let $\mathbb{P}^1(T_r)$ denote the space of polynomials of total degree less than or equal to 1 on T_r. We now introduce the following finite-dimensional spaces:

$$V^h = \left\{ w^h \in [C(\overline{\Omega})]^d \ : \ w^h_{|T_r} \in [\mathbb{P}^1(T_r)]^d,\ w^h = 0 \text{ on } \Gamma_1 \right\} \subset V,$$

$$Q^h = \left\{ \eta^h \in C(\overline{\Omega}) \ : \ \eta^h_{|T_r} \in \mathbb{P}^1(T_r),\ \eta^h = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \right\} \subset Q,$$

which approximate the spaces V and Q, respectively. For the temporal discretization, we consider a uniform partition $0 = t_0 < t_1 < \cdots < t_n = T$ of the interval $[0,T]$, where the time step size is given by $k = \frac{T}{N}$. For a continuous function g, we denote $g(t_n) = g_n$, while for a discrete sequence $\{y_n\}_{n=0}^N$, we define the backward difference quotient

$$\delta y_n = \frac{y_n - y_{n-1}}{k}.$$

Let u_0^h, v_0^h, θ_0^h, and φ_0^h denote suitable finite element approximations of the initial data u_0, v_0, θ_0, and φ_0, respectively. Finally, to simplify the notation, we introduce the velocity field v defined by

$$u(t) = u_0 + \int_0^t v(s)\,ds, \quad \forall t \in [0,T].$$

We formulate the fully discrete version of Problem 3.1 by using the backward Euler time-stepping scheme as follows.

Problem 5.1. *Find a discrete displacement field $u^{hk} = \{u_n^{hk}\}_{n=0}^N \subset V^h$ and a discrete temperature field $\theta^{hk} = \{\theta_n^{hk}\}_{n=0}^N \subset Q^h$ such that for all $w^h \in V^h$ and $\eta^h \in Q^h$*

$$\begin{aligned}(\delta v_n^{hk}, w^h)_H &+ (\mathscr{A}\varepsilon(v_n^{hk}), \varepsilon(w^h))_{\mathscr{H}} + (\mathscr{F}\varepsilon(u_{n-1}^{hk}), \varepsilon(w^h))_{\mathscr{H}} \\ &- (\theta_{n-1}^{hk}\mathscr{M}, \varepsilon(w^h))_{\mathscr{H}} + j(u_{n-1}^{hk}, w^h) = (f_n, w^h)_{V^*\times V},\end{aligned} \tag{5.1}$$

$$\begin{aligned}(\delta\theta_n^{hk}, \eta^h)_{L^2(\Omega)} &+ (\mathscr{K}\nabla\theta_n^{hk}, \nabla\eta^h)_H - (R(v_n^{hk}), \eta^h)_{L^2(\Omega)} + K_c(\theta_{n-1}^{hk}, \eta^h) \\ &= (q_{th_n}, \eta^h)_{Q^*\times Q},\end{aligned} \tag{5.2}$$

$$u_0^{hk} = u_0^h, \qquad v_0^{hk} = v_0^h, \qquad \theta_0^{hk} = \theta_0^h. \tag{5.3}$$

Here, the discrete displacement field and the velocity field $v^{hk} = \{v_n^{hk}\}_{n=0}^N$ are connected through the relations

$$u_n^{hk} = \delta v_n^{hk}, \qquad u_n^{hk} = u_0^h + k\sum_{i=0}^{N} v_i^{hk}, \quad n = 1,\dots,N.$$

Using similar arguments as in the previous section, one can show that Problem 5.1 admits a unique solution $(u^{hk},\theta^{hk}) \in V^h \times Q^h$. The objective here is to derive error estimates for the numerical approximations, namely $\|u_n - u_n^{hk}\|_V$, $\|v_n - v_n^{hk}\|_H$, $\|\theta_n - \theta_n^{hk}\|_{L^2(\Omega)}$, $\|\theta_n - \theta_n^{hk}\|_Q$.

