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A REFINEMENT OF THE STABILITY ANALYSIS OF QUADRATIC MINIMIZATION PROBLEMS

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Abstract. In this paper, we consider a general quadratic minimization problem. We introduce a third approach to the stability of the quadratic problem via fixed-point methods. Bounds are then discussed in terms of their sharpness. Finally, we investigate the stability under small variations in all the data of the problem, where the solution map is proven to be Hölder continuous with respect to the perturbation.
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1. INTRODUCTION

Quadratic optimization is an important branch of optimization theory. Minimizing quadratic functions with constraints is motivated wide applications stemming from various problems in engineering, economics, and data science. Such optimization problems offer a versatile mathematical structure in both finite- and infinite-dimensional spaces. In this context, a relevant problems is how to manage the phenomena of perturbations and variations on the data of the minimization problem under consideration, which is of course intimately related to stability and sensitivity analysis of optimal solutions, particularly when the variations are due to the involvement of parameters either at the level of the function or the underlying constraints. These issues can be rigorously addressed via the notion of metric regularity, a fundamental concept in variational analysis developed extensively by many researchers; see, e.g., [5] and the references therein.

In this work, we consider the following general quadratic minimization problem, treated in [5],

$$\text{minimize } \frac{1}{2}\langle x, Ax\rangle - \langle v, x\rangle \text{ over } x \in C, \tag{1.1}$$

where $A : X \to X$ is a linear and bounded operator with $\mathrm{dom}(A) = X$ and C is a nonempty, closed, and convex subset of a Hilbert space X, and v is a fixed element in X. The notation $\langle\cdot,\cdot\rangle$ denotes the inner product in X, with the associated norm $\|x\| = \sqrt{\langle x, x\rangle}$. Recall that A is

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assumed to be self-adjoint, i.e., $\langle x, Ay \rangle = \langle y, Ax \rangle$ for all $x, y \in X$, and satisfies $\langle x, Ax \rangle \geq \mu \|x\|^2$ for all $x \in C - C$ and a constant $\mu > 0$.

We first present a stability result similar to the one given in [5, Theorem 6H.1] by leveraging recent and different arguments based on stable abstract equilibrium problems as developed by Ait Mansour and Riahi [2]. Our contribution relies on connecting classical results to broader settings by using analytical techniques under the same conditions of [5]. To make a connection with [2], we have to consider a normed vector space X, two subsets M and Λ of another two normed spaces, a family of real-valued bifunctions $\{\varphi(.,.,\mu)\}_{\mu \in \Lambda}$ defined on $X \times X$, and a closed and convex-valued set-valued map $K : M \rightrightarrows X$, $\lambda \longmapsto K_\lambda := K(\lambda)$. Then, given a pair $(\bar{\mu}, \bar{\lambda}) \in \Lambda \times M$, we define an equilibrium problem as follows: Seek a solution $\bar{u} \in K_\lambda$ such that $\varphi(\bar{u}, v, \bar{\mu}) \geq 0$ for all $v \in K_{\bar{\lambda}}$. This problem has been studied with respect to variations in parameters μ and λ in [2]. We emphasize that the well-posedness and the stability of solutions of this abstract equilibrium problem requires key some assumptions, such as the Hölder property, strong monotonicity, and Lipschitz continuity with respect to different arguments. These conditions ensure the uniqueness of solutions and their continuous dependence on perturbations. Note that the stability theorem presented in [2] establishes explicit bounds on the variation of solutions with respect to changes in μ and λ, offering valuable insights into the behavior of the equilibrium problem. In addition, we explore the connection between the equilibrium problem and variational inequalities. In this context, we recall a fundamental characterization of minimizers in terms of the normal cone condition.

Theorem 1.1. [7, Theorem 2A.6]. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function defined on an open convex set $O \subset \mathbb{R}^n$, and let C be a closed convex subset of O. When minimizing g over C, a necessary condition for $x \in C$ to be a local minimizer is that it satisfies the variational inequality $\nabla g(x) + N_C(x) \ni 0$, or, equivalently, $-\nabla g(x) \in N_C(x)$, where $N_C(x)$ denotes the normal cone to C at x. If g is convex, this condition is both necessary and sufficient for x to be a global minimizer.*

This classical result highlights how variational inequalities provide a natural framework for expressing first-order optimality conditions. In particular, quadratic minimization problem (1.1) can be reformulated in such terms; see Theorem 1.2 below. In the second step, for an interesting discussion of the close literature, we include a detailed proof of [5, Theorem 6H.1] on the same problem, (1.1), which was only briefly described in [5]. Besides, there is an alternative proof in Rockafellar & Wets [7, Theorem 9.43]. This demonstrates the existence and uniqueness of the solution and establishes the Lipschitz continuity of the solution mapping trough a characterization of a variational inequality problem. Precisely, the central result that we rely on provides key insights into the behavior of the solution under given conditions, is as follows.

