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A THEORETICAL STUDY ON DIRECTIONAL NECESSARY AND SUFFICIENT OPTIMALITY CONDITIONS FOR MATHEMATICAL PROGRAMS WITH EQUILIBRIUM CONSTRAINTS

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Abstract. This paper studies directional optimality conditions for mathematical programs with equilibrium constraints (MPECs), a class of optimization problems with constraints of equilibrium type such as complementarity conditions. Due to the non-smooth and non-convex nature of MPECs, classical optimality conditions are usually not applicable. To overcome these difficulties, we introduce a directional analysis framework that provides a more accurate characterization of stationarity and optimality. We first present the basic building blocks of variational analysis, which are needed for directional analysis. We next derive general directional necessary optimality conditions for a non-linear optimization problem. Building on this foundation, we develop and compare several notions of directional stationarity, namely DW-, DM-, DC-, and DS-stationarity, emphasizing their theoretical distinctions and practical implications. Finally, we provide directional sufficient conditions that ensure global optimality, thus finishing a full optimality framework for MPECs. The findings present an insightful view of the solution structure of MPECs and provide a theoretical basis for future algorithmic developments.

Keywords. Directional optimality conditions; Directional derivatives; Directional subdifferentials; Directional stationarity conditions; Equilibrium constraints.
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1. INTRODUCTION

This study is devoted to the analysis of a class of optimization problems characterized by the following structure:

$$\begin{array}{rclrcl} f(x) & \to & \min & & & \\ g(x) & \leq & 0 & h(x) & = & 0 \\ G(x) & \geq & 0 & H(x) & \geq & 0 \\ & & & G(x)^\top H(x) & = & 0, \end{array} \qquad \text{(MPEC)}$$

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where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^p$, $h : \mathbb{R}^n \to \mathbb{R}^q$, $G : \mathbb{R}^n \to \mathbb{R}^m$, and $H : \mathbb{R}^n \to \mathbb{R}^m$. The superscript $^\top$ denotes the transpose operator. Due to the presence of the last constraint, this problem is called a mathematical program with equilibrium constraints (MPEC for short).

An MPEC is an optimization problem with equilibrium constraints and is usually described as a variational inequality problem with equilibrium constraints. This is the more general setting, but most of the literature has considered the case where these constraints are given in the form of complementarity conditions. Therefore, one often uses the term MPEC to denote the class of problems with complementarity constraints. Due to their structure, MPECs are also closely related to bilevel programming problems and are often referred to as generalized bilevel programs, where the equilibrium conditions are the lower-level problem, as discussed in [1, 2, 3, 4, 5]. For more theoretical and applied developments, we refer to [6, 7, 8, 9, 10] and the list of references given there.

In a recent contribution, Gfrerer [11] introduced a directional variant of the Karush-Kuhn-Tucker (KKT) necessary optimality condition for mathematical programs with generalized equation constraints, formulated via set-valued mappings. This framework is based on the notion of directional metric subregularity, a constraint qualification that is generally weaker than its non-directional counterpart. A notable advantage of the directional approach lies in its capacity to yield sharper and more informative optimality conditions by more precisely reflecting the local geometric structure of the feasible set. This offers a significant advantage over classical methods, where the non-directional nature of the constraint qualifications can lead to weaker and less refined characterizations of optimality.

Motivated by this approach, the present work develops a comprehensive methodology for analyzing MPECs through the perspective of directional optimality conditions. To this end, we begin by reviewing the foundational tools from variational analysis needed for this study, including directional derivatives, tangent cones, regular and limiting normal cones, and directional subdifferentials; see [12, 13, 14, 15, 16]. These constructs are essential for formulating optimality conditions in non-smooth optimization problems, thereby providing a rigorous analytical foundation. A crucial element of our methodology is the adoption of suitable constraint qualifications to guarantee the validity and robustness of the derived conditions. In particular, we employ the no non-zero abnormal multiplier constraint qualification, a well-established condition that plays a critical role in eliminating degenerate multipliers and facilitating meaningful stationarity characterizations. This ensures that the optimality conditions obtained are applicable under broad and practically relevant assumptions.

Our analysis begins with the formulation of directional necessary optimality conditions within a general framework that avoids imposing restrictive structural assumptions; see Section 3. These broad and flexible conditions provide a unified foundation for analyzing various forms of stationarity in MPECs. Building on this foundation, we systematically develop a hierarchy of directional stationarity concepts that capture varying degrees of constraint activity and regularity. This hierarchy begins with directional W-stationarity, a weak form of stationarity that holds under minimal assumptions. It then progresses to directional M-stationarity, which strengthens the conditions by incorporating specific multiplier requirements. Next is directional C-stationarity, which depends on the presence of generalized derivatives or suitable constraint qualifications. At the top of the hierarchy is directional S-stationarity, the most stringent and comprehensive of these conditions.

By systematically comparing the proposed directional stationarity concepts with their classical counterparts, see Figure 1, we reveal a clear hierarchical progression in the strength and analytical depth of the associated optimality conditions. Each successive level in this hierarchy offers increasingly precise and mathematically rigorous characterizations of stationarity, thereby enriching the theoretical understanding of equilibrium constraints in MPECs. This refined framework facilitates a more nuanced interpretation of constraint activity and optimality behaviour, particularly in complex and non-smooth optimization settings, where classical approaches may fall short in capturing the underlying structural subtleties.

Finally, we extend our analysis to the derivation of sufficient optimality conditions for MPECs. These results typically necessitate the use of generalized convexity frameworks. In this context, we focus on a class of generalized convex functions characterized by pseudoconvex sublevel sets. We establish that directional M-stationarity ensures strong global optimality when the the objective and the constraint functions possess pseudoconvex sublevel sets. This result constitutes a significant advancement in the theory of optimality conditions for MPECs, enhancing both the theoretical depth and practical relevance of the proposed framework.

The structure of the paper is as follows. Section 2 introduces the notations, essential preliminaries, and fundamental concepts from directional analysis that will be used in the subsequent developments. In Section 3, we establish the directional KKT conditions within a general framework for non-linear optimization problems. Section 4 extends this framework to problem (MPEC), wherein we formulate directional stationarity conditions and present the associated theoretical results. Section 5 is devoted to the derivation of sufficient optimality conditions for problem (MPEC), under generalized convexity assumptions. Finally, Section 6 summarizes the principal contributions of the study and provides concluding remarks.

2. Basic Tools

Initially, we present the notations employed in the paper. We denote $\mathbb{R}^n$ as the n-dimensional Euclidean space, equipped with the standard inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. For $\varepsilon>0$ and $\bar{x}\in\mathbb{R}^n$, we define the closed ε-ball around $\bar{x}$ as $\mathbb{B}(\bar{x},\varepsilon)=\{x\in\mathbb{R}^n : \|x-\bar{x}\|\le\varepsilon\}$. To make things simple, we denote by $\mathbb{B}$ the closed unit ball of $\mathbb{R}^n$. Let $\mathscr{A}$ be an arbitrary set of $\mathbb{R}^n$, $d\in\mathbb{R}^n$. We denote by $\operatorname{conv}\mathscr{A}$, $\operatorname{cone}\mathscr{A}$, and $|\mathscr{A}|$ the convex hull, the conic hull, and the cardinal of $\mathscr{A}$, respectively. We define $d_+=(\max\{d_1,0\},\dots,\max\{d_n,0\})^\top$ and $\operatorname{dist}(\mathscr{A},\bar{x})=\inf\{\|x-\bar{x}\| : x\in\mathscr{A}\}$. The set of perpendicular vectors to $\mathscr{A}$ in $\mathbb{R}^n$ is given by

$$\mathscr{A}^\perp=\{x^*\in\mathbb{R}^n : x^*\perp x\ \ \forall x\in\mathscr{A}\}.$$

We also define the notation $o(t)$ to indicate that $\lim_{t\to 0}\dfrac{o(t)}{t}=0$. For a set-valued mapping $\mathscr{Y}:\mathbb{R}^n\rightrightarrows\mathbb{R}^m$, the graph and domain of $\mathscr{Y}$ are denoted by $\operatorname{gph}\mathscr{Y}$ and $\operatorname{dom}\mathscr{Y}$, respectively, i.e., $\operatorname{gph}\mathscr{Y}=\{(z,v)\in\mathbb{R}^n\times\mathbb{R}^m : v\in\mathscr{Y}(z)\}$ and $\operatorname{dom}\mathscr{Y}=\{z\in\mathbb{R}^n : \mathscr{Y}(z)\neq\emptyset\}$. Additionally, we are interested in what are referred to as directional neighborhoods of specified directions [14]. They represent the sets of form

$$D(\bar{x},u;\varepsilon,\delta)=\mathbb{B}(\bar{x},\varepsilon)\cap(\bar{x}+\operatorname{cone}(u+\delta\mathbb{B})),$$

with $\bar{x}\in\mathbb{R}^n$, $u\in\mathbb{R}^n$ a direction vector, and $\varepsilon,\delta>0$.

Recently, Gfrerer [11] proposed the following equivalent notion of a directional neighborhood. Given a direction $u\in\mathbb{R}^n$ and positive numbers $\varepsilon,\rho>0$, the directional neighborhood of direction u is a set defined by

$$\mathscr{U}_{\varepsilon,\rho}(u)=\left\{v\in\varepsilon\mathbb{B}\ :\ \left\|\|u\|v-\|v\|u\right\|\leq\rho\|u\|\|v\|\right\}.$$

It is obvious that the directional neighborhood of a nonzero direction $u\neq 0$ is a smaller subset of $\varepsilon\mathbb{B}$ and that the directional neighborhood of the direction $u=0$ is just the open ball $\varepsilon\mathbb{B}$. In the following, we say that a set $\mathscr{U}\subset\mathbb{R}^n$ is a directional neighborhood of u if there exist $\varepsilon,\rho>0$ such that $\mathscr{U}_{\varepsilon,\rho}(u)\subset\mathscr{U}$.

Remark 2.1. Note that the two directional neighborhood concepts mentioned above are related in the sense that one implies the other. Indeed, let $0\neq u\in\mathbb{R}^n$.

- Suppose that a property $\mathscr{P}(x)$ holds in $D(\bar{x},u;\varepsilon,\delta)$. Pick $0<\rho\|u\|<\delta$, $\varepsilon>0$, and let $x\in\bar{x}+\mathscr{U}_{\varepsilon,\rho}(u)$. Then
$$x\in\mathbb{B}(\bar{x},\varepsilon)\ \text{ and }\ \left\|\|u\|(x-\bar{x})-\|x-\bar{x}\|u\right\|\leq\rho\|u\|\|x-\bar{x}\|.$$
Consequently, $x\in\mathbb{B}(\bar{x},\varepsilon)$ and $x-\bar{x}\in\dfrac{\|x-\bar{x}\|}{\|u\|}(u+\rho\|u\|\mathbb{B})$. Thus, $x\in D(\bar{x},u;\varepsilon,\delta)$. This implies that $\mathscr{P}(x)$ holds in $\bar{x}+\mathscr{U}_{\varepsilon,\rho}(u)$.
- Suppose that a property $\mathscr{P}(x)$ holds in $\bar{x}+\mathscr{U}_{\varepsilon,\rho}(u)$. Pick $0<\delta<\|u\|$, $\varepsilon>0$, and $x\in D(\bar{x},u;\varepsilon,\delta)$. Then $x\in\mathbb{B}(\bar{x},\varepsilon)$ and $x-\bar{x}=\alpha(u+\delta a)$, where $a\in\mathbb{B}$ and $\alpha\geq 0$. We possess the approximations $\|u\|-\delta\leq\|u+\delta a\|\leq\|u\|+\delta$ which leads to
$$\alpha=\frac{\|x-\bar{x}\|}{\|u+\delta a\|}=\frac{\|x-\bar{x}\|}{\|u\|+\zeta},\quad -\delta\leq\zeta\leq\delta.$$
Finally, putting all above argument, we see for $x\neq\bar{x}$ that
$$\frac{x-\bar{x}}{\|x-\bar{x}\|}-\frac{u}{\|u\|}=\left(\frac{1}{\|u\|+\zeta}-\frac{1}{\|u\|}\right)u+\frac{\delta}{\|u\|+\zeta}a.$$
Letting $\delta\to 0^+$, we obtain $\frac{x-\bar{x}}{\|x-\bar{x}\|}-\frac{u}{\|u\|}\to 0$. Using the definition of the limit, we conclude $\left\|\frac{x-\bar{x}}{\|x-\bar{x}\|}-\frac{u}{\|u\|}\right\|\leq\rho$. Hence, $x\in\bar{x}+\mathscr{U}_{\varepsilon,\rho}(u)$, which implies that $\mathscr{P}(x)$ holds in $D(\bar{x},u;\varepsilon,\delta)$.