Theorem 5.1. *Assume that the hypotheses of Theorem 3.1 are satisfied. Let (u,θ) and (u^{hk},θ^{hk}) be the solutions of Problem 3.1 and Problem 5.1, respectively. Suppose further that the following regularity conditions hold:*

$$v \in C(0,T;H^2(\Omega)^d) \cap H^1(0,T;V) \cap H^2(0,T;H), \quad v_{|\Gamma_3} \in C(0,T;H^2(\Gamma_3)^d), \tag{5.4}$$

$$\theta \in C(0,T;H^2(\Omega)) \cap H^1(0,T;Q) \cap H^2(0,T;L^2(\Omega)), \quad \dot\theta \in L^2(0,T;H^1(\Omega)). \tag{5.5}$$

Then, there exists a constant $c>0$, independent of h and k, such that

$$\max_{1\le n\le N}\Big\{\|u_n - u_n^{hk}\|_V + \|v_n - v_n^{hk}\|_H + \|\theta_n - \theta_n^{hk}\|_{L^2(\Omega)} + k\|\theta_n - \theta_n^{hk}\|_Q\Big\} \le c(h+k). \tag{5.6}$$

Proof. In what follows, we use the notation c to denote generic positive constants that may depend on the problem data, but are independent of the discretization parameters h and k. The value of c may change from one line to another.

We begin by deriving an error estimate for the velocity field. To this end, we evaluate (3.6) at time $t=t_n$ with $w = w^h \in V^h$ and subtract it from (5.1). This yields, for all $w^h \in V^h$:

$$\begin{aligned}(\dot v_n - \delta v_n^{hk}, w^h)_H &+ (\mathscr{A}\varepsilon(v_n - v_n^{hk}),\varepsilon(w^h))_{\mathscr{H}} + (\mathscr{F}\varepsilon(u_n - u_{n-1}^{hk}),\varepsilon(w^h))_{\mathscr{H}}\\ &- ((\theta_n - \theta_{n-1}^{hk})\mathscr{M},\varepsilon(w^h))_{\mathscr{H}} + j(u_n,w^h) - j(u_{n-1}^{hk},w^h) = 0.\end{aligned}$$

We now test the previous relation with $w^h - v_n^{hk}$, which yields, for all $w^h \in V^h$,

$$\begin{aligned}&(\delta(v_n - v_n^{hk}), v_n - v_n^{hk})_H + (\mathscr{A}\varepsilon(v_n - v_n^{hk}),\varepsilon(v_n - v_n^{hk}))_{\mathscr{H}}\\ &= (\delta(v_n - v_n^{hk}), v_n - w^h)_H + (\delta v_n - \dot v_n, w^h - v_n^{hk})_H\\ &\quad + (\mathscr{A}\varepsilon(v_n - v_n^{hk}),\varepsilon(v_n - w^h))_{\mathscr{H}} + (\mathscr{F}\varepsilon(u_n - u_{n-1}^{hk}),\varepsilon(v_n^{hk} - w^h))_{\mathscr{H}}\\ &\quad + ((\theta_n - \theta_{n-1}^{hk})\mathscr{M},\varepsilon(w^h - v_n^{hk}))_{\mathscr{H}} + j(u_{n-1}^{hk}, w^h - v_n^{hk}) - j(u_n, w^h - v_n^{hk}).\end{aligned} \tag{5.7}$$

From the properties of the function p, it follows that

$$\big|j(u_{n-1}^{hk}, w^h - v_n^{hk}) - j(u_n, w^h - v_n^{hk})\big| \le L_p c_0^2\,\|u_n - u_{n-1}^{hk}\|_V\,\|w^h - v_n^{hk}\|_V.$$

Moreover, recalling that

$$(\delta(v_n - v_n^{hk}), v_n - v_n^{hk})_H \ge \frac{1}{2k}\Big(\|v_n - v_n^{hk}\|_H^2 - \|v_{n-1} - v_{n-1}^{hk}\|_H^2\Big), \tag{5.8}$$

and using the coercivity of $\mathscr{A}$, together with the continuity of $\mathscr{A}$, $\mathscr{F}$, and $\mathscr{M}$, as well as the Cauchy–Schwarz and Young inequalities, we deduce from (5.7) that there exists a constant $c>0$

such that

$$\begin{aligned}
&\|v_n - v_n^{hk}\|_H^2 - \|v_{n-1} - v_{n-1}^{hk}\|_H^2 + k\|v_n - v_n^{hk}\|_V^2 \\
&\leq ck\Big\{\|\dot{v}_n - \delta v_n\|_H^2 + \|v_n - w^h\|_H^2 + \|v_n - w^h\|_V^2 + \|u_{n-1} - u_{n-1}^{hk}\|_V^2 \\
&\qquad + \|\theta_{n-1} - \theta_{n-1}^{hk}\|_{L^2(\Omega)}^2 + \|u_n - u_{n-1}\|_V^2 + \|\theta_n - \theta_{n-1}\|_{L^2(\Omega)}^2\Big\} \\
&\qquad + 2k(\delta(v_n - v_n^{hk}), v_n - w^h)_H.
\end{aligned} \tag{5.9}$$