Theorem 1.2. [5, Theorem 6H.1] *For problem (1.1) satisfying condition (6.7), there exists a unique solution x for each v. The solution mapping $S : v \mapsto x$ is therefore single-valued, with* $\operatorname{dom} S = X$. *Furthermore, this mapping S is Lipschitz continuous with the Lipschitz constant μ^{-1} and is characterized by $x = S(v) \Longleftrightarrow -v + Ax + N_C(x) \ni 0$.*

This formulation captures the fundamental properties of the solution and establishes a crucial link between optimality conditions and stability analysis. In the subsequent sections, we rigorously prove these properties, ensuring a thorough understanding of the solution behavior under perturbations in v. We further introduce a new approach to studying the stability of problem

(1.1) (different from the two previous methods) by involving fixed-point arguments, which provides another framework for addressing stability issues in mathematical analysis. To do this, we introduce the necessary background on fixed-point theory. Then, we define key concepts such as metric spaces, set distances and the extended Hausdorff metric. Additionally, we outline the conditions under which fixed points exist and their significance in terms of stability analysis. Building on these foundations, we leverage the Banach Fixed Point Theorem to justify the existence and uniqueness of solutions to problem (1.1). Then, we explore variational inequalities and their connection to our approach. The main results, including Theorem 5.1 and Corollary 5.1, provide critical insights into the stability of solutions under different conditions. By comparing the third method with the first two, we highlight its advantages, particularly in achieving a potentially tighter bound on the Lipschitz continuity of the solution mapping. This refinement offers a more precise understanding of stability for problem (1.1), making our contribution to the stability analysis of quadratic minimization more significant. Given that the sensitivity of solutions to parameter perturbations is fundamental in optimization theory for both numerical algorithms and applied reliable models, we continue our investigation for the stability of (1.1) with the more general case of variation at the level of all the data of the problems. This means that operator A, constraint set C, and vector v are simultaneously subject to variations. In this regard, we establish a central result demonstrating that the discrepancy between solutions $x_{\varepsilon'}$ and x_ε, corresponding to two distinct values of a small perturbation ε' and ε, is governed by a bound proportional to the square root of the norm of the difference between the two perturbing parameters. Specifically, we prove the existence of a constant $L \geq 0$ such that $\|x_{\varepsilon'} - x_\varepsilon\| \leq L|\varepsilon' - \varepsilon|^{1/2}$, where x_ε and $x_{\varepsilon'}$ denote the unique solutions to the perturbed problems (6.1) and (6.2), respectively. Here, $x_0 = x^*$, with x^* being the solution to the original problem (1.1). This inequality quantifies the robustness of the optimization framework, illustrating how variations in parameters are controlled and translated to bounded deviations in solutions. By analyzing the collective impact of perturbations across A, C, and the vector v, our result extends the previous stability of quadratic programs and underscores the inherent resilience of their solutions under global parameter changes.

The remainder of the paper is organized as follows. Section 2 is devoted to the stability of our quadratic minimization problem via an equilibrium problem formulation, where we first focus on transforming our specific problem into an appropriate abstract equilibrium problem. This reformulation allows us to leverage existing theoretical results in equilibrium theory to derive stability conditions and establish uniqueness of solutions. In Section 3, we present a detailed proof of the stability of quadratic problem (1.1), briefly described in [5]. Section 4 gives an alternative proof of the same stability for problem (1.1) via metric regularity characterization by using [7, Theorem 9.43]. In Section 5, we observe and underline a third approach to the stability of problem (1.1) by using fixed-point methods. The obtained bounds are then discussed in terms of their sharpness. Finally, in Section 6, we complete our investigation by studying the stability under small variations in all the data of the problem, where the solution map is proven to be Hölder continuous with respect to the perturbation.