Definition 2.1. We say that a sequence $\{x_k\}\subset\mathbb{R}^n$ converges to some $\bar{x}\in\mathbb{R}^n$ from a direction $u\in\mathbb{R}^n$, which is denoted as $x_k\xrightarrow{u}\bar{x}$, if there exist $t_k\downarrow 0$ and $u_k\to u$ with $x_k=\bar{x}+t_ku_k$, or, equivalently, if, for every directional neighborhood $\mathscr{U}_{\varepsilon,\rho}(u)$ of u, $x_k\in\bar{x}+\mathscr{U}_{\varepsilon,\rho}(u)$ for sufficiently large k.

We now explore some basic ideas and results in variational analysis that will be used later on. Let us fix a closed subset $\mathscr{A}\subset\mathbb{R}^n$ and a point $\bar{x}\in\mathscr{A}$. We employ

$$T_{\mathscr{A}}(\bar{x})=\{\zeta\in\mathbb{R}^n\ :\ \exists t_k\downarrow 0,\ \zeta_k\to\zeta\ \text{s.t. }\bar{x}+t_k\zeta_k\in\mathscr{A}\ \forall k\}$$

to indicate the tangent cone to $\mathscr{A}$ at $\bar{x}$. Moreover, we employ

$$\begin{aligned}\hat{N}_{\mathscr{A}}(\bar{x}) &= \{\eta\in\mathbb{R}^n\ :\ \langle\eta,x-\bar{x}\rangle\leq o(\|x-\bar{x}\|)\ \forall x\in\mathscr{A}\},\\ N_{\mathscr{A}}(\bar{x}) &= \left\{\eta\in\mathbb{R}^n\ :\ \exists x_k\xrightarrow{\mathscr{A}}\bar{x},\ \eta_k\to\eta\ \text{such that }\eta_k\in\hat{N}_{\mathscr{A}}(x_k)\ \forall k\right\},\end{aligned}$$

the regular (or Fréchet) and limiting (or Mordukhovich) normal cone to $\mathscr{A}$ at $\bar{x}$, where $x_k\xrightarrow{\mathscr{A}}\bar{x}$ means that $x_k\to\bar{x}$ and $x_k\in\mathscr{A}$. If $\bar{x}\notin\mathscr{A}$, we set $T_{\mathscr{A}}(\bar{x})=\emptyset$, $\hat{N}_{\mathscr{A}}(\bar{x})=\emptyset$ and $N_{\mathscr{A}}(\bar{x})=\emptyset$. In

[11, 13], directional versions of the above limiting constructions were presented for Banach spaces and repeated in an analogous manner for finite dimensional spaces.

Definition 2.2. Given a direction $u \in \mathbb{R}^n$, the limiting normal cone to $\mathscr{A}$ at $\bar{x}$ in direction u is defined by

$$N_{\mathscr{A}}(\bar{x},u) = \left\{ \eta \in \mathbb{R}^n \ : \begin{array}{l} \exists t_k \downarrow 0,\ \exists u_k \to u,\ \eta_k \to \eta \\ \text{such that } \eta_k \in \hat{N}_{\mathscr{A}}(\bar{x}+t_k u_k)\ \forall k \end{array} \right\}.$$

It is worth noting that $N_{\mathscr{A}}(\bar{x},u)$ is empty if u does not belong to $T_{\mathscr{A}}(\bar{x})$. If $\mathscr{A}$ is convex, then

$$N_{\mathscr{A}}(\bar{x},u) = N_{T_{\mathscr{A}}(\bar{x})}(u) = N_{\mathscr{A}}(\bar{x}) \cap \{u\}^{\perp},\ \forall u \in T_{\mathscr{A}}(\bar{x}). \tag{2.1}$$

Throughout this work, the notion of complementarity constraints is significant. However, any formulation of complementarity constraints leads to the following vector complementarity set:

$$\mathscr{X} = \left\{(a,b) \in \mathbb{R}^2 \ : \ a \geq 0,\ b \geq 0,\ ab = 0\right\}, \tag{2.2}$$

which is a union of the two polyhedral sets $\mathbb{R}^+ \times \{0\}$ and $\{0\} \times \mathbb{R}^+$.

The following result covers basic properties of vector complementarity set $\mathscr{X}$ in (2.2) which are derived from [17, Theorem 6.1] and [7, Lemma 4.1].

Proposition 2.1. *Consider the set $\mathscr{X}$ in (2.2). Let $(a,b) \in \mathscr{X}$ and $(u,v) \in T_{\mathscr{X}}(a,b)$.*

(1) The directional normal cone to $\mathscr{X}$ at (a,b) in direction (u,v) is given by:

$$\begin{array}{rll} (i)\ \textit{if} & a=0,\ b>0,\ \textit{then} & N_{\mathscr{X}}((a,b),(u,v)) = N_{\mathscr{X}}(a,b) = \mathbb{R} \times \{0\}, \\ (ii)\ \textit{if} & a>0,\ b=0,\ \textit{then} & N_{\mathscr{X}}((a,b),(u,v)) = N_{\mathscr{X}}(a,b) = \{0\} \times \mathbb{R}, \\ (iii)\ \textit{if} & a=0,\ b=0,\ \textit{then} & N_{\mathscr{X}}((a,b),(u,v)) = N_{\mathscr{X}}(u,v). \end{array}$$

(2) The contingent cone to $\mathscr{X}$ at (a,b) is given by

$$T_{\mathscr{X}}(a,b) = \left\{ (x,y) \in \mathbb{R}^2 \ : \begin{array}{ll} x = 0 & \textit{if } a=0,\ b>0 \\ x \geq 0,\ y \geq 0,\ xy = 0 & \textit{if } a=b=0 \\ y = 0 & \textit{if } a>0,\ b=0 \end{array} \right\}.$$

Remark 2.2. Observe that Proposition 2.1 remains valid with the opposite sign when the vector complementarity set has the following structure: $\mathscr{X} = \{(a,b) \in \mathbb{R}^2 \ : \ a \leq 0,\ b \leq 0,\ ab = 0\}$.

Let $\mathscr{Y} : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued mapping with a closed graph locally around $(\bar{x},\bar{y}) \in \operatorname{gph}\mathscr{Y}$. The graphical derivative of $\mathscr{Y}$ at $(\bar{x},\bar{y})$ is the mapping $D\mathscr{Y}(\bar{x},\bar{y}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ given by

$$\operatorname{gph} D\mathscr{Y}(\bar{x},\bar{y}) = T_{\operatorname{gph}\mathscr{Y}}(\bar{x},\bar{y}).$$

The coderivative of $\mathscr{Y}$ at $(\bar{x},\bar{y})$ is a multifunction $D^*\mathscr{Y}(\bar{x},\bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, with the value

$$D^*\mathscr{Y}(\bar{x},\bar{y})(v) = \left\{u \in \mathbb{R}^n \ : \ (u,-v) \in N_{\operatorname{gph}\mathscr{Y}}(\bar{x},\bar{y})\right\}, \quad \text{for } v \in \mathbb{R}^m.$$

Furthermore, for direction $(u,v) \in \mathbb{R}^n \times \mathbb{R}^m$, the coderivative of $\mathscr{Y}$ in direction (u,v) at $(\bar{x},\bar{y}) \in \operatorname{gph}\mathscr{Y}$ is defined as the multifunction $D^*\mathscr{Y}((\bar{x},\bar{y}),(u,v)) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ given by

$$D^*\mathscr{Y}((\bar{x},\bar{y}),(u,v))(\kappa) = \{\zeta \in \mathbb{R}^n \ : \ (\zeta,-\kappa) \in N_{\operatorname{gph}\mathscr{Y}}((\bar{x},\bar{y}),(u,v))\}, \quad \text{for } \kappa \in \mathbb{R}^m.$$

Clearly, one has $D^*\mathscr{Y}((\bar{x},\bar{y}),(0,0)) = D^*\mathscr{Y}(\bar{x},\bar{y})$. In the case where $\mathscr{Y}$ is single-valued at $\bar{x}$, with $\mathscr{Y}(\bar{x}) = \{\bar{y}\}$, we use the notation $D\mathscr{Y}(\bar{x}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and $D^*\mathscr{Y}(\bar{x}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ for brevity.

Remark 2.3. Let $\mathscr{Y} : \mathbb{R}^n \to \mathbb{R}^m$ be a single-valued function at $\bar{x}$.

(1) The graphical derivative of $\mathscr{Y}$ at $(\bar{x},\bar{y})$, with $\mathscr{Y}(\bar{x})=\bar{y}$, is reduced to

$$D\mathscr{Y}(\bar{x})(u)=\left\{v\in\mathbb{R}^m\ :\ \exists t_k\downarrow 0,\ \exists u_k\to u,\ \frac{\mathscr{Y}(\bar{x}+t_ku_k)-\bar{y}}{t_k}\to v\right\}.$$

(2) In the case that $\mathscr{Y}$ is continuously differentiable, with the jacobian $\nabla\mathscr{Y}(\bar{x})$. we have $D\mathscr{Y}(\bar{x})(u)=\nabla\mathscr{Y}(\bar{x})u$ and consequently, $D^*\mathscr{Y}(\bar{x},(u,v))(\kappa)\neq\emptyset$ if and only if $v=\nabla\mathscr{Y}(\bar{x})u$. This implies

$$D^*\mathscr{Y}(\bar{x},(u,v))(\kappa)=D^*\mathscr{Y}(\bar{x})(\kappa)=(\nabla\mathscr{Y}(\bar{x}))^\top\kappa,\quad\text{for }\kappa\in\mathbb{R}^m.$$

Definition 2.3. A multifunction $\Phi:\mathbb{R}^n\rightrightarrows\mathbb{R}^m$ is said to be directionally inner semicontinuous at $(\bar{x},\bar{y})\in\operatorname{gph}\Phi$ in a direction $u\in\mathbb{R}^n$ if, for any sequences $t_k\downarrow 0$, $u_k\to u$, there exists a sequence $y_k\in\Phi(\bar{x}+t_ku_k)$ converging to $\bar{y}$.

Next, we review directional variants of the Lipschitzian properties and the differentiability of mappings that will be employed in the subsequent analysis [15]. First, recall that a function $\varphi:\mathbb{R}^n\to\mathbb{R}^m$ is directionally differentiable at $\bar{x}$ in direction u if

$$\varphi'(\bar{x},u)=\lim_{t\downarrow 0}\frac{\varphi(\bar{x}+tu)-\varphi(\bar{x})}{t}\ \text{ exists.}$$

Definition 2.4. Let $\varphi:\mathbb{R}^n\to\mathbb{R}^m$ and $\bar{x}$, $u\in\mathbb{R}^n$.

(1) We say that φ is directionally Lipschitz continuous around $\bar{x}$ in the direction u with constant $l\geq 0$, if there exist $\varepsilon,\delta>0$ such that

$$\|\varphi(y)-\varphi(x)\|\leq l\|y-x\|,\ \text{ for all } x,y\in D(\bar{x},u;\varepsilon,\delta).$$

If, in the above definition, $y=\bar{x}$, we say that φ is directionally calm at $\bar{x}$ in direction $u\in\mathbb{R}^n$.

(2) φ is directionally strictly Fréchet differentiable at $\bar{x}$ in the direction u if there exists a linear operator $\mathscr{G}:\mathbb{R}^n\to\mathbb{R}^m$ such that, for any $\sigma>0$, there exist $\varepsilon>0$ and $\delta>0$ such that, for all x, $y\in D(\bar{x},u;\varepsilon,\delta)$,

$$\|\varphi(y)-\varphi(x)-\mathscr{G}(y-x)\|\leq\sigma\|y-x\|.$$

When $u=0$, the directional Lipschitz continuous property reduces to the usual Lipschitz continuous property. In this case, $D(\bar{x},u;\varepsilon,\delta)$ reduces to $\mathbb{B}(\bar{x},\varepsilon)$. Also, it can be observed that the concept of directional strict Fréchet differentiability simplifies to the standard strict Fréchet differentiability when $u=0$.

Our main concern throughout the study is the directionally strict differentiability. We collect some corresponding properties in the following lemma.

Lemma 2.1. *Let $\varphi:\mathbb{R}^n\to\mathbb{R}^m$ and $\bar{x}$, $u\in\mathbb{R}^n$.*

- *If φ is directionally strictly Fréchet differentiable at $\bar{x}$ in the direction u, then the linear operator $\mathscr{G}$ in Definition 2.4 is unique.*
- *If φ is directionally strictly Fréchet differentiable at $\bar{x}$ in the direction u, then φ is directionally Lipschitz continuous around $\bar{x}$ in the direction u.*

Remark 2.4. If the operator $\mathscr{G}$ in Definition 2.4 exists, we call it the directional strict Fréchet derivative of φ at $\bar{x}$ in the direction u, and we denote it by $\nabla\varphi(\bar{x},u)$.