Next, we turn to the estimation of the numerical error associated with the temperature field. We evaluate (3.7) at $t = t_n$ with $\eta = \eta^h \in Q^h$ and subtract it from (5.2). This yields, for all $\eta^h \in Q^h$, the relation

$$\begin{aligned}
(\dot{\theta}_n - \delta\theta_n^{hk}, \eta^h)_{L^2(\Omega)} + (\mathscr{K}\nabla(\theta_n - \theta_n^{hk}), \nabla\eta^h)_H &= (R(v_n) - R(v_n^{hk}), \eta^h)_{L^2(\Omega)} \\
&\quad + K_c(\theta_{n-1}^{hk}, \eta^h)_{L^2(\Gamma_3)} - K_c(\theta_n, \eta^h)_{L^2(\Gamma_3)}.
\end{aligned}$$

Substituting η^h by $\eta^h - \theta_n^{hk}$, we then obtain

$$\begin{aligned}
&(\delta(\theta_n - \theta_n^{hk}), \theta_n - \theta_n^{hk})_{L^2(\Omega)} + (\mathscr{K}\nabla(\theta_n - \theta_n^{hk}), \nabla(\theta_n - \theta_n^{hk}))_H \\
&\quad = (\delta\theta_n - \dot{\theta}_n, \eta^h - \theta_n^{hk})_{L^2(\Omega)} + (\delta(\theta_n - \theta_n^{hk}), \theta_n - \eta^h)_{L^2(\Omega)} \\
&\qquad + (\mathscr{K}\nabla(\theta_n - \theta_n^{hk}), \nabla(\theta_n - \eta^h))_H + (R(v_n) - R(v_n^{hk}), \eta^h - \theta_n^{hk})_{L^2(\Omega)} \\
&\qquad + K_c(\theta_{n-1}^{hk}, \eta^h - \theta_n^{hk})_{L^2(\Gamma_3)} - K_c(\theta_n, \eta^h - \theta_n^{hk})_{L^2(\Gamma_3)}.
\end{aligned} \tag{5.10}$$

Using an argument similar to (5.8), we obtain

$$(\delta(\theta_n - \theta_n^{hk}), \theta_n - \theta_n^{hk})_{L^2(\Omega)} \geq \frac{1}{2k}\Big(\|\theta_n - \theta_n^{hk}\|_{L^2(\Omega)}^2 - \|\theta_{n-1} - \theta_{n-1}^{hk}\|_{L^2(\Omega)}^2\Big). \tag{5.11}$$

By invoking the coercivity of $\mathscr{K}$, the continuity of $\mathscr{K}$ and R, together with the Cauchy–Schwarz inequality, and taking into account (5.11), we deduce from (5.10) that, for all $\eta^h \in Q^h$, there exists a constant $c > 0$ such that

$$\begin{aligned}
&\|\theta_n - \theta_n^{hk}\|_{L^2(\Omega)}^2 - \|\theta_{n-1} - \theta_{n-1}^{hk}\|_{L^2(\Omega)}^2 + k\|\theta_n - \theta_n^{hk}\|_Q^2 \\
&\quad \leq ck\Big(\|\dot{\theta}_n - \delta\theta_n\|_{L^2(\Omega)}^2 + \|\theta_n - \eta^h\|_Q^2 + \|v_n - v_n^{hk}\|_V^2 + \|u_n - u_n^{hk}\|_V^2\Big) \\
&\qquad + 2k(\delta(\theta_n - \theta_n^{hk}), \theta_n - \eta^h)_{L^2(\Omega)}.
\end{aligned} \tag{5.12}$$

