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ON AN INTERVAL-VALUED MULTIOBJECTIVE OPTIMIZATION PROBLEM AND APPROXIMATE VECTOR VARIATIONAL INEQUALITIES IN TERMS OF DIRECTIONAL CONVEXIFICATORS

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Abstract. In this paper, we develop approximate vector variational inequalities of both Stampacchia and Minty types by using the concept of directional convexificators (DCs), and establish their connection to an interval-valued multiobjective optimization problem (IVMOP). Our method employs a form of approximate convexity defined through DCs to derive necessary and sufficient conditions for a point to qualify as an approximate Pareto efficient solution of the IVMOP. Additionally, we explore the weak formulations of these approximate variational inequalities and present several results aimed at identifying approximate weak Pareto efficient solutions. To illustrate our approach and highlight the limitations of some existing work, we include some detailed examples. Our results demonstrate that DCs offer a powerful and flexible approach for handling complex optimization problems with interval-valued objectives.

Keywords. Approximate convex function; Directional convexificators; Interval-valued multiobjective optimization problem; Variational Inequalities.

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1. INTRODUCTION

Interval-valued multiobjective optimization problems (IVMOPs) represent an important extension of classical multiobjective optimization by incorporating uncertainty and imprecision directly into the objective functions. In traditional multiobjective optimization, we aim to simultaneously optimize multiple conflicting objective functions. However, in many practical scenarios, these objective values can only be estimated within certain ranges or intervals due to measurement errors or noise in data collection, incomplete information about the system, fluctuations in environmental or operational conditions, inherent variability in the processes being modeled, etc. IVMOPs formally acknowledge this uncertainty by representing each objective function as an interval rather than a single point value. This interval captures the range

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of possible outcomes for each objective under the given decision variables. IVMOPs find applications in diverse fields including generic algorithms [1], stock portfolio [2], transportation problems [3], economic management [4], decision making theories [5] etc. A convexificator is a mathematical tool used in optimization and variational analysis to approximate or replace non-convex functions with convex ones. Convexification is essential in optimization because convex problems are generally easier to solve than non-convex ones. Convexificators help in transforming non-convex functions while preserving key properties, enabling the application of powerful convex optimization techniques. In recent years, convexificators have been utilized to generalize several results in nonsmooth analysis and optimization. They can be considered as weaker form of subdifferentials, making them more adaptable for analysis and applications. Unlike subdifferentials, which are well-known for being convex and compact, convexificators are typically closed sets. For locally Lipschitz functions, most established subdifferentials serve as convexificators and may also include the convex hull of a convexificators.

The notion of directional convexificators extends the concept of convexificators by incorporating directional information, focusing on directions where the given function remains continuous. This approach is used to determine a convexificator for a lower semicontinuous (l.s.c.) function at a specific point.

1.1. **Literature survey.** Ishibuchi and Tanaka [6] first introduced an ordering relation between two closed intervals by separately considering maximization and minimization problems. Building on this, Wu [7] proposed two solution concepts for differentiable interval-valued scalar optimization problems (IVOP) and established the corresponding Karush-Kuhn-Tucker (KKT) conditions. Wu [8] extended this framework by formulating four types of differentiable IVOP and deriving their KKT conditions to support duality results, while Wu [9] developed a dual problem formulation using interval-valued Lagrangian functions. Further advancements were made by Chalco-Cano et al. [10], who applied the generalized Hukuhara (gH) derivative to interval-valued functions to derive KKT conditions. Singh et al. [11] addressed problems where both objectives and constraints are differentiable interval-valued functions. Zhang et al. [12] formulated KKT conditions for LU-preinvex and invex optimization problems under weakly continuous and Hukuhara differentiable settings, and Ahmad et al. [13] investigated invex interval-valued nonlinear programming using gH-derivatives. More recently, Guo et al. [14] introduced concepts of interval-valued symmetric invexity, pseudo-invexity, and quasi-invexity, examining optimality and duality in nonsmooth IVOP under generalized convexity. In the realm of multiobjective optimization, Wu [15] and Osuna-Gómez et al. [16] explored optimality conditions for interval-valued multiobjective optimization problems (IVMOP). Laha et al. [17] studied approximate solutions of nonsmooth IVMOP using convexificators, while Tung [18] and Tung [19] developed KKT conditions and duality results for semi-infinite programming with multiple interval-valued objectives, including convex cases. Hung et al. [20] and Dwivedi et al. [21] investigated optimality and duality for approximate solutions of nonsmooth interval-valued multiobjective semi-infinite programs using limiting and Clarke subdifferentials. Additionally, Tuyen [22] and Singh et al. [23] contributed to the development of IVOP and IVMOP by proposing new approaches, deriving optimality conditions, and highlighting practical applications. Demyanov [24] introduced the concept of convexificators as a generalization of the notions of upper convex and lower concave approximations. Demyanov and Jeyakumar [25] studied convexificators for positively homogeneous and locally Lipschitz functions. Jeyakumar and Luc [26] defined noncompact convexificators and presented several calculus rules for calculating convexificators. Wang and Jeyakumar [27], Luc [28], Li and Zhang [29, 30] extended various results in nonsmooth analysis and optimization by using convexificators. Recently, Golestani and Nobakhtian [31] and Luu [32] used convexificators to obtain optimality conditions for efficiency. Laha and Mishra [33] introduced the vector variational inequalities of the Stampacchia and Minty types through the concept of convexificators and utilized these inequalities to establish necessary and sufficient conditions for identifying a vector minimal point in a vector optimization problem (VOP). Dempe and Pilecka [34] introduced the notion of directional convexificators based on the notion of continuity directions. Directional convexificators were utilised by Gadhi et al. [35] to solve set-valued optimization problems. Gadhi [36] derived necessary and sufficient optimality conditions for a scalar optimization problem by using variational inequalities in terms of directional convexificators. For mathematical programs with equilibrium constraints, Lafhim and Kalmoun [37] obtained optimality conditions by using directional convexificators. Further, Gadhi and Odha [38] formulated the vector variational inequalities of Stampacchia and Minty types in terms of directional convexificators, and related them to a vector optimization problem. Sachan and Laha [39] employed higher-order vector variational inequalities formulated with directional convexificators to characterize strict and semi-strict minimizers in multiobjective optimization problems. Mohapatra et al. [40] derived optimality conditions for mathematical programs with vanishing constraints by employing directional convexificators. Sachan et al. [41] introduced the Wolfe-dual problem for MPEC and established weak and strong duality theorems using directional convexificators. Hartman and Stampacchia [42] introduced the concept of Variational Inequalities (VIs), which were later developed into two well-known forms: Minty [43] and Stampacchia [44]. In 1980, Giannessi [45] expanded this idea by introducing Vector Variational Inequalities (VVIs) in finite-dimensional Euclidean space. VVIs have become an essential tool in modern research, particularly in fields like optimal control, equilibrium problems, engineering, and economics [46, 47, 48]. They are widely used to analyze vector optimization problems. Researchers explored the link between VVIs and vector optimization, particularly for differentiable cases [49, 50, 51, 52]. For nonsmooth problems, Lee and Lee [53] formulated different VVIs by using subdifferentials for a nondifferentiable convex vector optimization problem, while Mishra and Laha [54] worked on approximate VVIs involving Fréchet subdifferentials and solved vector optimization problems with approximately star-shaped functions. Further advancements were made in the field: Mishra and Upadhyay [55] studied VVIs in nonsmooth multiobjective programming, Laha et al. [56] applied VVIs to vector optimization problems involving nonsmooth V-invex functions, Mishra and Laha [57] explored approximate VVIs by using Clarke subdifferentials. Al-Shamary et al. [58], Laha and Singh [59] examined VVIs in relation to multiobjective optimization under different convexity assumptions. Singh and Laha [60] investigated VVIs by using quasidifferentials and their applications in multiobjective optimization. VVIs were also studied in the context of interval-valued optimization problems. Zhang et al. [61] analyzed their role under LU-convexity assumptions, while Treanţă et al. [62] introduced geodesic LU-approximately convex functions on Hadamard manifolds. More recently, Upadhyay et al. [63], Ciontescu and Treanţă [64] deepened the understanding of VVIs by linking them to LU-optimal solutions and interval-valued optimization problems while Laha et al. [65] studied approximate VVIs by using quasidifferentials and connected them to approximate quasi Pareto optimal solutions within the frame work of IVMOP. Overall, VVIs continue to be a powerful mathematical tool in optimization and decision-making across multiple disciplines.

1.2. **Motivation and contribution.** This paper builds on the work of Laha and Mishra [33] and Gadhi and Odha [38] by extending their results to the interval-valued case and addressing the IVMOP in the context of l.s.c. functions involving solutions that are approximate rather than exact. The results from Laha and Mishra [33] focus on multiobjective optimization problems (MOPs) with locally Lipschitz functions, but they did not apply to interval-valued discontinuous functions with nonempty sets of continuity directions at optimal points. On the other hand, Gadhi and Ohda [38] investigated directional convexificators for discontinuous functions, and it is limited to efficient solutions in MOPs only. This paper aims to fill this gap by establishing connections between interval-valued multiobjective optimization problems (IVMOPs) and vector variational inequalities (VVIs) and $\mathscr{E}$-vector variational inequalities. These relations, formulated using directional convexificators, can address discontinuous functions with nonempty continuity direction sets. To our knowledge, no previous research has explored vector variational inequalities with directional convexificators for solving interval-valued multiobjective optimization problems involving discontinuous functions and having approximate solutions. Thus, this study makes a unique contribution to this area.

1.3. **Framework.** This paper is structured as follows. Section 2 introduces the key definitions and preliminary concepts that form the foundation of our study. In Section 3, we develop the approximate Minty and Stampacchia vector variational inequalities for IVMOP by using directional convexificators. We then establish necessary and sufficient conditions for a point to be an approximate solution of IVMOP, and support our findings with relevant examples. Finally, Section 4 summarizes the main results and outlines potential directions for future research.

2. Preliminaries

Let $\mathbb{R}^n$ denote the standard n-dimensional Euclidean space equipped with a norm $\|\cdot\|$. For any nonempty subset $\mathscr{X} \subseteq \mathbb{R}^n$, $cl\,\mathscr{X}$ and $conv\,\mathscr{X}$ represent its closure and convex hull, respectively. We write $\mathbb{R}^n_+$ for the non-negative orthant, $[\cdot,\cdot]$ for the closed line segment connecting any two points γ and θ in $\mathbb{R}^n$, and $\langle\cdot,\cdot\rangle$ for the inner product of any $\gamma,\theta \in \mathbb{R}^n$. Let $\bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ and assume $\beta : \mathbb{R}^n \to \bar{\mathbb{R}}$ is an extended real-valued function.

For vectors $\gamma := (\gamma_1,\dots,\gamma_n), \theta := (\theta_1,\dots,\theta_n) \in \mathbb{R}^n$, define the componentwise ordering as follows:

$$\gamma \leqq \theta \Leftrightarrow \gamma_i \leq \theta_i,\ \forall i = 1,\dots,n.$$
$$\gamma \leq \theta \Leftrightarrow \gamma_i \leq \theta_i,\ \forall i = 1,\dots,n \text{ and } \gamma \neq \theta.$$
$$\gamma < \theta \Leftrightarrow \gamma_i < \theta_i,\ \forall i = 1,\dots,n.$$

The definitions of the upper and lower Dini directional derivatives [26] are given as:

Definition 2.1. [26] Let $\delta \in \mathbb{R}^n$ be such that $\beta(\delta)$ is finite. The *upper and lower Dini derivatives* of β at δ in a direction $\vartheta \in \mathbb{R}^n$ are defined by

$$\beta^+(\delta,\vartheta) := \limsup_{t\downarrow 0} \frac{\beta(\delta + t\vartheta) - \beta(\delta)}{t},$$

and

$$\beta^{-}(\delta,\vartheta) := \liminf_{t\downarrow 0} \frac{\beta(\delta+t\vartheta)-\beta(\delta)}{t},$$

respectively.

Dempe and Pilecka [34] introduced a new approach by defining a directional convexificator, which helps to construct a potentially unbounded convexificator. This innovative tool is developed by restricting the conditions that define a convexificator to only those directions where the given function remains continuous. The concept of continuity directions plays a crucial role in the following analysis.

Definition 2.2. [34] A vector $\mathrm{d} \in \mathbb{R}^n$ is a *continuity direction (CD)* of β at $\delta \in \mathbb{R}^n$ iff, for all sequences $\{n_k\} \subset]0,+\infty[$ with $\{n_k\} \searrow 0$, $\lim_{k\to\infty} \beta(\delta + n_k \mathrm{d}) = \beta(\delta)$. The symbol $\mathscr{D}_\beta(\delta)$ stands for the collection of all CDs of β at δ.

The notion of directional convexificator (DC) is based on the notion of the CD.

Definition 2.3. [34] Let $\emptyset \neq \mathscr{D}$ be a cone of $\mathbb{R}^n$. Let $\mathscr{D}_\beta(\delta)$ be the collection of all CDs of β at δ. The function β admits

(a) a *directional upper convexificator (DUC)* at δ, denoted by $\partial^*_{\mathscr{D}}\beta(\delta) \subset \mathbb{R}^n$, iff $\mathscr{D} \subseteq \mathscr{D}_\beta(\delta)$, the set $\partial^*_{\mathscr{D}}\beta(\delta)$ is closed, and, for each $\mathrm{d} \in \mathscr{D}$, $\beta^{-}(\delta,\mathrm{d}) \leq \sup_{\delta^* \in \partial^*_{\mathscr{D}}\beta(\delta)} \langle \delta^*, \mathrm{d}\rangle$;

(b) a *directional lower convexificator (DLC)* at δ, denoted by $\partial^*_{\mathscr{D}}\beta(\delta) \subset \mathbb{R}^n$, iff $\mathscr{D} \subseteq \mathscr{D}_\beta(\delta)$, the set $\partial^*_{\mathscr{D}}\beta(\delta)$ is closed, and, for each $\mathrm{d} \in \mathscr{D}$, $\beta^{+}(\delta,\mathrm{d}) \geq \inf_{\delta^* \in \partial^*_{\mathscr{D}}\beta(\delta)} \langle \delta^*, \mathrm{d}\rangle$;

(c) a *directional convexificator (DC)* at δ, denoted by $\partial^*_{\mathscr{D}}\beta(\delta) \subset \mathbb{R}^n$, iff it is both a DUC and a DLC of β at δ.

Remark 2.1. If $\mathscr{D} = \mathbb{R}^n$, then the concept of convexificators and DCs merge.

Dempe and Pilecka [34] introduced the concept of convexity for a function β by considering its behavior along all continuity directions ($\mathrm{d} \in \mathscr{D}_\beta(\bar{\delta})$). The key idea is that even if the function is discontinuous, it should still maintain convexity along the directions where it remains continuous at a given reference point.

