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OPTIMALITY AND DUALITY CONDITIONS FOR NONDIFFERENTIABLE MULTIOBJECTIVE SEMI-INFINITE PROGRAMMING WITH EQUILIBRIUM CONSTRAINTS

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Abstract. In this paper, we investigate a nondifferentiable multiobjective semi-infinite mathematical programming problem with equilibrium constraints. We introduce a nonsmooth version of the ∂^T-Abadie constraint qualification (∂^T-ACQ) and propose ∂^T-strong stationarity conditions based on tangential subdifferentials. We demonstrate that strong stationarity serves as a necessary optimality condition under ∂^T-ACQ. Furthermore, we derive sufficient optimality conditions for this nonsmooth extremum problem, assuming that the involved functions are Dini generalized convex, characterized by their tangential subdifferentials. Additionally, we formulate the Mond–Weir dual problems and examine duality relations under Dini generalized convexity assumptions. To conclude, we present illustrative examples that demonstrate the effectiveness of our theoretical results.
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1. INTRODUCTION

Mathematical Programs with Equilibrium Constraints (MPECs) have garnered significant research interest due to their various applications in economics, engineering design, and transportation planning, and so on. For a deeper understanding of these applications and the latest advancements in theory and algorithms, we refer to [2, 16, 23]. In recent years, significant attention has been given to necessary optimality conditions and duality results for nonsmooth semi-infinite mathematical programming problems, as demonstrated in [1, 5, 6, 7, 8, 10, 12, 13, 14, 20, 21, 25, 29, 30, 31]. Nonsmooth semi-infinite mathematical programming problems with equilibrium constraints represent a nonsmooth optimization problem characterized by an infinite number of inequality constraints. This class of problems can be viewed as a generalization of constrained bilevel nonsmooth multiobjective programming and constrained nonsmooth vector multiobjective optimization problems, as highlighted in [3, 4, 16] and related studies.

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This investigation focuses on the subsequent multiobjective semi-infinite mathematical programming problem with equilibrium constraints $(\mathscr{MSMPEC})$ of the form:

$$(\mathscr{MSMPEC}): \quad \begin{cases} \min \Gamma(\mathfrak{z}) = \Big(\Gamma_1(\mathfrak{z}), \ldots, \Gamma_m(\mathfrak{z})\Big) \\ s.\,t. \begin{cases} c_s(\mathfrak{z}) \leq 0, \ \forall s \in \mathscr{S}, \\ d(\mathfrak{z}) = 0, \\ \mathscr{T}(\mathfrak{z}) \geq 0, \ \zeta(\mathfrak{z}) \geq 0, \\ \quad \big\langle \mathscr{T}(\mathfrak{z}), \zeta(\mathfrak{z}) \big\rangle = 0, \end{cases} \end{cases} \tag{1.1}$$

where the index set $\mathscr{S}$ is an arbitrary nonempty set that does not need to be finite, $\Gamma_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, $i \in \mathscr{I} = \{1, \ldots, m\}$, $c_s : \mathbb{R}^n \longrightarrow \mathbb{R}$, $d : \mathbb{R}^n \longrightarrow \mathbb{R}^q$, $\mathscr{T} : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ and $\zeta : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ are not necessarily convex nor differentiable. Let Λ be the feasible set of $(\mathscr{MSMPEC})$ defined by

$$\Lambda := \big\{\mathfrak{z} \in \mathbb{R}^n : c_s(\mathfrak{z}) \leq 0, d(\mathfrak{z}) = 0, \mathscr{T}(\mathfrak{z}) \geq 0, \ \zeta(\mathfrak{z}) \geq 0, \big\langle \mathscr{T}(\mathfrak{z}), \zeta(\mathfrak{z}) \big\rangle = 0\big\}.$$

The tangential subdifferential [15, 17, 18, 22, 27, 28], which generalizes both the Gâteaux derivative and the convex subdifferential, was introduced and applied to derive optimality conditions for nonlinear programming in [22]. Tung [28] demonstrated strong-KKT optimality conditions for Pareto efficient and weakly efficient solutions in semi-infinite multiobjective programming problems using the tangential subdifferential. Later, Jennane and Kalmoun [11] established optimality conditions for multiobjective semi-infinite programming with switching constraints based on tangential subdifferentials. Recently, Tripathi and Arora [26] developed optimality conditions and duality for fractional semi-infinite programming problems via tangential subdifferentials. The inclusion of Clarke regular subdifferentials and Michel-Penot regular subdifferentials within the tangential subdifferentials allows for more generalized results in the optimality conditions associated with these tangential subdifferentials.

To our knowledge, no existing research addresses the optimality conditions and duality relations of multiobjective semi-infinite mathematical programming problems with equilibrium constraints using tangential subdifferentials. Motivated by this gap in the literature, this article employs tangential subdifferentials as generalized derivatives to establish optimality conditions for Pareto solutions and to formulate Mond-Weir-type duality theorems for multiobjective semi-infinite programming.

The structure of the paper is as follows: In Section 2, we present the basic concepts and preliminaries. Section 3 introduces the concept of strong stationarity and demonstrates that ∂^T-strong stationarity conditions serve as necessary optimality conditions for $(\mathscr{MSMPEC})$. Section 4 discusses the sufficient optimality conditions for the $(\mathscr{MSMPEC})$ problem. Section 5 focuses on constructing a Mond–Weir-type dual problem $(\mathscr{MWSMPEC})$ for the primal problem $(\mathscr{MSMPEC})$ and establishes strong and weak duality theorems under appropriate assumptions of Dini-pseudoconvexity and Dini-quasiconvexity for the objective and constraint functions. Section 6, the last section, provides concluding remarks.

2. Preliminaries

In this section, we present several definitions, notations, and results that are referenced later in the paper. Throughout this work, we denote the standard n-dimensional Euclidean space as $\mathbb{R}^n$. The inner product is represented as $\langle \cdot, \cdot \rangle$. Let $\mathfrak{C}$ be a nonempty subset of $\mathbb{R}^n$. We define the convex hull of $\mathfrak{C}$ as $co\,\mathfrak{C}$, the interior of $\mathfrak{C}$ as $int\,\mathfrak{C}$, the convex cone (which includes the origin) of $\mathfrak{C}$ as $pos\,\mathfrak{C}$, the cone of $\mathfrak{C}$ as $\mathrm{cone}\,\mathfrak{C}$, and the closure of $\mathfrak{C}$ as $cl\,\mathfrak{C}$.

The orthogonal complement, the negative polar cone, and the strictly negative polar cone of $\mathfrak{C}$ are respectively (resp) defined as follows

$$\mathfrak{C}^{\perp} := \left\{\delta \in \mathbb{R}^n : \langle \delta, \mathfrak{z}\rangle = 0,\ \forall\, \mathfrak{z} \in \mathfrak{C}\right\},$$
$$\mathfrak{C}^{\circ} := \left\{\delta \in \mathbb{R}^n : \langle \delta, \mathfrak{z}\rangle \leq 0,\ \forall\, \mathfrak{z} \in \mathfrak{C}\right\},$$

and

$$\mathfrak{C}^{\circ\circ} := \left\{\delta \in \mathbb{R}^n : \langle \delta, \mathfrak{z}\rangle < 0,\ \forall\, \mathfrak{z} \in \mathfrak{C}\backslash\{0\}\right\}.$$

It is simple to demonstrate that $\mathfrak{C}^{\circ\circ} \subset \mathfrak{C}^{\circ}$, if $\mathfrak{C}^{\circ\circ} \neq \emptyset$, then $\mathfrak{C}^{\circ} = cl\,(\mathfrak{C}^{\circ\circ})$ and $\mathfrak{C}^{\perp} = \mathfrak{C}^{\circ} \cup (-\mathfrak{C})^{\circ}$. Moreover, at $\tilde{\mathfrak{z}} \in cl\,\mathfrak{C}$, the contingent cone and the linear hull of $\mathfrak{C}$ are defined resp by

$$\mathfrak{T}(\tilde{\mathfrak{z}}, \mathfrak{C}) := \left\{\delta \in \mathbb{R}^n : \exists\, u_n \downarrow 0,\ \exists\, \{\delta_n\} \subseteq \mathbb{R}^n,\ \delta_n \to \delta,\ \tilde{\mathfrak{z}} + u_n\delta_n \in \mathfrak{C}\right\}$$

and

$$span(\mathfrak{C}) := \left\{x = \sum_{i=1}^{s} \tau_i x_i : s \in \mathbb{N},\ \tau_i \in \mathbb{R}, x_i \in \mathfrak{C}, i = 1, 2, \ldots, s\right\}.$$

The Fréchet normal cone $\mathscr{N}(\mathfrak{z}, \mathfrak{C})$ to a set $\mathfrak{C}$ at a point $\tilde{\mathfrak{z}}$ is defined as

$$\mathscr{N}(\tilde{\mathfrak{z}}, \mathfrak{C}) = \left(\mathfrak{T}(\tilde{\mathfrak{z}}, \mathfrak{C})\right)^{\circ} = \left\{\mathfrak{z}^* \in \mathbb{R}^n : \langle \mathfrak{z}^*, \delta\rangle \leq 0,\ \forall\, \delta \in \mathfrak{T}(\tilde{\mathfrak{z}}, \mathfrak{C})\right\}.$$

Note that while the tangent cone $\mathfrak{T}(\tilde{\mathfrak{z}}, \mathfrak{C})$ is always closed, it is not necessarily convex. However, if $\mathfrak{C}$ is a convex set, then $\mathfrak{T}(\tilde{\mathfrak{z}}, \mathfrak{C})$ is convex, and

$$\mathscr{N}(\tilde{\mathfrak{z}}, \mathfrak{C}) = \left\{\mathfrak{z}^* \in \mathbb{R}^n : \langle \mathfrak{z}^*, \mathfrak{z} - \tilde{\mathfrak{z}}\rangle \leq 0,\ \forall\, \mathfrak{z} \in \mathfrak{C}\right\}.$$

Definition 2.1. ([27]) Let $\phi : \mathbb{R}^n \to \mathbb{R}$ and $\tilde{\mathfrak{z}} \in \mathbb{R}^n$ be given:

(i) The directional derivative of ϕ at $\tilde{\mathfrak{z}}$ in the direction $\delta \in \mathbb{R}^n$ is defined by

$$\phi'(\tilde{\mathfrak{z}}, \delta) = \lim_{t\downarrow 0} \frac{\phi(\tilde{\mathfrak{z}} + t\delta) - \phi(\tilde{\mathfrak{z}})}{t},$$

and if the directional derivative $\phi'(\tilde{\mathfrak{z}}, \delta)$ exists for all δ, then ϕ is called directionally differentiable at $\tilde{\mathfrak{z}}$.

(ii) The Hadamard directional derivative of ϕ at $\tilde{\mathfrak{z}}$ in the direction $\delta \in \mathbb{R}^n$ is defined by

$$\phi^H(\tilde{\mathfrak{z}}, \delta) = \lim_{t\downarrow 0\, \delta' \to \delta} \frac{\phi(\tilde{\mathfrak{z}} + t\delta') - \phi(\tilde{\mathfrak{z}})}{t},$$

and if the Hadamard directional derivative $\phi^H(\tilde{\mathfrak{z}}, \delta)$ exists for all δ, then ϕ is called Hadamard directionally differentiable at $\tilde{\mathfrak{z}}$.

Note that if the Hadamard directional derivative $\phi^H(\tilde{\mathfrak{z}}, \delta)$ exists, then the directional derivative $\phi'(\tilde{\mathfrak{z}}, \delta)$ also exists, and they are equivalent. Conversely, ϕ is Hadamard directionally differentiable at $\tilde{\mathfrak{z}}$ in δ if ϕ is locally Lipschitz at $\tilde{\mathfrak{z}}$ and directionally differentiable.

Next, the class of functions referred to as "tangentially convex" is defined, which was first proposed by Pshenichnyi [22] and termed by Lemaréchal [15].

Definition 2.2. [15, 22] A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is tangentially convex at $\tilde{\mathfrak{z}} \in \phi^{-1}(\mathbb{R})$ if its directional derivative $\phi'(\tilde{\mathfrak{z}}, \delta)$ exists, is finite for all directions $\delta \in \mathbb{R}^n$ and the function $\delta \mapsto \phi'(\tilde{\mathfrak{z}}, \delta)$ is convex.

Based on the definition of a tangentially convex function, Pshenichnyi [22] introduced the concept of its associated subdifferential, known as the tangential subdifferential (see also [17]).