Combining (5.9) and (5.12), we conclude that there exists a constant $c>0$ such that, for all sequences $\{w_i^h\}_{i=1}^N \subset V^h$ and $\{\eta_i^h\}_{i=1}^N \subset Q^h$, the following inequality holds:

$$
\begin{aligned}
&\|v_n - v_n^{hk}\|_H^2 + \|\theta_n - \theta_n^{hk}\|_{L^2(\Omega)}^2 + k\sum_{i=1}^N \left(\|v_i - v_i^{hk}\|_V^2 + \|\theta_i - \theta_i^{hk}\|_Q^2\right)\\
&\le \|v_0 - v_0^h\|_H^2 + \|\theta_0 - \theta_0^h\|_{L^2(\Omega)}^2 + c\left[k\sum_{i=1}^N \left(\|\dot v_n - \delta v_n\|_H^2 + \|\dot\theta_n - \delta\theta_n\|_{L^2(\Omega)}^2\right.\right.\\
&\quad \left. + \|v_n - w_i^h\|_H^2 + \|v_n - w_i^h\|_V^2 + \|\theta_i - \eta_i^h\|_Q^2\right)\\
&\quad + k\sum_{i=1}^N \left(\|u_i - u_i^{hk}\|_V^2 + \|\theta_i - \theta_i^{hk}\|_{L^2(\Omega)}^2\right)\\
&\quad \left. + k\sum_{i=1}^N \left(\|u_i - u_{i-1}\|_V^2 + \|\theta_i - \theta_{i-1}\|_{L^2(\Omega)}^2 + \|\theta_i - \theta_{i-1}\|_Q^2\right)\right]\\
&\quad + 2k\sum_{i=1}^N (\delta(v_i - v_i^{hk}), v_i - w_i^h)_H + 2k\sum_{i=1}^N (\delta(\theta_i - \theta_i^{hk}), \theta_i - \eta_i^h)_{L^2(\Omega)}.
\end{aligned} \tag{5.13}
$$

Furthermore, we recall that

$$
\begin{aligned}
2k\sum_{i=1}^N (\delta(v_i - v_i^{hk}), v_i - w_i^h)_H &\le \gamma_1 \|v_n - v_n^{hk}\|_H^2 + c\Big(\|v_n - w_n^h\|_H^2 + \|v_0 - v_0^h\|_H^2\\
&\quad + \|v_1 - w_1^h\|_H^2\Big) + k\sum_{i=1}^{n-1} \|v_i - v_i^{hk}\|_H^2 + \frac{1}{k}\sum_{i=1}^{n-1} \|v_i - w_i^h - (v_{i+1} - w_{i+1}^h)\|_H^2,
\end{aligned} \tag{5.14}
$$

and

$$
\begin{aligned}
2k\sum_{i=1}^N (\delta(\theta_i - \theta_i^{hk}), \theta_i - \eta_i^h)_{L^2(\Omega)} &\le \gamma_2 \|\theta_n - \theta_n^{hk}\|_{L^2(\Omega)}^2 + c\Big(\|\theta_n - \eta_n^h\|_{L^2(\Omega)}^2\\
&\quad + \|\theta_0 - \theta_0^h\|_{L^2(\Omega)}^2 + \|\theta_1 - \eta_1^h\|_{L^2(\Omega)}^2\Big) + k\sum_{i=1}^{n-1} \|\theta_i - \theta_i^{hk}\|_{L^2(\Omega)}^2\\
&\quad + \frac{1}{k}\sum_{i=1}^{n-1} \|\theta_i - \eta_i^h - (\theta_{i+1} - \eta_{i+1}^h)\|_{L^2(\Omega)}^2,
\end{aligned} \tag{5.15}
$$

where $\gamma_1 > 0$ and $\gamma_2 > 0$ are chosen sufficiently small.