Definition 2.4. [34] Let $\bar{\delta} \in \mathbb{R}^n$, and let $\emptyset \neq \mathscr{D} \subseteq \mathscr{D}_\beta(\bar{\delta})$ be a convex cone. The function β is said to be

(a) *convex* in all directions $\mathrm{d} \in \mathscr{D}$ at $\bar{\delta}$, iff, for any $\mathrm{d}_1, \mathrm{d}_2 \in \mathscr{D}$ and $\eta \in [0,1]$,

$$\beta(\eta\bar{\delta}_1 + (1-\eta)\bar{\delta}_2) \leq \eta\beta(\bar{\delta}_1) + (1-\eta)\beta(\bar{\delta}_2),$$

where $\bar{\delta}_1 = \bar{\delta} + \mathrm{d}_1$ and $\bar{\delta}_2 = \bar{\delta} + \mathrm{d}_2$;

(b) *strictly convex* in all directions $\mathrm{d} \in \mathscr{D}$ at $\bar{\delta}$, iff, for all $\mathrm{d}_1, \mathrm{d}_2 \in \mathscr{D}$, $\mathrm{d}_1 \neq \mathrm{d}_2$, and $\eta \in]0,1[$,

$$\beta(\eta\bar{\delta}_1 + (1-\eta)\bar{\delta}_2) < \eta\beta(\bar{\delta}_1) + (1-\eta)\beta(\bar{\delta}_2).$$

Gadhi [36] introduced the concepts of convexity and monotonicity for real-valued functions in terms of directional convexificators which is defined as follows:

Definition 2.5. [36] Let $\emptyset \neq \mathscr{D}$ be a cone of $\mathbb{R}^n$ and let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$. Let a real-valued function β be l.s.c. at $\bar{\delta} \in \Omega_{\mathscr{D}_\beta} := \{\delta \in \Omega : \mathscr{D} \subset \mathscr{D}_\beta(\delta)\}$. Suppose that β admits a bounded directional convexificator (BDC) $\partial^*_{\mathscr{D}}\beta(\bar{\delta})$ at $\bar{\delta}$. Then, β is said to be

(a) *$\partial^*_{\mathscr{D}}$-convex* at $\bar{\delta} \in \Omega_{\mathscr{D}_\beta}$ on $\Omega_{\mathscr{D}_\beta}$, iff, for any $\mathrm{d} \in \overrightarrow{\mathscr{D}}_{\Omega,\beta,\bar{\delta}}$ and $\bar{\delta}^* \in \partial^*_{\mathscr{D}}\beta(\bar{\delta})$, $\beta(\bar{\delta}+\mathrm{d}) - \beta(\bar{\delta}) \geq \langle \bar{\delta}^*, \mathrm{d}\rangle$, where $\overrightarrow{\mathscr{D}}_{\Omega,\beta,\delta} := \{\mathrm{d} \in \mathscr{D} : \delta + \mathrm{d} \in \Omega_{\mathscr{D}_\beta}\}$ for each $\delta \in \Omega_{\mathscr{D}_\beta}$;

(b) *strictly $\partial^*_{\mathscr{D}}$-convex* at $\bar{\delta} \in \Omega_{\mathscr{D}_\beta}$ on $\Omega_{\mathscr{D}_\beta}$, iff, for any $0 \neq \mathrm{d} \in \overrightarrow{\mathscr{D}}_{\Omega,\beta,\bar{\delta}}$ and $\bar{\delta}^* \in \partial^*_{\mathscr{D}}\beta(\bar{\delta})$, $\beta(\bar{\delta}+\mathrm{d}) - \beta(\bar{\delta}) > \langle \bar{\delta}^*, \mathrm{d}\rangle$.

Definition 2.6. [36] The DC $\partial^*_{\mathscr{D}}\beta$ is said to be

(a) *monotone* on $\Omega_{\mathscr{D}_\beta}$, iff, for any $\delta \in \Omega_{\mathscr{D}_\beta}$, $\mathrm{d} \in \overrightarrow{\mathscr{D}}_{\Omega,\beta,\delta}$, $\delta^* \in \partial^*_{\mathscr{D}}\beta(\delta)$ and $\rho^* \in \partial^*_{\mathscr{D}}\beta(\delta+\mathrm{d})$, $\langle \rho^* - \delta^*, \mathrm{d}\rangle \geq 0$;

(b) *strictly monotone* on $\Omega_{\mathscr{D}_\beta}$, iff, for any $\delta \in \Omega_{\mathscr{D}_\beta}$, $0 \neq \mathrm{d} \in \overrightarrow{\mathscr{D}}_{\Omega,\beta,\delta}$, $\delta^* \in \partial^*_{\mathscr{D}}\beta(\delta)$ and $\rho^* \in \partial^*_{\mathscr{D}}\beta(\delta+\mathrm{d})$, $\langle \rho^* - \delta^*, \mathrm{d}\rangle > 0$.

Gadhi [36] derived a proposition that establishes a necessary condition for a function to be both $\partial^*_{\mathscr{D}}$-convex and strictly $\partial^*_{\mathscr{D}}$-convex.

Proposition 2.1. [36] *Let $\mathscr{D}$ be a subspace of $\mathbb{R}^n$ and let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$. Let β be a l.s.c. function at $\delta \in \Omega_{\mathscr{D}_\beta}$. Suppose that $\Omega_{\mathscr{D}_\beta} \neq \emptyset$ is a convex set, and that, for any $\delta \in \Omega_{\mathscr{D}_\beta}$, the function β admits a DC $\partial^*_{\mathscr{D}}\beta(\delta)$ at δ.*

(a) If β is $\partial^*_{\mathscr{D}}$-convex on $\Omega_{\mathscr{D}_\beta}$, then β is convex in all directions $\mathrm{d} \in \overrightarrow{\mathscr{D}}_{\Omega,\beta,\delta}$ at every $\delta \in \Omega_{\mathscr{D}_\beta}$.

(b) If β is strictly $\partial^*_{\mathscr{D}}$-convex on $\Omega_{\mathscr{D}_\beta}$, then β is strictly convex in all directions $\mathrm{d} \in \overrightarrow{\mathscr{D}}_{\Omega,\beta,\delta}$ at every $\delta \in \Omega_{\mathscr{D}_\beta}$.

The following theorem gives the mean value theorem (MVT) using the DCs.

Theorem 2.1. [36, Theorem 3.1] *Let $\emptyset \neq \mathscr{D}$ be a cone of $\mathbb{R}^n$ and let $a_1, a_2 \in \mathbb{R}^n$. Assume that $-\mathscr{D} \subseteq \mathscr{D}$, that $a_2 - a_1 \in \mathscr{D}$, and that, for each $\delta \in [a_1, a_2]$, a real-valued function β admits a BDC $\partial^*_{\mathscr{D}}\beta(\delta)$ at δ. Then, there exist $a \in]a_1, a_2[$ and $a^* \in conv(\partial^*_{\mathscr{D}}\beta(a))$ satisfying $\beta(a_2) - \beta(a_1) = \langle a^*, a_2 - a_1\rangle$.*

The following theorem specifies the condition in which $\partial^*_{\mathscr{D}}$-convexity of a given function may be characterized by the monotonicity of its DC.

Theorem 2.2. [36] *Let $\mathscr{D}$ be a subspace of $\mathbb{R}^n$ and let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$. Let β be a l.s.c. function at $\delta \in \Omega_{\mathscr{D}_\beta}$. Suppose that $-\mathscr{D} \subseteq \mathscr{D}$, that $\Omega_{\mathscr{D}_\beta} \neq \emptyset$ is a convex set, and that, for any $\delta \in \Omega_{\mathscr{D}_\beta}$, the function β admits a BDC $\partial^*_{\mathscr{D}}\beta(\delta)$ at δ. Then, the function*

(a) β is $\partial^*_{\mathscr{D}}$-convex on $\Omega_{\mathscr{D}_\beta}$ iff $\partial^*_{\mathscr{D}}\beta$ is monotone on $\Omega_{\mathscr{D}_\beta}$;

(b) β is strictly $\partial^*_{\mathscr{D}}$-convex on $\Omega_{\mathscr{D}_\beta}$ iff $\partial^*_{\mathscr{D}}\beta$ is strictly monotone on $\Omega_{\mathscr{D}_\beta}$.

Now, we recall some fundamental notations from interval-valued analysis, as detailed in [66, 67, 68].

Let $\mathscr{K}_c := \{[\gamma^L, \gamma^U] : \gamma^L, \gamma^U \in \mathbb{R}, \gamma^L \leq \gamma^U\}$ be the class of all closed and bounded intervals in $\mathbb{R}$. Let $\Gamma := [\gamma^L, \gamma^U]$ and $\Theta := [\theta^L, \theta^U]$ be in $\mathscr{K}_c$. Then,

(i) $\Gamma + \Theta := \{\gamma + \theta : \gamma \in \Gamma, \theta \in \Theta\} = [\gamma^L + \theta^L, \gamma^U + \theta^U]$;

(ii) $\Gamma - \Theta := \{\gamma - \theta : \gamma \in \Gamma, \theta \in \Theta\} = [\gamma^L - \theta^U, \gamma^U - \theta^L]$;

(iii) for each $\alpha \in \mathbb{R}$, one has

$$\alpha\Gamma := \{\alpha\gamma : \gamma \in \Gamma\} = \begin{cases} [\alpha\gamma^L, \alpha\gamma^U], & \text{if } \alpha \geq 0, \\ [\alpha\gamma^U, \alpha\gamma^L], & \text{if } \alpha < 0. \end{cases}$$

If $\gamma^L = \gamma^U = \gamma$, then $\Gamma = [\gamma, \gamma] = \{\gamma\}$.

The various $LU-$ordering methods for comparing two intervals are defined as follows.

Definition 2.7. [69, Definition 3] Let $\Gamma = [\gamma^L, \gamma^U], \Theta = [\theta^L, \theta^U] \in \mathscr{K}_c$. We say that:

(i) $\Gamma \leq_{LU} \Theta$ iff $\gamma^L \leq \theta^L$ and $\gamma^U \leq \theta^U$,

(ii) $\Gamma <_{LU} \Theta$ iff $\Gamma \leq_{LU} \Theta$ and $\Gamma \neq \Theta$,
or, equivalently,
$\Gamma <_{LU} \Theta$ iff

$$\begin{cases} \gamma^L < \theta^L \\ \gamma^U \leq \theta^U \end{cases} or \begin{cases} \gamma^L \leq \theta^L \\ \gamma^U < \theta^U \end{cases} or \begin{cases} \gamma^L < \theta^L \\ \gamma^U < \theta^U \end{cases}$$

(iii) $\Gamma <^s_{LU} \Theta$ iff $\gamma^L < \theta^L$ and $\gamma^U < \theta^U$.

3. Approximate Stampacchia and Minty vector variational Inequalities

Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$, and let $\emptyset \neq \mathscr{D}$ be a cone of $\mathbb{R}^n$. Let $\mathfrak{f}_i : \mathbb{R}^n \to \mathbb{R}$ be the objective functions for any $i \in \mathscr{M} := \{1, 2, ..., m\}$, where every function $\mathfrak{f}_i$ is defined as an interval $\mathfrak{f}_i := [\mathfrak{f}_i^L, \mathfrak{f}_i^U]$. For each function $\mathfrak{f}_i$, let $\mathscr{D}_{\mathfrak{f}_i^L}(\delta)$, $\mathscr{D}_{\mathfrak{f}_i^U}(\delta)$ be the collections of all CDs of $\mathfrak{f}_i^L, \mathfrak{f}_i^U$ at $\delta \in \Omega$, respectively. We define the sets $\Omega_{\mathscr{D},\mathfrak{f}_i^L} := \{\delta \in \Omega : \mathscr{D} \subset \mathscr{D}_{\mathfrak{f}_i^L}(\delta)\}$ and $\Omega_{\mathscr{D},\mathfrak{f}_i^U} := \{\delta \in \Omega : \mathscr{D} \subset \mathscr{D}_{\mathfrak{f}_i^U}(\delta)\}$ for any $i \in \mathscr{M}$. The set $\Omega_{\mathscr{D},\mathfrak{f}}$ is defined as $\Omega_{\mathscr{D},\mathfrak{f}} := (\cap_{i=1}^m \Omega_{\mathscr{D},\mathfrak{f}_i^L}) \cap (\cap_{i=1}^m \Omega_{\mathscr{D},\mathfrak{f}_i^U})$. Finally, for each $\delta \in \Omega_{\mathscr{D},\mathfrak{f}}$, we define the set of feasible directions as $\overrightarrow{\mathscr{D}}_{\Omega,\mathfrak{f},\delta} := \{\mathrm{d} \in \mathscr{D} : \delta + \mathrm{d} \in \Omega_{\mathscr{D},\mathfrak{f}}\}$.

Now, for any $\bar{\delta} \in \Omega$, a IVMOP is given as follows:

$$\min \mathfrak{f}(\delta) := (\mathfrak{f}_1(\delta), ..., \mathfrak{f}_m(\delta)) \ s.t. \ \delta \in \Omega_{\mathscr{D},\mathfrak{f},\bar{\delta}} \qquad (IVMOP_{\mathscr{D},\bar{\delta}})$$

where $\Omega_{\mathscr{D},\mathfrak{f},\bar{\delta}} := \Omega_{\mathscr{D},\mathfrak{f}} \cap (\{\bar{\delta}\} + \mathscr{D})$.

Based on the definition presented by Hung et al. [70] in Definition 3.1, we write the concept of approximate Pareto efficient solutions for the problem $IVMOP_{\mathscr{D},\bar{\delta}}$.

Definition 3.1. Let $\mathscr{E}_i^L$, $\mathscr{E}_i^U, i \in \mathscr{M}$ be real-numbers satisfying $0 \leq \mathscr{E}_i^L \leq \mathscr{E}_i^U$ with $\mathscr{E}_i := [\mathscr{E}_i^L, \mathscr{E}_i^U]$ for all $i \in \mathscr{M}$ and let $\mathscr{E} := (\mathscr{E}_1, \ldots, \mathscr{E}_m)$. Then, $\bar{\delta} \in \Omega_{\mathscr{D},\mathfrak{f},\bar{\delta}}$ is said to be a

(i) *type-1 $\mathscr{E}$-quasi Pareto solution* of the IVMOP, denoted by $\bar{\delta} \in \mathscr{E} - \Omega^{1,q}_{\mathscr{D},\mathfrak{f},\bar{\delta}}(IVMOP)$, iff there is no $\delta \in \Omega_{\mathscr{D},\mathfrak{f},\bar{\delta}}$ such that

$$\mathfrak{f}_i(\delta) + \mathscr{E}_i \|\delta - \bar{\delta}\| \leq_{LU} \mathfrak{f}_i(\bar{\delta}), \forall i \in \mathscr{M},$$

and

$$\mathfrak{f}_k(\delta) + \mathscr{E}_k \|\delta - \bar{\delta}\| <_{LU} \mathfrak{f}_k(\bar{\delta}), \text{for at least one } k \in \mathscr{M};$$

(ii) *type-2 $\mathscr{E}$-quasi Pareto solution* of the IVMOP, denoted by $\bar{\delta} \in \mathscr{E} - \Omega^{2,q}_{\mathscr{D},\mathfrak{f},\bar{\delta}}(IVMOP)$, iff there is no $\delta \in \Omega_{\mathscr{D},\mathfrak{f},\bar{\delta}}$ such that

$$\mathfrak{f}_i(\delta) + \mathscr{E}_i \|\delta - \bar{\delta}\| \leq_{LU} \mathfrak{f}_i(\bar{\delta}), \forall i \in \mathscr{M},$$

and

$$\mathfrak{f}_k(\delta)+\mathscr{E}_k\|\delta-\bar{\delta}\| <_{LU}^{s} \mathfrak{f}_k(\bar{\delta}), \text{for at least one } k\in\mathscr{M};$$

(iii) *type-1 $\mathscr{E}$-quasi weakly Pareto solution* of the IVMOP, denoted by $\bar{\delta}\in\mathscr{E}-\Omega^{1,qw}_{\mathscr{D},\mathfrak{f},\bar{\delta}}(IVMOP)$, iff there is no $\delta\in\Omega_{\mathscr{D},\mathfrak{f},\bar{\delta}}$ such that

$$\mathfrak{f}_i(\delta)+\mathscr{E}_i\|\delta-\bar{\delta}\| <_{LU} \mathfrak{f}_i(\bar{\delta}), \forall i\in\mathscr{M}.$$

Following the concept of approximate convexity introduced by Gupta et al. [71] and Bhatia et al. [72] in terms of Clarke subdifferentials, we now extend this idea by defining approximate convexity by using directional convexificators.