Definition 2.3. [17, 22] Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a tangentially convex function at $\tilde{\mathfrak{z}} \in \phi^{-1}(\mathbb{R})$. It is said that the nonempty compact convex set $\partial^T \phi(\tilde{\mathfrak{z}}) \subset \mathbb{R}^n$ is the tangential subdifferential of ϕ at $\tilde{\mathfrak{z}}$, if, for every $\delta \in \mathbb{R}^n$, $\phi'(\tilde{\mathfrak{z}};\delta) = \max\limits_{\mathfrak{z}\in\partial^T\phi(\tilde{\mathfrak{z}})} \langle \mathfrak{z}, \delta\rangle$, which is equivalent to $\partial^T \phi(\tilde{\mathfrak{z}}) = \big\{\mathfrak{z} \in \mathbb{R}^n : \langle \mathfrak{z}, \delta\rangle \leq \phi'(\tilde{\mathfrak{z}};\delta),\ \forall \delta \in \mathbb{R}^n\big\}$.

Remark 2.1. Note that the definition of the tangential subdifferential is also equivalent to $\partial^T \phi(\tilde{\mathfrak{z}}) = \partial \phi'(\tilde{\mathfrak{z}},\cdot)(0)$, where ∂ denotes the subdifferential of a convex function in convex analysis. Additionally, according to Definition 2.3, if ϕ is tangentially convex at $\tilde{\mathfrak{z}} \in \phi^{-1}(\mathbb{R})$, then its tangential subdifferential at $\tilde{\mathfrak{z}}$ is nonempty, as follows from the sublinearity of $\phi'(\tilde{\mathfrak{z}};\cdot)$. Furthermore, from $\phi'(\tilde{\mathfrak{z}};\delta) = \max\limits_{\mathfrak{z}\in\partial^T\phi(\tilde{\mathfrak{z}})} \langle \mathfrak{z}, \delta\rangle$, $\phi'(\tilde{\mathfrak{z}};\cdot)$ acts as the support functional of the tangential subdifferential $\partial^T \phi(\tilde{\mathfrak{z}})$ of ϕ at $\tilde{\mathfrak{z}}$.

It can be noted that, among the various calculus rules applicable to tangential subdifferentials of tangentially convex functions, additivity is one of them. Specifically, if ϕ_1 and ϕ_2 are tangentially convex functions at a common point $\tilde{\mathfrak{z}}$, then the following holds

$$\begin{aligned}\partial^T\big(\phi_1+\phi_2\big)(\tilde{\mathfrak{z}}) = \partial\big(\phi_1+\phi_2\big)'(\tilde{\mathfrak{z}},\cdot)(0) &= \partial\big(\phi_1'(\tilde{\mathfrak{z}},\cdot)+\phi_2'(\tilde{\mathfrak{z}},\cdot)\big)(0)\\ &= \partial\phi_1'(\tilde{\mathfrak{z}},\cdot)(0)+\partial\phi_2'(\tilde{\mathfrak{z}},\cdot)(0) = \partial^T\phi_1(\tilde{\mathfrak{z}})+\partial^T\phi_2(\tilde{\mathfrak{z}}),\end{aligned}$$

demonstrating that the tangential subdifferential of the sum of two tangentially convex functions is the sum of their individual tangential subdifferentials at the common point $\tilde{\mathfrak{z}}$.

Definition 2.4. Let $\tilde{\mathfrak{z}} \in \Lambda$:

(i) $\tilde{\mathfrak{z}}$ is said to be a locally weakly Pareto solution of $(\mathscr{MSMPEC})$, denoted by $\tilde{\mathfrak{z}} \in LOWP(\mathscr{MSMPEC})$, if there exists a neighborhood $\mathscr{V}$ of $\tilde{\mathfrak{z}}$ such that

$$\Gamma(\mathfrak{z}) - \Gamma(\tilde{\mathfrak{z}}) \notin -int\,\mathbb{R}^n_+,\ \forall \mathfrak{z} \in \Lambda \cap \mathscr{V}.$$

(ii) $\tilde{\mathfrak{z}}$ is said to be a locally Pareto solution of $(\mathscr{MSMPEC})$, which is denoted by $\tilde{\mathfrak{z}} \in LOP(\mathscr{MSMPEC})$, if there exists a neighborhood $\mathscr{V}$ of $\tilde{\mathfrak{z}}$ such that

$$\Gamma(\mathfrak{z}) - \Gamma(\tilde{\mathfrak{z}}) \notin -\mathbb{R}^n_+\backslash\{0\},\ \forall \mathfrak{z} \in \Lambda \cap \mathscr{V}.$$

If $\mathscr{V} = \mathbb{R}^n$, the term "locally" is excluded, and in this case, the Pareto solution sets and the weakly Pareto solution sets are denoted as $P(\mathscr{MSMPEC})$ and $WP(\mathscr{MSMPEC})$, respectively. Clearly, $P(\mathscr{MSMPEC}) \subset WP(\mathscr{MSMPEC})$.

3. Necessary Optimality Condition

Given a feasible vector $\tilde{\mathfrak{z}} \in \Lambda$ for $(\mathscr{MSMPEC})$, we define the following index sets:

$$\mathscr{S}_c = \mathscr{S}_c(\tilde{\mathfrak{z}}) := \big\{s \in \mathscr{S} : c_s(\tilde{\mathfrak{z}}) = 0\big\},$$

$$\begin{aligned}\omega = \omega(\tilde{\mathfrak{z}}) &:= \big\{k = 1,\dots,p : \mathscr{T}_k(\tilde{\mathfrak{z}}) = 0,\ \zeta_k(\tilde{\mathfrak{z}}) > 0\big\},\\ \theta = \theta(\tilde{\mathfrak{z}}) &:= \big\{k = 1,\dots,p : \mathscr{T}_k(\tilde{\mathfrak{z}}) = 0,\ \zeta_k(\tilde{\mathfrak{z}}) = 0\big\},\\ \mu = \mu(\tilde{\mathfrak{z}}) &:= \big\{k = 1,\dots,p : \mathscr{T}_k(\tilde{\mathfrak{z}}) > 0,\ \zeta_k(\tilde{\mathfrak{z}}) = 0\big\}.\end{aligned}$$

If the biactive (or degenerate) index set θ is empty, the strict complementarity condition holds at $\tilde{\mathfrak{z}}$. However, our focus is on the non-strict case, where θ is nonempty. Let

$$\mathbb{R}^{|\mathscr{S}|} = \{\rho = (\rho_s)_{s\in\mathscr{S}} : \rho_s = 0,\ \forall s \in \mathscr{S},\ \text{but only finitely many } \rho_s \neq 0\}.$$

For $\mathbb{R}^{|\mathscr{S}|}$, the non-negative cone is given by

$$\mathbb{R}_+^{|\mathscr{S}|} = \{\rho = (\rho_s)_{s\in\mathscr{S}} \in \mathbb{R}^{|\mathscr{S}|} : \rho_s \geq 0,\ s \in \mathscr{S}\}.$$

The set of active constraint multipliers at $\tilde{\mathfrak{z}} \in \Lambda$ is

$$\Psi(\tilde{\mathfrak{z}}) = \{\rho \in \mathbb{R}_+^{|\mathscr{S}|} : \rho_s c_s(\tilde{\mathfrak{z}}) = 0,\ \forall s \in \mathscr{S}\}.$$

Now, assuming that $\tilde{\mathfrak{z}} \in \Lambda$, Γ_i, $i \in \mathscr{I} = \{1,\ldots,m\}$ is Hadamard directionally differentiable at $\tilde{\mathfrak{z}}$ and c_s, $s \in \mathscr{S}$, d_j, $-d_j$, $j \in \mathscr{J} = \{1,\ldots,q\}$, $\mathscr{T}_k$, $-\mathscr{T}_k$, ζ_k, and $-\zeta_k$, $k \in K_p = \{1,\ldots,p\}$ are tangentially convex at $\tilde{\mathfrak{z}}$.

We are now prepared to formulate ∂^T-Abadie-type constraint qualifications (∂^T-ACQ) using tangential subdifferentials.

Definition 3.1. Let $\tilde{\mathfrak{z}} \in \Lambda$ be the feasible point of $(\mathscr{MSMPEC})$. We say that the ∂^T-ACQ holds at $\tilde{\mathfrak{z}} \in \Lambda$ if $\Theta(\tilde{\mathfrak{z}}) \subseteq \mathfrak{T}(\tilde{\mathfrak{z}},\Lambda)$, where

$$\Theta(\tilde{\mathfrak{z}}) = \Big(\bigcup_{s\in\mathscr{S}_c} \partial^T c_s(\tilde{\mathfrak{z}})\Big)^{\circ} \cap \Big(\bigcup_{j\in\mathscr{J}} \partial^T d_j(\tilde{\mathfrak{z}})\Big)^{\perp} \cap \Big(\bigcup_{k\in\omega} \partial^T \mathscr{T}_k(\tilde{\mathfrak{z}})\Big)^{\perp}$$
$$\cap \Big(\bigcup_{k\in\mu} \partial^T \zeta_k(\tilde{\mathfrak{z}})\Big)^{\perp} \cap \Big(\bigcup_{k\in\theta} \partial^T(-\mathscr{T}_k)(\tilde{\mathfrak{z}})\Big)^{\circ} \cap \Big(\bigcup_{k\in\theta} \partial^T(-\zeta_k)(\tilde{\mathfrak{z}})\Big)^{\circ}.$$

Next, we present ∂^T-S-stationarity concepts for problem $(\mathscr{MSMPEC})$.

Definition 3.2. A feasible point $\tilde{\mathfrak{z}} \in \Lambda$ of $(\mathscr{MSMPEC})$ is said to be a ∂^T-strong stationary (∂^T-S-stationary) point if there exist multipliers $(\xi, \mathfrak{a}^c, \mathfrak{a}^d, \mathfrak{a}^{\mathscr{T}}, \mathfrak{a}^{\zeta}) \in \mathbb{R}_+^m \times \Psi(\tilde{\mathfrak{z}}) \times \mathbb{R}^q \times \mathbb{R}^{2p}$ with $\sum_{i=1}^m \xi_i = 1$, $\mathfrak{a}_\theta^{\mathscr{T}} \geq 0$, $\mathfrak{a}_\theta^{\zeta} \geq 0$, $\mathfrak{a}_\mu^{\mathscr{T}} = \mathfrak{a}_\omega^{\zeta} = 0$ such that

$$0 \in \sum_{i\in\mathscr{I}} \xi_i \partial^T \Gamma_i(\tilde{\mathfrak{z}}) + \sum_{s\in\mathscr{S}_c} \mathfrak{a}_s^c \partial^T c_s(\tilde{\mathfrak{z}}) + \sum_{j\in\mathscr{J}} \mathfrak{a}_j^d \partial^T d_j(\tilde{\mathfrak{z}})$$
$$+ \sum_{k\in K_p} \mathfrak{a}_k^{\mathscr{T}} \partial^T(-\mathscr{T}_k)(\tilde{\mathfrak{z}}) + \sum_{k\in K_p} \mathfrak{a}_k^{\zeta} \partial^T(-\zeta_k)(\tilde{\mathfrak{z}}). \quad (3.1)$$

Lemma 3.1. [9] *Let X, Y and Z be three arbitrary nonempty index sets, which may be infinite. Consider the mappings $h_x : X \to \mathbb{R}^n$, $h_y : Y \to \mathbb{R}^n$ and $h_z : Z \to \mathbb{R}^n$. If $co\{h_x, x \in X\} + cone\{h_y, y \in Y\} + span\{h_z, z \in Z\}$ is closed, then the following two assertions are equivalent:*

I: $\begin{cases} \langle h_x, \theta\rangle < 0,\ x \in X, \\ \langle h_y, \theta\rangle \leq 0,\ y \in Y, \\ \langle h_z, \theta\rangle = 0,\ z \in Z, \end{cases}$ *has no solution $\theta \in \mathbb{R}^n$;*

II: $0 \in co\{h_x, x \in X\} + cone\{h_y, y \in Y\} + span\{h_z, z \in Z\}$.

Lemma 3.2. [24] *Let $\{\Upsilon_r : r \in R\}$ represent an arbitrary collection of nonempty convex sets in $\mathbb{R}^n$, and $\vartheta = cone\big(\cup_{r\in R}\Upsilon_r\big)$. Then, every nonzero vector of ϑ can be written as a non-negative linear combination of at most n linear independent vectors, each belonging to a different Υ_r.*

This theorem outlines the main contribution of this work by establishing necessary Karush-Kuhn-Tucker (KKT)-type optimality conditions for the considered nonsmooth $(\mathscr{MSMPEC})$.