2. Stability via Equilibrium Problems Formulation

In this section, we convert problem (1.1) into an abstract equilibrium problem, as stated in [2]. Let us recall at first the framework and the main stability result of [2]. Let X be a normed vector topological space with norm $\|.\|$. Let also M and Λ be subsets of two normed spaces with norms also denoted by $\|\,.\,\|$ and $\{K_\lambda\}_{\lambda\in M}$ be a family of closed and convex subsets of X. We further consider a family of bifunctions $\{\varphi(.,.,\mu)\}_{\mu\in\Lambda}$ defined on $X\times X$. Given a pair $(\bar\mu,\bar\lambda)\in\Lambda\times M$, we consider the equilibrium problem: Find $u(\bar\mu,\bar\lambda):=\bar u\in K_\lambda$ such that

$$(EP_{\bar\mu,\bar\lambda})\qquad \varphi(\bar u,v,\bar\mu)\geqslant 0,\quad \forall v\in K_{\bar\lambda}.$$

The perturbed form of $(EP_{\bar\mu,\bar\lambda})$ is as follows: find $u(\mu,\lambda)\in K_\lambda$ such that

$$(EP_{\mu,\lambda})\qquad \varphi(u(\mu,\lambda),v,\mu)\geqslant 0,\quad \forall v\in K_\lambda.$$

Our key assumptions are as follows:

(H_0) Hölder property: $K(\lambda)=K_\lambda$ is Hölder at $\bar\lambda$, that is, for a neighborhood $\bar M$ of $\bar\lambda$ and some constant $L>0$, $K_\lambda\subset K_{\lambda'}+L\|\lambda-\lambda'\|^{\xi}B_X$ for all $\lambda,\lambda'\in\bar M$, where B_X stands for the unit ball of X.

(H_1) Strong monotonicity condition: assume that φ satisfies $\varphi(u,v,\mu)+\varphi(v,u,\mu)\leqslant -m\|u-v\|^{\alpha}$ for some $m>0,\alpha>0$, all $\mu\in\bar U$, and all $u,v\in X$.

(H_2) Lipschitz behavior with respect to second argument: there exist a neighborhood N of $\bar\mu$ and constants $R>0,\beta>0$ such that, for all $\mu\in N,u,v,v'\in X$, $|\varphi(u,v,\mu)-\varphi(u,v',\mu)|\leqslant R\|v-v'\|^{\beta}$.

(H_3) Lipschitz property with respect to parameter μ : there exist $\theta>0$, $\gamma>0$, and $\delta>0$ such that $|\varphi(u,v,\mu)-\varphi(u,v,\mu')|\leqslant\theta\|\mu-\mu'\|^{\gamma}\|v-u\|^{\delta}$ for all $u,v\in X$ and all μ,μ' in a neighborhood of $\bar\mu$.

(H_4) Control of data: assume that $\alpha>\delta$.

Remark 2.1. Assumption (H_1) ensures the uniqueness of solutions to both $(EP_{\bar\mu,\bar\lambda})$ and $(EP_{\mu,\lambda})$. For each $\lambda\in M$, let I_λ be the indicator function of K_λ, i.e.,

$$I_\lambda(x)=\begin{cases}0, & x\in K_\lambda,\\ +\infty, & x\notin K_\lambda.\end{cases}$$

Replacing φ in $\mathrm{EP}_{\mu,\lambda}$ by $\varphi+\Psi_\lambda$, where $\Psi_\lambda(u,v):=I_\lambda(v)-I_\lambda(u)$, yields

$$\varphi(u(\mu,\lambda),v,\mu)+\Psi_\lambda(u(\mu,\lambda),v)\geqslant 0,\quad \forall v\in X.$$

Let us now recall the main stability result of [2], which is useful for our purpose.

Theorem 2.1. [2, Theorem 2.2.1] *Let assumptions $(H_i)_{i=0,\ldots,4}$ hold. Then there exist constants $k_1,k_2>0$, and neighborhoods U_1 of $\bar\mu$,V_1 of $\bar\lambda$ such that*

(i) *For each $(\mu,\lambda)\in(\Lambda\cap U_1)\times(M\cap V_1)$, the solution $u(\mu,\lambda)$ to $\left(EP_{\mu,\lambda}\right)$ is unique*

(ii) *For all $(\mu,\lambda),(\mu',\lambda')\in(\Lambda\cap U_1)\times(M\cap V_1)$, it holds*

$$\left\|u(\mu,\lambda)-u\left(\mu',\lambda'\right)\right\|\leqslant k_1\left\|\mu-\mu'\right\|^{\gamma/\alpha-\delta}+k_2\left\|\lambda-\lambda'\right\|^{\beta\xi/\alpha}.$$

Corollary 2.1. *Let assumptions $(H_i)_{i=1,\ldots,4}$ hold. Then there exist a constant $k_1>0$ and a neighborhood U_1 of $\bar\mu$ such that*