Definition 3.2. Let $\varepsilon\geq 0$. Let $\emptyset\neq\mathscr{D}$ be a cone of $\mathbb{R}^n$ and let $\emptyset\neq\Omega\subseteq\mathbb{R}^n$. Let a real-valued function β be l.s.c. at $\bar{\delta}\in\Omega_{\mathscr{D}_\beta}:=\{\delta\in\Omega:\mathscr{D}\subset\mathscr{D}_{\beta(\delta)}\}$. Suppose that β admits a bounded directional convexificator (BDC) $\partial^*_{\mathscr{D}}\beta(\bar{\delta})$ at $\bar{\delta}$. Then, β is said to be

(a) $\varepsilon-\partial^*_{\mathscr{D}}-$convex at $\bar{\delta}\in\Omega_{\mathscr{D}_\beta}$ on $\Omega_{\mathscr{D}_\beta}$ iff, for any $\mathrm{d}\in\overrightarrow{\mathscr{D}}_{\Omega,\beta,\bar{\delta}}$ and $\bar{\delta}^*\in\partial^*_{\mathscr{D}}\beta(\bar{\delta})$,

$$\beta(\bar{\delta}+\mathrm{d})-\beta(\bar{\delta})\geq\langle\bar{\delta}^*,\mathrm{d}\rangle-\varepsilon\|\mathrm{d}\|;$$

(b) $\varepsilon-\partial^*_{\mathscr{D}}-$ strictly convex at $\bar{\delta}\in\Omega_{\mathscr{D}_\beta}$ on $\Omega_{\mathscr{D}_\beta}$, iff, for any $\mathrm{d}\in\overrightarrow{\mathscr{D}}_{\Omega,\beta,\bar{\delta}}$ and $\bar{\delta}^*\in\partial^*_{\mathscr{D}}\beta(\bar{\delta})$,

$$\beta(\bar{\delta}+\mathrm{d})-\beta(\bar{\delta})>\langle\bar{\delta}^*,\mathrm{d}\rangle-\varepsilon\|\mathrm{d}\|.$$

Remark 3.1.
- If the involved function is Lipschitz continuous, the above concepts reduce to the approximate convexity in terms of Clarke subdifferentials (see, e.g., [71, Theorem 1]).
- If the involved function is continuous, the above concepts reduce to the approximate convexity in terms of convexificators (see, e.g., [73, Definition 2.4]).
- For $\varepsilon=0$, the above concepts reduce to the convexity in terms of directional convexificators (see, e.g., [36, Definition 3.1]).
- For $\varepsilon=0$, and the involved function is Lipschitz continuous, the above concepts reduce to the convexity in terms of Clarke subdifferentials (see, e.g., [74, Definition 4.1]).
- For $\varepsilon=0$, and the involved function is continuous, the above concepts reduce to the convexity in terms of convexificators (see, e.g., [33, Definition 2.4]).

To solve $(IVMOP_{\mathscr{D},\bar{\delta}})$, we introduce approximate vector variational inequalities of the Stampacchia and Minty types. They are formulated by using directional convexificators, building on the VVIs proposed by Laha and Mishra [33], Khan et al. [73], and Gadhi and Odha [38].

Definition 3.3. Let $\mathscr{E}_i^L$, $\mathscr{E}_i^U, i\in\mathscr{M}$ be real numbers satisfying $0\leq\mathscr{E}_i^L\leq\mathscr{E}_i^U$ with $\mathscr{E}_i:=[\mathscr{E}_i^L,\mathscr{E}_i^U]$ for all $i\in\mathscr{M}$ and let $\mathscr{E}:=(\mathscr{E}_1,\ldots,\mathscr{E}_m)$. A vector $\bar{\delta}\in\Omega_{\mathscr{D},\mathfrak{f}}$ solves

(a) *$\mathscr{E}-$ Stampacchia $\partial^*_{\mathscr{D}}-$VVI* (in short $\mathscr{E}-$SVVI) iff, for any $\mathrm{d}\in\overrightarrow{\mathscr{D}}_{\Omega,\mathfrak{f},\bar{\delta}}$, there exists $\bar{\delta}^{*L}\in\partial^*_{\mathscr{D}}\mathfrak{f}^L(\bar{\delta})$ and $\bar{\delta}^{*U}\in\partial^*_{\mathscr{D}}\mathfrak{f}^U(\bar{\delta})$ such that

$$\begin{cases}\langle\bar{\delta}^{*L},\mathrm{d}\rangle_m+\mathscr{E}^L\|\mathrm{d}\|:=(\langle\bar{\delta}^{*L}_1,\mathrm{d}\rangle+\mathscr{E}^L_1\|\mathrm{d}\|,\ldots,\langle\bar{\delta}^{*L}_m,\mathrm{d}\rangle+\mathscr{E}^L_m\|\mathrm{d}\|)\notin-\mathbb{R}^m_+\setminus\{0\},\\ \langle\bar{\delta}^{*U},\mathrm{d}\rangle_m+\mathscr{E}^U\|\mathrm{d}\|:=(\langle\bar{\delta}^{*U}_1,\mathrm{d}\rangle+\mathscr{E}^U_1\|\mathrm{d}\|,\ldots,\langle\bar{\delta}^{*U}_m,\mathrm{d}\rangle+\mathscr{E}^U_m\|\mathrm{d}\|)\notin-\mathbb{R}^m_+\setminus\{0\}\end{cases}$$

where

$$\bar{\delta}^{*L} := (\bar{\delta}_1^{*L}, .., \bar{\delta}_m^{*L}) \in \partial^*_{\mathscr{D}} \mathfrak{f}^L(\bar{\delta}) := \partial^*_{\mathscr{D}} \mathfrak{f}_1^L(\bar{\delta}) \times ... \times \partial^*_{\mathscr{D}} \mathfrak{f}_m^L(\bar{\delta});$$
$$\bar{\delta}^{*U} := (\bar{\delta}_1^{*U}, .., \bar{\delta}_m^{*U}) \in \partial^*_{\mathscr{D}} \mathfrak{f}^U(\bar{\delta}) := \partial^*_{\mathscr{D}} \mathfrak{f}_1^U(\bar{\delta}) \times ... \times \partial^*_{\mathscr{D}} \mathfrak{f}_m^U(\bar{\delta});$$
$$\mathscr{E}^L \|\mathrm{d}\| := (\mathscr{E}_1^L \|\mathrm{d}\|, ..., \mathscr{E}_m^L \|\mathrm{d}\|);$$
$$\mathscr{E}^U \|\mathrm{d}\| := (\mathscr{E}_1^U \|\mathrm{d}\|, ..., \mathscr{E}_m^U \|\mathrm{d}\|);$$

(b) $\mathscr{E}-$ *weak Stampacchia* $\partial^*_{\mathscr{D}}-$*VVI* (in short $\mathscr{E}-$SWVVI) iff, for any $\mathrm{d} \in \overrightarrow{\mathscr{D}}_{\Omega,\mathfrak{f},\bar{\delta}}$, there exists $\bar{\delta}^{*L} \in \partial^*_{\mathscr{D}} \mathfrak{f}^L(\bar{\delta})$ and $\bar{\delta}^{*U} \in \partial^*_{\mathscr{D}} \mathfrak{f}^U(\bar{\delta})$ such that

$$\begin{cases} \langle \bar{\delta}^{*L}, \mathrm{d} \rangle_m + \mathscr{E}^L \|\mathrm{d}\| := (\langle \bar{\delta}_1^{*L}, \mathrm{d} \rangle + \mathscr{E}_1^L \|\mathrm{d}\|, ..., \langle \bar{\delta}_m^{*L}, \mathrm{d} \rangle + \mathscr{E}_m^L \|\mathrm{d}\|) \notin -int\mathbb{R}^m_+, \\ \langle \bar{\delta}^{*U}, \mathrm{d} \rangle_m + \mathscr{E}^U \|\mathrm{d}\| := (\langle \bar{\delta}_1^{*U}, \mathrm{d} \rangle + \mathscr{E}_1^U \|\mathrm{d}\|, ..., \langle \bar{\delta}_m^{*U}, \mathrm{d} \rangle + \mathscr{E}_m^U \|\mathrm{d}\|) \notin -int\mathbb{R}^m_+; \end{cases}$$

(c) $\mathscr{E}-$ *Minty* $\partial^*_{\mathscr{D}}-$*VVI* (in short $\mathscr{E}-$MVVI) iff, for any $\mathrm{d} \in \overrightarrow{\mathscr{D}}_{\Omega,\mathfrak{f},\bar{\delta}}$ and $\bar{\delta}^{*L} \in \partial^*_{\mathscr{D}} \mathfrak{f}^L(\bar{\delta}+\mathrm{d})$ and $\bar{\delta}^{*U} \in \partial^*_{\mathscr{D}} \mathfrak{f}^U(\bar{\delta}+\mathrm{d})$,

$$\begin{cases} \langle \bar{\delta}^{*L}, \mathrm{d} \rangle_m + \mathscr{E}^L \|\mathrm{d}\| := (\langle \bar{\delta}_1^{*L}, \mathrm{d} \rangle + \mathscr{E}_1^L \|\mathrm{d}\|, ..., \langle \bar{\delta}_m^{*L}, \mathrm{d} \rangle + \mathscr{E}_m^L \|\mathrm{d}\|) \notin -\mathbb{R}^m_+ \setminus \{0\}, \\ \langle \bar{\delta}^{*U}, \mathrm{d} \rangle_m + \mathscr{E}^U \|\mathrm{d}\| := (\langle \bar{\delta}_1^{*U}, \mathrm{d} \rangle + \mathscr{E}_1^U \|\mathrm{d}\|, ..., \langle \bar{\delta}_m^{*U}, \mathrm{d} \rangle + \mathscr{E}_m^U \|\mathrm{d}\|) \notin -\mathbb{R}^m_+ \setminus \{0\} \end{cases}$$

where

$$\bar{\delta}^{*L} := (\bar{\delta}_1^{*L}, .., \bar{\delta}_m^{*L}) \in \partial^*_{\mathscr{D}} \mathfrak{f}^L(\bar{\delta}+\mathrm{d}) = \partial^*_{\mathscr{D}} \mathfrak{f}_1^L(\bar{\delta}+\mathrm{d}) \times ... \times \partial^*_{\mathscr{D}} \mathfrak{f}_m^L(\bar{\delta}+\mathrm{d}),$$
$$\bar{\delta}^{*U} := (\bar{\delta}_1^{*U}, .., \bar{\delta}_m^{*U}) \in \partial^*_{\mathscr{D}} \mathfrak{f}^U(\bar{\delta}+\mathrm{d}) = \partial^*_{\mathscr{D}} \mathfrak{f}_1^U(\bar{\delta}+\mathrm{d}) \times ... \times \partial^*_{\mathscr{D}} \mathfrak{f}_m^U(\bar{\delta}+\mathrm{d}),$$
$$\mathscr{E}^L \|\mathrm{d}\| := (\mathscr{E}_1^L \|\mathrm{d}\|, ..., \mathscr{E}_m^L \|\mathrm{d}\|),$$
$$\mathscr{E}^U \|\mathrm{d}\| := (\mathscr{E}_1^U \|\mathrm{d}\|, ..., \mathscr{E}_m^U \|\mathrm{d}\|);$$

(d) $\mathscr{E}-$ *weak Minty* $\partial^*_{\mathscr{D}}-$*VVI* (in short $\mathscr{E}-$MWVVI) iff, for any $\mathrm{d} \in \overrightarrow{\mathscr{D}}_{\Omega,\mathfrak{f},\bar{\delta}}$ and $\bar{\delta}^{*L} \in \partial^*_{\mathscr{D}} \mathfrak{f}^L(\bar{\delta}+\mathrm{d})$ and $\bar{\delta}^{*U} \in \partial^*_{\mathscr{D}} \mathfrak{f}^U(\bar{\delta}+\mathrm{d})$,

$$\begin{cases} \langle \bar{\delta}^{*L}, \mathrm{d} \rangle_m + \mathscr{E}^L \|\mathrm{d}\| := (\langle \bar{\delta}_1^{*L}, \mathrm{d} \rangle + \mathscr{E}_1^L \|\mathrm{d}\|, ..., \langle \bar{\delta}_m^{*L}, \mathrm{d} \rangle + \mathscr{E}_m^L \|\mathrm{d}\|) \notin -int\mathbb{R}^m_+, \\ \langle \bar{\delta}^{*U}, \mathrm{d} \rangle_m + \mathscr{E}^U \|\mathrm{d}\| := (\langle \bar{\delta}_1^{*U}, \mathrm{d} \rangle + \mathscr{E}_1^U \|\mathrm{d}\|, ..., \langle \bar{\delta}_m^{*U}, \mathrm{d} \rangle + \mathscr{E}_m^U \|\mathrm{d}\|) \notin -int\mathbb{R}^m_+. \end{cases}$$

Remark 3.2.

- For $\mathscr{E}_i^L = \mathscr{E}_i^U = 0$, the above concepts reduce to the vector variational ineqaulities in terms of directional convexificators.
- If the involved functions are Lipschitz continuous, the above concepts reduce to the approximate vector variational inequalities in terms of Clarke subdifferentials (see, e.g., [75]).
- If $\mathfrak{f}_i^L = \mathfrak{f}_i^U = \mathfrak{f}_i$ and the involved functions are Lipschitz continuous, the above concepts reduce to the approximate vector variational inequalities in terms of Clarke subdifferentials (see, e.g., [76]).
- If the involved functions are continuous, the above concepts reduce to the approximate vector variational inequalities in terms of convexificators.
- If $\mathfrak{f}_i^L = \mathfrak{f}_i^U = \mathfrak{f}_i$ and the involved functions are continuous, the above concepts reduce to the approximate vector variational inequalities in terms of convexificators (see, e.g., [73, Definition 2.4]).

- For $\mathscr{E}_i^L = \mathscr{E}_i^U = 0$, and the involved functions are Lipschitz continuous, the above concepts reduce to the vector variational inequalities in terms of Clarke subdifferentials (see, e.g., [61]).
- For $\mathscr{E}_i^L = \mathscr{E}_i^U = 0$, and the involved function is continuous, the above concepts reduce to the vector variational inequalities in terms of convexificators (see, e.g., [77]).
- For $\mathscr{E}_i^L = \mathscr{E}_i^U = 0$, and $\mathfrak{f}_i^L = \mathfrak{f}_i^U = \mathfrak{f}_i$ the above concepts reduce to the vector variational ineqaulities in terms of directional convexificators (see, e.g., [38]).
- For $\mathscr{E}_i^L = \mathscr{E}_i^U = 0$, $\mathfrak{f}_i^L = \mathfrak{f}_i^U = \mathfrak{f}_i$ and the involved functions are Lipschitz continuous, the above concepts reduce to the vector variational ineqaulities in terms of clarke subdifferentials (see, e.g., [78]).
- For $\mathscr{E}_i^L = \mathscr{E}_i^U = 0$, $\mathfrak{f}_i^L = \mathfrak{f}_i^U = \mathfrak{f}_i$ and involved functions are continuous, the above concepts reduce to the vector variational ineqaulities in terms of convexificators (see, e.g., [33]).
- For $\mathscr{E}_i^L = \mathscr{E}_i^U = 0$, and the involved functions are continuously differentiable, the above concepts reduce to the vector variational inequalities in terms of Clarke subdifferentials (see, e.g., [61]).

Now we derive the relations among the different solution concepts in terms of directional convexificators.