Theorem 3.1. *Assume that $\tilde{\mathfrak{z}} \in LOWP(\mathscr{MSMPEC})$. If ∂^T-ACQ satisfes at $\tilde{\mathfrak{z}}$ and the set*

$$\Omega = cone\left(\bigcup_{s\in\mathscr{S}_c}\partial^T c_s(\tilde{\mathfrak{z}}) \cup \bigcup_{k\in\theta}\partial^T(-\mathscr{T}_k)(\tilde{\mathfrak{z}}) \cup \bigcup_{k\in\theta}\partial^T(-\zeta_k)(\tilde{\mathfrak{z}})\right) + span\left(\bigcup_{j\in\mathscr{J}}\partial^T d_j(\tilde{\mathfrak{z}}) \cup \bigcup_{k\in\omega}\partial^T\mathscr{T}_k(\tilde{\mathfrak{z}}) \cup \bigcup_{k\in\mu}\partial^T\zeta_k(\tilde{\mathfrak{z}})\right)$$

is closed, then $\tilde{\mathfrak{z}}$ is a ∂^T-S-stationary point of the $(\mathscr{MSMPEC})$.

Proof. Since $\tilde{\mathfrak{z}} \in LOWP(\mathscr{MSMPEC})$, by Definition 2.4, there exists a neighborhood $\mathscr{V}$ of $\tilde{\mathfrak{z}}$ such that $\Gamma(\mathfrak{z}) - \Gamma(\tilde{\mathfrak{z}}) \notin -int\,\mathbb{R}^n_+$, for all $\mathfrak{z} \in \Lambda\cap\mathscr{V}$. That is, there is no $\mathfrak{z} \in \Lambda\cap\mathscr{V}$ satisfying, for all $i \in \mathscr{I}$, $\Gamma_i(\mathfrak{z}) < \Gamma_i(\tilde{\mathfrak{z}})$. Now, we claim that

$$\left(\bigcup_{i\in\mathscr{I}}\partial^T\Gamma_i(\tilde{\mathfrak{z}})\right)^{\circ\circ} \cap \mathfrak{T}(\tilde{\mathfrak{z}},\Lambda) = \emptyset. \tag{3.2}$$

Indeed, suppose, contrary to the result, that there exists $\delta^* \in \left(\bigcup_{i\in\mathscr{I}}\partial^T\Gamma_i(\tilde{\mathfrak{z}})\right)^{\circ\circ} \cap \mathfrak{T}(\tilde{\mathfrak{z}},\Lambda)$. Then, $\delta^* \in \left(\bigcup_{i\in\mathscr{I}}\partial^T\Gamma_i(\tilde{\mathfrak{z}})\right)^{\circ\circ}$ and $\delta^* \in \mathfrak{T}(\tilde{\mathfrak{z}},\Lambda)$. From $\delta^* \in \left(\bigcup_{i\in\mathscr{I}}\partial^T\Gamma_i(\tilde{\mathfrak{z}})\right)^{\circ\circ}$, one has

$$\left\langle \alpha^*, \delta^*\right\rangle < 0,\ \forall \alpha^* \in \partial^T\Gamma_i(\tilde{\mathfrak{z}}), \forall i \in \mathscr{I}. \tag{3.3}$$

Since the function $\mathscr{F}_i : \partial^T\Gamma_i(\tilde{\mathfrak{z}}) \subset \mathbb{R}^n \to \mathbb{R}$, $i \in \mathscr{I}$ defined by $\mathscr{F}_i(\alpha^*) = \left\langle \alpha^*, \delta^*\right\rangle$, $\forall \alpha^* \in \partial^T\Gamma_i(\tilde{\mathfrak{z}})$, is continuous on the compact set $\partial^T\Gamma_i(\tilde{\mathfrak{z}})$, it follows that there exists a point $\alpha_i^* \in \partial^T\Gamma_i(\tilde{\mathfrak{z}})$ such that $\mathscr{F}_i(\bar{\alpha}_i^*) = \max_{\alpha^*\in\partial^T\Gamma_i(\tilde{\mathfrak{z}})}\left\langle \alpha^*, \delta^*\right\rangle$. Therefore, by (3.3), we obtain, for all $i \in \mathscr{I}$,

$$\Gamma_i'(\tilde{\mathfrak{z}}, \delta^*) = \max_{\alpha^*\in\partial^T\Gamma_i(\tilde{\mathfrak{z}})}\left\langle \alpha^*, \delta^*\right\rangle = \left\langle \bar{\alpha}_i^*, \delta^*\right\rangle < 0. \tag{3.4}$$

Note that $\delta^* \in \mathfrak{T}(\tilde{\mathfrak{z}},\Lambda)$ implies that there exist $t_r \downarrow 0$, $\delta_r^* \to \delta^*$ such that $\tilde{\mathfrak{z}} + t_r\delta_r^* \in \Lambda$ for all r. Since $\tilde{\mathfrak{z}} \in LOWP(\mathscr{MSMPEC})$, there exists $\tilde{\mathfrak{z}} + t_r\delta_r^* \in \mathfrak{B}(\tilde{\mathfrak{z}}, d)$, for some $d > 0$ and sufficiently large r, such that there exists $i_0 \in \mathscr{I}$ satisfying $\Gamma_{i_0}(\tilde{\mathfrak{z}} + t_r\delta_r^*) \geq \Gamma_{i_0}(\tilde{\mathfrak{z}})$. Therefore

$$\Gamma_{i_0}'(\tilde{\mathfrak{z}}, \delta^*) = \Gamma_{i_0}^H(\tilde{\mathfrak{z}}, \delta^*) = \lim_{r\to\infty}\frac{\Gamma_{i_0}(\tilde{\mathfrak{z}} + t_r\delta_r^*) - \Gamma_{i_0}(\tilde{\mathfrak{z}})}{t_r} \geq 0$$

$$\Longrightarrow \Gamma_{i_0}'(\tilde{\mathfrak{z}}, \delta^*) \geq 0,$$

which contradicts (3.4). Therefore, claim (3.2) is verified. Thus

$$\left(\bigcup_{i\in\mathscr{I}}\partial^T\Gamma_i(\tilde{\mathfrak{z}})\right)^{\circ\circ} \cap \mathfrak{T}(\tilde{\mathfrak{z}},\Lambda) = \emptyset.$$

By the ∂^T-Abadie-type constraint qualifications (∂^T-ACQ), we have

$$\left(\bigcup_{i\in\mathscr{I}}\partial^T\Gamma_i(\tilde{\mathfrak{z}})\right)^{\circ\circ} \cap \left(\bigcup_{s\in\mathscr{S}_c}\partial^T c_s(\tilde{\mathfrak{z}})\right)^{\circ} \cap \left(\bigcup_{j\in\mathscr{J}}\partial^T d_j(\tilde{\mathfrak{z}})\right)^{\perp} \cap \left(\bigcup_{k\in\omega}\partial^T\mathscr{T}_k(\tilde{\mathfrak{z}})\right)^{\perp} \cap \left(\bigcup_{k\in\mu}\partial^T\zeta_k(\tilde{\mathfrak{z}})\right)^{\perp} \cap \left(\bigcup_{k\in\theta}\partial^T(-\mathscr{T}_k)(\tilde{\mathfrak{z}})\right)^{\circ} \cap \left(\bigcup_{k\in\theta}\partial^T(-\zeta_k)(\tilde{\mathfrak{z}})\right)^{\circ} = \emptyset.$$

This implies that there is no $\delta \in \mathbb{R}^n$ such that

$$\begin{cases} \langle \partial^T \Gamma_i(\tilde{\mathfrak{z}}), \delta \rangle < 0, \ \forall i \in \mathscr{I}, \\ \langle \partial^T c_s(\tilde{\mathfrak{z}}), \delta \rangle \leq 0, \ \forall s \in \mathscr{S}_c, \\ \langle \partial^T d_j(\tilde{\mathfrak{z}}), \delta \rangle = 0, \ \forall j \in \mathscr{J}, \\ \langle \partial^T \mathscr{T}_k(\tilde{\mathfrak{z}}), \delta \rangle = 0, \ \forall k \in \omega, \\ \langle \partial^T \zeta_k(\tilde{\mathfrak{z}}), \delta \rangle = 0, \ \forall k \in \mu, \\ \langle \partial^T (-\mathscr{T}_k)(\tilde{\mathfrak{z}}), \delta \rangle \leq 0, \ \forall k \in \theta, \\ \langle \partial^T (-\zeta_k)(\tilde{\mathfrak{z}}), \delta \rangle \leq 0, \ \forall k \in \theta. \end{cases} \tag{3.5}$$

On the other hand, since $\partial^T \Gamma_i(\tilde{\mathfrak{z}})$ is compact for every $i \in \mathscr{I}$, $\bigcup_{i=1}^m \partial^T \Gamma_i(\tilde{\mathfrak{z}})$ is likewise compact. Therefore, we get that $\bigcup_{i=1}^m \partial^T \Gamma_i(\tilde{\mathfrak{z}}) + \Omega$ is closed. Hence, by Lemma 3.1, we conclude that

$$\begin{aligned} 0 \in co\Big(\bigcup_{i\in\mathscr{I}} \partial^T \Gamma_i(\tilde{\mathfrak{z}})\Big) + cone\left(\bigcup_{s\in\mathscr{S}_c} \partial^T c_s(\tilde{\mathfrak{z}}) \cup \bigcup_{k\in\theta} \partial^T(-\mathscr{T}_k)(\tilde{\mathfrak{z}}) \cup \bigcup_{k\in\theta} \partial^T(-\zeta_k)(\tilde{\mathfrak{z}})\right) \\ + span\left(\bigcup_{j\in\mathscr{J}} \partial^T d_j(\tilde{\mathfrak{z}}) \cup \bigcup_{k\in\omega} \partial^T \mathscr{T}_k(\tilde{\mathfrak{z}}) \cup \bigcup_{k\in\mu} \partial^T \zeta_k(\tilde{\mathfrak{z}})\right). \end{aligned}$$

It follows that

$$\begin{aligned} 0 \in co\Big(\bigcup_{i\in\mathscr{I}} \partial^T \Gamma_i(\tilde{\mathfrak{z}})\Big) + cone \bigcup_{s\in\mathscr{S}_c} \partial^T c_s(\tilde{\mathfrak{z}}) + cone \bigcup_{k\in\theta} \partial^T(-\mathscr{T}_k)(\tilde{\mathfrak{z}}) \\ + cone \bigcup_{k\in\theta} \partial^T(-\zeta_k)(\tilde{\mathfrak{z}}) + span \bigcup_{j\in\mathscr{J}} \partial^T d_j(\tilde{\mathfrak{z}}) \\ + span \bigcup_{k\in\omega} \partial^T \mathscr{T}_k(\tilde{\mathfrak{z}}) + span \bigcup_{k\in\mu} \partial^T \zeta_k(\tilde{\mathfrak{z}}). \end{aligned}$$

Furthermore, according to Lemma 3.2, there exists $(\xi, \mathfrak{a}^c, \mathfrak{a}^d, \mathfrak{a}^{\mathscr{T}}, \mathfrak{a}^{\zeta}) \in \mathbb{R}^m_+ \times \Psi(\tilde{\mathfrak{z}}) \times \mathbb{R}^q \times \mathbb{R}^{2p}$ with $\sum_{i=1}^m \xi_i = 1$, $\mathfrak{a}^{\mathscr{T}}_\theta \geq 0$, $\mathfrak{a}^{\zeta}_\theta \geq 0$, $\mathfrak{a}^{\mathscr{T}}_\mu = \mathfrak{a}^{\zeta}_\omega = 0$ such that

$$\begin{aligned} 0 \in \sum_{i\in\mathscr{I}} \xi_i \partial^T \Gamma_i(\tilde{\mathfrak{z}}) + \sum_{s\in\mathscr{S}_c} \mathfrak{a}^c_s \partial^T c_s(\tilde{\mathfrak{z}}) + \sum_{j\in\mathscr{J}} \mathfrak{a}^d_j \partial^T d_j(\tilde{\mathfrak{z}}) \\ + \sum_{k\in K_p} \mathfrak{a}^{\mathscr{T}}_k \partial^T(-\mathscr{T}_k)(\tilde{\mathfrak{z}}) + \sum_{k\in K_p} \mathfrak{a}^{\zeta}_k \partial^T(-\zeta_k)(\tilde{\mathfrak{z}}). \end{aligned}$$

Thus $\tilde{\mathfrak{z}}$ is a ∂^T-S-stationary point of $(\mathscr{MSMPEC})$. □

Now, we present the example of a nondifferentiable semi-infinite mathematical programming problem with equilibrium constraints to illustrate the necessary optimality conditions established in Theorem 3.1.