Lemma 2.2. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be strictly differentiable at $\bar{x} \in \mathbb{R}^n$ in direction $u \in \mathbb{R}^n$. Then, the directional derivative $\varphi'(\bar{x},u)$ exists and is computed by $\varphi'(\bar{x},u) = \nabla\varphi(\bar{x},u)^\top u$.*

Proof. Set $\zeta = \nabla\varphi(\bar{x},u)$. Let $\sigma > 0$. Then we can find $\varepsilon > 0$ and $\delta > 0$ such that, for all x, $y \in D(\bar{x},u;\varepsilon,\delta)$,

$$\|\varphi(y) - \varphi(x) - \langle \zeta, y-x\rangle\| \leq \sigma\|y-x\|.$$

For $x = \bar{x}$ and $y = \bar{x} + tu$ with $0 < t < \frac{\varepsilon}{\|u\|+\delta}$, $y \in D(\bar{x},u;\varepsilon,\delta)$,

$$\left\| \frac{\varphi(\bar{x}+tu) - \varphi(\bar{x})}{t} - \langle \zeta, u\rangle \right\| \leq \sigma\|u\|.$$

which implies that the directional derivative $\varphi'(\bar{x},u)$ exists and is computed by $\varphi'(\bar{x},u) = \nabla\varphi(\bar{x},u)^\top u$. □

After a brief introduction to regular and limiting subdifferentials, we define the so-called directional subdifferential of a function [11, 12, 13, 15]. Remarkably, the limiting subdifferential is known for being smaller than other often-used subdifferentials.

Definition 2.5. Let $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ be extended-real valued function and $\bar{x} \in \operatorname{dom}\varphi$. Moreover, let $u \in \mathbb{R}^n$ be a direction.

(i) The Fréchet (regular) subdifferential and the limiting subdifferential of φ at $\bar{x}$ are defined as follows:

$$\begin{aligned} \hat{\partial}\varphi(\bar{x}) &= \{\zeta \in \mathbb{R}^n : \varphi(x) \geq \varphi(\bar{x}) + \langle \zeta, x-\bar{x}\rangle + o(\|x-\bar{x}\|)\}, \\ \partial\varphi(\bar{x}) &= \left\{\zeta \in \mathbb{R}^n : \exists x_k \to \bar{x},\ \varphi(x_k) \to (\bar{x}),\ \zeta_k \to \zeta, \text{ such that } \zeta_k \in \hat{\partial}\varphi(x_k)\ \forall k\right\}. \end{aligned}$$

(ii) The analytic limiting subdifferential of φ at $\bar{x}$ in direction u is given by:

$$\partial\varphi(\bar{x},u) = \left\{\alpha \in \mathbb{R}^n : \exists x_k \xrightarrow{u} \bar{x},\ \alpha_k \to \alpha, \text{ such that } \varphi(x_k) \to \varphi(\bar{x}),\ \alpha_k \in \hat{\partial}\varphi(x_k)\ \forall k\right\}.$$

Inspired by directional coderivatives, Benko et al. in [12] considered a direction $(u,v) \in \mathbb{R}^{n+1}$ and defined the geometric limiting subdiferential of φ at $\bar{x}$ in direction (u,v) via a directional normal cone. That is,

$$\tilde{\partial}\varphi(\bar{x},(u,v)) = \left\{\xi \in \mathbb{R}^n : (\xi,-1) \in N_{\text{epi}\,\varphi}((\bar{x},\varphi(\bar{x})),(u,v))\right\}. \tag{2.3}$$

The following proposition gives the relationship between the analytic directional limiting subdifferential and the one defined in equation (2.3), provided that the function φ is directionally calm.

Proposition 2.2. *Let $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ be finite at $\bar{x}$ and consider a direction $u \in \mathbb{R}^n$. Furthermore, assume that φ is calm at $\bar{x}$ in direction u. Then $\partial\varphi(\bar{x},u) = \bigcup_{v \in D\varphi(\bar{x})(u)} \tilde{\partial}\varphi(\bar{x},(u,v))$. In addition, if φ is also directionally differentiable at $\bar{x}$ in direction u, then $D\varphi(\bar{x})(u) = \{\varphi'(\bar{x},u)\}$ and $\partial\varphi(\bar{x},u) = \tilde{\partial}\varphi(\bar{x},(u,\varphi'(\bar{x},u)))$.*

Furthermore, if φ is directionally Lipschitz continuous around $\bar{x}$ in the direction u, in [18], the authors defined the directional Clarke subdifferential of φ at $\bar{x}$ in the direction u as

$$\partial^c\varphi(\bar{x},u) = \operatorname{conv}(\partial\varphi(\bar{x},u)).$$

The result that follows combines a number of basic calculus ideas for directional limiting subdifferentials. Readers are directed to [15] and [19] for a more detailed discussion of the calculus properties of this subdifferential.

Definition 2.6. Let $\psi : \mathbb{R}^n \to \overline{\mathbb{R}}$ with $u \in \mathbb{R}^n$ and $\bar{x} \in \operatorname{dom} \psi$. We say that ψ is lower semicontinuous at $\bar{x}$ in direction u if $\psi(\bar{x}) \leq \liminf_{x \xrightarrow{u} \bar{x}} \psi(x)$.

Theorem 2.1. *Let $\chi, \psi : \mathbb{R}^n \to \overline{\mathbb{R}}$ with $u \in \mathbb{R}^n$ and $\bar{x} \in \operatorname{dom} \psi$. Let α, β be nonnegative scalars.*

(i) *Assume that χ is directionally strictly Fréchet differentiable at $\bar{x}$ in direction u. Then,*

$$\partial (\chi + \psi)(\bar{x}, u) = \nabla \chi (\bar{x}, u) + \partial \psi (\bar{x}, u).$$

(ii) *Suppose that χ be Lipschitz continuous around $\bar{x}$ in direction u and let ψ be lower semicontinuous in direction u. Then,*

$$\partial(\alpha\chi + \beta\psi)(\bar{x}, u) \subset \alpha\partial\chi(\bar{x}, u) + \beta\partial\psi(\bar{x}, u).$$

There are several different ideas on local regularity that are related to set-valued mappings. The primary focus of this study is the so-called directional metric sub-regularity, which comes from [12, Definition 2.1].

Definition 2.7. Let $\mathscr{Y} : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and $(\bar{x}, \bar{y}) \in \operatorname{gph} \mathscr{Y}$.

- We say that $\mathscr{Y}$ is metrically subregular at $(\bar{x}, \bar{y})$ provided there exist $\kappa > 0$ and a neighborhood $\mathscr{U}$ of $\bar{x}$ such that

$$\operatorname{dist}\left(\mathscr{Y}^{-1}(\bar{y}), x\right) \leq \kappa \operatorname{dist}(\mathscr{Y}(x), \bar{y}) \quad \forall x \in \mathscr{U}.$$

- Given $u \in \mathbb{R}^n$, we say that $\mathscr{Y}$ is metrically subregular in direction u at $(\bar{x}, \bar{y})$ if there exists a directional neighborhood $\mathscr{U}$ of u such that the above estimate holds for all $x \in \bar{x} + \mathscr{U}$.

At the end of this section, we present the following definition of the directional metric subregularity constraint qualification (MSCQ).

Definition 2.8. Given $\chi : \mathbb{R}^n \to \mathbb{R}^p$ and $\psi : \mathbb{R}^n \to \mathbb{R}^q$, let $\bar{x} \in \mathbb{R}^n$ be a solution to $\mathscr{P} = \{x \in \mathbb{R}^n \ : \ \chi(x) \leq 0, \ \psi(x) = 0\}$. The directional metric subregular constraint qualification (MSCQ) at $\bar{x}$ in a direction $u \in \mathbb{R}^n$ is said to be satisfied by system $\mathscr{P}$ if there exist $\varepsilon, \rho, \sigma > 0$ such that

$$\operatorname{dist}(\mathscr{P}, x) \leq \sigma \left(\left\|(\chi(x))_+\right\| + \|\psi(x)\| \right), \quad \forall x \in \bar{x} + \mathscr{U}_{\varepsilon,\rho}(u).$$

If $u = 0$, we say that system $\mathscr{P}$ satisfies the (MSCQ) at $\bar{x}$.

Remark 2.5. Keep in mind that the two definitions of directional metric subregularity given above are closely connected; in fact, let $\bar{x} \in \mathbb{R}^n$ be a solution of the system $\mathscr{P} = \{x \in \mathbb{R}^n \ : \ \chi(x) \leq 0, \ \psi(x) = 0\}$ and assume that the directional (MSCQ) holds at $\bar{x}$ in a direction $u \in \mathbb{R}^n$. Now, denote by $\Pi(x) = \mathbb{R}^p_- \times \mathbb{R}^q - (\chi(x), \psi(x))$ a multifunction associated with the system $\mathscr{P}$. Hence, Π is metrically subregular in direction u at $(\bar{x}, 0)$.

3. Directional Necessary Optimality Conditions in General Framework

In the current section, we develop the directional Karush–Kuhn–Tucker $(DKKT)$ conditions for the nonlinear optimization problem

$$\min_x \ \psi(x) \text{ s.t. } \varphi(x) \leq 0, \ \phi(x) = 0, \ x \in \Omega, \tag{NLP}$$

where $\psi : \mathbb{R}^n \to \mathbb{R}$, the constraint functions $\varphi : \mathbb{R}^n \to \mathbb{R}^p$ and $\phi : \mathbb{R}^n \to \mathbb{R}^q$ are continuous, and Ω is a closed set of $\mathbb{R}^n$.

The presence of nondegenerate multipliers is related to the validity of specific constraint qualification conditions while developing optimality conditions. Properties such as metric regularity and subregularity are examples of constraint qualification conditions. In our case, we will use the directed variant of the metric subregularity for (NLP). To proceed, we denote the feasible set of (NLP) by

$$\mathscr{F} = \{x \in \mathbb{R}^n \ : \ \varphi(x) \leq 0, \ \phi(x) = 0, \ x \in \Omega\}.$$

For $\bar{x} \in \mathscr{F}$, the set $I(\varphi, \bar{x}) = \{i \in \{1, \ldots, p\} \ : \ \varphi_i(\bar{x}) = 0\}$ represents the indexes of active constraints at $\bar{x}$.

Assuming that ψ is directionally Lipschitz continuous near $\bar{x}$ and directionally differentiable, while φ and ϕ are both directionally differentiable, we construct the critical cone associated to problem (NLP) at $\bar{x}$ as follows:

$$\mathscr{C}_{\mathscr{F}}(\bar{x}) = \left\{ u \in \mathbb{R}^n \ : \begin{array}{l} \psi'(\bar{x}, u) \leq 0, \ \varphi_i'(\bar{x}, u) \leq 0, \ i \in I(\varphi, \bar{x}) \\ \phi_i'(\bar{x}, u) = 0, \ u \in T_\Omega(\bar{x}) \end{array} \right\}.$$

We define the set-valued mapping Ψ as

$$\Psi(x, y) = \left\{ (\chi(x), z) \ : \ z \in \mathbb{R}^p_+ \times \{0_q\} \times \Omega, \ \chi(x) + z = y \right\},$$

where $\chi(x) = (\varphi(x), \phi(x), -x)$.

Theorem 3.1. *Let $\bar{x}$ be a local optimal solution to problem (NLP), and let $u \in \mathscr{C}_{\mathscr{F}}(\bar{x})$ be a critical direction at $\bar{x}$. Assume that ψ is Lipschitz continuous near $\bar{x}$ and directionally differentiable at $\bar{x}$ in the direction u and that φ and ϕ admit directional derivatives at $\bar{x}$ in the direction u. Further, suppose that both ϕ and ψ are directionally Lipschitz at $\bar{x}$ in the direction u. Moreover, assume that system $\mathscr{F}$ satisfies the directional (MSCQ) at $\bar{x}$ in the direction u, and that mapping Ψ is directionally inner semicontinuous at $(\bar{x}, 0, \chi(\bar{x}), -\chi(\bar{x}))$ in the direction $(u, 0)$. Then, there exist multipliers $v \in \mathbb{R}^p$ and $\gamma \in \mathbb{R}^q$ such that*

$$\begin{cases} 0 \in \partial \psi(\bar{x}, u) + \partial \langle v, \varphi \rangle (\bar{x}, u) + \partial \langle \gamma, \phi \rangle (\bar{x}, u) + N_\Omega(\bar{x}, u) \\ 0 \leq v \perp \varphi(\bar{x}), v \perp \varphi'(\bar{x}, u). \end{cases}$$

Proof. Besides, we associate with the mathematical program (NLP) two set-valued mappings

$$\begin{array}{llll} G : \mathbb{R}^n \rightrightarrows \mathbb{R}^p \times \mathbb{R}^q & \text{defined by} & G(x) & = & \chi(x) + \mathbb{R}^p_+ \times \{0_q\} \times \Omega, \\ M : \mathbb{R}^n \rightrightarrows \mathbb{R}^{p+q+1} & \text{given by} & M(x) & = & (\psi(x) - \mathbb{R}_-) \times G(x). \end{array}$$

Hence, problem (NLP) can be rewritten as

$$\min \{\psi(x) \ : \ 0 \in G(x)\}. \tag{3.1}$$

Consequently, u is also a critical direction of (3.1), that is, $(u, (0,0)) \in T_{\text{gph} M}(\bar{x}, (\psi(\bar{x}), 0))$. By considering the sequence $\{(u_k, (\alpha_k, \beta_k))\}$ defined by $u_k = u$, $\alpha_k = 0$, $\beta_k^1 = \frac{1}{t_k}(\varphi(\bar{x} + t_k u_k) - \varphi(\bar{x})) - \varphi'(\bar{x}, u)$, $\beta_k^2 = \frac{1}{t_k}\phi(\bar{x} + t_k u_k)$, and $\beta_k^3 = -u_k$ for all k, we obtain, for all $t_k \downarrow 0$, that

$$(\bar{x}, (\psi(\bar{x}), 0)) + t_k(u_k, (\alpha_k, \beta_k)) \in \text{gph} M, \ \beta_k = \left(\beta_k^1, \beta_k^2, \beta_k^3\right),$$

for all sufficiently large k. Hence, u is a critical direction for problem (3.1) at $\bar{x}$.