Theorem 3.1. *Let $\bar{\delta} \in \Omega_{\mathscr{D},\mathfrak{f}}$ and let each $\mathfrak{f}_i^L, \mathfrak{f}_i^U (i \in \mathscr{M})$ admits a BDC, $\partial_{\mathscr{D}}^* \mathfrak{f}_i^L(\bar{\delta}), \partial_{\mathscr{D}}^* \mathfrak{f}_i^U(\bar{\delta})$ at $\bar{\delta}$, respectively. Assume that $\mathscr{D} \neq \{0_{\mathbb{R}^n}\}$, that $\Omega_{\mathscr{D},\mathfrak{f}}$ is convex, and that each $\mathfrak{f}_i^L, \mathfrak{f}_i^U (i \in \mathscr{M})$ is $\mathscr{E} - \partial_D^*-$convex at $\bar{\delta}$ over $\Omega_{\mathscr{D},\mathfrak{f}}$. If $\bar{\delta}$ solves ∂_D^*-SVVI, then $\bar{\delta} \in \mathscr{E} - \Omega^{1,q}_{\mathscr{D},\mathfrak{f},\bar{\delta}}(IVMOP)$.*

Proof. Contrarily, assume that $\bar{\delta} \in \Omega_{\mathscr{D},\mathfrak{f},\bar{\delta}}$ is not a type-1 $\mathscr{E}-$ quasi pareto solution of $IVMOP_{\mathscr{D},\bar{\delta}}$. Then, there exists $\bar{\delta} \neq \tilde{\delta} \in \Omega_{\mathscr{D},\mathfrak{f},\bar{\delta}}$ such that

$$\begin{cases} \mathfrak{f}_i(\tilde{\delta}) + \mathscr{E}_i \|\tilde{\delta} - \bar{\delta}\| \leq_{LU} \mathfrak{f}_i(\bar{\delta}), \ \forall i \in \mathscr{M}, \\ \mathfrak{f}_k(\tilde{\delta}) + \mathscr{E}_k \|\tilde{\delta} - \bar{\delta}\| <_{LU} \mathfrak{f}_k(\bar{\delta}), \ \text{for at least one } k \in \mathscr{M}. \end{cases}$$

or equivalently,

$$\mathfrak{f}_i^L(\tilde{\delta}) + \mathscr{E}_i^L \|\tilde{\delta} - \bar{\delta}\| \leq \mathfrak{f}_i^L(\bar{\delta}) \text{ and } \mathfrak{f}_i^U(\tilde{\delta}) + \mathscr{E}_i^U \|\tilde{\delta} - \bar{\delta}\| \leq \mathfrak{f}_i^U(\bar{\delta}), \ \forall i \in \mathscr{M} \tag{3.1}$$

are satisfied and one of the following is satisfied for at least one index $k \in \mathscr{M}$

$$\left.\begin{array}{ll} \text{(i)} & \mathfrak{f}_k^L(\tilde{\delta}) + \mathscr{E}_k^L \|\tilde{\delta} - \bar{\delta}\| < \mathfrak{f}_k^L(\bar{\delta}) \text{ and } \mathfrak{f}_k^U(\tilde{\delta}) + \mathscr{E}_k^U \|\tilde{\delta} - \bar{\delta}\| < \mathfrak{f}_k^U(\bar{\delta}); \text{ or} \\ \text{(ii)} & \mathfrak{f}_k^L(\tilde{\delta}) + \mathscr{E}_k^L \|\tilde{\delta} - \bar{\delta}\| \leq \mathfrak{f}_k^L(\bar{\delta}) \text{ and } \mathfrak{f}_k^U(\tilde{\delta}) + \mathscr{E}_k^U \|\tilde{\delta} - \bar{\delta}\| < \mathfrak{f}_k^U(\bar{\delta}); \text{ or} \\ \text{(iii)} & \mathfrak{f}_k^L(\tilde{\delta}) + \mathscr{E}_k^L \|\tilde{\delta} - \bar{\delta}\| < \mathfrak{f}_k^L(\bar{\delta}) \text{ and } \mathfrak{f}_k^U(\tilde{\delta}) + \mathscr{E}_k^U \|\tilde{\delta} - \bar{\delta}\| \leq \mathfrak{f}_k^U(\bar{\delta}). \end{array}\right\} \tag{3.2}$$

Since $\tilde{\delta} \in \Omega_{\mathscr{D},\mathfrak{f},\bar{\delta}}$, then there exists $\tilde{\mathrm{d}} \in \mathscr{D}$ such that $\tilde{\delta} = \bar{\delta} + \tilde{\mathrm{d}}$. Hence, $\tilde{\mathrm{d}} \in \overrightarrow{\mathscr{D}}_{\Omega,\mathfrak{f},\bar{\delta}}$. By the $\mathscr{E}_i^L - \partial_D^*-$convexity of $\mathfrak{f}_i^L(i \in \mathscr{M})$ and $\mathscr{E}_i^U - \partial_D^*-$convexity of $\mathfrak{f}_i^U(i \in \mathscr{M})$ at $\bar{\delta}$ over $\Omega_{\mathscr{D},\mathfrak{f},\bar{\delta}}$, it follows that

$$\left.\begin{array}{l} \langle \bar{\delta}_i^{*L}, \tilde{\mathrm{d}} \rangle \leq \mathfrak{f}_i^L(\tilde{\delta}) - \mathfrak{f}_i^L(\bar{\delta}) + \mathscr{E}_i^L \|\tilde{\mathrm{d}}\|, \forall \bar{\delta}_i^{*L} \in \partial_{\mathscr{D}}^* \mathfrak{f}_i^L(\bar{\delta}), \ \forall i \in \mathscr{M}, \\ \langle \bar{\delta}_i^{*U}, \tilde{\mathrm{d}} \rangle \leq \mathfrak{f}_i^U(\tilde{\delta}) - \mathfrak{f}_i^U(\bar{\delta}) + \mathscr{E}_i^U \|\tilde{\mathrm{d}}\|, \forall \bar{\delta}_i^{*U} \in \partial_{\mathscr{D}}^* \mathfrak{f}_i^U(\bar{\delta}), \ \forall i \in \mathscr{M}. \end{array}\right\} \tag{3.3}$$

Using inequalities (3.1) and (3.2), together with the $\mathscr{E}_i^L - \partial_D^*-$convexity of $\mathfrak{f}_i^L (i \in \mathscr{M})$ and the $\mathscr{E}_i^U - \partial_D^*-$convexity of $\mathfrak{f}_i^U (i \in \mathscr{M})$ at $\bar{\delta}$ over $\Omega_{\mathscr{D},\mathfrak{f},\bar{\delta}}$ given in (3.3), one has

$$\left.\begin{array}{l}\langle \bar{\delta}_i^{*L}, \tilde{\mathrm{d}} \rangle \leq 0, \forall \bar{\delta}_i^{*L} \in \partial_{\mathscr{D}}^* \mathfrak{f}_i^L(\bar{\delta}), \ \forall i \in \mathscr{M}, \\ \langle \bar{\delta}_i^{*U}, \tilde{\mathrm{d}} \rangle \leq 0, \forall \bar{\delta}_i^{*U} \in \partial_{\mathscr{D}}^* \mathfrak{f}_i^U(\bar{\delta}), \ \forall i \in \mathscr{M},\end{array}\right\} \tag{3.4}$$

and for at least one index $k \in \mathscr{M}$

$$\left.\begin{array}{ll}\text{(i)} & \langle \bar{\delta}_k^{*L}, \tilde{\mathrm{d}} \rangle < 0, \forall \bar{\delta}_k^{*L} \in \partial_{\mathscr{D}}^* \mathfrak{f}_k^L(\bar{\delta}) \text{ and } \langle \bar{\delta}_k^{*U}, \tilde{\mathrm{d}} \rangle < 0, \forall \bar{\delta}_k^{*U} \in \partial_{\mathscr{D}}^* \mathfrak{f}_k^U(\bar{\delta}); or \\ \text{(ii)} & \langle \bar{\delta}_k^{*L}, \tilde{\mathrm{d}} \rangle \leq 0, \forall \bar{\delta}_k^{*L} \in \partial_{\mathscr{D}}^* \mathfrak{f}_k^L(\bar{\delta}) \text{ and } \langle \bar{\delta}_k^{*U}, \tilde{\mathrm{d}} \rangle < 0, \forall \bar{\delta}_k^{*U} \in \partial_{\mathscr{D}}^* \mathfrak{f}_k^U(\bar{\delta}); or \\ \text{(i)} & \langle \bar{\delta}_k^{*L}, \tilde{\mathrm{d}} \rangle < 0, \forall \bar{\delta}_k^{*L} \in \partial_{\mathscr{D}}^* \mathfrak{f}_k^L(\bar{\delta}) \text{ and } \langle \bar{\delta}_k^{*U}, \tilde{\mathrm{d}} \rangle \leq 0, \forall \bar{\delta}_k^{*U} \in \partial_{\mathscr{D}}^* \mathfrak{f}_k^U(\bar{\delta}).\end{array}\right\} \tag{3.5}$$

Combining inequalities (3.4) with inequalities (3.5)(i), (3.5)(ii), and (3.5)(iii), we obtain

$$\begin{array}{rll}\text{(i)} & ((\langle \bar{\delta}_1^{*L}, \tilde{\mathrm{d}} \rangle, \ldots, \langle \bar{\delta}_m^{*L}, \tilde{\mathrm{d}} \rangle) \in -\mathbb{R}_+^m \setminus \{0\}, & ((\langle \bar{\delta}_1^{*U}, \tilde{\mathrm{d}} \rangle, \ldots, \langle \bar{\delta}_m^{*U}, \tilde{\mathrm{d}} \rangle) \in -\mathbb{R}_+^m \setminus \{0\}; \text{or} \\ \text{(ii)} & ((\langle \bar{\delta}_1^{*L}, \tilde{\mathrm{d}} \rangle, \ldots, \langle \bar{\delta}_m^{*L}, \tilde{\mathrm{d}} \rangle) \in -\mathbb{R}_+^m, & ((\langle \bar{\delta}_1^{*U}, \tilde{\mathrm{d}} \rangle, \ldots, \langle \bar{\delta}_m^{*U}, \tilde{\mathrm{d}} \rangle) \in -\mathbb{R}_+^m \setminus \{0\}; \text{or} \\ \text{(iii)} & ((\langle \bar{\delta}_1^{*L}, \tilde{\mathrm{d}} \rangle, \ldots, \langle \bar{\delta}_m^{*L}, \tilde{\mathrm{d}} \rangle) \in -\mathbb{R}_+^m \setminus \{0\}, & ((\langle \bar{\delta}_1^{*U}, \tilde{\mathrm{d}} \rangle, \ldots, \langle \bar{\delta}_m^{*U}, \tilde{\mathrm{d}} \rangle) \in -\mathbb{R}_+^m,\end{array}$$

respectively. This contradicts our assumption that $\bar{\delta}$ solves Stampacchia $\partial_{\mathscr{D}}^*-$VVI. □

Remark 3.3. If $\mathscr{E}_i^L = \mathscr{E}_i^U = 0$ and $\mathfrak{f}_i^L = \mathfrak{f}_i^U = \mathfrak{f}_i$, then Theorem 3.1 reduces to [38, Proposition 3.1].

Theorem 3.2. *Let $\bar{\delta} \in \Omega_{\mathscr{D},\mathfrak{f}}$ and let each $\mathfrak{f}_i^L, \mathfrak{f}_i^U (i \in \mathscr{M})$ admits a BDC, $\partial_{\mathscr{D}}^* \mathfrak{f}_i^L(\bar{\delta}), \partial_{\mathscr{D}}^* \mathfrak{f}_i^U(\bar{\delta})$ at $\bar{\delta}$, respectively. Assume that $\mathscr{D} \neq \{0_{\mathbb{R}^n}\}$, that $\Omega_{\mathscr{D},\mathfrak{f}}$ is convex, and that each $\mathfrak{f}_i^L, \mathfrak{f}_i^U (i \in \mathscr{M})$ is ∂_D^*-convex at $\bar{\delta}$ over $\Omega_{\mathscr{D},\mathfrak{f}}$. If $\bar{\delta}$ solves $\mathscr{E} - \partial_D^*-$SVVI, then $\bar{\delta} \in \mathscr{E} - \Omega_{\mathscr{D},\mathfrak{f},\bar{\delta}}^{1,q}(IVMOP)$.*

Proof. Contrarily, assume that $\bar{\delta} \in \Omega_{\mathscr{D},\mathfrak{f},\bar{\delta}}$ is not a type-1 $\mathscr{E}-$ quasi pareto solution of $IVMOP_{\mathscr{D},\bar{\delta}}$. Then, there exists $\bar{\delta} \neq \tilde{\delta} \in \Omega_{\mathscr{D},\mathfrak{f},\bar{\delta}}$ such that

$$\left.\begin{array}{l}\mathfrak{f}_i(\tilde{\delta}) + \mathscr{E}_i \|\tilde{\delta} - \bar{\delta}\| \leq_{LU} \mathfrak{f}_i(\bar{\delta}), \ \forall i \in \mathscr{M}, \\ \mathfrak{f}_k(\tilde{\delta}) + \mathscr{E}_k \|\tilde{\delta} - \bar{\delta}\| <_{LU} \mathfrak{f}_k(\bar{\delta}), \ \text{for at least one } k \in \mathscr{M}.\end{array}\right\}.$$

or equivalently,

$$\mathfrak{f}_i^L(\tilde{\delta}) + \mathscr{E}_i^L \|\tilde{\delta} - \bar{\delta}\| \leq \mathfrak{f}_i^L(\bar{\delta}) \text{ and } \mathfrak{f}_i^U(\tilde{\delta}) + \mathscr{E}_i^U \|\tilde{\delta} - \bar{\delta}\| \leq \mathfrak{f}_i^U(\bar{\delta}), \ \forall i \in \mathscr{M} \tag{3.6}$$

are satisfied and one of the following is satisfied for at least one index $k \in \mathscr{M}$

$$\left.\begin{array}{ll}\text{(i)} & \mathfrak{f}_k^L(\tilde{\delta}) + \mathscr{E}_k^L \|\tilde{\delta} - \bar{\delta}\| < \mathfrak{f}_k^L(\bar{\delta}) \text{ and } \mathfrak{f}_k^U(\tilde{\delta}) + \mathscr{E}_k^U \|\tilde{\delta} - \bar{\delta}\| < \mathfrak{f}_k^U(\bar{\delta}); \text{ or} \\ \text{(ii)} & \mathfrak{f}_k^L(\tilde{\delta}) + \mathscr{E}_k^L \|\tilde{\delta} - \bar{\delta}\| \leq \mathfrak{f}_k^L(\bar{\delta}) \text{ and } \mathfrak{f}_k^U(\tilde{\delta}) + \mathscr{E}_k^U \|\tilde{\delta} - \bar{\delta}\| < \mathfrak{f}_k^U(\bar{\delta}); \text{ or} \\ \text{(iii)} & \mathfrak{f}_k^L(\tilde{\delta}) + \mathscr{E}_k^L \|\tilde{\delta} - \bar{\delta}\| < \mathfrak{f}_k^L(\bar{\delta}) \text{ and } \mathfrak{f}_k^U(\tilde{\delta}) + \mathscr{E}_k^U \|\tilde{\delta} - \bar{\delta}\| \leq \mathfrak{f}_k^U(\bar{\delta}).\end{array}\right\} \tag{3.7}$$

Since $\tilde{\delta} \in \Omega_{\mathscr{D},\mathfrak{f},\bar{\delta}}$, then there exists $\tilde{\mathrm{d}} \in \mathscr{D}$ such that $\tilde{\delta} = \bar{\delta} + \tilde{\mathrm{d}}$ and hence $\tilde{\mathrm{d}} \in \overrightarrow{\mathscr{D}}_{\Omega,\mathfrak{f},\bar{\delta}}$. By the ∂_D^*-convexity of $\mathfrak{f}_i^L (i \in \mathscr{M})$ and $\mathfrak{f}_i^U (i \in \mathscr{M})$ at $\bar{\delta}$ over $\Omega_{\mathscr{D},\mathfrak{f},\bar{\delta}}$, it follows that

$$\left.\begin{array}{l}\langle \bar{\delta}_i^{*L}, \tilde{\mathrm{d}} \rangle \leq \mathfrak{f}_i^L(\tilde{\delta}) - \mathfrak{f}_i^L(\bar{\delta}), \forall \bar{\delta}_i^{*L} \in \partial_{\mathscr{D}}^* \mathfrak{f}_i^L(\bar{\delta}), \ \forall i \in \mathscr{M}, \\ \langle \bar{\delta}_i^{*U}, \tilde{\mathrm{d}} \rangle \leq \mathfrak{f}_i^U(\tilde{\delta}) - \mathfrak{f}_i^U(\bar{\delta}), \forall \bar{\delta}_i^{*U} \in \partial_{\mathscr{D}}^* \mathfrak{f}_i^U(\bar{\delta}), \ \forall i \in \mathscr{M}.\end{array}\right\} \tag{3.8}$$