Example 3.1. Consider the following semi-infinite mathematical programming problems with equilibrium constraints

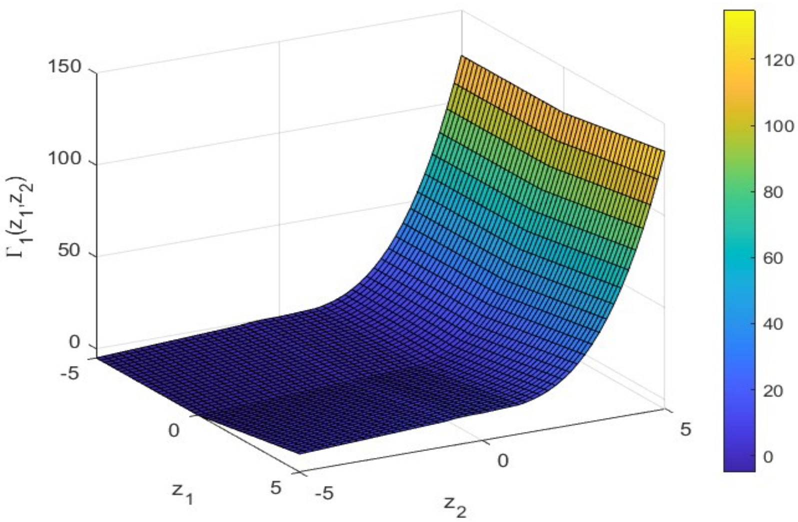

FIGURE 1. The graph of the objective function $\Gamma_1(\mathfrak{z}_1,\mathfrak{z}_2)$ of example 3.1.

$$(\mathscr{EXMSMPEC}): \quad \begin{cases} min\, \Gamma(\mathfrak{z}_1,\mathfrak{z}_2) = \Big(\Gamma_1(\mathfrak{z}_1,\mathfrak{z}_2), \Gamma_2(\mathfrak{z}_1,\mathfrak{z}_2)\Big) \\ s.\,t. \begin{cases} c_s(\mathfrak{z}_1,\mathfrak{z}_2) = |\mathfrak{z}_1| - s \le 0, \ \forall s \in \big[0,1\big], \\ \mathscr{T}_1(\mathfrak{z}_1,\mathfrak{z}_2) = \mathfrak{z}_2 \ge 0, \\ \zeta_1(\mathfrak{z}_1,\mathfrak{z}_2) = \begin{cases} \mathfrak{z}_1 \text{ if } \mathfrak{z}_2 \ge 0, \\ 2\mathfrak{z}_2 \text{ elsewhere} \end{cases} \ge 0, \\ \quad \mathscr{T}_1(\mathfrak{z}_1,\mathfrak{z}_2)\zeta_1(\mathfrak{z}_1,\mathfrak{z}_2) = 0, \end{cases} \end{cases}$$

where $\Gamma_1(\mathfrak{z}_1,\mathfrak{z}_2) = \mathfrak{z}_1 + |\mathfrak{z}_1| + \max(\mathfrak{z}_2, \mathfrak{z}_2^3)$ and $\Gamma_2(\mathfrak{z}_1,\mathfrak{z}_2) = |\mathfrak{z}_1| + \mathfrak{z}_2$. Then, the feasible region of the $(\mathscr{EXMSMPEC})$ is given by

$$\Lambda_{\mathscr{EXMSMPEC}} = \Big\{(\mathfrak{z}_1,\mathfrak{z}_2) : \mathfrak{z}_1 = 0, \mathfrak{z}_2 \ge 0\Big\}$$

and $\omega = \mu = \emptyset$, $\theta = \{1\}$. It is evident that $(\tilde{\mathfrak{z}}_1,\tilde{\mathfrak{z}}_2) = (0,0) \in \Lambda_{\mathscr{EXMSMPEC}}$ is an Pareto solution to $(\mathscr{EXMSMPEC})$. It is easy to see that the sets

$$\partial^T\Gamma_1(\tilde{\mathfrak{z}}_1,\tilde{\mathfrak{z}}_2) = [0,2]\times[0,1],\ \partial^T\Gamma_2(\tilde{\mathfrak{z}}_1,\tilde{\mathfrak{z}}_2) = [-1,1]\times\{1\},$$

$$\partial^T c_s(\tilde{\mathfrak{z}}_1,\tilde{\mathfrak{z}}_2) = \{(-1,0),\,(1,0)\},\ \partial^T(\mathscr{T}_1)(\tilde{\mathfrak{z}}_1,\tilde{\mathfrak{z}}_2) = \{(0,1)\},$$

$$\partial^T(\zeta_1)(\tilde{\mathfrak{z}}_1,\tilde{\mathfrak{z}}_2) = \{(1,0)\},$$

are the tangential subdifferentials of Γ_1, Γ_2, c_s, $\mathscr{T}_1$, ζ_1 at $(\tilde{\mathfrak{z}}_1,\tilde{\mathfrak{z}}_2)$.

By direct calculations, one has

$$\mathfrak{T}\big((\tilde{\mathfrak{z}}_1,\tilde{\mathfrak{z}}_2),\Lambda\big) = \Lambda_{\mathscr{EXMSMPEC}},\ \Big(\bigcup_{s\in\mathscr{S}_c}\partial^T c_s(\tilde{\mathfrak{z}})\Big)^\circ = \{(\mathfrak{z}_1,\mathfrak{z}_2) : \mathfrak{z}_1 = 0\},$$

$$\Big(\bigcup_{k\in\theta}\partial^T(-\mathscr{T}_k)(\tilde{\mathfrak{z}})\Big)^\circ = \{(\mathfrak{z}_1,\mathfrak{z}_2) : \mathfrak{z}_2 \ge 0\},\ \Big(\bigcup_{k\in\theta}\partial^T(-\zeta_k)(\tilde{\mathfrak{z}})\Big)^\circ = \{(\mathfrak{z}_1,\mathfrak{z}_2) : \mathfrak{z}_1 \ge 0\}.$$

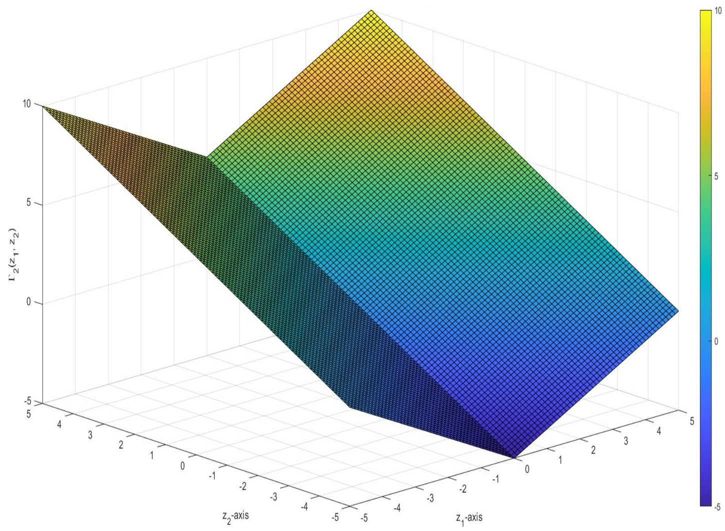

FIGURE 2. The graph of the objective function $\Gamma_2(\mathfrak{z}_1,\mathfrak{z}_2)$ of example 3.1.

Therefore

$$\Big(\bigcup_{s\in\mathscr{S}_c}\partial^T c_s(\tilde{\mathfrak{z}})\Big)^{\circ}\cap\Big(\bigcup_{k\in\theta}\partial^T(-\mathscr{T}_k)(\tilde{\mathfrak{z}})\Big)^{\circ}\cap\Big(\bigcup_{k\in\theta}\partial^T(-\zeta_k)(\tilde{\mathfrak{z}})\Big)^{\circ}=\Big\{(\mathfrak{z}_1,\mathfrak{z}_2):\mathfrak{z}_1=0,\mathfrak{z}_2\geq 0\Big\}.$$

We deduce that

$$\Big(\bigcup_{s\in\mathscr{S}_c}\partial^T c_s(\tilde{\mathfrak{z}})\Big)^{\circ}\cap\Big(\bigcup_{k\in\theta}\partial^T(-\mathscr{T}_k)(\tilde{\mathfrak{z}})\Big)^{\circ}\cap\Big(\bigcup_{k\in\theta}\partial^T(-\zeta_k)(\tilde{\mathfrak{z}})\Big)^{\circ}\subseteq\mathfrak{T}\big((\tilde{\mathfrak{z}}_1,\tilde{\mathfrak{z}}_2),\Lambda\big),$$

which demonstrates that ∂^T-ACQ is satisfied at $(\tilde{\mathfrak{z}}_1,\tilde{\mathfrak{z}}_2)$. Moreover,

$$\Omega = cone\left(\bigcup_{s\in\mathscr{S}_c}\partial^T c_s(\tilde{\mathfrak{z}})\cup\bigcup_{k\in\theta}\partial^T(-\mathscr{T}_k)(\tilde{\mathfrak{z}})\cup\bigcup_{k\in\theta}\partial^T(-\zeta_k)(\tilde{\mathfrak{z}})\right) = \{(\mathfrak{z}_1,\mathfrak{z}_2)\ :\ \mathfrak{z}_2\ \leq\ 0\}$$

is closed. Thus, if we set $\xi_1=\frac{1}{3}$, $\xi_2=\frac{2}{3}$, $\mathfrak{a}^c=\frac{1}{6}$, $\mathfrak{a}_1^{\mathscr{T}}=1$ and $\mathfrak{a}_1^{\zeta}=\frac{3}{2}$, since $(2,-1)\in\Gamma_1(\tilde{\mathfrak{z}}_1,\tilde{\mathfrak{z}}_2)$ and $(1,1)\in\Gamma_2(\tilde{\mathfrak{z}}_1,\tilde{\mathfrak{z}}_2)$, we get

$$(0,0)=\frac{1}{3}(2,1)+\frac{2}{3}(1,1)+\frac{1}{6}(1,0)+1(0,-1)+\frac{3}{2}(-1,0),$$

which shows that $(\tilde{\mathfrak{z}}_1,\tilde{\mathfrak{z}}_2)$ is a ∂^T-S-stationary point of $(\mathscr{EXMSMPEC})$.

4. SUFFICIENT OPTIMALITY CONDITIONS

In this section, we analyze and prove the sufficiency of the necessary optimality conditions developed in the previous section. We establish sufficient conditions for optimality for the $(\mathscr{MSMPEC})$ problem under study within the framework of generalized convexity, formulated in terms of the tangential subdifferentials of the involved functions. Specifically, we demonstrate that the ∂^T-S-stationary concept, introduced earlier, serves as a global sufficient optimality condition when the problem satisfies certain generalized convexity assumptions.

Now, we recall the definitions of Dini-convexity, Dini-pseudoconvexity and Dini-quasiconvexity of a function formulated in terms of tangential subdifferential. The aforesaid definitions have been derived by Tung [28].

Definition 4.1. [28] Let $\mathscr{S} \subset \mathbb{R}^n$ be a nonempty convex set and $\bar{q} \in \mathscr{S}$ be given. Further, assume that $\varphi : \mathscr{S} \to \mathbb{R}$ is a tangentially convex at $\bar{q}$. Then:

- φ is Dini-convex at $\bar{q}$ on $\mathscr{S}$ if the relation
$$\varphi(q) - \varphi(\bar{q}) \geq \langle \vartheta,\ q - \bar{q} \rangle,\ \forall\ \vartheta \in \partial^T \varphi(\bar{q})$$
hold for all $q \in \mathscr{S}$.
- φ is Dini-pseudoconvex at $\bar{q}$ on $\mathscr{S}$ if the relation
$$\varphi(q) < \varphi(\bar{q}) \Longrightarrow \langle \vartheta,\ q - \bar{q} \rangle < 0,\ \forall\ \vartheta \in \partial^T \varphi(\bar{q})$$
hold for all $q \in \mathscr{S}$.
- φ is strictly Dini-pseudoconvex at $\bar{q}$ on $\mathscr{Q}$ if the relation
$$\varphi(q) \leq \varphi(\bar{q}) \Longrightarrow \langle \vartheta,\ q - \bar{q} \rangle < 0,\ \forall\ \vartheta \in \partial^T \varphi(\bar{q})$$
holds for all $q \in \mathscr{Q} \backslash \{\bar{q}\}$.
- φ is Dini-quasiconvex at $\bar{q}$ on $\mathscr{S}$ if the relation
$$\varphi(q) \leq \varphi(\bar{q}) \Longrightarrow \langle \vartheta,\ q - \bar{q} \rangle \leq 0,\ \forall\ \vartheta \in \partial^T \varphi(\bar{q})$$
hold for all $q \in \mathscr{S}$.