Under the fact that G has a closed graph and is metrically subregular in direction u at $(\bar{x},0)$, we conclude from [11, Theorem 7] that there exist $v\in\mathbb{R}^p$, $\gamma\in\mathbb{R}^q$, and $\zeta\in\mathbb{R}^n$ such that

$$0\in\partial\psi(\bar{x},u)+D^*G((\bar{x},0),(u,0))(v,\gamma,\zeta).$$

Since Ψ is directionally inner semicontinuous at $(\bar{x},0,\chi(\bar{x}),-\chi(\bar{x}))$ in direction $(u,0)$, then, in this case, Ψ is directionally inner semicontinuous at $(\bar{x},0,\chi(\bar{x}),-\chi(\bar{x}))$ in direction $(u,0,\chi'(\bar{x},u),-\chi'(\bar{x},u))$ in the sense of [16]. Hence, by applying the calculus rule from [16, Corollary 3.10] for $F_1=\chi$ and $F_2=\mathbb{R}^p_+\times\{0_q\}\times\Omega$, we have

$$\begin{aligned}0 \;\in\;& \partial\psi(\bar{x},u)+D^*\chi(\bar{x},(u,\chi'(\bar{x},u)))((v,\gamma,\zeta))\\ &+D^*F_2((\bar{x},-\chi(\bar{x})),(u,-\chi'(\bar{x},u)))((v,\gamma,\zeta)).\end{aligned} \tag{3.2}$$

Because $\operatorname{gph}F_2=\mathbb{R}^n\times\mathbb{R}^p_+\times\{0_q\}\times\Omega$, we have

$$(w,-v,-\gamma,-\zeta)\in N_{\mathbb{R}^n\times\mathbb{R}^p_+\times\{0_q\}\times\Omega}\left((\bar{x},-\chi(\bar{x})),\left(u,-\chi'(\bar{x},u)\right)\right),$$

for any $w\in D^*F_2((\bar{x},-\chi(\bar{x})),(u,-\chi'(\bar{x},u)))(v,\gamma,\zeta)$. Exploiting [17, Proposition 3.3], we get

$$\gamma\text{ free},\quad w\in N_{\mathbb{R}^n}(\bar{x},u),\qquad -v\in N_{\mathbb{R}^p_+}\left(-\varphi(\bar{x}),-\varphi'(\bar{x},u)\right)\;\text{ and }\;-\zeta\in N_\Omega(\bar{x},u).$$

Moreover, since $\mathbb{R}^n$ and $\mathbb{R}^p_+$ are convex sets, $T_{\mathbb{R}^n}(\bar{x})=\mathbb{R}^n$ and $-\varphi'(\bar{x},u)\in T_{\mathbb{R}^p_+}(-\varphi(\bar{x}))$, we deduce from (2.1) that

$$w=0\;\text{ and }\;v\in\left\{v\in\mathbb{R}^p\;:\;0\le v\perp\varphi(\bar{x}),\;v\perp\varphi'(\bar{x},u)\right\}. \tag{3.3}$$

Because φ and ϕ are directionally Lipschitz and directionally differentiable at $\bar{x}$ in direction u, we obtain from [12, Proposition 5.1] that

$$D^*\chi\left(\bar{x},\left(u,\chi'(\bar{x},u)\right)\right)(v,\gamma,\zeta)=\partial\langle v,\varphi\rangle(\bar{x},u)+\partial\langle\gamma,\phi\rangle(\bar{x},u)-\zeta. \tag{3.4}$$

In conclusion, substituting relations (3.3) in (3.2) while considering (3.4), we arrive at

$$\begin{cases}0\in\partial\psi(\bar{x},u)+\partial\langle v,\varphi\rangle(\bar{x},u)+\partial\langle\gamma,\phi\rangle(\bar{x},u)+N_\Omega(\bar{x},u)\\ 0\le v\perp\varphi(\bar{x}),v\perp\varphi'(\bar{x},u).\end{cases}$$

Thus, the proof is complete. □

Next, we describe a special case where ψ, φ, and ϕ are directionally strictly differentiable with respect to critical directions.

Theorem 3.2. *Let $\bar{x}$ be a locally optimal solution to problem (NLP), and let $u\in\mathscr{C}_{\mathscr{F}}(\bar{x})$ be a critical direction at $\bar{x}$. Assume that ψ is Lipschitz continuous near $\bar{x}$. Moreover, assume that system $\mathscr{F}$ satisfies the directional (MSCQ) at $\bar{x}$ in the direction u and that mapping Ψ is directionally inner semicontinuous at $(\bar{x},0,\chi(\bar{x}),-\chi(\bar{x}))$ in the direction $(u,0)$. Then, there exist multipliers $v\in\mathbb{R}^p$ and $\gamma\in\mathbb{R}^q$ such that*

$$\begin{cases}0\in\nabla\psi(\bar{x},u)+\nabla\varphi(\bar{x},u)^\top v+\nabla\phi(\bar{x},u)^\top\gamma+N_\Omega(\bar{x},u)\\ 0\le v\perp\varphi(\bar{x}),v\perp\varphi'(\bar{x},u).\end{cases}$$

Proof. This is a consequence of Theorem 3.1 and is achieved by applying the sum principles of Theorem 2.1 to functions ψ, φ, and ϕ while using Lemma 2.2. □

Remark 3.1. According to [12, Corollary 5.3], if φ and ϕ are Lipschitz continuous at $\bar{x}$, then the results of Theorems 3.1 and 3.2 remain valid even without the assumption of directional inner semicontinuity of Ψ.

4. Directional Stationary Conditions for MPEC

In this section, we focus on directional stationarity conditions and associated results. It includes the basis for our investigation, Theorem 4.2.

For a given feasible vector $\bar{x}$ of (MPEC), we define the following index sets:

$$\begin{array}{rcl} I_g &=& \{i : g_i(\bar{x}) = 0\}, \\ \alpha &=& \{i : G_i(\bar{x}) = 0,\ H_i(\bar{x}) > 0\}, \\ \beta &=& \{i : G_i(\bar{x}) = 0,\ H_i(\bar{x}) = 0\}, \\ \gamma &=& \{i : G_i(\bar{x}) > 0,\ H_i(\bar{x}) = 0\}. \end{array}$$

Further, we define the linearized cone of (MPEC) by

$$\mathscr{L}_{MPEC}(\bar{x}) = \left\{ u \in \mathbb{R}^n \ : \begin{array}{l} \nabla g_i(\bar{x},u)^\top u \leq 0, i \in I_g \\ \nabla G_i(\bar{x},u)^\top u = 0, i \in \alpha, \quad \nabla H_i(\bar{x},u)^\top u = 0, i \in \gamma \\ \nabla G_i(\bar{x},u)^\top u \geq 0, i \in \beta, \quad \nabla H_i(\bar{x},u)^\top u \geq 0, i \in \beta \\ (\nabla G_i(\bar{x},u)^\top u) \cdot (\nabla H_i(\bar{x},u)^\top u) = 0, i \in \beta \\ \nabla h_i(\bar{x},u)^\top u = 0, i = 1 \cdots, q \end{array} \right\},$$

and the cone of critical directions by

$$\mathscr{C}_{MPEC}(\bar{x}) = \{u \in \mathscr{L}_{MPEC}(\bar{x}) \ : \ \nabla f(\bar{x},u)^\top u \leq 0\}.$$

Unlike standard non-linear programming, which has a single dual stationarity condition (the Karush-Kuhn-Tucker condition), the MPEC encompasses multiple stationarity concepts. Below, we study the directional version of these concepts and explore their interconnections.

4.1. **Directional W-stationary conditions.** Let us consider the following tightened formulation of the (MPEC) program under consideration.

$$\begin{cases} \min f(x) \\ \text{s.t } g(x) \leq 0,\ h(x) = 0, \\ \quad G_i(x) = 0,\ i \in \alpha, \quad H_i(x) = 0, i \in \gamma, \\ \quad G_i(x) = 0,\ H_i(x) = 0,\ i \in \beta. \end{cases} \tag{TMPEC}$$

Let $\mathscr{W}$ denote the feasible region of (TMPEC). For $\bar{x} \in \mathscr{W}$, the critical cone of (TMPEC) at $\bar{x}$ is defined by

$$\mathscr{C}_{\mathscr{W}}(\bar{x}) = \left\{ u \in \mathbb{R}^n \ : \begin{array}{ll} \nabla f(\bar{x},u)^\top u \leq 0 & \\ \nabla g_i(\bar{x},u)^\top u \leq 0, i \in I_g, & \nabla h(\bar{x},u)^\top u = 0 \\ \nabla G_i(\bar{x},u)^\top u = 0,\ i \in \alpha, & \nabla H_i(\bar{x},u)^\top u = 0,\ i \in \gamma \\ \nabla G_i(\bar{x},u)^\top u = 0,\ i \in \beta, & \nabla H_i(\bar{x},u)^\top u = 0,\ i \in \beta \end{array} \right\}.$$

We shall use the following kind of directional metric subregularity constraint qualification for MPEC. We say that the system $\mathscr{W}$ satisfies the directional metric subregularity holds at the feasible point $\bar{x} \in \mathscr{W}$ if there exist $\varepsilon, \rho, \sigma > 0$ such that

$$\operatorname{dist}(x, \mathscr{W}) \leq \sigma \left(\|(g(x))_+\| + \left\|G_\beta(x)\right\| + \left\|H_\beta(x)\right\| + \|h(x)\| + \|G_\alpha(x)\| + \left\|H_\gamma(x)\right\| \right),$$

for all $x \in \bar{x} + \mathscr{U}_{\varepsilon,\rho}(u)$.

In the following theorem, we establish new W-stationary conditions for a local minimum of (MPEC) under the directional metric subregularity condition.

Theorem 4.1. *Let $\bar{x}$ be a local optimal solution to problem (MPEC), and let $u \in \mathscr{C}_{\mathscr{W}}(\bar{x})$. Assume that the function f is Lipschitz continuous near $\bar{x}$ and directionally strictly differentiable at $\bar{x}$ in the direction u. Suppose further that the functions g, h, G_α, G_γ, H_β, and H_γ are all Lipschitz continuous and directionally strictly differentiable at $\bar{x}$ in the direction u. Moreover, assume that the feasible region $\mathscr{W}$ of (TMPEC) is metrically subregular at $\bar{x}$ in the direction u. Then, there exist multipliers $\lambda^g \in \mathbb{R}^p$, $\lambda^h \in \mathbb{R}^q$, $\lambda^G \in \mathbb{R}^m$, and $\lambda^H \in \mathbb{R}^m$ such that*

$$\begin{aligned} 0 \;=\; & \nabla f(\bar{x}, u) + \textstyle\sum_{i \in I_g} \lambda_i^g \nabla g_i(\bar{x}, u) + \sum_{i=1}^q \lambda_i^h \nabla h_i(\bar{x}, u) \\ & - \textstyle\sum_{i=1}^m \left(\lambda_i^G \nabla G_i(\bar{x}, u) + \lambda_i^H \nabla H_i(\bar{x}, u) \right), \end{aligned} \tag{4.1}$$

$$\lambda_{I_g}^g \geq 0, \quad \lambda^g \perp g'(\bar{x}, u), \quad \lambda_\gamma^G = 0, \quad \lambda_\alpha^H = 0. \tag{4.2}$$

Proof. The proof relies fundamentally on Theorem 3.2. Specifically, we set $\psi = f$, $\varphi = g$, $\phi = \left(h, G_{\alpha\cup\beta}, H_{\gamma\cup\beta}\right)$, and $\Omega = \mathbb{R}^n$. Under the assumptions of the current theorem and Ramark 3.1, the conclusions (4.1) and (4.2) follow directly from Theorem 3.2, while taking into account that $N_\Omega(\bar{x}, u) = N_{\mathbb{R}^n}(\bar{x}) \cap \{u\}^\perp = \{0\}$. □

Definition 4.1 (Directional W-stationary point)**.** We say that a feasible point $\bar{x}$ of problem (MPEC) is directionally weakly stationary (DW-stationary point) if, for all $u \in \mathscr{C}_{\mathscr{W}}(\bar{x})$, there exist multipliers $\lambda^g \in \mathbb{R}^p$, $\lambda^h \in \mathbb{R}^q$, $\lambda^G \in \mathbb{R}^m$, and $\lambda^H \in \mathbb{R}^m$ such that conditions (4.1) and (4.2) hold.