Finally, using the estimate (see [6])

$$
\|u_i - u_i^{hk}\|_V^2 \le ck\sum_{j=1}^i \|v_j - v_j^{hk}\|_V^2 + c(h^2 + k^2), \quad i = 1, \dots, N, \tag{5.16}
$$

and combining (5.13), (5.14), (5.15) and (5.16), together with a discrete version of Gronwall's inequality, we arrive at the desired error estimates for all $\{w_i^h\}_{i=1}^N \subset V^h$ and $\{\eta_i^h\}_{i=1}^N \subset Q^h$.

$$\begin{aligned}
&\max_{1\le n\le N}\left\{\left\|v_n - v_n^{hk}\right\|_H^2 + \left\|\theta_n - \theta_n^{hk}\right\|_{L^2(\Omega)}^2\right\} + k\sum_{i=1}^N\Bigg(\left\|v_i - v_i^{hk}\right\|_V^2 \\
&+\left\|\theta_i - \theta_i^{hk}\right\|_Q^2\Bigg) \le c\Bigg[\left\|v_0 - v_0^h\right\|_H^2 + \left\|\theta_0 - \theta_0^h\right\|_{L^2(\Omega)}^2 \\
&+\max_{1\le n\le N}\left\{\left\|v_n - w_n^h\right\|_H^2 + \left\|\theta_n - \eta_n^h\right\|_{L^2(\Omega)}^2\right\} \\
&+k\sum_{i=1}^N\left(\|\dot v_i - \delta v_i\|_H^2 + \|\dot\theta_i - \delta\theta_i\|_{L^2(\Omega)}^2 + \left\|v_i - w_i^h\right\|_V^2 + \left\|\theta_i - \eta_i^h\right\|_Q^2\right) \\
&+k\sum_{i=1}^N\left(\|u_i - u_{i-1}\|_V^2 + \|\theta_i - \theta_{i-1}\|_{L^2(\Omega)}^2\right) + \left\|v_1 - w_1^h\right\|_H^2 + \left\|\theta_1 - \eta_1^h\right\|_{L^2(\Omega)}^2 \\
&+\frac{1}{k}\sum_{i=1}^{N-1}\left\|v_i - w_i^h - (v_{i+1} - w_{i+1}^h)\right\|_H^2 \\
&+\frac{1}{k}\sum_{i=1}^{N-1}\left\|\theta_i - \eta_i^h - (\theta_{i+1} - \eta_{i+1}^h)\right\|_{L^2(\Omega)}^2 + h^2 + k^2\Bigg].
\end{aligned} \tag{5.17}$$

Next, we denote by $\Pi_{\mathscr{Z}}^h$ the standard finite element interpolation operator associated with the space $\mathscr{Z}$. We set $w_n^h = \Pi_V^h v_n$ and $\eta_n^h = \Pi_Q^h \theta_n$, i.e., the finite element interpolants of v_n and θ_n, respectively. From the classical finite element interpolation error bounds [15], it follows that

$$\|v_n - w_n^h\|_V \le ch\|v\|_{C(0,T;H^2(\Omega)^d)}, \tag{5.18}$$

$$\|v_n - w_n^h\|_H \le ch^2\|v\|_{C(0,T;H^2(\Omega)^d)}, \tag{5.19}$$

$$\|\theta_n - \eta_n^h\|_Q \le ch\|\theta\|_{C(0,T;H^2(\Omega))}. \tag{5.20}$$

The discrete initial conditions u_0^h, v_0^h and θ_0^h are chosen as

$$u_0^h = \Pi_V^h u_0, \quad v_0^h = \Pi_V^h v_0, \quad \theta_0^h = \Pi_Q^h \theta_0, \tag{5.21}$$

and satisfy (see [15, 18]).

$$\begin{aligned}
&\|u_0 - u_0^h\|_V + \|v_0 - v_0^h\|_H + \|\theta_0 - \theta_0^h\|_{L^2(\Omega)} \\
&\qquad \le ch\Big(\|u\|_{C(0,T;H^2(\Omega)^d)} + \|u\|_{C^1(0,T;V)} + \|\theta\|_{C(0,T;Q)}\Big).
\end{aligned} \tag{5.22}$$

Moreover, from (5.4) and (5.5), one easily verifies that

$$k\sum_{i=1}^N\left(\|\dot v_i - \delta v_i\|_H^2 + \|\dot\theta_i - \delta\theta_i\|_{L^2(\Omega)}^2\right) \le ck^2\Big(\|v\|_{H^2(0,T;H)}^2 + \|\theta\|_{H^2(0,T;L^2(\Omega))}^2\Big). \tag{5.23}$$