For $\tilde{\mathfrak{z}} \in \Lambda$ and $(\mathfrak{a}^c, \mathfrak{a}^d, \mathfrak{a}^{\mathscr{T}}, \mathfrak{a}^{\zeta}) \in \mathbb{R}_+^{|\mathscr{S}|} \times \mathbb{R}^q \times \mathbb{R}^{2p}$, we define

$$\mathscr{S}_c^+(\tilde{\mathfrak{z}}) := \{ s \in \mathscr{S}_c(\tilde{\mathfrak{z}}) : \mathfrak{a}_s^c > 0 \},$$
$$\mathscr{J}_d^+(\tilde{\mathfrak{z}}) := \{ j \in \mathscr{J} : \mathfrak{a}_j^d > 0 \},\ \mathscr{J}_d^-(\tilde{\mathfrak{z}}) := \{ j \in \mathscr{J} : \mathfrak{a}_j^d < 0 \},$$
$$\theta^+(\tilde{\mathfrak{z}}) := \{ k \in \theta(\tilde{\mathfrak{z}}) : \mathfrak{a}_k^{\mathscr{T}} > 0,\ \mathfrak{a}_k^{\zeta} > 0 \},$$
$$\theta_{\mathscr{T}}^+(\tilde{\mathfrak{z}}) := \{ k \in \theta(\tilde{\mathfrak{z}}) : \mathfrak{a}_k^{\mathscr{T}} = 0,\ \mathfrak{a}_k^{\zeta} > 0 \},\ \theta_{\mathscr{T}}^-(\tilde{\mathfrak{z}}) := \{ k \in \theta(\tilde{\mathfrak{z}}) : \mathfrak{a}_k^{\mathscr{T}} = 0,\ \mathfrak{a}_k^{\zeta} < 0 \},$$
$$\theta_{\zeta}^+(\tilde{\mathfrak{z}}) := \{ k \in \theta(\tilde{\mathfrak{z}}) : \mathfrak{a}_k^{\mathscr{T}} > 0,\ \mathfrak{a}_k^{\zeta} = 0 \},\ \theta_{\zeta}^-(\tilde{\mathfrak{z}}) := \{ k \in \theta(\tilde{\mathfrak{z}}) : \mathfrak{a}_k^{\mathscr{T}} < 0,\ \mathfrak{a}_k^{\zeta} = 0 \},$$
$$\omega^+(\tilde{\mathfrak{z}}) := \{ k \in \omega(\tilde{\mathfrak{z}}) : \mathfrak{a}_k^{\mathscr{T}} > 0 \},\ \omega^-(\tilde{\mathfrak{z}}) := \{ k \in \omega(\tilde{\mathfrak{z}}) : \mathfrak{a}_k^{\mathscr{T}} < 0 \},$$

and

$$\mu^+(\tilde{\mathfrak{z}}) := \{ k \in \mu(\tilde{\mathfrak{z}}) : \mathfrak{a}_k^{\zeta} > 0 \},\ \mu^-(\tilde{\mathfrak{z}}) := \{ k \in \mu(\tilde{\mathfrak{z}}) : \mathfrak{a}_k^{\zeta} < 0 \}.$$

Now, under the generalized convexity assumption, we establish sufficient optimality conditions for the $(\mathscr{MSMPEC})$.

Theorem 4.1. *Let $\tilde{\mathfrak{z}} \in \Lambda$ be a ∂^T-S-stationary point of $(\mathscr{MSMPEC})$. Assume that $\Gamma_i\,(i \in \mathscr{I})$ is Dini-pseudoconvex at $\tilde{\mathfrak{z}}$, and $\omega^- \cup \mu^- = \emptyset$, $c_s\,(s \in \mathscr{S}_c)$, $d_j\,(j \in \mathscr{J}_d^+)$, $-d_j\,(j \in \mathscr{J}_d^-)$, $-\mathscr{T}_k\,(k \in \theta^+ \cup \theta_{\zeta}^+ \cup \omega^+)$, $-\zeta_k\,(k \in \theta^+ \cup \theta_{\mathscr{T}}^+ \cup \mu^+)$ are Dini-quasiconvex at $\tilde{\mathfrak{z}}$. Then $\tilde{\mathfrak{z}}$ is a weakly Pareto solution of the $(\mathscr{MSMPEC})$.*

Proof. Since $\tilde{\mathfrak{z}} \in \Lambda$ is a ∂^T-S-stationary point of $(\mathscr{M}\mathscr{S}\mathscr{M}\mathscr{P}\mathscr{E}\mathscr{C})$ and $\omega^- \cup \mu^- = \emptyset$, there exists $(\xi, \mathfrak{a}^c, \mathfrak{a}^d, \mathfrak{a}^{\mathscr{T}}, \mathfrak{a}^\zeta) \in \mathbb{R}^m_+ \times \mathbb{R}^{|\mathscr{S}|}_+ \times \mathbb{R}^q \times \mathbb{R}^{2p}$ with $\sum_{i=1}^m \xi_i = 1$, $\mathfrak{a}^{\mathscr{T}}_\theta \geq 0$, $\mathfrak{a}^\zeta_\theta \geq 0$, $\mathfrak{a}^{\mathscr{T}}_\mu = \mathfrak{a}^\zeta_\omega = 0$ and $\eta_i^\Gamma \in \partial^T \Gamma_i(\tilde{\mathfrak{z}})\,(i \in \mathscr{I})$, $\eta_s^c \in \partial^T c_s(\tilde{\mathfrak{z}})\,(s \in \mathscr{S}_c)$, $\eta_j^d \in \partial^T d_j(\tilde{\mathfrak{z}})\,(j \in \mathscr{J})$, $\eta_k^{\mathscr{T}} \in \partial^T(-\mathscr{T}_k)(\tilde{\mathfrak{z}})\,(k \in \omega \cup \theta)$, $\eta_k^\zeta \in \partial^T(-\zeta_k)(\tilde{\mathfrak{z}})\,(k \in \mu \cup \theta)$ such that

$$0 = \sum_{i \in \mathscr{I}} \xi_i \eta_i^\Gamma + \sum_{s \in \mathscr{S}_c} \mathfrak{a}_s^c \eta_s^c + \sum_{j \in \mathscr{J}} \mathfrak{a}_j^d \eta_j^d + \sum_{k \in \omega \cup \theta} \mathfrak{a}_k^{\mathscr{T}} \eta_k^{\mathscr{T}} + \sum_{k \in \mu \cup \theta} \mathfrak{a}_k^\zeta \eta_k^\zeta. \tag{4.1}$$

Let $\mathfrak{z}_0 \in \Lambda$ be any arbitrary feasible point of $(\mathscr{M}\mathscr{S}\mathscr{M}\mathscr{P}\mathscr{E}\mathscr{C})$. Then, for any $s \in \mathscr{S}_c$, $c_s(\mathfrak{z}_0) \leq 0 = c_s(\tilde{\mathfrak{z}})$. Since $c_s\,(s \in \mathscr{S}_c)$ is Dini-quasiconvex at $\mathfrak{z}$, by Definition 4.1, we see that $\left\langle \eta_s^c, \mathfrak{z}_0 - \tilde{\mathfrak{z}} \right\rangle \leq 0$, for all $s \in \mathscr{S}_c$, for any $\eta_s^c \in \partial^T c_s(\tilde{\mathfrak{z}})$. As $\mathfrak{a}_s^c \geq 0$, we have

$$\left\langle \sum_{s \in \mathscr{S}_c} \mathfrak{a}_s^c \eta_s^c, \mathfrak{z}_0 - \tilde{\mathfrak{z}} \right\rangle \leq 0, \tag{4.2}$$

for any $\eta_s^c \in \partial^T c_s(\tilde{\mathfrak{z}})$. Since $d_j(\mathfrak{z}_0) = d_j(\tilde{\mathfrak{z}}) = 0,\ \forall\, j \in \mathscr{J}$, we have

$$d_j(\mathfrak{z}_0) \leq d_j(\tilde{\mathfrak{z}}),\ \forall\, j \in \mathscr{J}_d^+ \text{ and } -d_j(\mathfrak{z}_0) \leq -d_j(\tilde{\mathfrak{z}}),\ \forall\, j \in \mathscr{J}_d^-.$$

Since $d_j\,(j \in \mathscr{J}_d^+)$ and $-d_j\,(j \in \mathscr{J}_d^-)$ are Dini-quasiconvex at $\tilde{\mathfrak{z}}$, by Definition 4.1, we get that

$$\left\langle \eta_j^d, \mathfrak{z}_0 - \tilde{\mathfrak{z}} \right\rangle \leq 0,\ \forall\ j \in \mathscr{J}_d^+ \text{ and } \left\langle -\eta_j^d, \mathfrak{z}_0 - \tilde{\mathfrak{z}} \right\rangle \leq 0,\ \forall\ j \in \mathscr{J}_d^-,$$

holds for any $\eta_j^d \in \partial^T d_j(\tilde{\mathfrak{z}})$. Considering the definitions of $\mathscr{J}_d^+$, $\mathscr{J}_d^-$, this provides us

$$\left\langle \sum_{j \in \mathscr{J}} \mathfrak{a}_j^d \eta_j^d, \mathfrak{z}_0 - \tilde{\mathfrak{z}} \right\rangle \leq 0. \tag{4.3}$$

For $\mathfrak{z}_0 \in \Lambda$, it holds that $-\mathscr{T}_k(\mathfrak{z}_0) \leq 0, \forall k \in K_p$. Consequently, for indices $k \in \theta^+ \cup \theta_\zeta^+ \cup \omega^+$, we have $-\mathscr{T}_k(\mathfrak{z}_0) \leq -\mathscr{T}_k(\tilde{\mathfrak{z}})$. Since $-\mathscr{T}_k\,(k \in \theta^+ \cup \theta_\zeta^+ \cup \omega^+)$ is Dini-quasiconvex at $\tilde{\mathfrak{z}}$, by Definition 4.1, we see that $\left\langle \eta_k^{\mathscr{T}}, \mathfrak{z}_0 - \tilde{\mathfrak{z}} \right\rangle \leq 0$, for any $\eta_k^{\mathscr{T}} \in \partial^T \mathscr{T}_k(\tilde{\mathfrak{z}})$. The definition of $\theta^+ \cup \theta_\zeta^+ \cup \omega^+$implies that

$$\left\langle \sum_{k \in \theta^+ \cup \theta_\zeta^+ \cup \omega^+} \mathfrak{a}_k^{\mathscr{T}} \eta_k^{\mathscr{T}}, \mathfrak{z}_0 - \tilde{\mathfrak{z}} \right\rangle \leq 0. \tag{4.4}$$

In a similar fashion, we can substantiate that

$$\left\langle \sum_{k \in \theta^+ \cup \theta_{\mathscr{T}}^+ \cup \mu^+} \mathfrak{a}_k^\zeta \eta_k^\zeta, \mathfrak{z}_0 - \tilde{\mathfrak{z}} \right\rangle \leq 0 \tag{4.5}$$

for any $\eta_k^\zeta \in \partial^T \zeta_k(\tilde{\mathfrak{z}})$. Therefore, we deduce from equations (4.1), (4.2), (4.3), (4.4), (4.5), and the fact that $\omega^- \cup \mu^- = \emptyset$ that, for any $\mathfrak{z}_0 \in \Lambda$,

$$\left\langle \sum_{i \in \mathscr{I}} \xi_i \eta_i^\Gamma, \mathfrak{z}_0 - \tilde{\mathfrak{z}} \right\rangle = -\left\langle \sum_{s \in \mathscr{S}_c} \mathfrak{a}_s^c \eta_s^c + \sum_{j \in \mathscr{J}} \mathfrak{a}_j^d \eta_j^d + \sum_{k \in K_p} \mathfrak{a}_k^{\mathscr{T}} \eta_k^{\mathscr{T}} + \sum_{k \in K_p} \mathfrak{a}_k^\zeta \eta_k^\zeta, \mathfrak{z}_0 - \tilde{\mathfrak{z}} \right\rangle \geq 0.$$

$$\Longrightarrow \left\langle \sum_{i \in \mathscr{I}} \xi_i \eta_i^\Gamma, \mathfrak{z}_0 - \tilde{\mathfrak{z}} \right\rangle \geq 0. \tag{4.6}$$

Assume, for the sake of contradiction, that $\tilde{\mathfrak{z}}$ is not a weakly Pareto solution of $(\mathscr{MSMPEC})$. This implies the existence of a feasible point $\mathfrak{z}^* \in \Lambda$ such that $\Gamma_i(\mathfrak{z}^*) < \Gamma_i(\tilde{\mathfrak{z}})$, for all $i \in \mathscr{I}$. Since $\Gamma_i\,(i \in \mathscr{I})$ is Dini-pseudoconvex at $\tilde{\mathfrak{z}}$, we see by Definition 4.1 that $\langle \eta_i^\Gamma,\, \mathfrak{z}^* - \tilde{\mathfrak{z}}\rangle < 0$ for all $i \in \mathscr{I}$. As $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}_+^m$, with $\sum_{i=1}^m \xi_i = 1$, we have

$$\left\langle \sum_{i\in\mathscr{I}} \xi_i \eta_i^\Gamma,\, \mathfrak{z}^* - \tilde{\mathfrak{z}} \right\rangle < 0,$$

for any $\eta_i^\Gamma \in \partial^T \Gamma_i(\mathfrak{x})$. This contradicts inequality (4.6). Therefore, the theorem is proven. □