(i) *For each $\mu\in\Lambda\cap U_1$, the solution $u(\mu)$ to $\left(EP_\mu\right)$ is unique.*

(ii) *For all $\mu,\mu'\in\Lambda\cap U_1$, it holds $\|u(\mu)-u(\mu'\|\leqslant k_1\|\mu-\mu'\|^{\gamma/\alpha-\delta}$.*

Remark 2.2. Corollary 2.1 can be applied straightforwardly to the classical variational inequality problem as a special case of equilibrium problems. Precisely, let $C \subset \mathbb{R}^n$ be a convex and closed set and $g : \mathbb{R}^n \to \mathbb{R}^n$ a given mapping. Then $\nabla g(x) + N_C(x) \ni 0$ can be viewed as an equilibrium problem over the constraint C with the bifunction φ defined on $C \times C$ by $\varphi(x,y) = \langle g(x), y-x\rangle$, for all $x,y \in C$.

Our analysis requires certain monotonicity properties, which we recall in the following.

Definition 2.1 (Monotonicity)**.** A mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be monotone on a convex set $C \subset \operatorname{dom} f$ if $\langle f(x') - f(x), x' - x\rangle \geq 0$ for all $x, x' \in C$. If there exists $\mu > 0$ such that $\langle f(x') - f(x), x' - x\rangle \geq \mu \|x' - x\|^2$ for all $x, x' \in C$, then f is said to be strongly monotone on C with modulus μ.

The term monotonicity can be interpreted in the classical sense along line segments: for any $\hat{x} \in C$ and any unit vector $w \in \mathbb{R}^n$, the scalar function $\varphi(\tau) := \langle f(\hat{x} + \tau w), w\rangle$ is nondecreasing for all τ such that $\hat{x} + \tau w \in C$. Moreover, f is strongly monotone with modulus $\mu > 0$ if and only if $\varphi(\tau') - \varphi(\tau) \geq \mu|\tau' - \tau|$ whenever $\tau' > \tau$.

We now provide another viewpoint than Theorem 1.2 for the stability of our problem (1.1) with respect to variations of vector v. Precisely, we suggest to regard the vector v in problem (1.1) as a perturbing parameter and then look for continuity properties of the solutions with respect to v. In this way, we state the following theorem.

Theorem 2.2. *Assume that the operator A is coercive on the set $C - C$, that is,*

$$\langle x, Ax\rangle \geq \mu\|x\|^2 \quad \text{for all } x \in C - C, \text{ where } \mu \in (0, +\infty). \tag{2.1}$$

Let C be bounded. Then, there exists a constant $\kappa > 0$ such that, for any $v \in C$, the solution $u(v)$ to problem (1.1) is unique. Furthermore, for all $v, v' \in C$, $\|u(v) - u(v')\| \leq \kappa\|v - v'\|$.

Proof. Our key idea is to infer the required stability result from Corollary 2.1, so we have to check that our quadratic problem verifies all the conditions of Corollary 2.1 by precision of the coefficients α, γ, δ. Let $x, y, v \in C$. Recall that $N_C(x) = \{x^* \in H : \langle x^*, y - x\rangle \leq 0,\ \forall y \in C\}$. Assume now that $x \in C$ is a solution to problem (1.1) and observe that

$$\begin{aligned} -v + Ax + N_C(x) \ni 0 \quad &\Leftrightarrow \quad N_C(x) \ni v - Ax \\ &\Leftrightarrow \quad \langle v - Ax, y - x\rangle \leq 0, \qquad \forall y \in C \\ &\Leftrightarrow \quad \langle Ax - v, y - x\rangle \geq 0, \qquad \forall y \in C. \end{aligned}$$

Therefore, it suffices to consider the bifunction defined by $f_v(x,y) = \langle Ax - v, y - x\rangle$ for every $x, y \in C$. We will prove (see the proof of Proposition 5.1) that there exists $x \in C$ such that, for all y in C, $\langle Ax - v, y - x\rangle \geq 0$. Thus, for any $v \in X$, there exists $x \in C$ such that, for all $y \in C$, $f_v(x,y) \geqslant 0$. Now, let us show that f_v satisfies the hypotheses (H_i) of Corollary 2.1, for all $i \in \{1,2,3,4\}$. In view of the condition (2.1), we have