Example 4.1. We consider the (MPEC) with the following data: $f(x) = |x_1| + x_2^2$, $g(x) = x_2 - 1$, $h(x) = x_1 + x_2 = 0$, $G(x) = x_1$, $H(x) = |x_3^3|$, and $G(x)^\top H(x) = x_1 \cdot |x_3^3|$. The point $\bar{x} = (0,0,0)$ is a local optimal solution to problem (MPEC). One can verify that the direction $u = (0,0,2) \in \mathscr{C}_{\mathscr{W}}(\bar{x})$

- The function f is Lipschitz continuous near $\bar{x}$ and directionally strictly differentiable at $\bar{x}$ in the direction u.
- The functions g, h, G_α, G_γ, H_β, and H_γ are all continuous and directionally strictly differentiable at $\bar{x}$ in the direction u, with $\beta = \{1\}$, and $\gamma = \alpha = \emptyset$.
- The feasible region $\mathscr{W}$ of (TMPEC) is metrically subregular at $\bar{x}$ in the direction u, with $\varepsilon = \dfrac{1}{2}$, $\rho = 2$ and $\sigma = 3$.

Then, for the multipliers $\lambda^g = 3$, $\lambda^h = 0$, $\lambda^G = 0$, and $\lambda^H = 0$, conditions (4.1) and (4.2) are satisfied.

Note that the conditions (4.1) and (4.2) correspond to the (DKKT) conditions for the tightened problem (TMPEC) at $\bar{x}$, which generally has a locally smaller feasible set than (MPEC).

In our result, we adopt a weaker constraint qualification, which acts as the directional counterpart of the well-known constraint qualification. Additionally, we derive a directional stationary condition that is generally sharper than its non-directional counterpart.

4.2. **Directional M-stationary conditions.** In this subsection, for (MPEC), including strictly directionally differentiable functions, we prove a new necessary optimality condition. There are several approaches to reformulate the problem, but in this case, we offer an easier option using slack variables. That is,

$$\begin{cases} \min\limits_{x,y,z} f(x) \\ \text{s.t.} \;\; g(x) \le 0,\; h(x)=0, \\ \qquad G(x)-y=0,\; H(x)-z=0, \\ \qquad (y,z)\in\Omega, \end{cases} \tag{QMPEC}$$

where Ω is the vector complementarity set version of $\mathscr{X}$ in (2.2) which is giving by

$$\Omega = \left\{ (y,z)\in\mathbb{R}^{2m} \;:\; (y_i,z_i)\in\mathscr{X} \;\text{ for all }\; i=1,\cdots,m \right\} = \mathscr{X}^m.$$

In what follows, we denote by $\mathscr{M}$ the set of feasible points of (QMPEC). The critical cone of (QMPEC) at $(\bar{x},\bar{y},\bar{z})\in\mathscr{M}$ is defined by

$$\mathscr{C}_{\mathscr{M}}(\bar{x},\bar{y},\bar{z}) = \left\{ (u,v,w)\in\mathbb{R}^{n+2m} \;:\; \begin{array}{l} (v,w)\in T_{\Omega}(\bar{y},\bar{z}),\;\; \nabla f(\bar{x},u)^\top u\le 0 \\ \nabla g_i(\bar{x},u)^\top u\le 0,\; i\in I_g \\ \nabla h_i(\bar{x},u)^\top u=0,\; i=1\cdots,q \\ \nabla G_i(\bar{x},u)^\top u - v_i=0,\; i=1\cdots,m \\ \nabla H_i(\bar{x},u)^\top u - w_i=0, i=1\cdots,m \end{array} \right\},$$

and let us introduce

$$\begin{array}{rcl} \alpha^+(u) &=& \left\{ i\in\beta \;:\; \nabla G_i(\bar{x},u)^\top u=0,\; \nabla H_i(\bar{x},u)^\top u>0 \right\}, \\ \beta^+(u) &=& \left\{ i\in\beta \;:\; \nabla G_i(\bar{x},u)^\top u=0,\; \nabla H_i(\bar{x},u)^\top u=0 \right\}, \\ \gamma^+(u) &=& \left\{ i\in\beta \;:\; \nabla G_i(\bar{x},u)^\top u>0,\; \nabla H_i(\bar{x},u)^\top u=0 \right\}, \end{array}$$

the so-called directional index sets.

Theorem 4.2. *Let $\bar{x}$ be a local optimal solution for* (MPEC)*, and let $(u,v,w)\in\mathscr{C}_{\mathscr{M}}(\bar{x},\bar{y},\bar{z})$, with $\bar{y}=G(\bar{x})$ and $\bar{z}=H(\bar{x})$. Assume that the function f is Lipschitz continuous near $\bar{x}$ and directionally strictly differentiable at $\bar{x}$ in the direction u. Suppose further that the functions g, h, G, and H are all Lipschitz continuous and directionally strictly differentiable at $\bar{x}$ in the direction u. Moreover, suppose that $\mathscr{M}$ is metrically subregular at $(\bar{x},\bar{y},\bar{z})$ in direction (u,v,w). Then, there exist $\lambda^g\in\mathbb{R}^p$, $\lambda^h\in\mathbb{R}^q$, $\lambda^G\in\mathbb{R}^m$, and $\lambda^H\in\mathbb{R}^m$ such that* (4.1), (4.2)*, and the following conditions hold:*

$$\lambda^G_{\gamma^+(u)}=0 \text{ and } \lambda^H_{\alpha^+(u)}=0, \tag{4.3}$$

$$\lambda_i^G>0, \lambda_i^H>0 \text{ or } \lambda_i^G\lambda_i^H=0 \text{ for all } i\in\beta^+(u). \tag{4.4}$$

Proof. Since $\bar{x}$ is a local optimal solution to (MPEC), then $(\bar{x},\bar{y},\bar{z})$ is a local solution to (QMPEC). Because $\mathscr{M}$ is a directional metric subregular, we have from Theorem 3.2 and Remark 3.1 that there exist $(\lambda^g,\lambda^h,\lambda^G,\lambda^H)\in\mathbb{R}^{p+q+2m}$ and $(\tau,\xi)\in N_\Omega((\bar{y},\bar{z}),(v,w))$ such that

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}\nabla f(\bar{x},u)\\0\\0\end{bmatrix} + \textstyle\sum_{i\in I_g}\lambda_i^g \begin{bmatrix}\nabla g_i(\bar{x},u)\\0\\0\end{bmatrix} + \textstyle\sum_{i=1}^q \lambda_i^h \begin{bmatrix}\nabla h_i(\bar{x},u)\\0\\0\end{bmatrix}$$

$$-\textstyle\sum_{i=1}^m \lambda_i^G \begin{bmatrix}\nabla G_i(\bar{x},u)\\-e_i\\0\end{bmatrix} - \textstyle\sum_{i=1}^m \lambda_i^H \begin{bmatrix}\nabla H_i(\bar{x},u)\\0\\-e_i\end{bmatrix} + \begin{bmatrix}0\\\tau\\\xi\end{bmatrix},$$

with $\lambda_i^g\geq 0$ for all $i\in I_g$ and $\lambda^g\perp g'(\bar{x},u)$, where e_i denotes the unit vector whose ith component is equal to 1. It consequently follows that

$$\begin{aligned}
&0=\nabla f(\bar{x},u)+\sum_{i\in I_g}\lambda_i^g\nabla g_i(\bar{x},u)+\sum_{i=1}^q\lambda_i^h\nabla h_i(\bar{x},u)-\sum_{i=1}^m\lambda_i^G\nabla G_i(\bar{x},u)-\sum_{i=1}^m\lambda_i^H\nabla H_i(\bar{x},u),\\
&\lambda_i^g\geq 0,\ \text{for all } i\in I_g,\ \lambda^g\perp g'(\bar{x},u),\\
&\tau+\lambda^G=0,\ \xi+\lambda^H=0.
\end{aligned}$$

First, note that $\Omega=\mathscr{X}^m$ applying [17, Proposition 3.3], the directioanl normal cone $N_\Omega((\bar{y},\bar{z}),(v,w))$ can be expressed as

$$N_\Omega((\bar{y},\bar{z}),(v,w))=\prod_{i=1}^m N_{\mathscr{X}}\left((\bar{y}_i,\bar{z}_i),(v_i,w_i)\right).$$

Secondly, because $(\tau,\xi)\in N_\Omega((\bar{y},\bar{z}),(v,w))$ and $(\bar{y},\bar{z})=(G(\bar{x}),H(\bar{x}))$, three cases should be considered.

- **•:** Suppose that $i\in\beta$. Then, $(\bar{y}_i,\bar{z}_i)=(0,0)$. By Proposition 2.1-($iii$), we get $(\tau_i,\xi_i)\in N_{\mathscr{X}}(v_i,w_i)$. From the following characterization of non-directional Mordukhovich normal cone

$$N_\Omega(v,w)=\left\{(\tau,\xi):\begin{array}{l}\tau_i=0\quad\text{if } v_i>0\\ \xi_i=0\quad\text{if } w_i>0\\ \text{either } \tau_i<0,\xi_i<0 \text{ or } \tau_i\xi_i=0 \text{ if } v_i=w_i=0\end{array}\right\}.$$

  $(u,v,w)\in\mathscr{C}_{\mathscr{M}}(\bar{x},\bar{y},\bar{z})$, we have that $v_i=\nabla G_i(\bar{x},u)^\top u$ and $w_i=\nabla H_i(\bar{x},u)^\top u$, for all $i=1\ldots,m$. Hence, the conditions $\tau+\lambda^G=0$ and $\xi+\lambda^H=0$ become $\lambda^G_{\gamma^+(u)}=0$, $\lambda^H_{\alpha^+(u)}=0$ and $\lambda_i^G>0,\lambda_i^H>0$ or $\lambda_i^G\lambda_i^H=0$ for all $i\in\beta^+(u)$.
- **•:** If $i\in\alpha$, one has $(\bar{y}_i,\bar{z}_i)\neq(0,0)$, since $\bar{z}_i>0$. Then, again from Proposition 2.1-(i) we obtain $(\tau_i,\xi_i)\in\mathbb{R}^m\times\{0\}$. It follows that $\lambda_\alpha^H=0$.
- **•:** Assume that $i\in\gamma$. Then $(\bar{y}_i,\bar{z}_i)\neq(0,0)$, since $\bar{y}_i>0$. Using the same argument as above, we get $\lambda_\gamma^G=0$.

Now putting all the above together, we arrive at the result. □

Example 4.2. We consider the (MPEC) with the following data: $f(x)=\sqrt{x^4+4}-x^2$, $g(x)=x+3$, $h(x)=(x+4)^2$, $G(x)=(x+4)^2$, and $H(x)=|x+4|^3$. The point $\bar{x}=-4$ is a local optimal solution to problem (MPEC). One can verify that the direction $(u,v,w)=(1,0,0)\in\mathscr{C}_{\mathscr{M}}(\bar{x},\bar{y},\bar{z})$ with $\bar{y}=\bar{z}=0$.

- **•:** f is Lipschitz continuous near $\bar{x}$ and directionally strictly differentiable at $\bar{x}$ in the direction u.

- **•:** g, h, G, and H are all continuous and directionally strictly differentiable at $\bar{x}$ in the direction u.
- **•:** $\mathscr{M}$ is metrically subregular at $(\bar{x},\bar{y},\bar{z})$ in direction (u,v,w),

For $\lambda^g = 0$, $\lambda^h = 3$, $\lambda^G = 1$, and $\lambda^H = 2$, the directional optimality conditions (4.1), (4.2), (4.3), and (4.4) are satisfied.

Remark 4.1. Clearly, if u is a critical direction of (MPEC), then $(u, \nabla G(\bar{x},u)^\top u, \nabla H(\bar{x},u)^\top u)$ is a critical direction of the equivalent problem, (QMPEC). Indeed, this is an easy consequence of the structure of the tangent cone $T_\Omega(G(\bar{x}),H(\bar{x}))$ in Proposition 2.1. Furthermore, $\{\alpha^+(u), \beta^+(u), \gamma^+(u)\}$ is a partition of the index set β.

Definition 4.2 (Directional M-stationary point)**.** We say that a feasible point $\bar{x}$ of the problem of (MPEC) is a directional M-stationary point (DM-stationary point) if, for all $u \in \mathscr{C}_{MPEC}(\bar{x})$, there exist multipliers $\lambda^g \in \mathbb{R}^p$, $\lambda^h \in \mathbb{R}^q$, $\lambda^G \in \mathbb{R}^m$, and $\lambda^H \in \mathbb{R}^m$ such that conditions (4.1), (4.2), (4.3), and (4.4) hold.