Theorem 4.2. *Let $\tilde{\mathfrak{z}} \in \Lambda$ be a ∂^T-S-stationary point of $(\mathscr{MSMPEC})$. Assume that $\Gamma_i\,(i \in \mathscr{I})$ is strictly Dini-pseudoconvex at $\tilde{\mathfrak{z}}$, and $\omega^- \cup \mu^- = \emptyset$, $c_s\,(s \in \mathscr{S}_c)$, $d_j\,(j \in \mathscr{J}_d^+)$, $-d_j\,(j \in \mathscr{J}_d^-)$, $-\mathscr{T}_k\,(k \in \theta^+ \cup \theta_\zeta^+ \cup \omega^+)$, $-\zeta_k\,(k \in \theta^+ \cup \theta_{\mathscr{T}}^+ \cup \mu^+)$ are Dini-quasiconvex at $\tilde{\mathfrak{z}}$. Then $\tilde{\mathfrak{z}}$ is a Pareto solution of the $(\mathscr{MSMPEC})$.*

Proof. By contrary, assume that $\tilde{\mathfrak{z}}$ is not a Pareto solution of $(\mathscr{MSMPEC})$. Then there exists feasible point $\mathfrak{z}^* \in \Lambda$ and $i_0 \in \mathscr{I}$ such that

$$\begin{cases} \Gamma_i(\mathfrak{z}^*) \leq \Gamma_i(\tilde{\mathfrak{z}}),\ \forall\, i \in \mathscr{I}\backslash\{i_0\}, \\ \Gamma_{i_0}(\mathfrak{z}^*) \leq \Gamma_{i_0}(\tilde{\mathfrak{z}}). \end{cases}$$

Consequently, $\mathfrak{z}^* \neq \tilde{\mathfrak{z}}$. Due to the strictly Dini-pseudoconvex of $\Gamma_i\,(i \in \mathscr{I})$ at $\tilde{\mathfrak{z}}$, it follows that $\langle \eta_i^\Gamma,\, \mathfrak{z}^* - \tilde{\mathfrak{z}}\rangle < 0$, for all $i \in \mathscr{I}$. As $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}_+^m$, we have

$$\left\langle \sum_{i\in\mathscr{I}} \xi_i \eta_i^\Gamma,\, \mathfrak{z}^* - \tilde{\mathfrak{z}} \right\rangle < 0.$$

This contradicts inequality (4.6). This completes the proof of this theorem. □

Now, we present the example of an semi-infinite mathematical programming problems with equilibrium constraints to illustrate the optimality conditions established in this section.

Example 4.1. Consider the following semi-infinite mathematical programming problems with equilibrium constraints

$$(\mathscr{EXMSMPEC}): \quad \begin{cases} \min\, \Gamma(\mathfrak{z}_1, \mathfrak{z}_2) = \Big(\Gamma_1(\mathfrak{z}_1, \mathfrak{z}_2), \Gamma_2(\mathfrak{z}_1, \mathfrak{z}_2)\Big) \\ s.\,t. \begin{cases} c_s(\mathfrak{z}_1, \mathfrak{z}_2) = s\mathfrak{z}_1 \leq 0,\ \forall s \in \mathscr{S} = \mathbb{N} = \{1,2\ldots\}, \\ \mathscr{T}_1(\mathfrak{z}_1, \mathfrak{z}_2) = \mathfrak{z}_1 \geq 0, \\ \zeta_1(\mathfrak{z}_1, \mathfrak{z}_2) = \frac{1}{3}\mathfrak{z}_1 + \mathfrak{z}_2 \geq 0, \\ \quad \mathscr{T}_1(\mathfrak{z}_1, \mathfrak{z}_2)\zeta_1(\mathfrak{z}_1, \mathfrak{z}_2) = 0, \end{cases} \end{cases}$$

where $\Gamma_1(\mathfrak{z}_1, \mathfrak{z}_2) = \max(\mathfrak{z}_1, \mathfrak{z}_2)$ and $\Gamma_2(\mathfrak{z}_1, \mathfrak{z}_2) = \frac{1}{2}|\mathfrak{z}_1| + \frac{1}{2}|\mathfrak{z}_2|$. The set of feasible solutions $\Lambda_{\mathscr{EXMSMPEC}}$ is given by $\Lambda_{\mathscr{EXMSMPEC}} = \big\{(\mathfrak{z}_1, \mathfrak{z}_2) : \mathfrak{z}_1 = 0, \mathfrak{z}_2 \geq 0\big\}$ and $\omega = \mu = \emptyset$, $\theta = \{1\}$. It is evident that $(\tilde{\mathfrak{z}}_1, \tilde{\mathfrak{z}}_2) = (0,0) \in \Lambda_{\mathscr{EXMSMPEC}}$ is a feasible point of $(\mathscr{EXMSMPEC})$. By Definition 2.3, $\partial^T \Gamma_1(\tilde{\mathfrak{z}}_1, \tilde{\mathfrak{z}}_2) = \big\{(1,0), (0,1)\big\}$, $\partial^T \Gamma_2(\tilde{\mathfrak{z}}_1$ and $\tilde{\mathfrak{z}}_2) = \big[-\frac{1}{2}, \frac{1}{2}\big] \times \big[-\frac{1}{2}, \frac{1}{2}\big]$ are the tangential subdifferentials of Γ_1, Γ_2. It can be shown, by Definition 4.1, that the objective functions Γ_1 and Γ_2 are Dini-pseudoconvex at $(\tilde{\mathfrak{z}}_1, \tilde{\mathfrak{z}}_2)$. By Definition 2.3, the sets $\partial^T c_s(\tilde{\mathfrak{z}}_1, \tilde{\mathfrak{z}}_2) = \big\{(s,0)\big\}, s \in \mathscr{S}$, $\partial^T(-\mathscr{T}_1)(\tilde{\mathfrak{z}}_1, \tilde{\mathfrak{z}}_2) = \big\{(-1,0)\big\}$ and $\partial^T(-\zeta_1)(\tilde{\mathfrak{z}}_1, \tilde{\mathfrak{z}}_2) = \big\{(-\frac{1}{3}, -1)\big\}$ are the tangential subdifferentials of c_s, $-\mathscr{T}_1$ and $-\zeta_1$ at $(\tilde{\mathfrak{z}}_1, \tilde{\mathfrak{z}}_2)$. Further, it can be shown, by Definition

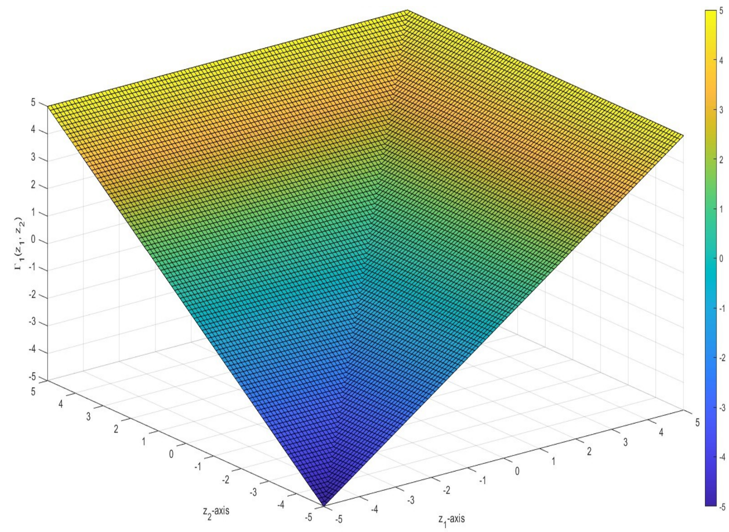

FIGURE 3. The graph of the objective function $\Gamma_1(\mathfrak{z}_1, \mathfrak{z}_2)$ of Example 4.1.

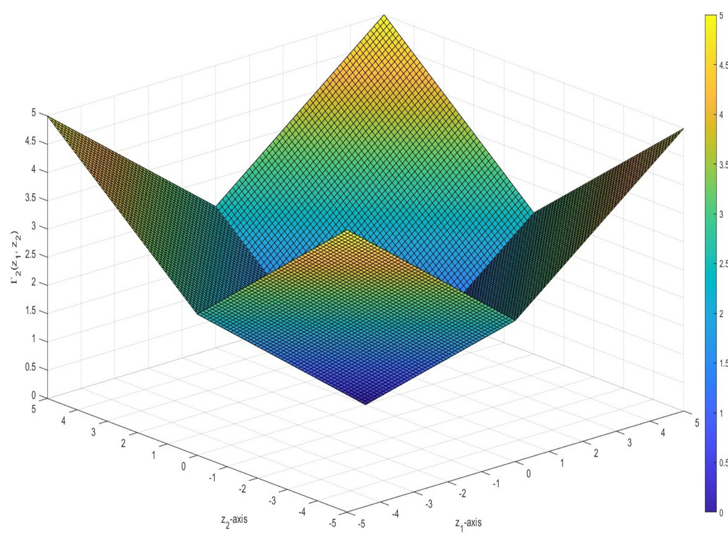

FIGURE 4. The graph of the objective function $\Gamma_2(\mathfrak{z}_1, \mathfrak{z}_2)$ of Example 4.1.

4.1, that c_s, $-\mathscr{T}_1$ and $-\zeta_1$ are Dini-quasiconvex at $(\tilde{\mathfrak{z}}_1, \tilde{\mathfrak{z}}_2)$. It can be shown that the ∂^T-ACQ holds at $(\tilde{\mathfrak{z}}_1, \tilde{\mathfrak{z}}_2)$. Indeed, by Definition 3.1, one has

$$\Big(\bigcup_{s\in\mathscr{S}_c} \partial^T c_s(\tilde{\mathfrak{z}})\Big)^{\circ} = \big\{(\mathfrak{z}_1, \mathfrak{z}_2) : \mathfrak{z}_1 \leq 0\big\}, \Big(\bigcup_{k\in\theta} \partial^T(-\mathscr{T}_k)(\tilde{\mathfrak{z}})\Big)^{\circ} = \big\{(\mathfrak{z}_1, \mathfrak{z}_2) : \mathfrak{z}_2 \geq 0\big\}$$

and

$$\Big(\bigcup_{k\in\theta} \partial^T(-\zeta_k)(\tilde{\mathfrak{z}})\Big)^{\circ} = \big\{(\mathfrak{z}_1, \mathfrak{z}_2) : \mathfrak{z}_1 \geq 0, \mathfrak{z}_2 \geq 0\big\}.$$

Thus

$$\Theta(\tilde{\mathfrak{z}}_1, \tilde{\mathfrak{z}}_2) = \big\{(\mathfrak{z}_1, \mathfrak{z}_2) : \mathfrak{z}_1 = 0, \mathfrak{z}_2 \geq 0\big\} = \mathfrak{T}\big((\tilde{\mathfrak{z}}_1, \tilde{\mathfrak{z}}_2), \Lambda\big).$$

For $\xi_1 = \frac{1}{5}$, $\xi_2 = \frac{4}{5}$, $\mathfrak{a}_1^{\mathscr{T}} = \frac{22}{15}$, $\mathfrak{a}_1^{\zeta} = \frac{2}{5}$, let $\mathfrak{a}^c : \mathscr{S} \to \mathbb{R}$ be defined by

$$\mathfrak{a}^c(s) = \begin{cases} 1 \text{ if } s = 1, \\ 0 \text{ otherwise.} \end{cases}$$

It follows that

$$(0,0) = \frac{1}{5}(1,0) + \frac{4}{5}\Big(\frac{1}{2}, \frac{1}{2}\Big) + 1(1,0) + \frac{22}{15}\Big(-1, 0\Big) + \frac{2}{5}\Big(-\frac{1}{3}, -1\Big),$$

which means that $(\tilde{\mathfrak{z}}_1, \tilde{\mathfrak{z}}_2)$ is a ∂^T-S-stationary point. Hence, by Theorem 4.1, it follows that $(\tilde{\mathfrak{z}}_1, \tilde{\mathfrak{z}}_2)$ is a weakly Pareto solution of $(\mathscr{E}\mathscr{X}\mathscr{M}\mathscr{S}\mathscr{M}\mathscr{P}\mathscr{E}\mathscr{C})$.

5. Mond–Weir Duality

In this section, we develop the Mond-Weir [19] type dual formulation for the nondifferentiable multiobjective semi-infinite mathematical programming problem with equilibrium constraints $(\mathscr{M}\mathscr{S}\mathscr{M}\mathscr{P}\mathscr{E}\mathscr{C})$. Under suitable generalized convexity assumptions on the involved functions, we derive duality relationships between the primal problem $(\mathscr{M}\mathscr{S}\mathscr{M}\mathscr{P}\mathscr{E}\mathscr{C})$ and its corresponding dual.