$$\begin{aligned} f_v(x,y) + f_v(y,x) &= \langle Ax - v, y - x\rangle + \langle Ay - v, x - y\rangle \\ &= \langle Ax, y - x\rangle + \langle Ay, x - y\rangle \\ &= -\langle Ax - Ay, x - y\rangle \\ &= -\langle A(x - y), x - y\rangle \\ &\leq -\mu\|x - y\|^2, \end{aligned}$$

which proves that f_v is strongly monotone with constant μ. Concerning(H_2), fix an element $v \in C$, take $x,y,y' \in C$, and observe that $|f_v(x,y) - f_v(x,y')| \leqslant \|Ax - v\| \|y - y'\|$ so $|f_v(x,y) - f_v(x,y')| \leqslant (\|x\|\|A\| + \|v\|)\|y - y'\|$. The set C being bounded, it is enough to choice a real number R such that $\|x\|\|A\| + \|v\| < R$ for any $x \in C$. Therefore, f_v satisfies (H_2).

Next, we verify that f_v satisfies (H_3): the Lipschitz property with respect to parameter v. Let $x,y,v \in C$.

$$|f_v(x,y) - f_{v'}(x,y)| = \langle Ax - v, y - x\rangle - \langle Ax - v', y - x\rangle = \langle v' - v, y - x\rangle \leqslant \|v' - v\|\|y - x\|.$$

Therefore, f_v satisfies (H_3) with $\theta = 1$, $\gamma = 1$, and $\delta = 1$. Regarding (H_4), clearly we see that $\alpha = 2 > 1 = \delta$. Hence, by directly applying Corollary 2.1, one sees that there exists a constant $\kappa > 0$ such that, for each $v \in C$, the solution $u(v)$ to the problem (1.1) is unique, and for all $v, v' \in C$, we have $\|u(v) - u(v')\| \leqslant \kappa \|v - v'\|^{\gamma/\alpha - \delta}$, where

$$\kappa = \left(\frac{\theta}{m}\right)^{\frac{1}{(\alpha-\delta)}} = \frac{1}{\mu}, \text{ and } \frac{\gamma}{\alpha - \delta} = 1.$$

Consequently,

$$\|u(v) - u(v')\| \leqslant \mu^{-1} \left\|v - v'\right\|.$$

□

3. The Proof to the Stability of Quadratic Problem (1.1)

In this section, we give our detailed proof of Theorem 1.2 [5, Theorem 6H.1]. This result states the existence and uniqueness of the solution to the quadratic problem (1.1) under condition (6.7). Furthermore, it ensures that the solution mapping $S : v \mapsto x$ is well-defined, single-valued, and Lipschitz continuous with a constant μ^{-1}. The theorem under consideration also characterizes the solution through a variational inequality, providing a strong foundation for the stability analysis of the problem. Before proceeding with the proof, we first recall two important related results from the literature.

Theorem 3.1. [7, Theorem 2F.5]. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a mapping, and let $C \subset \operatorname{dom} f$ be a nonempty, closed, and convex set on which f is continuous (without assuming monotonicity). Suppose that there exist a point $\hat{x} \in C$ and a constant $\rho > 0$ such that*

$$\forall x \in C \text{ with } \|x - \hat{x}\| \geq \rho, \quad \langle f(x), x - \hat{x}\rangle > 0.$$

Then the variational inequality

$$f(x) + N_C(x) \ni 0 \tag{3.1}$$

has at least one solution, and every solution x of (3.1) satisfies

$$\|x - \hat{x}\| < \rho.$$

Proposition 3.1. [7, Proposition 2G.4 (tilted minimization of strongly convex functions)] *Let $g : \mathbb{R}^n \to \mathbb{R}$ be C^1 on an open set $O \subset \mathbb{R}^n$. If $C \subset O$ is nonempty, closed, and convex, and g is μ-strongly convex on C for some $\mu > 0$, then, for all $v \in \mathbb{R}^n$, problem (3.1) admits a unique solution $S(v)$, and S is globally Lipschitz with constant μ^{-1}.*

These two results provide general insights into the solvability and stability of variational inequalities and strongly convex minimization problems. Although we do not directly apply them in our proof of Theorem 1.2, they illustrate equivalent conclusions and situate our analysis within the broader theoretical framework of variational inequality theory and convex optimization. To rigorously prove Theorem 1.2, we proceed in a structured manner in key mathematical steps as follows:

(1) Existence and Uniqueness of the Solution
- We show that, for each v, there exists a unique x satisfying the optimality conditions of problem (1.1).
- This will be done by leveraging the convexity of the function and properties of monotone operators.