Recall that, for a smooth MPEC program, the no non-zero abnormal multiplier constraint qualification (NNAMCQ) is a natural constraint qualification. In the same way, we propose a directional version of (NNAMCQ) for (MPEC).

Definition 4.3. Let $\bar{x}$ be a feasible point in (MPEC) where all functions are directionally strictly differentiable in direction u. We state that the (NNAMCQ) is satisfied at $\bar{x}$ in direction u if there are no non-zero vectors $\lambda^g \in \mathbb{R}^p$, $\lambda^h \in \mathbb{R}^q$, $\lambda^G \in \mathbb{R}^m$, and $\lambda^H \in \mathbb{R}^m$ such that conditions (4.1), (4.2), (4.3), and (4.4) hold.

We next demonstrate that the directional (NNAMCQ) serves as a sufficient condition for the existence of a metrically subregular, utilizing a recent and advanced approach in non-directional generalized differentiation.

Proposition 4.1. *Let $\bar{x}$ be a feasible point for (MPEC), and assume that the functions g, h, G, and H are continuous, directionally Lipschitz, and directionally differentiable at $\bar{x}$ in direction u. Suppose that the (NNAMCQ) is satisfied at $\bar{x}$ in direction $u \in \mathscr{L}_{MPEC}(\bar{x})$. Then, system $\mathscr{M}$ is metrically subregular at $\bar{x}$ in direction u.*

Proof. Consider a auxiliary function $\Upsilon : \mathbb{R}^n \to \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^{2m}$, defined

$$\Upsilon(x) = (g(x), h(x), -G(x), -H(x)).$$

We claim that the (NNAMCQ) is satisfied at $\bar{x}$ in a critical direction u, which signifies that there exists no $\varsigma \neq 0$ such that

$$0 \in D^*\Upsilon(\bar{x},(u,\Upsilon'(\bar{x},u)))(\varsigma) = \partial\langle\varsigma,\Upsilon\rangle(\bar{x},u), \quad \varsigma \in N_{\mathbb{R}^p_- \times \{0_q\} \times \Omega}(\Upsilon(\bar{x}),\Upsilon'(\bar{x},u)),$$

with $\Upsilon'(\bar{x},u) \in T_{\mathbb{R}^p_- \times \{0_q\} \times \Omega}(\Upsilon(\bar{x}))$ and $\Omega = \mathscr{X}^m$.

First, in view of

$$\Upsilon'(\bar{x},u) = (g'(\bar{x},u), h'(\bar{x},u), -G'(\bar{x},u), -H'(\bar{x},u)),$$

one gets

$$\begin{aligned} T_{\mathbb{R}^p_- \times \{0_q\} \times \Omega}(\Upsilon(\bar{x})) &= T_{\mathbb{R}^p_-}(g(\bar{x})) \times T_{\{0_q\}}(h(\bar{x})) \times T_\Omega((-G(\bar{x}), -H(\bar{x}))) \\ &= T_{\mathbb{R}^p_-}(g(\bar{x})) \times T_{\{0_q\}}(h(\bar{x})) \times \prod_{i=1}^m T_{\mathscr{X}}(-G_i(\bar{x}), -H_i(\bar{x})). \end{aligned}$$

Since $T_{\mathbb{R}^p_-}(g(\bar{x})) = \prod_{i=1}^p T_{\mathbb{R}_-}(g_i(\bar{x}))$, with $T_{\mathbb{R}_-}(g_i(\bar{x})) = \mathbb{R}_-$ if $i \in I_g$ and $T_{\mathbb{R}_-}(g_i(\bar{x})) = \mathbb{R}$ if $i \notin I_g$, we have $g'(\bar{x},u) \in T_{\mathbb{R}^p_-}(g(\bar{x}))$ for any $u \in \mathscr{L}_{MPEC}(\bar{x})$. Since, $h(\bar{x}) = 0$, we have $T_{\{0_q\}}(h(\bar{x})) = \mathbb{R}^q$ and we obtain $h'(\bar{x},u) \in T_{\{0_q\}}(h(\bar{x}))$. Now, using the expression of the tangent cone given in Proposition 2.1, one can show that $(-G_i'(\bar{x},u), -H_i'(\bar{x},u)) \in T_{\mathscr{X}}(-G_i(\bar{x}), -H_i(\bar{x}))$ for all $i = 1, \cdots, m$. Putting all the above together, we arrive at $\Upsilon'(\bar{x},u) \in T_{\mathbb{R}^p_- \times \{0_q\} \times \Omega}(\Upsilon(\bar{x}))$.

Secondly, by exploiting [17, Proposition 3.3], we get

$$\begin{aligned} N_{\mathbb{R}^p_- \times \{0_q\} \times \Omega}(\Upsilon(\bar{x}), \Upsilon'(\bar{x},u)) &= N_{\mathbb{R}^p_-}(g(\bar{x}), g'(\bar{x},u)) \times N_{\{0_q\}}(h(\bar{x}), h'(\bar{x},u)) \\ &\times N_{\Omega}((-G(\bar{x}), -H(\bar{x})), (-G'(\bar{x},u), -H'(\bar{x},u))). \end{aligned}$$

From equality (2.1) and $u \in \mathscr{L}_{MPEC}(\bar{x})$, we have

$$\begin{aligned} N_{\mathbb{R}^p_-}(g(\bar{x}), g'(\bar{x},u)) &= \{\lambda^g \in \mathbb{R}^p \ : \ \lambda_i^g \geq 0, \text{ for all } i \in I_g,\ \lambda^g \perp g'(\bar{x},u)\}, \\ N_{\{0_q\}}(h(\bar{x}), h'(\bar{x},u)) &= \mathbb{R}^q. \end{aligned}$$

Applying again the equality (2.1) together with Proposition 2.1, we have

$$N_{\Omega}\left((-G(\bar{x}), -H(\bar{x})), \left(-G'(\bar{x},u), -H'(\bar{x},u)\right)\right) = \left\{ (\lambda^G, \lambda^H) \in \mathbb{R}^{2m} \ : \begin{array}{l} \lambda^G_{\gamma^+(u)\cup\gamma} = 0, \lambda^H_{\alpha^+(u)\cup\alpha} = 0 \\ \left(\lambda_i^G > 0, \lambda_i^H > 0 \text{ or } \lambda_i^G \lambda_i^H = 0\right) \text{ for all } i \in \beta^+(u) \end{array} \right\}.$$

Moreover, since Υ is directionally Lipschitz and directionally differentiable at $\bar{x}$ in direction u, from Proposition 2.2 and [12, Proposition 5.1], we conclude that $D^*\Upsilon(\bar{x},(u,\Upsilon'(\bar{x},u)))(\lambda) = \partial\langle\lambda,\Upsilon\rangle(\bar{x},u)$.

Consequently, our claim is now obtained by combining the aforementioned arguments.

Furthermore, it follows directly from [12, Proposition 2.2] that (NNAMCQ) is satisfied at $\bar{x}$ in a critical direction u, which implies that the set-valued map $F(x) = \mathbb{R}^p \times \{0_q\} \times \Omega - \Upsilon(x)$ is metrically subregular at $(\bar{x},0)$ in direction u. Or equivalently, system $\mathscr{M}$ is metrically subregular at $\bar{x}$ in direction u. This completes the proof. ☐

4.3. **Directional C-stationary conditions.** The non-directional Clarke stationary conditions for (MPEC) were introduced by Scheel and Scholtes [8]. This outcome is derived from Clarke's (1976) stationarity condition applicable to programs involving locally Lipschitz functions.

The directional Clarke subdifferential is a more refined version of the Clarke subdifferential, making the directional stationary condition more precise than its non-directional counterpart.

This section is dedicated to formulating a directional Clarke-stationary condition for problem (MPEC). To this end, we shall focus attention on an equivalent non-smooth formulation of the complementarity constraints as

$$\begin{cases} \min\limits_x f(x) \\ \text{s.t. } \ g(x) \leq 0,\ h(x) = 0, \\ \qquad G_i(x) = 0,\ i \in \alpha,\ \ H_i(x) = 0,\ i \in \gamma, \\ \qquad \theta_i(x) = 0,\ i \in \beta, \end{cases} \tag{CMPEC}$$

where, for each $i \in \beta$, $\theta_i(x) = \min\{\theta_{i_1}(x), \theta_{i_2}(x)\}$, with $\theta_{i_1}(x) = G_i(x)$ and $\theta_{i_2}(x) = H_i(x)$. Let $\bar{x}, u \in \mathbb{R}^n$ and $v \in \mathbb{R}$, consider the sets

$$\begin{aligned} I^i(\bar{x}) &= \left\{ j \in \{1,2\} \; : \; \theta_i(\bar{x}) = \theta_{i_j}(\bar{x}) \right\}, \\ I^i(\bar{x},(u,v)) &= \left\{ j \in \{1,2\} \; : \; \theta_i(\bar{x}) = \theta_{i_j}(\bar{x}) \;\; \text{and} \;\; v = \theta_{i_j}'(\bar{x},u) \right\}, \\ I_0^i(\bar{x},(u,v)) &= \left\{ j \in I^i(\bar{x},(u,v)) \; : \; \partial\theta_{i_j}(\bar{x},u) \neq \emptyset \right\}, \end{aligned}$$

and let $\mathscr{C}$ denote the feasible set of (CMPEC). For $\bar{x} \in \mathscr{C}$, the critical cone of (CMPEC) at $\bar{x}$ is defined by

$$\mathscr{C}_{\mathscr{C}}(\bar{x}) = \left\{ u \in \mathbb{R}^n \; : \; \begin{array}{ll} \nabla f(\bar{x},u)^\top u \le 0, & \\ \nabla g_i(\bar{x},u)^\top u \le 0, i \in I_g, & \nabla h(\bar{x},u)^\top u = 0 \\ \nabla G_i(\bar{x},u)^\top u = 0, \; i \in \alpha, & \nabla H_i(\bar{x},u)^\top u = 0, \; i \in \gamma \\ \theta_i^\circ(\bar{x},u) \le 0, \; i \in \beta & \end{array} \right\},$$

where $\theta_i^\circ(\bar{x},u)$ is the Clarke generalized directional derivative, which is defined by:

$$\theta_i^\circ(\bar{x},u) = \limsup_{t \to 0^+, y \to \bar{x}} \frac{\theta_i(y+tu) - \theta_i(y)}{t}.$$

Based on Theorem 3.1 and chain rules of directional limiting subdifferentials, we construct an original directional Clarke stationary condition for (MPEC). In order to arrive at the key result, we need the following lemma about the formulas for computing directional derivatives of composition.

Lemma 4.1. *Let $F = (f_1, \ldots, f_s)$ such that $f_k : \mathbb{R}^n \to \mathbb{R}$ as a directional derivative at $\bar{x}$ in direction u, $k = 1, \ldots, s$; and let $g : \mathbb{R}^s \to \mathbb{R}$ be a given functional that has a directional derivative at $F(\bar{x})$ in direction $F'(\bar{x},u)$. Assume that g is locally Lipschitz continuous. Then, $g \circ F$ is directionally differentiable at $\bar{x}$ in direction u and $(g \circ F)'(\bar{x},u) = g'(F(\bar{x}), F'(\bar{x},u))$.*

Proof. First, since F is directionally differentiable at $\bar{x}$ in direction u, then $F(\bar{x}+tu) = F(\bar{x}) + tF'(\bar{x},u) + o(t)$. Hence,

$$\begin{aligned} g(F(\bar{x}+tu)) &= g(F(\bar{x}) + tF'(\bar{x},u) + o(t)) \\ &\overset{(1)}{=} g(F(\bar{x}) + tF'(\bar{x},u)) + o(t) \\ &\overset{(2)}{=} g(F(\bar{x})) + tg'(F(\bar{x}), F'(\bar{x},u)) + o(t), \end{aligned}$$

where (1) results from the Lipschitz property of g and (2) corresponds to the definition of the directional derivative of g. Then, it follows immediately

$$(g \circ F)'(\bar{x},u) = \lim_{t \to 0} \frac{g(F(\bar{x}+tu)) - g(F(\bar{x}))}{t} = g'(F(\bar{x}), F'(\bar{x},u)).$$