Using inequalities (3.6) and (3.7), together with the ∂_D^*-convexity of $\mathfrak{f}_i^L(i\in\mathscr{M})$ and the ∂_D^*-convexity of $\mathfrak{f}_i^U(i\in\mathscr{M})$ at $\bar{\delta}$ over $\Omega_{\mathscr{D},\mathfrak{f},\bar{\delta}}$ given in (3.8), one has

$$\left.\begin{aligned}\langle\bar{\delta}_i^{*L},\tilde{\mathrm{d}}\rangle+\mathscr{E}_i^L\|\tilde{\mathrm{d}}\|\leq 0,\forall\bar{\delta}_i^{*L}\in\partial_{\mathscr{D}}^*\mathfrak{f}_i^L(\bar{\delta}),\ \forall i\in\mathscr{M},\\ \langle\bar{\delta}_i^{*U},\tilde{\mathrm{d}}\rangle+\mathscr{E}_i^U\|\tilde{\mathrm{d}}\|\leq 0,\forall\bar{\delta}_i^{*U}\in\partial_{\mathscr{D}}^*\mathfrak{f}_i^U(\bar{\delta}),\ \forall i\in\mathscr{M},\end{aligned}\right\}\tag{3.9}$$

and for at least one index $k\in\mathscr{M}$

$$\left.\begin{array}{ll}\text{(i)} & \langle\bar{\delta}_k^{*L},\tilde{\mathrm{d}}\rangle+\mathscr{E}_i^L\|\tilde{\mathrm{d}}\|<0,\forall\bar{\delta}_k^{*L}\in\partial_{\mathscr{D}}^*\mathfrak{f}_k^L(\bar{\delta})\text{ and }\langle\bar{\delta}_k^{*U},\tilde{\mathrm{d}}\rangle+\mathscr{E}_i^U\|\tilde{\mathrm{d}}\|<0,\forall\bar{\delta}_k^{*U}\in\partial_{\mathscr{D}}^*\mathfrak{f}_k^U(\bar{\delta});or\\ \text{(ii)} & \langle\bar{\delta}_k^{*L},\tilde{\mathrm{d}}\rangle+\mathscr{E}_i^L\|\tilde{\mathrm{d}}\|\leq 0,\forall\bar{\delta}_k^{*L}\in\partial_{\mathscr{D}}^*\mathfrak{f}_k^L(\bar{\delta})\text{ and }\langle\bar{\delta}_k^{*U},\tilde{\mathrm{d}}\rangle+\mathscr{E}_i^U\|\tilde{\mathrm{d}}\|<0,\forall\bar{\delta}_k^{*U}\in\partial_{\mathscr{D}}^*\mathfrak{f}_k^U(\bar{\delta});or\\ \text{(iii)} & \langle\bar{\delta}_k^{*L},\tilde{\mathrm{d}}\rangle+\mathscr{E}_i^L\|\tilde{\mathrm{d}}\|<0,\forall\bar{\delta}_k^{*L}\in\partial_{\mathscr{D}}^*\mathfrak{f}_k^L(\bar{\delta})\text{ and }\langle\bar{\delta}_k^{*U},\tilde{\mathrm{d}}\rangle+\mathscr{E}_i^U\|\tilde{\mathrm{d}}\|\leq 0,\forall\bar{\delta}_k^{*U}\in\partial_{\mathscr{D}}^*\mathfrak{f}_k^U(\bar{\delta}),\end{array}\right\}\tag{3.10}$$

Combining inequalities (3.9) with inequalities (3.10)(i), (3.10)(ii), and (3.10)(iii), we obtain

(i)

$$\begin{aligned}&(\langle\bar{\delta}_1^{*L},\tilde{\mathrm{d}}\rangle+\mathscr{E}_1^L\|\tilde{\mathrm{d}}\|,\ldots,\langle\bar{\delta}_m^{*L},\tilde{\mathrm{d}}\rangle+\mathscr{E}_m^L\|\tilde{\mathrm{d}}\|)\in-\mathbb{R}_+^m\setminus\{0\},\\ &(\langle\bar{\delta}_1^{*U},\tilde{\mathrm{d}}\rangle+\mathscr{E}_1^U\|\tilde{\mathrm{d}}\|,\ldots,\langle\bar{\delta}_m^{*U},\tilde{\mathrm{d}}\rangle+\mathscr{E}_m^U\|\tilde{\mathrm{d}}\|)\in-\mathbb{R}_+^m\setminus\{0\};\text{or}\end{aligned}$$

(ii)

$$\begin{aligned}&(\langle\bar{\delta}_1^{*L},\tilde{\mathrm{d}}\rangle+\mathscr{E}_1^L\|\tilde{\mathrm{d}}\|,\ldots,\langle\bar{\delta}_m^{*L},\tilde{\mathrm{d}}\rangle+\mathscr{E}_m^L\|\tilde{\mathrm{d}}\|)\in-\mathbb{R}_+^m,\\ &(\langle\bar{\delta}_1^{*U},\tilde{\mathrm{d}}\rangle+\mathscr{E}_1^U\|\tilde{\mathrm{d}}\|,\ldots,\langle\bar{\delta}_m^{*U},\tilde{\mathrm{d}}\rangle+\mathscr{E}_m^U\|\tilde{\mathrm{d}}\|)\in-\mathbb{R}_+^m\setminus\{0\};\text{or}\end{aligned}$$

(iii)

$$\begin{aligned}&(\langle\bar{\delta}_1^{*L},\tilde{\mathrm{d}}\rangle+\mathscr{E}_1^L\|\tilde{\mathrm{d}}\|,\ldots,\langle\bar{\delta}_m^{*L},\tilde{\mathrm{d}}\rangle+\mathscr{E}_m^L\|\tilde{\mathrm{d}}\|)\in-\mathbb{R}_+^m\setminus\{0\},\\ &(\langle\bar{\delta}_1^{*U},\tilde{\mathrm{d}}\rangle+\mathscr{E}_1^U\|\tilde{\mathrm{d}}\|,\ldots,\langle\bar{\delta}_m^{*U},\tilde{\mathrm{d}}\rangle+\mathscr{E}_m^U\|\tilde{\mathrm{d}}\|)\in-\mathbb{R}_+^m,\end{aligned}$$

respectively. This contradicts our assumption that $\bar{\delta}$ solves $\mathscr{E}-$Stampacchia $\partial_{\mathscr{D}}^*-$VVI. □

Remark 3.4. The main difference between Theorem 3.1 and Theorem 3.2 is in the assumptions and the type of variational inequalities used. In Theorem 3.1, the functions are only approximately convex, but even the usual (non-approximate) variational inequalities are enough to get an approximate solution of the IVMOP. On the other hand, in Theorem 3.2, the functions are convex, but we need to use approximate variational inequalities to obtain an approximate solution. So, one result uses weaker assumptions on functions with stronger conditions on variational inequalities, while the other uses stronger assumptions on functions with weaker conditions on variational inequalities.

Theorem 3.3. *Let $\bar{\delta}\in\Omega_{\mathscr{D},\mathfrak{f}}$ and let each $\mathfrak{f}_i^L,\mathfrak{f}_i^U,i\in\mathscr{M}$ admits a BDC $\partial_{\mathscr{D}}^*\mathfrak{f}_i(\bar{\delta})$ at $\bar{\delta}\in\Omega_{\mathscr{D},\mathfrak{f},\bar{\delta}}$. Assume that $-\mathscr{D}\subseteq\mathscr{D}$, that $\Omega_{\mathscr{D},\mathfrak{f}}\neq\emptyset$ is convex, and that each $\mathfrak{f}_i^L,\mathfrak{f}_i^U,i\in\mathscr{M}$ is $\partial_{\mathscr{D}}^*-$convex at $\bar{\delta}$ over $\Omega_{\mathscr{D},\mathfrak{f}}$. If $\bar{\delta}$ solves $\mathscr{E}-\partial_D^*-$SVVI, then $\bar{\delta}$ also solves $\mathscr{E}-\partial_{\mathscr{D}}^*-$MVVI.*

Proof. Since $\bar{\delta}$ solves $\mathscr{E}-\partial_D^*-$SVVI, then, for any $\mathrm{d}\in\overrightarrow{\mathscr{D}}_{\Omega,\mathfrak{f},\bar{\delta}}$, there exist $\bar{\delta}^{*L}\in\partial_{\mathscr{D}}^*\mathfrak{f}^L(\bar{\delta})$ and $\bar{\delta}^{*U}\in\partial_{\mathscr{D}}^*\mathfrak{f}^U(\bar{\delta})$, such that

$$\left.\begin{aligned}\langle\bar{\delta}^{*L},\mathrm{d}\rangle_m+\mathscr{E}^L\|\mathrm{d}\|:=(\langle\bar{\delta}_1^{*L},\mathrm{d}\rangle+\mathscr{E}_1^L\|\mathrm{d}\|,...,\langle\bar{\delta}_m^{*L},\mathrm{d}\rangle+\mathscr{E}_m^L\|\mathrm{d}\|)\notin-\mathbb{R}_+^m\setminus\{0\},\\ \langle\bar{\delta}^{*U},\mathrm{d}\rangle_m+\mathscr{E}^U\|\mathrm{d}\|:=(\langle\bar{\delta}_1^{*U},\mathrm{d}\rangle+\mathscr{E}_1^U\|\mathrm{d}\|,...,\langle\bar{\delta}_m^{*U},\mathrm{d}\rangle+\mathscr{E}_m^U\|\mathrm{d}\|)\notin-\mathbb{R}_+^m\setminus\{0\}.\end{aligned}\right\}\tag{3.11}$$

Since each $\mathfrak{f}_i^L, \mathfrak{f}_i^U, i \in \mathscr{M}$ is ∂_D^*-convex on $\Omega_{\mathscr{D},\mathfrak{f}}$, by Theorem 2.2, each $\partial_{\mathscr{D}}^* \mathfrak{f}_i^L, \partial_{\mathscr{D}}^* \mathfrak{f}_i^L, i \in \mathscr{M}$ is monotone on $\Omega_{\mathscr{D},\mathfrak{f}}$, which implies that, for any $\mathrm{d} \in \overrightarrow{\mathscr{D}}_{\Omega,\mathfrak{f},\bar{\delta}}, \delta^{*L} \in \partial_D^* \mathfrak{f}^L(\bar{\delta}+\mathrm{d})$ and $\delta^{*U} \in \partial_D^* \mathfrak{f}^U(\bar{\delta}+\mathrm{d})$, one has

$$\langle \delta^{*L} - \bar{\delta}^{*L}, \mathrm{d}\rangle \geq 0 \text{ and } \langle \delta^{*U} - \bar{\delta}^{*U}, \mathrm{d}\rangle \geq 0,$$

which implies that

$$\langle \delta^{*L}, \mathrm{d}\rangle \geq \langle \bar{\delta}^{*L}, \mathrm{d}\rangle \text{ and } \langle \delta^{*U}, \mathrm{d}\rangle \geq \langle \bar{\delta}^{*U}, \mathrm{d}\rangle. \tag{3.12}$$

From equations (3.11) and (3.12), we obtain

$$\left.\begin{array}{l}\langle \delta^{*L}, \mathrm{d}\rangle_m + \mathscr{E}^L\|\mathrm{d}\| := (\langle \delta_1^{*L}, \mathrm{d}\rangle + \mathscr{E}_1^L\|\mathrm{d}\|, \ldots, \langle \delta_m^{*L}, \mathrm{d}\rangle + \mathscr{E}_m^L\|\mathrm{d}\|) \notin -\mathbb{R}_+^m \setminus \{0\}, \\ \langle \delta^{*U}, \mathrm{d}\rangle_m + \mathscr{E}^U\|\mathrm{d}\| := (\langle \delta_1^{*U}, \mathrm{d}\rangle + \mathscr{E}_1^U\|\mathrm{d}\|, \ldots, \langle \delta_m^{*U}, \mathrm{d}\rangle + \mathscr{E}_m^U\|\mathrm{d}\|) \notin -\mathbb{R}_+^m \setminus \{0\}.\end{array}\right\}$$

Hence, $\bar{\delta}$ also solves $\mathscr{E}-\partial_{\mathscr{D}}^*-$MVVI. □

Theorem 3.4. *Let $\bar{\delta} \in \Omega_{\mathscr{D},\mathfrak{f}}$. Assume that $-\mathscr{D} \subseteq \mathscr{D}$, that $\Omega_{\mathscr{D},\mathfrak{f}} \neq \emptyset$ is convex, and that for any $\delta \in \Omega_{\mathscr{D},\mathfrak{f}}$, the function $\mathfrak{f}_i^L, \mathfrak{f}_i^U, i \in \mathscr{M}$ admits a BDC $\partial_{\mathscr{D}}^* \mathfrak{f}_i^L(\delta), \partial_{\mathscr{D}}^* \mathfrak{f}_i^U(\delta)$, respectively, at δ. Suppose that each $\mathfrak{f}_i^L, \mathfrak{f}_i^U, i \in \mathscr{M}$ is $\partial_{\mathscr{D}}^*-$convex on $\Omega_{\mathscr{D},\mathfrak{f}}$. Then, $\bar{\delta} \in \mathscr{E} - \Omega_{\mathscr{D},\mathfrak{f},\bar{\delta}}^{2,q}(IVMOP)$ if and only if $\bar{\delta}$ solves Minty $\mathscr{E}-\partial_{\mathscr{D}}^*-$VVI.*

Proof. Contrarily, assume that $\bar{\delta} \in \Omega_{\mathscr{D},\mathfrak{f},\bar{\delta}}$ does not solve Minty $\mathscr{E}-\partial_{\mathscr{D}}^*-$VVI. Then, there exist $\tilde{\mathrm{d}} \in \overrightarrow{\mathscr{D}}_{\Omega,\mathfrak{f},\bar{\delta}}$ and $\tilde{\delta}_i^{*L} \in \partial_{\mathscr{D}}^* \mathfrak{f}_i^L(\bar{\delta}+\tilde{\mathrm{d}}), \tilde{\delta}_i^{*U} \in \partial_{\mathscr{D}}^* \mathfrak{f}_i^U(\bar{\delta}+\tilde{\mathrm{d}})$ such that

$$\left.\begin{array}{l}\langle \tilde{\delta}_i^{*L}, \tilde{\mathrm{d}}\rangle + \mathscr{E}_i^L\|\tilde{\mathrm{d}}\| \leq 0, \ \forall i \in \mathscr{M}, \\ \langle \tilde{\delta}_k^{*L}, \tilde{\mathrm{d}}\rangle + \mathscr{E}_k^L\|\tilde{\mathrm{d}}\| < 0, \text{ for at least one } k \in \mathscr{M}\end{array}\right\} \tag{3.13}$$

and

$$\left.\begin{array}{l}\langle \tilde{\delta}_i^{*U}, \tilde{\mathrm{d}}\rangle + \mathscr{E}_i^U\|\tilde{\mathrm{d}}\| \leq 0, \ \forall i \in \mathscr{M}, \\ \langle \tilde{\delta}_k^{*U}, \tilde{\mathrm{d}}\rangle + \mathscr{E}_k^U\|\tilde{\mathrm{d}}\| < 0, \text{ for at least one } k \in \mathscr{M}.\end{array}\right\} \tag{3.14}$$

Since each $\mathfrak{f}_i^L, \mathfrak{f}_i^U, i \in \mathscr{M}$ is $\partial_{\mathscr{D}}^*-$convex on $\Omega_{\mathscr{D},\mathfrak{f}}$, it follows that

$$\mathfrak{f}_i^L(\bar{\delta}) - \mathfrak{f}_i^L(\bar{\delta}+\tilde{\mathrm{d}}) \geq \langle \tilde{\delta}_i^{*L}, -\tilde{\mathrm{d}}\rangle, \quad \forall i \in \mathscr{M}, \tag{3.15}$$

$$\mathfrak{f}_i^U(\bar{\delta}) - \mathfrak{f}_i^U(\bar{\delta}+\tilde{\mathrm{d}}) \geq \langle \tilde{\delta}_i^{*U}, -\tilde{\mathrm{d}}\rangle, \quad \forall i \in \mathscr{M}. \tag{3.16}$$

From inclusions (3.13) and (3.15), we obtain

$$\left.\begin{array}{l}\mathfrak{f}_i^L(\bar{\delta}+\tilde{\mathrm{d}}) - \mathfrak{f}_i^L(\bar{\delta}) + \mathscr{E}_i^L\|\tilde{\mathrm{d}}\| \leq 0, \ \forall i \in \mathscr{M}, \\ \mathfrak{f}_k^L(\bar{\delta}+\tilde{\mathrm{d}}) - \mathfrak{f}_k^L(\bar{\delta}) + \mathscr{E}_k^L\|\tilde{\mathrm{d}}\| < 0, \ \text{for at least one } k \in \mathscr{M}.\end{array}\right\}$$

From inclusions (3.14) and (3.16), we conclude

$$\left.\begin{array}{l}\mathfrak{f}_i^U(\bar{\delta}+\tilde{\mathrm{d}}) - \mathfrak{f}_i^U(\bar{\delta}) + \mathscr{E}_i^U\|\tilde{\mathrm{d}}\| \leq 0, \ \forall i \in \mathscr{M}, \\ \mathfrak{f}_k^U(\bar{\delta}+\tilde{\mathrm{d}}) - \mathfrak{f}_k^U(\bar{\delta}) + \mathscr{E}_k^U\|\tilde{\mathrm{d}}\| < 0, \ \text{for at least one } k \in \mathscr{M},\end{array}\right\}$$

which is a contradiction that $\bar{\delta} \in \mathscr{E} - \Omega_{\mathscr{D},\mathfrak{f},\bar{\delta}}^{2,q}(IVMOP)$ and hence the result.