Proceeding as in [13], we further obtain

$$\begin{aligned}\frac{1}{k}\sum_{i=1}^{N-1}&\Big(\|v_i-w_i^h-(v_{i+1}-w_{i+1}^h)\|_H^2+\|\theta_i-\eta_i^h-(\theta_{i+1}-\eta_{i+1}^h)\|_{L^2(\Omega)}^2\Big)\\&\le ch^2\Big(\|v\|_{H^1(0,T;V)}^2+\|\theta\|_{H^1(0,T;L^2(\Omega))}^2\Big),\end{aligned}\tag{5.24}$$

while from [18] we deduce

$$k\sum_{i=1}^{N}\Big(\|u_i-u_{i-1}\|_V^2+\|\theta_i-\theta_{i-1}\|_{L^2(\Omega)}^2\Big)\le ck^2\Big(\|u\|_{H^1(0,T;V)}^2+\|\theta\|_{H^1(0,T;L^2(\Omega))}^2\Big).\tag{5.25}$$

Combining estimates (5.18)–(5.25) with (5.17), we infer that there exists a constant $c>0$ such that

$$\begin{aligned}\max_{1\le n\le N}&\Big\{\|v_n-v_n^{hk}\|_H^2+\|\theta_n-\theta_n^{hk}\|_{L^2(\Omega)}^2\Big\}\\&+\sum_{i=1}^{N}k\Big(\|v_i-v_i^{hk}\|_V^2+\|\theta_i-\theta_i^{hk}\|_Q^2\Big)\le c(h^2+k^2).\end{aligned}\tag{5.26}$$

Finally, combining (5.26) with (5.16) yields the desired error estimate (5.6). □

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REFERENCES

[1] B. Aharrouch, M. Bouallala, Optimal control and mathematical analysis of thermo-viscoelastic systems under friction and contact constraints, Int. J. Syst. Assur. Eng. Manag. (2026) doi: 10.1007/s13198-025-03133-4

[2] M. Alaoui, M. Bouallala, E. H. Essoufi, A. Ouaanabi, Analysis of a quasistatic thermo-electro-viscoelastic contact problem modeled by variational-hemivariational and hemivariational inequalities, Math. Methods Appl. Sci. 48 (2025), 1809-1830.

[3] M. Alaoui, E. H. Essoufi, A. Ouaanabi, M. Bouallala, Analysis of a contact problem modeled by hemivariational inequalities in thermo-piezoelectricity, J. Nonlinear Model. Anal. 7 (2025), 739-763.

[4] M. Alaoui, E. H. Essoufi, A. Ouaanabi, M. Bouallala, On the dynamic Coulomb's frictional contact problem for thermo-piezoelectric materials, Z. Angew. Math. Mech. 104 (2024), e202300891.

[5] M. Alaoui, E. H. Essoufi, A. Ouaanabi, M. Bouallala, On the dynamic frictionless contact problem with normal compliance in thermo-piezoelectricity, Eur. J. Math. Comput. Appl. 12 (1) (2024), 4-27.

[6] M. Barboteu, J. R. Fernández, R. Tarraf, Numerical analysis of a dynamic piezoelectric contact problem arising in viscoelasticity, Comput. Methods Appl. Mech. Engrg. 197 (2008), 3724-3732.

[7] K. Bartosz, T. Janiczko, P. Szafraniec, M. Shillor, Dynamic thermoviscoelastic thermistor problem with contact and nonmonotone friction, Appl. Anal. 97 (8) (2018), 1432-1453.

[8] A. Bendarag, M. Bouallala, M. Chnigli, Dynamic contact response of a thermo-viscoelastic rod with damping effects, J. Nonlinear Math. Phys. 32 (2025), 99.

[9] M. Bouallala, Variational and error estimation for a frictionless contact problem in thermo-viscoelasticity with time fractional derivatives, Commun. Anal. Comput. 2 (2024), 1-18.

[10] M. Bouallala, E. H. Essoufi, Analysis for unilateral contact problem with Coulomb's friction in thermoelectro-visco-elasticity, Filomat 36 (13) (2022), 4443-4457.

[11] M. Bouallala, E.H. Essoufi, Analysis results for dynamic contact problem with friction in thermoviscoelasticity, Methods Funct. Anal. Topol. 26 (2020), 317-326.