This completes the proof. □

Theorem 4.3. *Let $\bar{x}$ be a local optimal solution for (MPEC), and let $u \in \mathscr{C}_{\mathscr{C}}(\bar{x})$. Assume that the function f is Lipschitz continuous near $\bar{x}$ and directionally strictly differentiable at $\bar{x}$ in the direction u. Suppose further that the functions g, h, G_α, H_γ, and θ_β are all Lipschitz continuous and directionally strictly differentiable at $\bar{x}$ in the direction u. Moreover, suppose that $\mathscr{C}$ is* *metrically subregular at $\bar{x}$ in direction u. Then, there exist $\lambda^g \in \mathbb{R}^p$, $\lambda^h \in \mathbb{R}^q$, $\lambda^G \in \mathbb{R}^m$, and $\lambda^H \in \mathbb{R}^m$ such that (4.1) and (4.2) and the following condition hold:*

$$\lambda^G_\beta \lambda^H_\beta \geq 0. \tag{4.5}$$

Proof. Fix $i \in \beta$ and define the functions $P_i : \mathbb{R}^2 \to \mathbb{R}$ and $E_i : \mathbb{R}^n \to \mathbb{R}^2$ via

$$P_i(x_1, x_2) = \min\{x_1, x_2\}, \quad E_i(x) = (\theta_{i_1}(x), \theta_{i_2}(x)).$$

Note that $(P_i \circ E_i)(x) = \theta_i(x)$. Since P_i is a minimum of linear functions, then it is concave and locally Lipschitz continuous. Hence, we see from Lemma 4.1 that $\theta_i'(\bar{x}, u) = \min_{j \in I^i(\bar{x})} \theta_{i_j}'(\bar{x}, u)$. By Theorem 3.1 and Remark 3.1, there exist $\lambda^g \in \mathbb{R}^p$, $\lambda^h \in \mathbb{R}^q$, $\lambda^G \in \mathbb{R}^{|\alpha|}$, $\lambda^H \in \mathbb{R}^{|\gamma|}$, and $\lambda^\theta \in \mathbb{R}^{|\beta|}$ such that $0 \leq \lambda^g \perp g(\bar{x})$, $\lambda^g \perp g'(\bar{x}, u)$ and

$$\begin{aligned} 0 \in\ & \nabla f(\bar{x}, u) + \sum_{i \in I_g} \lambda_i^g \nabla g_i(\bar{x}, u) + \sum_{i=1}^q \lambda_i \nabla h_i(\bar{x}, u) + \sum_{i \in \alpha} \lambda_i^G \nabla G_i(\bar{x}, u) \\ & + \sum_{i \in \gamma} \lambda_i^H \nabla H_i(\bar{x}, u) + \sum_{i \in \beta} \lambda_i^\theta \partial \theta_i(\bar{x}, u). \end{aligned} \tag{4.6}$$

Now, given a direction $v \in \mathbb{R}$, applying [12, Proposition 4.4] to the geometric limiting subdiferential of θ_i, we get

$$\tilde{\partial} \theta_i(\bar{x}, (u, v)) \subset \bigcup_{j \in I_0^i(\bar{x}, (u, v))} \tilde{\partial} \theta_{i_j}(\bar{x}, (u, v)), \quad \text{for all } i \in \beta. \tag{4.7}$$

Choosing $v = \theta_i'(\bar{x}, u)$ and exploiting Proposition 2.2, relationship (4.7) becomes

$$\partial \theta_i(\bar{x}, u) \subset \bigcup_{j \in I^i(\bar{x})} \partial \theta_{i_j}(\bar{x}, u), \quad \text{for all } i \in \beta.$$

Passing to the clarke directional subdiferential, we then have

$$\partial \theta_i(\bar{x}, u) \subset \partial_C \theta_i(\bar{x}, u) \subset \operatorname{conv}\left(\bigcup_{j \in I^i(\bar{x})} \partial \theta_{i_j}(\bar{x}, u) \right), \quad \text{for all } i \in \beta.$$

Applying Carathéodory's theorem while taking into account that all the functions are directionally strictly differentiable gives us that, for any $\xi_i \in \partial_C \theta_i(\bar{x}, u)$, there exist $(\lambda_i^G, \lambda_i^H) \in \mathbb{R}^2_+$ with $\lambda_i^G + \lambda_i^H = 1$ such that $\xi_i = \lambda_i^G \nabla G_i(\bar{x}, u) + \lambda_i^H \nabla H_i(\bar{x}, u))$. Substituting these into (4.6) yields

$$\begin{aligned} 0 \in\ & \nabla f(\bar{x}, u) + \sum_{i \in I_g} \lambda_i^g \nabla g_i(\bar{x}, u) + \sum_{i=1}^q \lambda_i \nabla h_i(\bar{x}, u) + \sum_{i \in \alpha} \lambda_i^G \nabla G_i(\bar{x}, u) \\ & + \sum_{i \in \gamma} \lambda_i^H \nabla H_i(\bar{x}, u) + \textstyle\sum_{i \in \beta} \lambda_i^\theta \left(\lambda_i^G \nabla G_i(\bar{x}, u) + \lambda_i^H \nabla H_i(\bar{x}, u) \right). \end{aligned}$$

Setting $\lambda^G_\beta = \lambda^\theta_\beta \lambda^{G,\beta}$ and $\lambda^H_\beta = \lambda^\theta_\beta \lambda^{G,\beta}$, we see that λ^G_β and λ^H_β have the same sign as λ^θ_β and

$$\begin{aligned} 0 \in\ & \nabla f(\bar{x}, u) + \sum_{i \in I_g} \lambda_i^g \nabla g_i(\bar{x}, u) + \sum_{i=1}^q \lambda_i \nabla h_i(\bar{x}, u) + \textstyle\sum_{i \in \alpha} \lambda_i^G \nabla G_i(\bar{x}, u) \\ & + \textstyle\sum_{i \in \gamma} \lambda_i^H \nabla H_i(\bar{x}, u) + \displaystyle\sum_{i \in \beta} \left(\lambda_i^G \nabla G_i(\bar{x}, u) + \lambda_i^H \nabla H_i(\bar{x}, u) \right). \end{aligned}$$

Hence, it is not difficult to see that the last expression becomes

$$\begin{aligned} 0 \ \in \ & \nabla f(\bar{x},u) + \sum_{i \in I_g} \lambda_i^g \nabla g_i(\bar{x},u) + \sum_{i=1}^{q} \lambda_i \nabla h_i(\bar{x},u) \\ & + \sum_{i=1}^{m} \left(\lambda_i^G \nabla G_i(\bar{x},u) + \lambda_i^H \nabla H_i(\bar{x},u) \right), \end{aligned}$$

with $\lambda_\gamma^G = 0$, $\lambda_\alpha^H = 0$ and $\lambda_\beta^G \lambda_\beta^H \geq 0$ which proves the result. □

Definition 4.4 (Directional C-stationary point)**.** We say that a feasible point $\bar{x}$ of (MPEC) is a directional C-stationary point (DC-stationary point) if, for all $u \in \mathscr{C}_{\mathscr{C}}(\bar{x})$, there exist multipliers $\lambda^g \in \mathbb{R}^p$, $\lambda^h \in \mathbb{R}^q$, $\lambda^G \in \mathbb{R}^m$, and $\lambda^H \in \mathbb{R}^m$ such that conditions (4.1), (4.2), and (4.5) hold.

4.4. **Directional S-stationary conditions.** In this subsection, we consider the following relaxed formulation of the initial (MPEC) program

$$\begin{cases} \min\limits_{x} f(x) \\ \text{s.t.} \ \ g(x) \leq 0, \ h(x) = 0 \\ \qquad G_i(x) = 0, \ i \in \alpha, \ \ H_i(x) = 0, \ i \in \gamma \\ \qquad G_i(x) \geq 0, \ H_i(x) \geq 0, i \in \beta. \end{cases} \tag{RMPEC}$$

Let $\mathscr{S}$ denote the feasible region of (RMPEC). For $\bar{x} \in \mathscr{S}$, the critical cone at $\bar{x}$ is defined by

$$\mathscr{C}_{\mathscr{S}}(\bar{x}) = \left\{ u \in \mathbb{R}^n \ : \begin{array}{ll} \nabla f(\bar{x},u)^\top u \leq 0 & \\ \nabla g_i(\bar{x},u)^\top u \leq 0, i \in I_g, & \nabla h(\bar{x},u)^\top u = 0 \\ \nabla G_i(\bar{x},u)^\top u = 0, \ i \in \alpha, & \nabla H_i(\bar{x},u)^\top u = 0, \ i \in \gamma \\ \nabla G_i(\bar{x},u)^\top u \geq 0, \ i \in \beta, & \nabla H_i(\bar{x},u)^\top u \geq 0, \ i \in \beta \end{array} \right\}.$$

We shall use the following kind of directional metric subregularity constraint qualification for (MPEC). We say that system $\mathscr{S}$ satisfies the directional metric subregularity holds at the feasible point $\bar{x} \in \mathscr{S}$ if $\varepsilon, \rho, \sigma > 0$ such that

$$\operatorname{dist}(x,\mathscr{S}) \leq \sigma \left(\|(g(x))_+\| + \left\|(-G_\beta(x))_+\right\| + \left\|(-H_\beta(x))_+\right\| + \|h(x)\| + \|G_\alpha(x)\| + \left\|H_\gamma(x)\right\| \right),$$

for all $x \in \bar{x} + \mathscr{U}_{\varepsilon,\rho}(u)$.

In the following theorem, we establish new DS-stationary conditions for a local minimum of (MPEC) under the directional metric subregularity condition.

Theorem 4.4. *Let $\bar{x}$ be a local optimal solution for* (MPEC)*, and let $u \in \mathscr{C}_{\mathscr{S}}(\bar{x})$. Assume that the function f is Lipschitz continuous near $\bar{x}$ and directionally strictly differentiable at $\bar{x}$ in the direction u. Suppose that the functions g, h, G_α, G_β, H_β, and H_γ are all Lipschitz continuous and directionally strictly differentiable at $\bar{x}$ in direction u. Moreover, suppose that $\mathscr{S}$ is metrically subregular at $\bar{x}$ in direction u. Then, there exist $\lambda^g \in \mathbb{R}^p$, $\lambda^h \in \mathbb{R}^q$, $\lambda^G \in \mathbb{R}^m$, and $\lambda^H \in \mathbb{R}^m$ such that* (4.1) *and* (4.2) *and the following conditions hold:*

$$\lambda_\beta^G \perp G'_\beta(\bar{x},u), \quad \lambda_\beta^H \perp H'_\beta(\bar{x},u), \tag{4.8}$$

$$\lambda_i^G \geq 0, \ \lambda_i^H \geq 0, \ \textit{for all } i \in \beta. \tag{4.9}$$

Proof. The proof follows essentially from Theorem 3.2. Let us define $\psi(x) = f(x)$, $\varphi(x) = \big(g(x), -G_\beta(x), -H_\beta(x)\big)$, and $\phi(x) = \big(h(x), G_\alpha(x), H_\gamma(x)\big)$. Under the given assumptions, the functions ψ, φ, and ϕ are directionally strictly differentiable at $\bar{x}$ in the direction u. Moreover, set $\Omega = \mathbb{R}^n$. Since $u \in \mathscr{C}_{\mathscr{S}}(\bar{x})$ and the system $\mathscr{S}$ is metrically subregular at $\bar{x}$ in the direction u, we can apply Theorem 3.2 and Remark 3.1 to obtain the relations (4.1), (4.2), (4.8), and (4.9), while taking into account that $N_\Omega(\bar{x}, u) = N_{\mathbb{R}^n}(\bar{x}) \cap \{u\}^\perp = \{0\}$. □

Definition 4.5 (Directional S-stationary point)**.** We say that a feasible point $\bar{x}$ of (MPEC) is a directional S-stationary point (DS-stationary point) if, for all $u \in \mathscr{C}_{\mathscr{S}}(\bar{x})$, there exist multipliers $\lambda^g \in \mathbb{R}^p$, $\lambda^h \in \mathbb{R}^q$, $\lambda^G \in \mathbb{R}^m$, and $\lambda^H \in \mathbb{R}^m$ such that conditions (4.1), (4.2), (4.8), and (4.9) hold.