Conversely, assume that $\bar{\delta} \notin \mathscr{E} - \Omega^{2,q}_{\mathscr{D},\mathfrak{f},\bar{\delta}}(IVMOP)$. Then, there exist $\tilde{\delta} \in \Omega_{\mathscr{D},\mathfrak{f},\bar{\delta}}$ with $\tilde{\delta} \neq \bar{\delta}$ such that

$$\left.\begin{array}{l} \mathfrak{f}_i^L(\bar{\delta}+\tilde{\mathrm{d}}) - \mathfrak{f}_i^L(\bar{\delta}) + \mathscr{E}_i^L\|\tilde{\mathrm{d}}\| \leq 0,\ \mathfrak{f}_i^U(\bar{\delta}+\tilde{\mathrm{d}}) - \mathfrak{f}_i^U(\bar{\delta}) + \mathscr{E}_i^U\|\tilde{\mathrm{d}}\| \leq 0,\ \forall i \in \mathscr{M}, \\ \mathfrak{f}_k^L(\bar{\delta}+\tilde{\mathrm{d}}) - \mathfrak{f}_k^L(\bar{\delta}) + \mathscr{E}_k^L\|\tilde{\mathrm{d}}\| < 0,\ \mathfrak{f}_k^U(\bar{\delta}+\tilde{\mathrm{d}}) - \mathfrak{f}_k^U(\bar{\delta}) + \mathscr{E}_k^U\|\tilde{\mathrm{d}}\| < 0 \text{ for at least one } k \in \mathscr{M}, \end{array}\right\} \tag{3.17}$$

Since $\tilde{\delta} \in \Omega_{\mathscr{D},\mathfrak{f},\bar{\delta}}$, then there exists $\tilde{\mathrm{d}} \in \mathscr{D}$ such that $\tilde{\delta} = \bar{\delta} + \tilde{\mathrm{d}} \in \Omega_{\mathscr{D},\mathfrak{f}}$ and hence $\tilde{\mathrm{d}} \in \overrightarrow{\mathscr{D}}_{\Omega,\mathfrak{f},\bar{\delta}}$. Since $\Omega_{\mathscr{D},\mathfrak{f}}$ is a convex set, then $\delta(\eta) := \eta\tilde{\delta} + (1-\eta)\bar{\delta} = \bar{\delta} + \eta\tilde{\mathrm{d}} \in \Omega_{\mathscr{D},\mathfrak{f}}$ and $\delta(\eta) - \bar{\delta} = \eta\tilde{\mathrm{d}} \in \mathscr{D}$ for every $\eta \in [0,1]$. Since each $\mathfrak{f}_i^L, \mathfrak{f}_i^U, i \in \mathscr{M}$ is $\partial^*_{\mathscr{D}}-$convex on $\Omega_{\mathscr{D},\mathfrak{f}}$, one sees by Proposition 2.1 that each $\mathfrak{f}_i^L, \mathfrak{f}_i^L, i \in \mathscr{M}$ is convex in all directions $\mathrm{d} \in \overrightarrow{\mathscr{D}}_{\Omega,\mathfrak{f},\delta}$ at every $\delta \in \Omega_{\mathscr{D},\mathfrak{f}}$, i.e., for any $i \in \mathscr{M}$ and $\eta \in (0,1)$, one has

$$\mathfrak{f}_i^L(\bar{\delta} + \eta(\tilde{\delta} - \bar{\delta})) - \mathfrak{f}_i^L(\bar{\delta}) \leq \eta(\mathfrak{f}_i^L(\tilde{\delta}) - \mathfrak{f}_i^L(\bar{\delta})),$$
$$\mathfrak{f}_i^U(\bar{\delta} + \eta(\tilde{\delta} - \bar{\delta})) - \mathfrak{f}_i^U(\bar{\delta}) \leq \eta(\mathfrak{f}_i^U(\tilde{\delta}) - \mathfrak{f}_i^U(\bar{\delta})).$$

By the mean value Theorem 2.1 involving DCs, for any $i \in \mathscr{M}$ and $\eta \in (0,1)$, there exist $\hat{\eta}_i \in (0,\eta)$, $\hat{\delta}_i^{*L} \in conv\partial^*_{\mathscr{D}}\mathfrak{f}_i^L(\delta(\hat{\eta}_i))$ and $\hat{\delta}_i^{*U} \in conv\partial^*_{\mathscr{D}}\mathfrak{f}_i^U(\delta(\hat{\eta}_i))$ such that

$$\mathfrak{f}_i^L(\bar{\delta} + \eta(\tilde{\delta} - \bar{\delta})) - \mathfrak{f}_i^L(\bar{\delta}) = \langle \hat{\delta}_i^{*L}, \eta(\tilde{\delta} - \bar{\delta})\rangle,$$
$$\mathfrak{f}_i^U(\bar{\delta} + \eta(\tilde{\delta} - \bar{\delta})) - \mathfrak{f}_i^U(\bar{\delta}) = \langle \hat{\delta}_i^{*U}, \eta(\tilde{\delta} - \bar{\delta})\rangle,$$

which implies from (3.17) that

$$\left.\begin{array}{l} \langle \hat{\delta}_i^{*L}, \eta(\tilde{\delta} - \bar{\delta})\rangle + \mathscr{E}_i^L\|\eta\tilde{\mathrm{d}}\| \leq 0,\ \langle \hat{\delta}_i^{*U}, \eta(\tilde{\delta} - \bar{\delta})\rangle + \mathscr{E}_i^U\|\eta\tilde{\mathrm{d}}\| \leq 0,\ \forall i \in \mathscr{M}, \\ \langle \hat{\delta}_k^{*L}, \eta(\tilde{\delta} - \bar{\delta})\rangle + \mathscr{E}_k^L\|\eta\tilde{\mathrm{d}}\| < 0,\ \langle \hat{\delta}_k^{*U}, \eta(\tilde{\delta} - \bar{\delta})\rangle + \mathscr{E}_k^U\|\eta\tilde{\mathrm{d}}\| < 0, \text{ for at least one } k \in \mathscr{M}. \end{array}\right\}$$

or equivalently,

$$\langle \hat{\delta}_i^{*L}, \tilde{\delta} - \bar{\delta}\rangle + \mathscr{E}_i^L\|\tilde{\mathrm{d}}\| \leq 0,\ \langle \hat{\delta}_i^{*U}, \tilde{\delta} - \bar{\delta}\rangle + \mathscr{E}_i^U\|\tilde{\mathrm{d}}\| \leq 0,\ \forall i \in \mathscr{M},$$

$$\langle \hat{\delta}_k^{*L}, \tilde{\delta} - \bar{\delta}\rangle + \mathscr{E}_k^L\|\tilde{\mathrm{d}}\| < 0,\ \langle \hat{\delta}_k^{*U}, \tilde{\delta} - \bar{\delta}\rangle + \mathscr{E}_k^U\|\tilde{\mathrm{d}}\| < 0, \text{ for at least one } k \in \mathscr{M}.$$

Suppose that $\hat{\eta}_i = \hat{\eta}$ for all $i \in \mathscr{M}$. After both sides of the above inequality are multiplied by $\hat{\eta}$, one has

$$\langle \hat{\delta}_i^{*L}, \delta(\hat{\eta}) - \bar{\delta}\rangle + \mathscr{E}_i^L\|\hat{\eta}\tilde{\mathrm{d}}\| \leq 0, \langle \hat{\delta}_i^{*U}, \delta(\hat{\eta}) - \bar{\delta}\rangle + \mathscr{E}_i^U\|\hat{\eta}\tilde{\mathrm{d}}\| \leq 0,\ \forall i \in \mathscr{M},$$

$$\langle \hat{\delta}_k^{*L}, \delta(\hat{\eta}) - \bar{\delta}\rangle + \mathscr{E}_k^L\|\hat{\eta}\tilde{\mathrm{d}}\| \leq 0, \langle \hat{\delta}_k^{*U}, \delta(\hat{\eta}) - \bar{\delta}\rangle + \mathscr{E}_k^U\|\hat{\eta}\tilde{\mathrm{d}}\| \leq 0, \text{ for at least one } k \in \mathscr{M}.$$

Since $\delta(\hat{\eta}) - \bar{\delta} = \hat{\eta}\tilde{\mathrm{d}} \in \overrightarrow{\mathscr{D}}_{\Omega,\mathfrak{f},\bar{\delta}}$, it contradicts that $\bar{\delta}$ solves Minty $\mathscr{E} - \partial^*_{\mathscr{D}}-$VVI. Suppose that $\hat{\eta}_i, i \in \mathscr{M}$ are not all equal. Since $\mathfrak{f}_i^L, \mathfrak{f}_i^U, i \in \mathscr{M}$ are $\partial^*_{\mathscr{D}}-$convex on $\Omega_{\mathscr{D},\mathfrak{f}}$, by Theorem 2.2, $conv\partial^*_{\mathscr{D}}\mathfrak{f}_i^L, conv\partial^*_{\mathscr{D}}\mathfrak{f}_i^U, i \in \mathscr{M}$ are monotone on $\Omega_{\mathscr{D},\mathfrak{f}}$ and by setting $\bar{\eta} := \min\{\hat{\eta}_1, \ldots, \hat{\eta}_m\}$, we can find $\bar{\delta}_i^{*L} \in conv\partial^*_{\mathscr{D}}\mathfrak{f}_i^L(\delta(\bar{\eta})), \bar{\delta}_i^{*U} \in conv\partial^*_{\mathscr{D}}\mathfrak{f}_i^U(\delta(\bar{\eta}))$ such that

$$\langle \hat{\delta}_i^{*L}, \delta(\bar{\eta}) - \bar{\delta}\rangle + \mathscr{E}_i^L\|\bar{\eta}\tilde{\mathrm{d}}\| \leq 0, \langle \hat{\delta}_i^{*U}, \delta(\bar{\eta}) - \bar{\delta}\rangle + \mathscr{E}_i^U\|\bar{\eta}\tilde{\mathrm{d}}\| \leq 0, \quad \forall i \in \mathscr{M},$$

with strict inequality for at least one $i \in \mathscr{M}$. Since $\delta(\bar{\eta}) - \bar{\delta} = \bar{\eta}\tilde{\mathrm{d}} \in \overrightarrow{\mathscr{D}}_{\Omega,\mathfrak{f},\bar{\delta}}$, it contradicts that $\bar{\delta}$ solves Minty $\mathscr{E} - \partial^*_{\mathscr{D}}-$VVI. Hence, the result is obtained. □

Theorem 3.5. *Let $\bar{\delta} \in \Omega_{\mathscr{D},\mathfrak{f}}$ and let each $\mathfrak{f}_i^L, \mathfrak{f}_i^U (i \in \mathscr{M})$ admits a BDC, $\partial_{\mathscr{D}}^* \mathfrak{f}_i^L(\bar{\delta}), \partial_{\mathscr{D}}^* \mathfrak{f}_i^U(\bar{\delta})$ at $\bar{\delta}$, respectively. Assume that $\mathscr{D} \neq \{0_{\mathbb{R}^n}\}$, that $\Omega_{\mathscr{D},\mathfrak{f}}$ is convex, and that each $\mathfrak{f}_i^L, \mathfrak{f}_i^U (i \in \mathscr{M})$ is $\mathscr{E} - \partial_D^* -$ strictly convex at $\bar{\delta}$ over $\Omega_{\mathscr{D},\mathfrak{f}}$. If $\bar{\delta}$ solves ∂_D^*-WSVVI, then $\bar{\delta} \in \mathscr{E} - \Omega^{1,qw}_{\mathscr{D},\mathfrak{f},\bar{\delta}}(IVMOP)$.*

Proof. Contrarily, assume that $\bar{\delta} \in \Omega_{\mathscr{D},\mathfrak{f},\bar{\delta}}$ is not a type-1 $\mathscr{E}-$ quasi weak pareto solution of $IVMOP_{\mathscr{D},\bar{\delta}}$. Then, there exists $\bar{\delta} \neq \tilde{\delta} \in \Omega_{\mathscr{D},\mathfrak{f},\bar{\delta}}$ such that $\mathfrak{f}_i(\tilde{\delta}) + \mathscr{E}_i\|\tilde{\delta} - \bar{\delta}\| <_{LU} \mathfrak{f}_i(\bar{\delta}),\ \forall i \in \mathscr{M}$, or equivalently, $\forall i \in \mathscr{M}$

$$\left.\begin{array}{l}\mathfrak{f}_i^L(\tilde{\delta}) + \mathscr{E}_i^L\|\tilde{\delta} - \bar{\delta}\| < \mathfrak{f}_i^L(\bar{\delta}), \text{ and } \mathfrak{f}_i^U(\tilde{\delta}) + \mathscr{E}_i^U\|\tilde{\delta} - \bar{\delta}\| < \mathfrak{f}_i^U(\bar{\delta}); or\\ \mathfrak{f}_i^L(\tilde{\delta}) + \mathscr{E}_i^L\|\tilde{\delta} - \bar{\delta}\| \leq \mathfrak{f}_i^L(\bar{\delta}), \text{ and } \mathfrak{f}_i^U(\tilde{\delta}) + \mathscr{E}_i^U\|\tilde{\delta} - \bar{\delta}\| < \mathfrak{f}_i^U(\bar{\delta}); or\\ \mathfrak{f}_i^L(\tilde{\delta}) + \mathscr{E}_i^L\|\tilde{\delta} - \bar{\delta}\| < \mathfrak{f}_i^L(\bar{\delta}), \text{ and } \mathfrak{f}_i^U(\tilde{\delta}) + \mathscr{E}_i^U\|\tilde{\delta} - \bar{\delta}\| \leq \mathfrak{f}_i^U(\bar{\delta}).\end{array}\right\} \quad (3.18)$$

Since $\tilde{\delta} \in \Omega_{\mathscr{D},\mathfrak{f},\bar{\delta}}$, then there exists $\tilde{\mathrm{d}} \in \mathscr{D}$ such that $\tilde{\delta} = \bar{\delta} + \tilde{\mathrm{d}}$ and hence $\tilde{\mathrm{d}} \in \overrightarrow{\mathscr{D}}_{\Omega,\mathfrak{f},\bar{\delta}}$. By the $\mathscr{E}_i^L - \partial_D^* -$ strictly convexity of $\mathfrak{f}_i^L(i \in \mathscr{M})$ and $\mathscr{E}_i^U - \partial_D^* -$ strictly convexity of $\mathfrak{f}_i^U(i \in \mathscr{M})$ at $\bar{\delta}$ over $\Omega_{\mathscr{D},\mathfrak{f},\bar{\delta}}$, it follows that

$$\left.\begin{array}{l}\langle \bar{\delta}_i^{*L}, \tilde{\mathrm{d}}\rangle < \mathfrak{f}_i^L(\tilde{\delta}) - \mathfrak{f}_i^L(\bar{\delta}) + \mathscr{E}_i^L\|\tilde{\mathrm{d}}\|, \forall \bar{\delta}_i^{*L} \in \partial_{\mathscr{D}}^* \mathfrak{f}_i^L(\bar{\delta}),\ \forall i \in \mathscr{M},\\ \langle \bar{\delta}_i^{*U}, \tilde{\mathrm{d}}\rangle < \mathfrak{f}_i^U(\tilde{\delta}) - \mathfrak{f}_i^U(\bar{\delta}) + \mathscr{E}_i^U\|\tilde{\mathrm{d}}\|, \forall \bar{\delta}_i^{*U} \in \partial_{\mathscr{D}}^* \mathfrak{f}_i^U(\bar{\delta}),\ \forall i \in \mathscr{M}.\end{array}\right\} \quad (3.19)$$

From equation (3.18) and (3.19), we have

$$\left.\begin{array}{l}\langle \bar{\delta}_i^{*L}, \tilde{\mathrm{d}}\rangle < 0, \forall \bar{\delta}_i^{*L} \in \partial_{\mathscr{D}}^* \mathfrak{f}_i^L(\bar{\delta}),\ \forall i \in \mathscr{M},\\ \langle \bar{\delta}_i^{*U}, \tilde{\mathrm{d}}\rangle < 0, \forall \bar{\delta}_i^{*U} \in \partial_{\mathscr{D}}^* \mathfrak{f}_i^U(\bar{\delta}),\ \forall i \in \mathscr{M}.\end{array}\right\}$$

which contradicts that $\bar{\delta}$ solves weak Stampacchia $\partial_{\mathscr{D}}^*-$VVI. □

The following example illustrates the above results.