(2) Characterization via a Variational Inequality
- We will demonstrate that the solution $x = S(v)$ satisfies the variational inequality: $-v + Ax + N_C(x) \ni 0$.
- This step relies on necessary and sufficient conditions for optimality, incorporating elements of convex analysis.

(3) Lipschitz Continuity of the Solution Mapping
- We establish that the mapping S is Lipschitz continuous by proving that the difference between two solutions is bounded by a multiple of the difference in their respective inputs.
- This involves proving that the inverse of operator A is well-behaved under certain norm constraints, ensuring stability.

Each of these steps is based on the theoretical framework provided in [5, Theorem 6H.1] to ensure a complete and rigorous proof of Theorem 1.2.

Proof. First, let us prove the existence of solutions to problem (1.1). To do that, let us fix a vector v in X. For each real number α sufficiently large, we consider the set C_α defined by

$$C_\alpha := \Big\{x \in C : \frac{1}{2}\langle x, Ax\rangle - \langle v, x\rangle \leq \alpha\Big\}.$$

We set $f_v(x) := \frac{1}{2}\langle x, Ax\rangle - \langle v, x\rangle$.

i) C_α is nonempty and closed. Indeed, since A is bounded and C is nonempty, there exists a vector x_0 in C. We choose $\alpha_0 \in \mathbb{R}$ such that $f_v(x_0) \leq \|x_0\|^2 \times \|A\| + \|v\|\|x_0\| \leq \alpha_0$. For any $\alpha \in \mathbb{R}$ such that α is largely greater than α_0, set C_α is nonempty. Moreover, since $C_\alpha = C \cap f_v^{-1}(]-\infty, \alpha])$, C is closed and f_v is continuous, it follows that C_α is closed.

ii) Convexity: First, we recall that a function $f : H \to \mathbb{R}$ defined on a Hilbert space H is said to be strongly convex with modulus $\mu > 0$ if, for all $x_1, x_2 \in H$ and $t \in [0,1]$, $f\big(tx_1 + (1-t)x_2\big) \leq tf(x_1) + (1-t)f(x_2) - \frac{\mu}{2}t(1-t)\|x_2 - x_1\|^2$.

We now check that C_α is a convex set. For this, it is sufficient to note that f_v is strongly convex. Specifically, for x_1 and x_2 in C_α and $t \in [0,1]$, we have

$$f_v\big(tx_1 + (1-t)x_2\big) = \frac{1}{2}\big\langle tx_1 + (1-t)x_2, A[tx_1 + (1-t)x_2]\big\rangle - \langle v, tx_1 + (1-t)x_2\rangle.$$

Expanding this expression yields

$$f_v\big(tx_1+(1-t)x_2\big)=\frac{1}{2}t^2\langle x_1,Ax_1\rangle - t\langle v,x_1\rangle+\frac{1}{2}(1-t)^2\langle x_2,Ax_2\rangle-(1-t)\langle v,x_2\rangle$$
$$+t(1-t)\langle x_1,Ax_2\rangle+t(1-t)\langle x_2,Ax_1\rangle.$$

Given that A satisfies $\langle x,Ax\rangle \geq \mu\|x\|^2$ for all $x\in C-C$ and for some constant $\mu>0$, one finds $\langle x_1,Ax_1\rangle+\langle x_2,Ax_2\rangle-2\langle x_1,Ax_2\rangle \geq \mu\|x_2-x_1\|^2$ for all $x_1,x_2\in C$, which is equivalent to $\langle x_1,Ax_2\rangle \leq \frac{1}{2}\langle x_1,Ax_1\rangle+\frac{1}{2}\langle x_2,Ax_2\rangle-\frac{1}{2}\mu\|x_2-x_1\|^2$. This inequality allows us to bound the cross-term $\langle x_1,Ax_2\rangle$ in the expansion of $f_v(tx_1+(1-t)x_2)$. Substituting this bound into the previous expansion ensures that the combination of x_1 and x_2 in f_v does not exceed the linear interpolation of $f_v(x_1)$ and $f_v(x_2)$ minus a strictly positive quadratic term. This is precisely the defining property of strong convexity in a Hilbert space: the function lies below the secant line by a quadratic margin, controlled by μ. Hence, we obtain

$$f_v\big(tx_1+(1-t)x_2\big)\leq tf_v(x_1)+(1-t)f_v(x_2)-\frac{1}{2}\mu\|x_2-x_1\|^2.$$