The Figure 1 that follows provides an overview of the connections that exist between the dual stationary notions that we have been discussing:

$$
\begin{array}{ccc}
\text{Directional S-stationary point} & \Rightarrow & \text{S-stationary point} \\
\oplus\, u \in \mathscr{C}_{\mathscr{M}}(\bar{x}) & & \\
\Downarrow & & \Downarrow \\
\text{Directional M-stationary point} & \Rightarrow & \text{M-stationary point} \\
\oplus\, u \in \mathscr{C}_{\mathscr{C}}(\bar{x}) & & \\
\Downarrow & & \Downarrow \\
\text{Directional C-stationary point} & \Rightarrow & \text{C-stationary point} \\
\oplus\, u \in \mathscr{C}_{\mathscr{W}}(\bar{x}) & & \\
\Downarrow & & \Downarrow \\
\text{Directional W-stationary point} & \Rightarrow & \text{W-stationary point}
\end{array}
$$

FIGURE 1. *Since, $\mathscr{C}_{\mathscr{W}}(\bar{x}) \subset \mathscr{C}_{\mathscr{C}}(\bar{x}) \subset \mathscr{C}_{\mathscr{M}}(\bar{x}) \subset \mathscr{C}_{\mathscr{S}}(\bar{x})$, we use the symbol $\oplus$ here to indicate that the directional stationary point is taking over each critical set when the symbol $\oplus$ is set beside it.*

5. Sufficient Optimality Conditions

This section concerns sufficient optimality conditions for (MPEC). In general, sufficient optimality results require the use of a type of generalized convexity. Here, we exploit an MPEC generalized convexity condition associated with the pseudoconvex sublevel sets to appearing functions. Mainly, we prove that directional M-stationarity conditions ensure strict global optimality when the objective and the constraint functions admit pseudoconvex sublevel sets. To proceed, denote the sublevel set of a function $\pi : \mathbb{R}^n \to \mathbb{R}$, defined on a set $\mathscr{D} \subset \mathbb{R}^n$, at $\bar{x} \in \mathscr{D}$ by

$$L(\pi, \bar{x}, \mathscr{D}) = \{x \in \mathscr{D} \ : \ \pi(x) \leq \pi(\bar{x})\}.$$

Definition 5.1. [20, Definition 1] Let π be a finite-valued real function defined on the set $\mathscr{D}$. We say that π is a function with pseudoconvex sublevel sets with respect to the tangent cone if

$$L(\pi, \bar{x}, \mathscr{D}) \subset \bar{x} + T_{L(\pi, \bar{x}, \mathscr{D})}(\bar{x}) \quad \text{for all } \bar{x} \in \mathscr{D}.$$

Definition 5.2. Let π be a finite-valued real function defined on the set $\mathscr{D}$. We say that the sublevel set $L(\pi,\bar{x},\mathscr{D})$ is strongly pseudoconvex at $\bar{x}$ if for all $x \in L(\pi,\bar{x},\mathscr{D}) \setminus \{\bar{x}\}$ there exist sequences $t_k \downarrow 0$ and $u_k \to x-\bar{x}$ such that $\pi(\bar{x}+t_k u_k) \leq \pi(\bar{x}) - \varepsilon t_k$ for some $\varepsilon > 0$.

Let $\mathscr{Z}$ be the feasible set of (MPEC). Now, for a feasible point $\bar{x}$ of (MPEC), we consider the following sets:

$$\begin{aligned} Lev(\bar{x}) \;=\; T_{L(f,\bar{x},\mathscr{Z})}(\bar{x}) \;&\cap\; \bigcap\nolimits_{i\in I_g} T_{L(g_i,\bar{x},\mathscr{Z})}(\bar{x}) \cap \bigcap\nolimits_{i=1}^{q} T_{L(h_i,\bar{x},\mathscr{Z})}(\bar{x}) \\ &\cap\; \bigcap\nolimits_{i\in\alpha\cup\beta} T_{L(-G_i,\bar{x},\mathscr{Z})}(\bar{x}) \cap \bigcap\nolimits_{i\in\gamma\cup\beta} T_{L(-H_i,\bar{x},\mathscr{Z})}(\bar{x}), \end{aligned}$$

and

$$\mathscr{B}_G(\bar{x}) = \left\{ u \in \mathbb{R}^n \;:\; \begin{array}{l} \exists t_k \downarrow 0,\; u_k \to u \;\text{ such that:} \\ G_i(\bar{x}+t_k u_k) < 0 \;\text{ for some } i \in \alpha\cup\beta \end{array} \right\},$$

$$\mathscr{B}_H(\bar{x}) = \left\{ u \in \mathbb{R}^n \;:\; \begin{array}{l} \exists t_k \downarrow 0,\; u_k \to u \;\text{ such that:} \\ H_i(\bar{x}+t_k u_k) < 0 \;\text{ for some } i \in \gamma\cup\beta \end{array} \right\},$$

$$\Xi(\bar{x}) = \left\{ u \in Lev(\bar{x}) \;:\; f'(\bar{x},u) \leq 0 \right\} \cap \left[\mathscr{B}_G(\bar{x}) \cup \mathscr{B}_H(\bar{x})\right].$$

Now, corresponding to each $\left(\lambda^h,\lambda^G,\lambda^H\right) \in \mathbb{R}^{q+2m}$, we define:

$$\begin{aligned} \alpha^- &= \left\{ i \in \alpha \;:\; \lambda_i^G < 0 \right\}, & J^+ &= \left\{ i = 1,\ldots,q \;:\; \lambda_i^h > 0 \right\}, \\ \beta^- &= \left\{ i \in \beta \;:\; \lambda_i^G < 0 \text{ or } \lambda_i^H < 0 \right\}, & J^- &= \left\{ i = 1,\ldots,q \;:\; \lambda_i^h < 0 \right\}, \\ \gamma^- &= \left\{ i \in \gamma \;:\; \lambda_i^H < 0 \right\}. \end{aligned}$$

We now present the subsequent corollary of Theorem 4.2.

Corollary 5.1. *Let $\bar{x}$ be a local optimal solution for (MPEC) and $u \in \mathscr{C}_{MPEC}(\bar{x})$ such that the assumptions of Theorem 4.2 are satisfied. Then, there exist $\lambda^g \in \mathbb{R}^p$, $\lambda^h \in \mathbb{R}^q$, $\lambda^G \in \mathbb{R}^m$, and $\lambda^H \in \mathbb{R}^m$ such that (4.2), (4.3), (4.4), and the following condition hold:*

$$f'(\bar{x},u) + \sum_{i\in I_g} \lambda_i^g g'(\bar{x},u) + \sum_{i=1}^{q} \lambda_i^h h'(\bar{x},u) - \sum_{i=1}^{m} \left(\lambda_i^G G_i'(\bar{x},u) + \lambda_i^H H_i'(\bar{x},u) \right) \geq 0. \tag{5.1}$$

Proof. From Theorem 4.2, we deduce that conditions (4.1), (4.2), (4.3), and (4.4) are satisfied. Furthermore, in view of [14, Theorem 3.6], the properties of the sup function, and the validity of condition (4.1), we conclude that condition (5.1) also holds. This completes the proof. □

We are now in a position to derive sufficient optimality conditions for problem (MPEC), as stated in the following theorem.

Theorem 5.1. *Let $\bar{x}$ be a feasible point of (MPEC). Assume that the functions $g_i, i \in I_g$, h, G, and H are directionally strictly differentiable at $\bar{x}$ in each direction $u \in Lev(\bar{x})$. Moreover, suppose that f, g_i, $i \in I_g$, h_i, $i \in J^+$, $-h_i$, $i \in J^-$, G_i, $i \in \alpha\cup\beta$, H_i, $i \in \gamma\cup\beta$, are functions with pseudoconvex sublevel sets at $\bar{x}$. Suppose that for all $u \in Lev(\bar{x})$, there exist multipliers satisfying the directional M-stationary conditions (4.2), (4.3), (4.4), and (5.1). Assume further that at least one of the sets $L(f,\bar{x},\mathscr{Z})$, $L(g_i,\bar{x},\mathscr{Z})$, $i \in I_g$ where $\lambda_i^g > 0$, $L(-G_i,\bar{x},\mathscr{Z})$, $i \in \alpha\cup\beta$ where $\lambda_i^G > 0$, $L(-H_i,\bar{x},\mathscr{Z})$, $i \in \gamma\cup\beta$ where $\lambda_i^H > 0$, is strongly pseudoconvex. Then, if $\alpha^-\cup\beta^-\cup\gamma^- = \emptyset$ and $\Xi(\bar{x}) = \emptyset$, the point $\bar{x}$ is a strict global optimal solution of (MPEC).*

Proof. Suppose on the contrary, i.e., there exists a feasible point x of (MPEC) such that $x \neq \bar{x}$ and $f(x) \leq f(\bar{x})$. Since $L(f,\bar{x},\mathscr{Z})$ is pseudoconvex, then $x-\bar{x} \in T_{L(f,\bar{x},\mathscr{Z})}(\bar{x})$. Hence, there exist sequences $t_k \downarrow 0$, $u_k \to x-\bar{x}$ with $\bar{x}+t_k u_k \in L(f,\bar{x},\mathscr{Z})$. Furthermore, we have $f(\bar{x}+t_k u_k) \leq f(\bar{x})$, which implies that

$$f'(\bar{x},u) \leq 0, \tag{5.2}$$

where $u = x-\bar{x}$.

It obvious that $g_i(x) \leq g_i(\bar{x})$, for all $i \in I_g$. Let $i \in I_g$. It follows from the pseudoconvexity of $L(g_i,\bar{x},\mathscr{Z})$ that $u \in T_{L(g_i,\bar{x},\mathscr{Z})}(\bar{x})$. This in turn gives us sequences $t_k^i \downarrow 0$ and $u_k^i \to u$ such that $\bar{x}+t_k^i u_k^i \in L(g_i,\bar{x},\mathscr{Z})$, and consequently $g_i(\bar{x}+t_k^i u_k^i) \leq g_i(\bar{x})$. Then

$$g_i'(\bar{x},u) \leq 0, \quad \text{for all } i \in I_g. \tag{5.3}$$

Let $i = 1,\ldots,q$ be arbitrary. Using similar arguments, we obtain from the pseudoconvexity of $L(h_i,\bar{x},\mathscr{Z})$ and $L(-h_i,\bar{x},\mathscr{Z})$ at $\bar{x}$ that $u \in T_{L(h_i,\bar{x},\mathscr{Z})}(\bar{x}) \cap T_{L(-h_i,\bar{x},\mathscr{Z})}(\bar{x})$. Repeating now the above process, we found $s_k^i \downarrow 0$, $\mu_k^i \downarrow 0$, $v_k^i \to u$ and $w_k^i \to u$ such that $\bar{x}+s_k^i v_k^i \in L(h_i,\bar{x},\mathscr{Z})$ and $\bar{x}+\mu_k^i w_k^i \in L(-h_i,\bar{x},\mathscr{Z})$. Hence

$$\begin{aligned} h_i\left(\bar{x}+s_k^i v_k^i\right) &\leq h_i(\bar{x}) \quad \text{for all } i \in J^+, \\ -h_i\left(\bar{x}+\mu_k^i w_k^i\right) &\leq -h_i(\bar{x}) \quad \text{for all } i \in J^-. \end{aligned}$$

It follows that

$$h_i'(\bar{x},u) \leq 0 \quad \text{for all } i \in J^+, \tag{5.4}$$

$$h_i'(\bar{x},u) \geq 0 \quad \text{for all } i \in J^-. \tag{5.5}$$

Since $G_i(\bar{x}) = 0 \leq G_i(x)$, $i \in \alpha \cup \beta$, $H_i(\bar{x}) = 0 \leq H_i(x)$, $i \in \gamma \cup \beta$, $\Xi(\bar{x}) = \emptyset$, and $-G_i$, $i \in \alpha \cup \beta$, $-H_i$, $i \in \gamma \cup \beta$, have pseudoconvex sublevel sets, it easy to see that

$$-G_i'(\bar{x},u) \leq 0, \quad \forall i \in \alpha \cup \beta, \tag{5.6}$$

$$-H_i'(\bar{x},u) \leq 0, \quad \forall i \in \gamma \cup \beta. \tag{5.7}$$

Since $u \in Lev(\bar{x})$, then inequality (5.1) is satisfied. On the other hand, note that one of (5.2)-(5.7) is in fact strict due to the satisfaction of the strong pseudoconvexity assumption. By multiplying (5.3), (5.4), (5.5), (5.6), and (5.7) by $\lambda_i^g \geq 0$, $i \in I_g$, $\lambda_i^h > 0$, $i \in J^+$, $-\lambda_i^h > 0$, $i \in J^-$, $\lambda_i^G > 0$, $i \in \alpha \cup \beta$, $\lambda_i^H > 0$, $i \in \gamma \cup \beta$ respectively, and adding with the inequality (5.2), we get

$$f'(\bar{x},u) + \sum_{i \in I_g} \lambda_i^g g'(\bar{x},u) + \sum_{i=1}^{q} \lambda_i^h h'(\bar{x},u) - \sum_{i=1}^{m} \left(\lambda_i^G G_i'(\bar{x},u) + \lambda_i^H H_i'(\bar{x},u)\right) < 0,$$

while taking account that $\alpha^- \cup \beta^- \cup \gamma^- = \emptyset$. This contradicts the validity of inequality (5.1). Consequently, $\bar{x}$ is strcit global optimal solution of (MPEC). □

6. Conclusion

In summary, this work established a comprehensive and flexible framework for the study of directional stationarity conditions in mathematical programs with equilibrium constraints. By deriving directional necessary optimality conditions for a broad class of nonsmooth and constrained problems, we introduced a hierarchy of directional stationarity concepts, directional W-, M-, C-, and S-stationarity, tailored to MPECs. Our comparison with classical stationarity notions underscores the greater accuracy of these directional conditions in characterizing the feasible set's geometry. Moreover, we extended the framework to sufficient optimality conditions, proving that, under suitable generalized convexity assumptions, specifically the pseudoconvexity of both the objective and the constraint sublevel sets, directional M-stationarity guarantees strong global optimality. These findings provide a solid theoretical foundation for further advances in the analysis and solution of complex equilibrium-constrained optimization problems.

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