Example 3.1. Let $\mathfrak{f}_1^L, \mathfrak{f}_1^U, \mathfrak{f}_2^L, \mathfrak{f}_2^U : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$\mathfrak{f}_1^L(\delta) = \delta_1 + \delta_2$$

$$\mathfrak{f}_1^U(\delta) = \begin{cases} 1 + (\delta_1 + \delta_2)^2 + \delta_1, & \text{if } \delta_1 > 0 \ \& \ \delta_2 > 0,\\ \delta_1^2 + \delta_1\delta_2 + \delta_2^2 + \delta_1 + \delta_2, & \text{otherwise,}\end{cases}$$

$$\mathfrak{f}_2^L(\delta) = \delta_1 + \delta_2 - 1,$$

and

$$\mathfrak{f}_2^U(\delta) = \begin{cases} 1 + \delta_1 + \delta_2, & \text{if } \delta_1 > 0 \ \& \ \delta_2 > 0,\\ \delta_1^2 + \delta_2^2 - \delta_1, & \text{otherwise,}\end{cases}$$

respectively. The collection of all CDs of $\mathfrak{f}_1^L, \mathfrak{f}_1^U, \mathfrak{f}_2^L$, and $\mathfrak{f}_2^U$ at any $\delta := (\delta_1, \delta_2) \in \mathbb{R}^2$ are given by

$$\mathscr{D}_{\mathfrak{f}_1^L}(\delta) = \mathscr{D}_{\mathfrak{f}_2^L}(\delta) = \mathbb{R}^2,$$

$$\mathscr{D}_{\mathfrak{f}_1^U}(\delta) = \mathscr{D}_{\mathfrak{f}_2^U}(\delta) = \begin{cases} \mathbb{R}^- \times \mathbb{R}, & \text{if } \delta_1 = 0 \ \& \ \delta_2 > 0,\\ \mathbb{R} \times \mathbb{R}^-, & \text{if } \delta_1 > 0 \ \& \ \delta_2 = 0,\\ \mathscr{D}(0_{\mathbb{R}^2}), & \text{if } \delta_1 = 0 \ \& \ \delta_2 = 0,\\ \mathbb{R}^2, & \text{otherwise,}\end{cases}$$

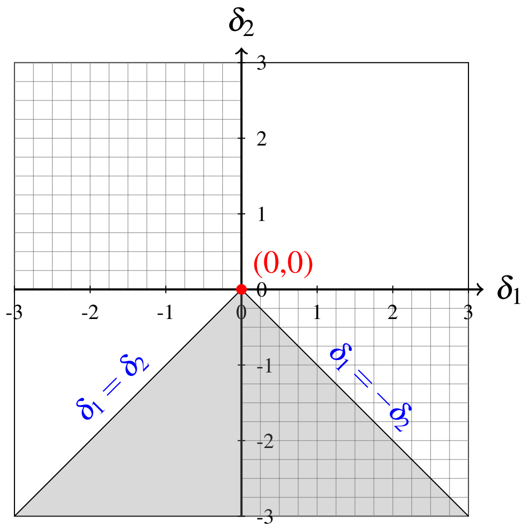

FIGURE 1. $\Omega_{\hat{\mathscr{D}},\mathfrak{f},\bar{\delta}}$ for $\bar{\delta} = 0_{\mathbb{R}^2}$ in the Example 3.1

where

$$\mathscr{D}(0_{\mathbb{R}^2}) = \{\mathrm{d} := (\mathrm{d}_1, \mathrm{d}_2) \in \mathbb{R}^2 : \mathrm{d}_1 \leq 0 \text{ or } \mathrm{d}_2 \leq 0\}.$$

We consider nonempty cone $\hat{\mathscr{D}} = (\mathbb{R}^- \times \mathbb{R}^+) \cup (\mathscr{R}^+ \times \mathbb{R}^-)$ such that $-\hat{\mathscr{D}} \subseteq \hat{\mathscr{D}}$ as shown by the gridded region in Fig. 1 and nonconvex feasible region given by

$$\Omega := \{(\delta_1, \delta_2) \in \mathbb{R}^2 : \delta_1\delta_2 = 0 \textit{ and } \delta_1 \geq 0, \delta_2 \geq 0\} \cup \{(\delta_1, \delta_2) \in \mathbb{R}^2 : \delta_2 \leq -|\delta_1|\}.$$

Then,

$$\Omega_{\hat{\mathscr{D}},\mathfrak{f}_1^L} := \{\delta \in \Omega : \hat{\mathscr{D}} \subseteq \mathscr{D}_{\mathfrak{f}_1^L}(\delta)\} = \Omega,$$

$$\Omega_{\hat{\mathscr{D}},\mathfrak{f}_1^U} := \{\delta \in \Omega : \hat{\mathscr{D}} \subseteq \mathscr{D}_{\mathfrak{f}_1^U}(\delta)\} = \{(\delta_1, \delta_2) \in \mathbb{R}^2 : \delta_2 \leq -|\delta_1|\}$$

and

$$\Omega_{\hat{\mathscr{D}},\mathfrak{f}_2^L} := \{\delta \in \Omega : \hat{\mathscr{D}} \subseteq \mathscr{D}_{\mathfrak{f}_2^L}(\delta)\} = \Omega,$$

$$\Omega_{\hat{\mathscr{D}},\mathfrak{f}_2^U} := \{\delta \in \Omega : \hat{\mathscr{D}} \subseteq \mathscr{D}_{\mathfrak{f}_2^U}(\delta)\} = \{(\delta_1, \delta_2) \in \mathbb{R}^2 : \delta_2 \leq -|\delta_1|\}$$

which gives

$$\Omega_{\hat{\mathscr{D}},\mathfrak{f}} := \Omega_{\hat{\mathscr{D}},\mathfrak{f}_1^L} \cap \Omega_{\hat{\mathscr{D}},\mathfrak{f}_1^U} \cap \Omega_{\hat{\mathscr{D}},\mathfrak{f}_2^L} \cap \Omega_{\hat{\mathscr{D}},\mathfrak{f}_2^U} = \{(\delta_1, \delta_2) \in \mathbb{R}^2 : \delta_2 \leq -|\delta_1|\}$$

which is a convex set. For $\bar{\delta} = 0_{\mathbb{R}^2}$,

$$\Omega_{\hat{\mathscr{D}},\mathfrak{f},\bar{\delta}} = \Omega_{\hat{\mathscr{D}},\mathfrak{f}} \cap (\bar{\delta} + \hat{\mathscr{D}}) = \{(\delta_1, \delta_2) \in \mathbb{R}^2 : \delta_2 \leq -\delta_1 \text{ and } \delta_1 \geq 0\}$$

as represented by the intersection of the gridded and the shaded region in Fig. 1. Since there exists some $\delta \in \Omega_{\hat{\mathscr{D}},\mathfrak{f},\bar{\delta}}$ (take $\delta = (\frac{1}{5}, -\frac{3}{10})$) such that $\mathfrak{f}_i^L(\delta) < \mathfrak{f}_i^L(\bar{\delta})$ and $\mathfrak{f}_i^U(\delta) < \mathfrak{f}_i^U(\bar{\delta})$ for all $i = \{1,2\}$, $\bar{\delta} \neq \delta \in \Omega_{\hat{\mathscr{D}},\mathfrak{f},\bar{\delta}}$ is not a *type-1 quasi Pareto solution* of the $(MOP_{\hat{\mathscr{D}},\bar{\delta}})$. If we choose $[\mathscr{E}_1^L, \mathscr{E}_1^U] = [2,3]$ and $[\mathscr{E}_2^L, \mathscr{E}_2^U] = [2,3]$, $\bar{\delta}$ is a *type-1 $\mathscr{E}$-quasi Pareto solution* of the $(MOP_{\hat{\mathscr{D}},\bar{\delta}})$.

Now, for $\bar{\delta} = 0_{\mathbb{R}^2} \in \Omega_{\hat{\mathscr{D}},\mathfrak{f}}$, one has

$$\overrightarrow{\hat{\mathscr{D}}}_{\Omega,\mathfrak{f},\bar{\delta}} = \{(\mathrm{d}_1, \mathrm{d}_2) \in \mathbb{R}^2 : \mathrm{d}_2 \leq -\mathrm{d}_1 \text{ and } \mathrm{d}_1 \geq 0\}$$

The sets $\Omega_{\hat{\mathscr{D}},\mathfrak{f},\bar{\delta}}$ and $\overrightarrow{\hat{\mathscr{D}}}_{\Omega,\mathfrak{f},\bar{\delta}}$ are convex. For $\bar{\delta}=(0,0)\in\Omega_{\hat{\mathscr{D}},\mathfrak{f}}$, the functions $\mathfrak{f}_1^L,\mathfrak{f}_1^U,\mathfrak{f}_2^L$ and $\mathfrak{f}_2^U$ admit

$$\partial^*_{\hat{\mathscr{D}}}\mathfrak{f}_1^L(\bar{\delta})=\{((1,1)\},\ \partial^*_{\hat{\mathscr{D}}}\mathfrak{f}_1^U(\bar{\delta})=\{(1,1)\}$$

and

$$\partial^*_{\hat{\mathscr{D}}}\mathfrak{f}_2^L(\bar{\delta})=\{(1,1)\},\ \partial^*_{\hat{\mathscr{D}}}\mathfrak{f}_2^U(\bar{\delta})=\{(0,0)\}$$

as directional convexificators. Let $\mathrm{d}\in\overrightarrow{\hat{\mathscr{D}}}_{\Omega,\mathfrak{f},\delta}$ and $\bar{\delta}^*\in\partial^*_{\hat{\mathscr{D}}}\mathfrak{f}(\bar{\delta})$. Note that

$$\begin{aligned}\mathfrak{f}_1^L(\bar{\delta}+\mathrm{d})-\mathfrak{f}_1^L(\bar{\delta})-\langle\bar{\delta}_1^{*L},\mathrm{d}\rangle&=\mathrm{d}_1+\mathrm{d}_2-\langle(1,1),(\mathrm{d}_1,\mathrm{d}_2)\rangle\\&=\mathrm{d}_1+\mathrm{d}_2-\mathrm{d}_1-\mathrm{d}_2=0,\\ \mathfrak{f}_1^U(\bar{\delta}+\mathrm{d})-\mathfrak{f}_1^U(\bar{\delta})-\langle\bar{\delta}_1^{*U},\mathrm{d}\rangle&=\mathrm{d}_1^2+\mathrm{d}_2^2+\mathrm{d}_1\mathrm{d}_2\\&=(\mathrm{d}_1+\frac{\mathrm{d}_2}{2})^2+\frac{3\mathrm{d}_2^2}{4},\\ \mathfrak{f}_2^L(\bar{\delta}+\mathrm{d})-\mathfrak{f}_2^L(\bar{\delta})-\langle\bar{\delta}_2^{*L},\mathrm{d}\rangle&=\mathrm{d}_1+\mathrm{d}_2-\langle(1,1),(\mathrm{d}_1,\mathrm{d}_2)\rangle\\&=\mathrm{d}_1+\mathrm{d}_2-\mathrm{d}_1-\mathrm{d}_2=0,\end{aligned}$$

and

$$\begin{aligned}\mathfrak{f}_2^U(\bar{\delta}+\mathrm{d})-\mathfrak{f}_2^U(\bar{\delta})-\langle\bar{\delta}_2^{*U},\mathrm{d}\rangle&=\mathrm{d}_1^2+\mathrm{d}_2^2-\mathrm{d}_1-\langle(-1,0),(\mathrm{d}_1,\mathrm{d}_2)\rangle\\&=\mathrm{d}_1^2+\mathrm{d}_2^2.\end{aligned}$$

Consequently, $\mathfrak{f}_1^L,\mathfrak{f}_2^L,\mathfrak{f}_1^U$, and $\mathfrak{f}_2^U$ are $\partial^*_{\hat{\mathscr{D}}}$-convex on $\Omega_{\hat{\mathscr{D}},\mathfrak{f}}$. For any $\mathrm{d}\in\overrightarrow{\mathscr{D}}_{\Omega,\mathfrak{f},\bar{\delta}}$, $[\mathscr{E}_1^L,\mathscr{E}_1^U]=[2,3]$ and $[\mathscr{E}_2^L,\mathscr{E}_2^U]=[2,3]$, one has

$$\begin{aligned}\langle\bar{\delta}^{*L},\mathrm{d}\rangle_2+\mathscr{E}^L\|\mathrm{d}\|&=(\langle\bar{\delta}_1^{*L},\mathrm{d}\rangle+\mathscr{E}_1^L\|\mathrm{d}\|,\langle\bar{\delta}_2^{*L},\mathrm{d}\rangle+\mathscr{E}_2^L\|\mathrm{d}\|)\\&=(\langle(1,1),(\mathrm{d}_1,\mathrm{d}_2)\rangle+\mathscr{E}_1^L\sqrt{\mathrm{d}_1^2+\mathrm{d}_2^2},\langle(1,1),(\mathrm{d}_1,\mathrm{d}_2)\rangle+\mathscr{E}_2^L\sqrt{\mathrm{d}_1^2+\mathrm{d}_2^2})\\&=(\mathrm{d}_1+\mathrm{d}_2+2\sqrt{\mathrm{d}_1^2+\mathrm{d}_2^2},\mathrm{d}_1+\mathrm{d}_2+2\sqrt{\mathrm{d}_1^2+\mathrm{d}_2^2})\notin-\mathbb{R}^2_+\setminus\{0\},\\&\qquad\forall\bar{\delta}_1^{*L}\in\partial^*_{\mathscr{D}}\mathfrak{f}_1^L(\bar{\delta}),\ \forall\bar{\delta}_2^{*L}\in\partial^*_{\mathscr{D}}\mathfrak{f}_2^L(\bar{\delta}),\end{aligned}$$

and

$$\begin{aligned}\langle\bar{\delta}^{*U},\mathrm{d}\rangle_2+\mathscr{E}^U\|\mathrm{d}\|&=(\langle\bar{\delta}_1^{*U},\mathrm{d}\rangle+\mathscr{E}_1^U\|\mathrm{d}\|,\langle\bar{\delta}_2^{*U},\mathrm{d}\rangle+\mathscr{E}_2^U\|\mathrm{d}\|)\\&=(\langle(1,1),(\mathrm{d}_1,\mathrm{d}_2)\rangle+\mathscr{E}_1^U\sqrt{\mathrm{d}_1^2+\mathrm{d}_2^2},\langle(0,0),(\mathrm{d}_1,\mathrm{d}_2)\rangle+\mathscr{E}_2^U\sqrt{\mathrm{d}_1^2+\mathrm{d}_2^2})\\&=(\mathrm{d}_1+\mathrm{d}_2+3\sqrt{\mathrm{d}_1^2+\mathrm{d}_2^2},3\sqrt{\mathrm{d}_1^2+\mathrm{d}_2^2})\notin-\mathbb{R}^2_+\setminus\{0\},\\&\qquad\forall\bar{\delta}_1^{*U}\in\partial^*_{\mathscr{D}}\mathfrak{f}_1^U(\bar{\delta}),\ \forall\bar{\delta}_2^{*U}\in\partial^*_{\mathscr{D}}\mathfrak{f}_2^U(\bar{\delta}).\end{aligned}$$

Therefore $\bar{\delta}$ is a solution to $\mathscr{E}-$ Stampacchia $\partial^*_{\mathscr{D}}-VVI$.