To facilitate our discussion, we use the following notations for $\mathfrak{h}, \mathfrak{k} \in \mathbb{R}^n$:

$$\mathfrak{h} \prec \mathfrak{k} \Leftrightarrow \mathfrak{h}_i < \mathfrak{k}_i, \text{ for all } i \in \mathscr{I},$$

$$\mathfrak{h} \preceq \mathfrak{k} \Leftrightarrow \begin{cases} \mathfrak{h}_i \leq \mathfrak{k}_i, \text{ for all } i \in \mathscr{I}, \\ \mathfrak{h}_{i^*} < \mathfrak{k}_{i^*} \text{ for at least one } i^* \in \mathscr{I}, \end{cases}$$

and $\mathfrak{h} \not\prec \mathfrak{k}$ and $\mathfrak{h} \not\preceq \mathfrak{k}$ are the negations of $\mathfrak{h} \prec \mathfrak{k}$ and $\mathfrak{h} \preceq \mathfrak{k}$, respectively. For $\tilde{\mathfrak{z}} \in \Lambda$, we present the corresponding Mond-Weir-type dual problem associated with the $(\mathscr{M}\mathscr{S}\mathscr{M}\mathscr{P}\mathscr{E}\mathscr{C})$ as follows:

$$\begin{aligned}
\big(\mathscr{M}\mathscr{W}\mathscr{S}\mathscr{M}\mathscr{P}\mathscr{E}\mathscr{C}(\tilde{\mathfrak{z}})\big) : &\max \Gamma(\mathfrak{x}) \\
s.t\ 0 \in &\sum_{i \in \mathscr{I}} \xi_i \partial^T \Gamma_i(\mathfrak{x}) + \sum_{s \in \mathscr{S}_c} \mathfrak{a}_s^c \partial^T c_s(\mathfrak{x}) + \sum_{j \in \mathscr{J}} \mathfrak{a}_j^d \partial^T d_j(\mathfrak{x}) \\
&+ \sum_{k \in K_p} \mathfrak{a}_k^{\mathscr{T}} \partial^T(-\mathscr{T}_k)(\mathfrak{x}) + \sum_{k \in K_p} \mathfrak{a}_k^{\zeta} \partial^T(-\zeta_k)(\mathfrak{x}), \\
&c_s(\mathfrak{x}) \geq 0\,(s \in \mathscr{S}_c),\, d_j(\mathfrak{x}) = 0\,(j \in \mathscr{J}), \\
&-\mathscr{T}_k(\mathfrak{x}) \geq 0\,(k \in \omega \cup \theta),\, -\zeta_k(\mathfrak{x}) \geq 0\,(k \in \mu \cup \theta), \\
&\sum_{i \in \mathscr{I}} \xi_i = 1,\, \mathfrak{a}^c_{\mathscr{S} \setminus \mathscr{S}_c} = 0,\, \mathfrak{a}^{\mathscr{T}}_{\mu} = 0,\, \mathfrak{a}^{\mathscr{T}}_{\theta} \geq 0,\, \mathfrak{a}^{\zeta}_{\omega} = 0,\, \mathfrak{a}^{\zeta}_{\theta} \geq 0, \\
&(\mathfrak{x}, \xi, \mathfrak{a}^c, \mathfrak{a}^d, \mathfrak{a}^{\mathscr{T}}, \mathfrak{a}^{\zeta}) \in \mathbb{R}^n \times \mathbb{R}^m_+ \times \mathbb{R}^{|\mathscr{S}|}_+ \times \mathbb{R}^q \times \mathbb{R}^{2p}.
\end{aligned}$$

The set of all feasible solutions $(\mathfrak{x}, \xi, \mathfrak{a}^c, \mathfrak{a}^d, \mathfrak{a}^{\mathscr{T}}, \mathfrak{a}^{\zeta})$ for problem $\left(\mathscr{MWSMPEC}\right)$ is denoted by $\Lambda_{\mathscr{MWSMPEC}}$ and defined by

$$\begin{aligned}\Lambda_{\mathscr{MWSMPEC}} = \Big\{ &(\mathfrak{x}, \xi, \mathfrak{a}^c, \mathfrak{a}^d, \mathfrak{a}^{\mathscr{T}}, \mathfrak{a}^{\zeta}) \in \mathbb{R}^n \times \mathbb{R}^m_+ \times \mathbb{R}^{|\mathscr{S}|}_+ \times \mathbb{R}^q \times \mathbb{R}^{2p} : \\ &0 \in \sum_{i\in\mathscr{I}} \xi_i \partial^T \Gamma_i(\mathfrak{x}) + \sum_{s\in\mathscr{S}_c} \mathfrak{a}^c_s \partial^T c_s(\mathfrak{x}) + \sum_{j\in\mathscr{J}} \mathfrak{a}^d_j \partial^T d_j(\mathfrak{x}) \\ &\quad + \sum_{k\in K_p} \mathfrak{a}^{\mathscr{T}}_k \partial^T(-\mathscr{T}_k)(\mathfrak{x}) + \sum_{k\in K_p} \mathfrak{a}^{\zeta}_k \partial^T(-\zeta_k)(\mathfrak{x}), \\ &c_s(\mathfrak{x}) \geq 0\,(s\in\mathscr{S}_c),\, d_j(\mathfrak{x}) = 0\,(j\in\mathscr{J}),\, -\mathscr{T}_k(\mathfrak{x}) \geq 0\,(k\in\omega\cup\theta), \\ &-\zeta_k(\mathfrak{x}) \geq 0\,(k\in\mu\cup\theta), \sum_{i\in\mathscr{I}} \xi_i = 1,\, \mathfrak{a}^c_{\mathscr{S}\setminus\mathscr{S}_c} = 0,\, \mathfrak{a}^{\mathscr{T}}_{\mu} = 0,\, \mathfrak{a}^{\mathscr{T}}_{\theta} \geq 0,\, \mathfrak{a}^{\zeta}_{\omega} = 0,\, \mathfrak{a}^{\zeta}_{\theta} \geq 0 \Big\}.\end{aligned}$$

Under the assumptions of Dini-pseudoconvexity and Dini-quasiconvexity, we prove the following weak duality theorem connecting our primal problem $(\mathscr{MSMPEC})$ with $(\mathscr{MWSMPEC})$.

Theorem 5.1. ***(Weak Duality):*** *Let $\mathfrak{z}_0 \in \Lambda$ and $(\mathfrak{x}, \xi, \mathfrak{a}^c, \mathfrak{a}^d, \mathfrak{a}^{\mathscr{T}}, \mathfrak{a}^{\zeta}) \in \Lambda_{\mathscr{MWSMPEC}}$. If $\Gamma_i\,(i\in\mathscr{I})$ is Dini-pseudoconvex at $\mathfrak{x}$, $\omega^- \cup \mu^- = \emptyset$, $c_s\,(s\in\mathscr{S})$, $d_j\,(j\in\mathscr{J}^+_d)$, $-d_j\,(j\in\mathscr{J}^-_d)$, $-\mathscr{T}_k\,(k\in\theta^+\cup\theta^+_{\zeta}\cup\omega^+)$, $-\zeta_k\,(k\in\theta^+\cup\theta^+_{\mathscr{T}}\cup\mu^+)$ are Dini-quasiconvex at $\mathfrak{x}$, then $\Gamma(\mathfrak{z}_0) \not\leq \Gamma(\mathfrak{x})$.*

Proof. Since $\mathfrak{z}_0 \in \Lambda$, we have

$$c_s(\mathfrak{z}_0) \leq 0\ (s\in\mathscr{S}),\ d_j(\mathfrak{z}_0) = 0\ (j\in\mathscr{J}),\ \mathscr{T}_k(\mathfrak{z}_0) \geq 0,$$
$$\zeta_k(\mathfrak{z}_0) \geq 0,\ \mathscr{T}_k(\mathfrak{z}_0)\zeta_k(\mathfrak{z}_0) = 0\ (k\in K_p),$$

and $(\mathfrak{x}, \xi, \mathfrak{a}^c, \mathfrak{a}^d, \mathfrak{a}^{\mathscr{T}}, \mathfrak{a}^{\zeta}) \in \Lambda_{\mathscr{MWSMPEC}}$. Note that $(\xi, \mathfrak{a}^c, \mathfrak{a}^d, \mathfrak{a}^{\mathscr{T}}, \mathfrak{a}^{\zeta}) \in \mathbb{R}^m_+ \times \mathbb{R}^{|\mathscr{S}|}_+ \times \mathbb{R}^q \times \mathbb{R}^{2p}$ with $\sum_{i\in\mathscr{I}} \xi_i = 1$, $\mathfrak{a}^c_{\mathscr{S}\setminus\mathscr{S}_c} = 0$, $\mathfrak{a}^{\mathscr{T}}_{\mu} = 0$, $\mathfrak{a}^{\mathscr{T}}_{\theta} \geq 0$, $\mathfrak{a}^{\zeta}_{\omega} = 0$, $\mathfrak{a}^{\zeta}_{\theta} \geq 0$ and that there exist $\eta^{\Gamma}_i \in \partial^T\Gamma_i(\mathfrak{x})\,(i\in\mathscr{I})$, $\eta^c_s \in \partial^T c_s(\mathfrak{x})\,(s\in\mathscr{S}_c)$, $\eta^d_j \in \partial^T d_j(\mathfrak{x})\,(j\in\mathscr{J})$, $\eta^{\mathscr{T}}_k \in \partial^T(-\mathscr{T}_k)(\mathfrak{x})\,(k\in k\in K_p)$, $\eta^{\zeta}_k \in \partial^T(-\zeta_k)(\mathfrak{x})\,(k\in K_p)$ such that

$$0 = \sum_{i\in\mathscr{I}} \xi_i \eta^{\Gamma}_i + \sum_{s\in\mathscr{S}_c} \mathfrak{a}^c_s \eta^c_s + \sum_{j\in\mathscr{J}} \mathfrak{a}^d_j \eta^d_j + \sum_{k\in K_p} \mathfrak{a}^{\mathscr{T}}_k \eta^{\mathscr{T}}_k + \sum_{k\in K_p} \mathfrak{a}^{\zeta}_k \eta^{\zeta}_k \tag{5.1}$$

$$c_s(\mathfrak{x}) \geq 0\,(s\in\mathscr{S}_c),\, d_j(\mathfrak{x}) = 0\,(j\in\mathscr{J}),\, -\mathscr{T}_k(\mathfrak{x}) \geq 0\,(k\in\omega\cup\theta),$$
$$-\zeta_k(\mathfrak{x}) \geq 0\,(k\in\mu\cup\theta), \sum_{i\in\mathscr{I}} \xi_i = 1,\, \mathfrak{a}^c_{\mathscr{S}\setminus\mathscr{S}_c} = 0,\, \mathfrak{a}^{\mathscr{T}}_{\mu} = 0,\, \mathfrak{a}^{\mathscr{T}}_{\theta} \geq 0,\, \mathfrak{a}^{\zeta}_{\omega} = 0,\, \mathfrak{a}^{\zeta}_{\theta} \geq 0.$$

These imply

$$c_s(\mathfrak{z}_0) \leq 0 \leq c_s(\mathfrak{x}),\ \forall s\in\mathscr{S}_c,$$
$$d_j(\mathfrak{z}_0) = d_j(\mathfrak{x}) = 0,\ \forall j\in\mathscr{J}^+_d \cup \mathscr{J}^-_d,$$
$$-\mathscr{T}_k(\mathfrak{z}_0) \leq 0 \leq -\mathscr{T}_k(\mathfrak{x}),\ \forall k\in\theta^+\cup\theta^+_{\zeta}\cup\omega^+,$$

and

$$-\zeta_k(\mathfrak{z}_0) \leq 0 \leq -\zeta_k(\mathfrak{x}),\ \forall k\in\theta^+\cup\theta^+_{\mathscr{T}}\cup\mu^+.$$