This implies that f_v is strongly convex, which in turn leads to $f_v\big(tx_1+(1-t)x_2\big)\leq \alpha-\frac{1}{2}\mu\|x_2-x_1\|^2\leq\alpha$. Accordingly, C_α is convex for all sufficiently large $\alpha\in\mathbb{R}$.

iii) In this step, we justify that C_α is bounded. To do that, we show the existence of a positive number M such that, for any $x\in C_\alpha$, $\|x\|\leq M$. Take $x\in C_\alpha$, and choose $x_0\in C$ such that $\|x_0\|=\min_{x\in C}\|x\|$, which is well-defined since every nonempty closed convex subset K of a Hilbert space X has a unique element of minimal norm. Observe that $\mu\|x-x_0\|^2\leq\langle x-x_0,Ax-Ax_0\rangle$, which implies $\mu\|x-x_0\|^2\leq\langle x,Ax\rangle+\langle x_0,Ax_0\rangle-2\langle x,Ax_0\rangle$. Since x belongs to C_α, we have $\langle x,Ax\rangle\leq 2\alpha+2\langle v,x\rangle$, which leads to $\mu\|x-x_0\|^2\leq 2\alpha+2\langle v,x\rangle+\langle x_0,Ax_0\rangle-2\langle x,Ax_0\rangle$. Hence

$$\mu\|x-x_0\|^2\leq 2\alpha+2\|v\|\times\|x\| \ +\|A\|\times\|x_0\|^2+2\|x\|\times\|A\|\|x_0\|.$$

Now, let us expand $\mu\|x-x_0\|^2$ and obtain

$$\mu(\|x\|^2+\|x_0\|^2-2Re\langle x,x_0\rangle)\leq 2\alpha+2\|v\|\times\|x\| \ +\|A\|\times\|x_0\|^2+2\|x\|\times\|A\|\|x_0\|.$$

Thus $\mu(\|x\|^2+\|x_0\|^2)\leq 2\alpha+2\|v\|\times\|x\| \ +\|A\|\times\|x_0\|^2+2\|x\|\times\|A\|\|x_0\|+2\|x\|\|x_0\|$. Consequently,

$$\mu\|x\|^2+\mu\|x_0\|^2-2\alpha-2\|v\|\times\|x\| \ -\|A\|\times\|x_0\|^2-2\|x\|\times\|A\|\|x_0\|-2\|x\|\|x_0\|\leq 0.$$

Note that

$$\mu\|x\|^2-2\|x\|(\|v\|+\|A\|\|x_0\| \ +\|x_0\| \ )+\mu\|x_0\|^2-2\alpha-\|A\|\|x_0\|^2\leq 0.$$

Set $b=-2\big(\|v\|+\|A\|\,\|x_0\|+\|x_0\|\big)$ and $c=\mu\,\|x_0\|^2-2\alpha-\|A\|\,\|x_0\|^2$. Then, $\mu\,\|x\|^2+b\,\|x\|+c\leq 0$. Furthermore, since $\mu>0$, it follows that $\|x\|$ is bounded.

iv) Intersection and a global minimizer.

Equivalent form of the strong convexity in Hilbert spaces. Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. It is known that the strong convexity, with a modulus $\mu>0$, of a Fréchet differentiable function $f:H\to\mathbb{R}$ is equivalent to

$$f(y)\ \geq\ f(x)+\langle\nabla f(x),y-x\rangle+\tfrac{\mu}{2}\|y-x\|^2,\qquad \forall x,y\in H.$$

Define $T_\infty := \cap_{\alpha\geq\alpha_0} C_\alpha$ and $C_\alpha := \{x \in C : f_v(x) \leq \alpha\}$. Since we are in a Hilbert space (which is reflexive), the Banach–Alaoglu theorem together with the Eberlein–Šmulian theorem ensures that closed, bounded, and convex subsets are weakly compact. Hence, each C_α is weakly compact. Moreover, the family $\{C_\alpha\}_{\alpha\geq\alpha_0}$ is nested (i.e., $\alpha_1 < \alpha_2 \implies C_{\alpha_1} \subseteq C_{\alpha_2}$) and nonempty. By the finite intersection property of nested weakly compact sets, it follows that T_∞ is nonempty. Furthermore, T_∞ is closed, convex, and weakly compact. Observe that $cl(\bigcup_{\alpha\geq\alpha_0} C_\alpha) = C$, since every $x \in C$ belongs to some sublevel set C_α. By weak compactness of T_∞ and weak lower semicontinuity of f_v, the Weierstrass-type theorem guarantees the existence of $a^* \in T_\infty$ such that $f_v(a^*) = \min_{x\in T_\infty} f_v(x) =: m_v$.