[12] M. Bouallala, E. H. Essoufi, A. Ouaanabi, Y. Ouafik, Variational analysis and numerical simulation for a dynamic contact problem with Coulomb's friction in thermo-viscoelasticity, Appl. Set-Valued Anal. Optim. 8 (2026), 147-166.

[13] M. Campo, J. R. Fernández, Numerical analysis of a quasistatic thermoviscoelastic frictional contact problem, Comput. Mech. 35 (2005), 459-469.

[14] O. Chau, R. Oujja, M. Rochdi, A mathematical analysis of a dynamical frictional contact model in thermo-viscoelasticity, Discrete Contin. Dyn. Syst. Ser. S (2008), 61-70.

[15] P. G. Ciarlet, The finite element method for elliptic problems, in: Handbook of Numerical Analysis, P. G. Ciarlet, J.-L. Lions (eds.), Vol. II, pp. 17-352, North-Holland (1991).

[16] A. Djabi, A frictional contact problem with normal damped response conditions and thermal effects for a thermo-electro-viscoelastic material, Arab J. Math. Sci. 31 (2025), 61-81.

[17] C. Eck, J. Jarušek, M. Krbec, Unilateral Contact Problems: Variational Methods and Existence Theorems, CRC Press, 2005.

[18] J. R. Fernández, K. L. Kuttler, A dynamic thermoviscoelastic problem: numerical analysis and computational experiments, Q. J. Mech. Appl. Math. 63 (2010), 295-314.

[19] A. Kulig, S. Migórski, Solvability and continuous dependence results for second order nonlinear inclusion with Volterra-type operator, Nonlinear Anal. 75 (2012), 4729-4746.

[20] S. Migórski, M. Sofonea, Dynamic frictional contact problems with normal compliance and memory, Nonlinear Anal. Real World Appl. 14 (2013), 1027-1040.

[21] S. Migórski, P. Szafraniec, A class of dynamic frictional contact problems governed by a system of hemivariational inequalities in thermoviscoelasticity, Nonlinear Anal. Real World Appl. 15 (2014), 158-171.

[22] S. Migórski, A. Zeng, A class of dynamic viscoelastic contact problems with nonmonotone friction, J. Math. Anal. Appl. 412 (2014), 295-318.

[23] A. Ouaanabi, M. Alaoui, M. Bouallala, E. H. Essoufi, Continuous dependence result for a class of evolutionary variational-hemivariational inequalities with application to a dynamic thermo-viscoelastic contact problem, Acta Appl. Math. 195 (2025), 2.

[24] A. Ouaanabi, M. Alaoui, M. Bouallala, E. H. Essoufi, Coupled time-fractional hemivariational inequalities in frictional thermo-viscoelastic contact problem, Math. Model. Control 5 (2025), 400-409.

[25] A. Ouaanabi, M. Alaoui, M. Bouallala, E. H. Essoufi, Mathematical analysis for a dynamic thermo-electro-viscoelastic contact problem governed by a system of variational–hemivariational inequalities, SeMA J. (2025), 1-24.

[26] A. Ouaanabi, M. Alaoui, M. Bouallala, E. H. Essoufi, Optimal error estimates for a class of history-dependent quasi variational-hemivariational inequalities with constraints, Commun. Nonlinear Sci. Numer. Simul. 161 (2026), 110031.

[27] A. Ouaanabi, M. Alaoui, M. Bouallala, E. H. Essoufi, Variational and Numerical Analysis of a Dynamic Tresca Frictional Contact Problem for Thermo-Electro-Viscoelastic Materials, ZAMM-J. Appl. Math. Mech. 106 (2026), e70383.

[28] A. Ouaanabi, M. Alaoui, E. H. Essoufi, M. Bouallala, Existence and uniqueness of solutions for a dynamic thermoviscoelastic contact problem governed by an evolutionary variational hemivariational inequality with unilateral constraints, Comput. Appl. Math. 44 (2025), 311.

[29] A. Ouaanabi, M. Alaoui, E. H. Essoufi, M. Bouallala, Mathematical analysis of a dynamic Signorini's contact problem governed by a system of variational-hemivariational inequalities in thermo-viscoelasticity, Iran. J. Sci. 50 (2025), 171-182.

[30] M. Sofonea, W. Han, M. Shillor, Analysis and Approximation of Contact Problems with Adhesion or Damage, Chapman and Hall/CRC, 2005.