Since $c_s\,(s\in\mathscr{S})$, $d_j\,(j\in\mathscr{J}^+_d)$, $-d_j\,(j\in\mathscr{J}^-_d)$, $-\mathscr{T}_k\,(k\in\theta^+\cup\theta^+_{\zeta}\cup\omega^+)$, $-\zeta_k\,(k\in\theta^+\cup\theta^+_{\mathscr{T}}\cup\mu^+)$ are Dini-quasiconvex at $\mathfrak{x}$, by Definition 4.1, the inequalities above yield, respectively, for any $\eta_s^c \in \partial^T c_s(\mathfrak{x})\,(s \in \mathscr{S})$, $\eta_j^d \in \partial^T d_j(\mathfrak{x})\,(j \in \mathscr{J})$, $\eta_k^{\mathscr{T}} \in \partial^T(-\mathscr{T}_k)(\mathfrak{x})\,(k \in \omega \cup \theta)$, $\eta_k^{\zeta} \in \partial^T(-\zeta_k)(\mathfrak{x})\,(k \in \mu \cup \theta)$, along with the definitions of the index sets that

$$\left\langle \eta_s^c,\, \mathfrak{z}_0 - \mathfrak{x} \right\rangle \leq 0,\ \forall\, s \in \mathscr{S}_c, \tag{5.2}$$

$$\left\langle \eta_j^d,\, \mathfrak{z}_0 - \mathfrak{x} \right\rangle \leq 0,\ \forall\, j \in \mathscr{J}_d^+, \tag{5.3}$$

$$\left\langle -\eta_j^d,\, \mathfrak{z}_0 - \mathfrak{x} \right\rangle \leq 0,\ \forall\, j \in \mathscr{J}_d^-, \tag{5.4}$$

$$\left\langle \eta_k^{\mathscr{T}},\, \mathfrak{z}_0 - \mathfrak{x} \right\rangle \leq 0,\ \forall\, k \in \theta^+ \cup \theta_{\zeta}^+ \cup \omega^+, \tag{5.5}$$

and

$$\left\langle \eta_k^{\zeta},\, \mathfrak{z}_0 - \mathfrak{x} \right\rangle \leq 0,\ \forall\, k \in \theta^+ \cup \theta_{\mathscr{T}}^+ \cup \mu^+. \tag{5.6}$$

The aforementioned inequalities, along with $\omega^- \cup \mu^- = \emptyset$ and $\mathfrak{a}^c_{\mathscr{S}\setminus\mathscr{S}_c} = 0$, lead to the conclusion that

$$\left\langle \sum_{s\in\mathscr{S}_c} \mathfrak{a}_s^c\,\eta_s^c + \sum_{j\in\mathscr{J}} \mathfrak{a}_j^d\,\eta_j^d + \sum_{k\in K_p} \mathfrak{a}_k^{\mathscr{T}}\,\eta_k^{\mathscr{T}} + \sum_{k\in K_p} \mathfrak{a}_k^{\zeta}\,\eta_k^{\zeta},\, \mathfrak{z}_0 - \mathfrak{x} \right\rangle \leq 0, \tag{5.7}$$

holds for any $\eta_s^c \in \partial^T c_s(\mathfrak{x})\,(s \in \mathscr{S})$, $\eta_j^d \in \partial^T d_j(\mathfrak{x})\,(j \in \mathscr{J})$, $\eta_k^{\mathscr{T}} \in \partial^T(-\mathscr{T}_k)(\mathfrak{x})\,(k \in \omega \cup \theta)$, $\eta_k^{\zeta} \in \partial^T(-\zeta_k)(\mathfrak{x})\,(k \in \mu \cup \theta)$. From inequality (5.7) and equation (5.1), we have

$$\left\langle \sum_{i\in\mathscr{I}} \xi_i\,\eta_i^{\Gamma},\, \mathfrak{z}_0 - \mathfrak{x} \right\rangle = -\left\langle \sum_{s\in\mathscr{S}_c} \mathfrak{a}_s^c\,\eta_s^c + \sum_{j\in\mathscr{J}} \mathfrak{a}_j^d\,\eta_j^d + \sum_{k\in K_p} \mathfrak{a}_k^{\mathscr{T}}\,\eta_k^{\mathscr{T}} + \sum_{k\in K_p} \mathfrak{a}_k^{\zeta}\,\eta_k^{\zeta},\, \mathfrak{z}_0 - \mathfrak{x} \right\rangle \geq 0,$$

$$\Rightarrow \left\langle \sum_{i\in\mathscr{I}} \xi_i\,\eta_i^{\Gamma},\, \mathfrak{z}_0 - \mathfrak{x} \right\rangle \geq 0. \tag{5.8}$$

Contrary to the result, suppose that $\Gamma(\mathfrak{z}_0) \preceq \Gamma(\mathfrak{x})$. This is equivalent to

$$\begin{cases} \Gamma_i(\mathfrak{z}_0) \leq \Gamma_i(\mathfrak{x}),\ \forall\, i \in \mathscr{I}, \\ \Gamma_{i^*}(\mathfrak{z}_0) < \Gamma_{i^*}(\mathfrak{x}),\ \text{for at least one } i^* \in \mathscr{I}. \end{cases}$$

Since $\Gamma_i\,(i \in \mathscr{I})$ is Dini-pseudoconvex at $\mathfrak{x}$, therefore, by Definition 4.1, we get that $\langle \eta_i^{\Gamma},\, \mathfrak{z}^* - \mathfrak{x}\rangle < 0$, for all $i \in \mathscr{I}$. As $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m_+$, with $\sum_{i=1}^m \xi_i = 1$, we have

$$\left\langle \sum_{i\in\mathscr{I}} \xi_i\,\eta_i^{\Gamma},\, \mathfrak{z}^* - \tilde{\mathfrak{z}} \right\rangle < 0,$$

for any $\eta_i^{\Gamma} \in \partial^T \Gamma_i(\mathfrak{x})$. This contradicts inequality (5.8). Therefore, we conclude that $\Gamma(\mathfrak{z}_0) \npreceq \Gamma(\mathfrak{x})$. This concludes the proof. $\square$

The following theorem establishes strong duality relations between problem $(\mathscr{MSMPEC})$ and its corresponding Mond-Weir-type dual formulation $(\mathscr{MWSMPEC})$.

Theorem 5.2. ***(Strong Duality):*** *Let $\tilde{\mathfrak{z}} \in LOWP(\mathscr{MSMPEC})$. If ∂^T-ACQ satisfes at $\tilde{\mathfrak{z}}$ and the set Ω is closed, then there exist $(\tilde{\xi}, \tilde{\mathfrak{a}}^c, \tilde{\mathfrak{a}}^d, \tilde{\mathfrak{a}}^{\mathscr{T}}, \tilde{\mathfrak{a}}^{\zeta}) \in \mathbb{R}^m_+ \times \mathbb{R}^{|\mathscr{S}|}_+ \times \mathbb{R}^q \times \mathbb{R}^{2p}$ with $\sum_{i\in\mathscr{I}} \tilde{\xi}_i = 1$, $\tilde{\mathfrak{a}}^{\mathscr{T}}_{\mu} = 0$, $\tilde{\mathfrak{a}}^{\mathscr{T}}_{\theta} \geq 0$, $\tilde{\mathfrak{a}}^{\zeta}_{\omega} = 0$, $\tilde{\mathfrak{a}}^{\zeta}_{\theta} \geq 0$ such that $(\tilde{\mathfrak{z}}, \tilde{\xi}, \tilde{\mathfrak{a}}^c, \tilde{\mathfrak{a}}^d, \tilde{\mathfrak{a}}^{\mathscr{T}}, \tilde{\mathfrak{a}}^{\zeta}) \in \Lambda_{\mathscr{MWSMPEC}}$. Moreover, assume that $\Gamma_i\,(i \in \mathscr{I})$ is Dini-pseudoconvex at $\tilde{\mathfrak{z}}$, $\omega^- \cup \mu^- = \emptyset$, $c_s\,(s \in \mathscr{S})$, $d_j\,(j \in \mathscr{J}_d^+)$, $-d_j\,(j \in \mathscr{J}_d^-)$, $-\mathscr{T}_k\,(k \in \theta^+ \cup \theta_{\zeta}^+ \cup \omega^+)$, $-\zeta_k\,(k \in \theta^+ \cup \theta_{\mathscr{T}}^+ \cup \mu^+)$ are Dini-quasiconvex at $\tilde{\mathfrak{z}}$. Then $(\tilde{\mathfrak{z}}, \tilde{\xi}, \tilde{\mathfrak{a}}^c, \tilde{\mathfrak{a}}^d, \tilde{\mathfrak{a}}^{\mathscr{T}}, \tilde{\mathfrak{a}}^{\zeta}) \in WP\big(\mathscr{MWSMPEC}(\tilde{\mathfrak{z}})\big)$.*

Proof. Considering Theorem 3.1, there exist multipliers $(\tilde{\xi}, \tilde{\mathfrak{a}}^c, \tilde{\mathfrak{a}}^d, \tilde{\mathfrak{a}}^{\mathscr{T}}, \tilde{\mathfrak{a}}^{\zeta}) \in \mathbb{R}^m_+ \times \Psi(\tilde{\mathfrak{z}}) \times \mathbb{R}^q \times \mathbb{R}^{2p}$ with $\sum_{i\in\mathscr{I}} \tilde{\xi}_i = 1$, $\tilde{\mathfrak{a}}^{\mathscr{T}}_\mu = 0$, $\tilde{\mathfrak{a}}^{\mathscr{T}}_\theta \geq 0$, $\tilde{\mathfrak{a}}^{\zeta}_\omega = 0$, $\tilde{\mathfrak{a}}^{\zeta}_\theta \geq 0$ such that the following conditions are satisfied:

$$0 \in \sum_{i\in\mathscr{I}} \tilde{\xi}_i \partial^T \Gamma_i(\tilde{\mathfrak{z}}) + \sum_{s\in\mathscr{S}_c} \tilde{\mathfrak{a}}^c_s \partial^T c_s(\tilde{\mathfrak{z}}) + \sum_{j\in\mathscr{J}} \tilde{\mathfrak{a}}^d_j \partial^T d_j(\tilde{\mathfrak{z}}) + \sum_{k\in K_p} \tilde{\mathfrak{a}}^{\mathscr{T}}_k \partial^T(-\mathscr{T}_k)(\tilde{\mathfrak{z}}) + \sum_{k\in K_p} \tilde{\mathfrak{a}}^{\zeta}_k \partial^T(-\zeta_k)(\tilde{\mathfrak{z}}).$$

Since $\tilde{\mathfrak{a}}^c \in \Psi(\tilde{\mathfrak{z}})$, we know that $c_s(\tilde{\mathfrak{z}}) = 0$ for $s \in \mathscr{S}_c$, while $c_s(\tilde{\mathfrak{z}}) < 0$, $\forall s \in \mathscr{S}\backslash\mathscr{S}_c$. This, together with $\tilde{\mathfrak{a}}^c_s c_s(\tilde{\mathfrak{z}}) = 0$ for $s \in \mathscr{S}_c$ implies that $\tilde{\mathfrak{a}}^c_{\mathscr{S}\backslash\mathscr{S}_c} = 0$ and the inclusion of $\tilde{\mathfrak{z}} \in \Lambda$ ensures that $d_j(\tilde{\mathfrak{z}}) = 0 \; \forall j \in \mathscr{J}$. Furthermore, since $\tilde{\mathfrak{a}}^{\mathscr{T}}_\mu = 0$, for all $k \in \omega \cup \theta$, $\mathscr{T}_k(\tilde{\mathfrak{z}}) = 0$, it follows that $\sum_{k\in K_p} \tilde{\mathfrak{a}}^{\mathscr{T}}_k \mathscr{T}_k(\tilde{\mathfrak{z}}) = 0$. Similarly, since $\tilde{\mathfrak{a}}^{\zeta}_\omega = 0$, for all $k \in \mu \cup \theta$, $\zeta_k(\mathfrak{x}) = 0$, we conclude that $\sum_{k\in K_p} \tilde{\mathfrak{a}}^{\zeta}_k \zeta_k(\tilde{\mathfrak{z}}) = 0$. Therefore, $(\tilde{\mathfrak{z}}, \tilde{\xi}, \tilde{\mathfrak{a}}^c, \tilde{\mathfrak{a}}^d, \tilde{\mathfrak{a}}^{\mathscr{T}}, \tilde{\mathfrak{a}}^{\zeta}) \in \Lambda_{\mathscr{MWSMPEC}}$, ensuring the two objective values are equal.

Now, we assume that there exists $(\mathfrak{x}, \xi, \mathfrak{a}^c, \mathfrak{a}^d, \mathfrak{a}^{\mathscr{T}}, \mathfrak{a}^{\zeta}) \in \Lambda_{\mathscr{MWSMPEC}}$ such that $\Gamma(\tilde{\mathfrak{z}}) \preceq \Gamma(\mathfrak{x})$. This assumption contradicts Theorem 5.1. Hence, the proof is complete. □

6. Concluding Remarks

This paper presents a detailed analysis of a nondifferentiable multiobjective semi-infinite mathematical programming problem with equilibrium constraints. By utilizing the ∂^T-Abadie constraint qualification, the necessary Karush-Kuhn-Tucker (KKT) optimality conditions are established. The paper also derives sufficient optimality conditions based on Dini generalized convexity and explores duality relations by formulating the Mond–Weir dual problems. The practical applicability of the theoretical results is demonstrated through illustrative examples, highlighting the effectiveness of the proposed methods. The findings presented in this paper generalize several previously established results from the existing literature [13, 29]. Future research may focus on extending these results to incorporating approximate solution concepts for the $(\mathscr{MSMPEC})$.

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