- By Theorem 3.2, $\bar{\delta}$ is a *type-1 $\mathscr{E}$-quasi Pareto solution* of IVMOP.
- By Theorem 3.3, $\bar{\delta}$ also solves $\mathscr{E}-$ Minty $\partial^*_{\mathscr{D}}-VVI$.

In the example below, we consider that all the functions are discontinuous and the feasible region is convex.

Example 3.2. Let $\mathfrak{f}_1^L, \mathfrak{f}_1^U, \mathfrak{f}_2^L, \mathfrak{f}_2^U : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$\mathfrak{f}_1^L(\delta) = \begin{cases} -\delta_1, & \delta_1 \geq 0 \\ \delta_1 - 1, & \delta_1 < 0, \end{cases}$$

$$\mathfrak{f}_1^U(\delta) = \begin{cases} \delta_1^2 + \delta_2^2 - \delta_1, & \delta_1 \geq 0, \\ \delta_1^2 + 1, & \delta_1 < 0, \end{cases}$$

$$\mathfrak{f}_2^L(\delta) = \begin{cases} -\delta_1, & \delta_1 \geq 0 \\ \delta_1 + 1, & \delta_1 < 0, \end{cases}$$

and

$$\mathfrak{f}_2^U(\delta) = \begin{cases} -\delta_1 + \delta_2^2, & \delta_1 \geq 0 \\ \delta_1^2 + 1, & \delta_1 < 0, \end{cases}$$

respectively. The collection of all CDs of $\mathfrak{f}_1^L, \mathfrak{f}_1^U, \mathfrak{f}_2^L$, and $\mathfrak{f}_2^U$ at any $\delta := (\delta_1, \delta_2) \in \mathbb{R}^2$ are given by

$$\mathscr{D}_{\mathfrak{f}_1^L}(\delta) = \mathscr{D}_{\mathfrak{f}_2^L}(\delta) = \mathscr{D}_{\mathfrak{f}_1^U}(\delta) = \mathscr{D}_{\mathfrak{f}_2^U}(\delta) = \begin{cases} \mathbb{R}^+ \times \mathbb{R}, & \text{if } \delta_1 = 0 \ \& \ \delta_2 \in \mathbb{R}, \\ \mathbb{R}^2, & \text{otherwise}, \end{cases}$$

where

$$\mathscr{D}(0_{\mathbb{R}^2}) = \mathbb{R}^+ \times \mathbb{R}.$$

We consider nonempty cone $\hat{\mathscr{D}} = \{0\} \times \mathbb{R}^+$ such that $-\hat{\mathscr{D}} \subseteq \hat{\mathscr{D}}$ and feasible region given by

$$\Omega := \{(\delta_1, \delta_2) \in \mathbb{R}^2 : \delta_2 \geq |\delta_1|\}.$$

Then,

$$\Omega_{\hat{\mathscr{D}}, \mathfrak{f}_1^L} := \{\delta \in \Omega : \hat{\mathscr{D}} \subseteq \mathscr{D}_{\mathfrak{f}_1^L}(\delta)\} = \Omega,$$

$$\Omega_{\hat{\mathscr{D}}, \mathfrak{f}_1^U} := \{\delta \in \Omega : \hat{\mathscr{D}} \subseteq \mathscr{D}_{\mathfrak{f}_1^U}(\delta)\} = \Omega$$

and

$$\Omega_{\hat{\mathscr{D}}, \mathfrak{f}_2^L} := \{\delta \in \Omega : \hat{\mathscr{D}} \subseteq \mathscr{D}_{\mathfrak{f}_2^L}(\delta)\} = \Omega,$$

$$\Omega_{\hat{\mathscr{D}}, \mathfrak{f}_2^U} := \{\delta \in \Omega : \hat{\mathscr{D}} \subseteq \mathscr{D}_{\mathfrak{f}_2^U}(\delta)\} = \Omega,$$

which gives

$$\Omega_{\hat{\mathscr{D}}, \mathfrak{f}} := \Omega_{\hat{\mathscr{D}}, \mathfrak{f}_1^L} \cap \Omega_{\hat{\mathscr{D}}, \mathfrak{f}_1^U} \cap \Omega_{\hat{\mathscr{D}}, \mathfrak{f}_2^L} \cap \Omega_{\hat{\mathscr{D}}, \mathfrak{f}_2^U} = \Omega.$$

which is a convex set. For $\bar{\delta} = 0_{\mathbb{R}^2}$,

$$\Omega_{\hat{\mathscr{D}}, \mathfrak{f}, \bar{\delta}} = \Omega_{\hat{\mathscr{D}}, \mathfrak{f}} \cap (\bar{\delta} + \hat{\mathscr{D}}) = \{(\delta_1, \delta_2) \in \mathbb{R}^2 : \delta_1 = 0 \text{ and } \delta_2 \geq 0\}$$

as represented by the red line in Fig. 2. Since there exists some $\delta \in \Omega_{\hat{\mathscr{D}}, \mathfrak{f}, \bar{\delta}}$ (take $\delta = (\frac{1}{2}, 0)$) such that $\mathfrak{f}_i^L(\delta) < \mathfrak{f}_i^L(\bar{\delta})$ and $\mathfrak{f}_i^U(\delta) < \mathfrak{f}_i^U(\bar{\delta})$, for all $i = \{1, 2\}$. Hence $\bar{\delta} \neq \delta \in \Omega_{\hat{\mathscr{D}}, \mathfrak{f}, \bar{\delta}}$ is not a *type-1 quasi Pareto solution* of the $(MOP_{\hat{\mathscr{D}}, \bar{\delta}})$. But if we choose $[\mathscr{E}_1^L, \mathscr{E}_1^U] = [1, 2]$ and $[\mathscr{E}_2^L, \mathscr{E}_2^U] = [1, 2]$, $\bar{\delta}$ is a *type-1* $\mathscr{E}$*-quasi Pareto solution* of the $(MOP_{\hat{\mathscr{D}}, \bar{\delta}})$. Now, for $\bar{\delta} = 0_{\mathbb{R}^2} \in \Omega_{\hat{\mathscr{D}}, \mathfrak{f}}$, one has

$$\overrightarrow{\hat{\mathscr{D}}}_{\Omega, \mathfrak{f}, \bar{\delta}} = \{(\mathrm{d}_1, \mathrm{d}_2) \in \mathbb{R}^2 : \mathrm{d}_1 = 0 \text{ and } \mathrm{d}_2 \geq 0\}.$$

The sets $\Omega_{\hat{\mathscr{D}}, \mathfrak{f}, \bar{\delta}}$ and $\overrightarrow{\hat{\mathscr{D}}}_{\Omega, \mathfrak{f}, \bar{\delta}}$ are convex. For $\bar{\delta} = (0, 0) \in \Omega_{\hat{\mathscr{D}}, \mathfrak{f}}$, the functions $\mathfrak{f}_1^L, \mathfrak{f}_1^U, \mathfrak{f}_2^L$ and $\mathfrak{f}_2^U$ admit

$$\partial^*_{\hat{\mathscr{D}}} \mathfrak{f}_1^L(\bar{\delta}) = \{((-1, 0)\}, \ \partial^*_{\hat{\mathscr{D}}} \mathfrak{f}_1^U(\bar{\delta}) = \{(-1, 0)\}$$

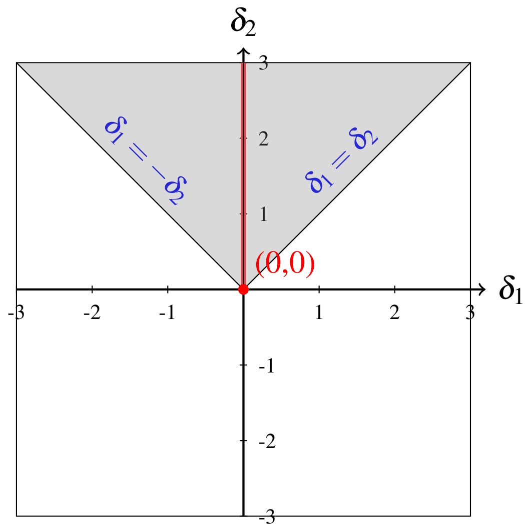

FIGURE 2. The shaded region represents the feasible region Ω and the red line represents $\Omega_{\hat{\mathscr{D}},\mathfrak{f},\bar{\delta}}$ for $\bar{\delta} = 0_{\mathbb{R}^2}$ in Example 3.2.

and

$$\partial^*_{\hat{\mathscr{D}}}\mathfrak{f}^L_2(\bar{\delta}) = \{(-1,0)\},\ \partial^*_{\hat{\mathscr{D}}}\mathfrak{f}^U_2(\bar{\delta}) = \{(-1,0)\}$$

as directional convexificators. Let $\mathrm{d} \in \overrightarrow{\mathscr{D}}_{\Omega,\mathfrak{f},\delta}$ and $\bar{\delta}^* \in \partial^*_{\hat{\mathscr{D}}}\mathfrak{f}(\bar{\delta})$. Note that

$$\mathfrak{f}^L_1(\bar{\delta}+\mathrm{d}) - \mathfrak{f}^L_1(\bar{\delta}) - \langle \bar{\delta}^{*L}_1, \mathrm{d}\rangle = 0 - \langle(-1,0),(0,\mathrm{d}_2)\rangle = 0,$$

$$\mathfrak{f}^U_1(\bar{\delta}+\mathrm{d}) - \mathfrak{f}^U_1(\bar{\delta}) - \langle \bar{\delta}^{*U}_1, \mathrm{d}\rangle = \mathrm{d}_2^2 + \langle(-1,0),(0,\mathrm{d}_2)\rangle = \mathrm{d}_2^2,$$

$$\mathfrak{f}^L_2(\bar{\delta}+\mathrm{d}) - \mathfrak{f}^L_2(\bar{\delta}) - \langle \bar{\delta}^{*L}_2, \mathrm{d}\rangle = 0 - \langle(-1,0),(0,\mathrm{d}_2)\rangle = 0,$$

and

$$\mathfrak{f}^U_2(\bar{\delta}+\mathrm{d}) - \mathfrak{f}^U_2(\bar{\delta}) - \langle \bar{\delta}^{*U}_2, \mathrm{d}\rangle = \mathrm{d}_2^2 - \langle(-1,0),(0,\mathrm{d}_2)\rangle = \mathrm{d}_2^2.$$

Consequently, $\mathfrak{f}^L_1, \mathfrak{f}^L_2, \mathfrak{f}^U_1$, and $\mathfrak{f}^U_2$ are $\partial^*_{\hat{\mathscr{D}}}$-convex on $\Omega_{\hat{\mathscr{D}},\mathfrak{f}}$. For any $\mathrm{d} \in \overrightarrow{\mathscr{D}}_{\Omega,\mathfrak{f},\bar{\delta}}$, $[\mathscr{E}^L_1, \mathscr{E}^U_1] = [1,2]$ and $[\mathscr{E}^L_2, \mathscr{E}^U_2] = [1,2]$, one has

$$\begin{aligned}\langle \bar{\delta}^{*L}, \mathrm{d}\rangle_2 + \mathscr{E}^L\|\mathrm{d}\| &= (\langle \bar{\delta}^{*L}_1, \mathrm{d}\rangle + \mathscr{E}^L_1\|\mathrm{d}\|, \langle \bar{\delta}^{*L}_2, \mathrm{d}\rangle + \mathscr{E}^L_2\|\mathrm{d}\|)\\ &= (\langle(-1,0),(0,\mathrm{d}_2)\rangle + \mathscr{E}^L_1\sqrt{\mathrm{d}_2^2}, \langle(-1,0),(0,\mathrm{d}_2)\rangle + \mathscr{E}^L_2\sqrt{\mathrm{d}_2^2})\\ &= (\sqrt{\mathrm{d}_2^2}, \sqrt{\mathrm{d}_2^2}) \notin -\mathbb{R}^2_+ \setminus \{0\},\\ &\qquad \forall \bar{\delta}^{*L}_1 \in \partial^*_{\mathscr{D}}\mathfrak{f}^L_1(\bar{\delta}),\ \forall \bar{\delta}^{*L}_2 \in \partial^*_{\mathscr{D}}\mathfrak{f}^L_2(\bar{\delta}),\end{aligned}$$

and

$$\begin{aligned}\langle \bar{\delta}^{*U}, \mathrm{d}\rangle_2 + \mathscr{E}^U\|\mathrm{d}\| &= (\langle \bar{\delta}^{*U}_1, \mathrm{d}\rangle + \mathscr{E}^U_1\|\mathrm{d}\|, \langle \bar{\delta}^{*U}_2, \mathrm{d}\rangle + \mathscr{E}^U_2\|\mathrm{d}\|)\\ &= (\langle(-1,0),(0,\mathrm{d}_2)\rangle + \mathscr{E}^U_1\sqrt{\mathrm{d}_2^2}, \langle(-1,0),(0,\mathrm{d}_2)\rangle + \mathscr{E}^U_2\sqrt{\mathrm{d}_2^2})\\ &= (2\sqrt{\mathrm{d}_2^2}, 2\sqrt{\mathrm{d}_2^2}) \notin -\mathbb{R}^2_+ \setminus \{0\},\\ &\qquad \forall \bar{\delta}^{*U}_1 \in \partial^*_{\mathscr{D}}\mathfrak{f}^U_1(\bar{\delta}),\ \forall \bar{\delta}^{*U}_2 \in \partial^*_{\mathscr{D}}\mathfrak{f}^U_2(\bar{\delta}).\end{aligned}$$

Therefore $\bar{\delta}$ is a solution of $\mathscr{E}-$ Stampacchia $\partial^*_{\mathscr{D}} - VVI$.

- By Theorem 3.2, $\bar{\delta}$ is a *type-1 $\mathscr{E}$-quasi Pareto solution* of IVMOP.
- By Theorem 3.3, $\bar{\delta}$ also solves $\mathscr{E}-$ Minty $\partial^*_{\mathscr{D}} - VVI$.

4. Conclusion

This research establishes a unified theoretical framework that connects approximate vector variational inequalities with interval-valued multiobjective optimization problems through the innovative use of directional convexificators (DCs). By developing Stampacchia and Minty-type formulations within this framework, we provided both necessary and sufficient conditions for characterizing approximate Pareto and weak Pareto efficient solutions. The use of DCs introduces a flexible alternative to traditional convexity assumptions, enabling the treatment of a wider class of optimization problems where standard methods may fail. We provided illustrative examples to demonstrate the practical effectiveness of the approach and highlight limitations of existing methods. Overall, the DC-based framework offers both theoretical depth and practical adaptability, paving the way for further advancements in interval-valued optimization.

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