Finally, let $x \in C \setminus T_\infty$. Then, by definition of T_∞, there exists some α such that $f_v(x) > \alpha$. Since $a^* \in T_\infty \subseteq C_\alpha$, we have $f_v(a^*) \leq \alpha < f_v(x)$. Thus $f_v(x) \geq m_v$ for all $x \in C$. Hence, the point a^* of T_∞ is a global minimizer of f_v over C.

v) The uniqueness of the solution can been seen via standard convexity arguments as follows. Suppose that there exist two distinct solutions, denoted as x_1 and x_2. Then $f_v(x_1) = f_v(x_2) = m_v$. For all $x \in C$, we have $m_v \leq f_v(x)$. Since f_v is strongly convex, it follows that

$$f_v\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right) \leq \frac{1}{2}f_v(x_1) + \frac{1}{2}f_v(x_2) - \frac{1}{8}\mu\|x_1 - x_2\|^2.$$

We have $\frac{1}{2}x_1 + \frac{1}{2}x_2 \in C$, so

$$m_v \leq f_v\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right) \leq m_v - \frac{1}{8}\mu\|x_1 - x_2\|^2 < m_v,$$

which leads to a contradiction, implying that $x_1 = x_2$.

vi) Let us prove $x = S(v) \iff -v + Ax + N_C(x) \ni 0$.t For the direct implication, we have to show that $x = S(v) \implies -v + Ax + N_C(x) \ni 0$. We organize in the following steps.

Step 0: Optimality condition from convex analysis. Let $g : X \to \mathbb{R} \cup \{+\infty\}$ be a convex functional on a Hilbert space X, and let $C \subset X$ be a nonempty closed convex set. Then

$$x \in C \text{ minimizes } g \text{ over } C \quad \iff \quad 0 \in \partial g(x) + N_C(x),$$

where $\partial g(x) = \{\xi \in X : g(y) \geq g(x) + \langle \xi, y - x\rangle \ \ \forall y \in X\}$ is the convex subdifferential of g at x, and $N_C(x) = \{w \in X : \langle w, y - x\rangle \leq 0 \ \ \forall y \in C\}$ is the normal cone of C at x.

Subdifferential of the indicator function. Let $\iota_C : X \to \mathbb{R} \cup \{+\infty\}$ be the indicator function of C:

$$\iota_C(y) := \begin{cases} 0, & y \in C, \\ +\infty, & y \notin C. \end{cases}$$

Then the subdifferential of ι_C at $x \in C$ is exactly the normal cone:

$$\partial \iota_C(x) = \{\xi \in X : \langle \xi, y - x\rangle \leq 0 \ \ \forall y \in C\} = N_C(x).$$

Since $x = S(v)$ minimizes $f_v(x) := \langle v, x\rangle + \frac{1}{2}\langle Ax, x\rangle$ over C, we obtain by the optimality condition that

$$0 \in \partial f_v(x) + \partial \iota_C(x) = \partial f_v(x) + N_C(x). \tag{3.2}$$

On the other hand, since the functional f_v is differentiable with $\nabla f_v(x) = v + Ax$, then $\partial f_v(x) = \{\nabla f_v(x)\} = \{v + Ax\}$. Thus pluging the subdifferential expression into optimality condition (3.2) yields

$$0 \in \partial f_v(x) + N_C(x) = \{v + Ax\} + N_C(x) \quad \Longrightarrow \quad 0 \in v + Ax + N_C(x),$$

which means $-v + Ax + N_C(x) \ni 0$. Hence, if $x = S(v)$, then x satisfies the variational inequality: $-v + Ax + N_C(x) \ni 0$. Let us now investigate the converse assertion. The mapping S being well-defined from X to C and single-valued, let $x \in C$ such that $-v + Ax + N_C(x) \ni 0$. We have to prove that $\{x\} = S(v)$. For any $x' \in C$, we can write $\langle \nabla f_v(x), x' - x\rangle \geqslant 0$. Given that f_v is convex over C, we immediately obtain

$$f_v(x') - f_v(x) \geqslant \langle \nabla f_v(x), x' - x\rangle \geqslant 0.$$

Consequently, $f_v(x') \geqslant f_v(x)$ for all $x' \in C$. This implies that x is a minimum of f_v over C. In other words, $x \in S(v)$. Note that S is single-valued, so we can write $S(v) = x$.

vii) For the Lipschitz property of S, we take two vectors v_0 and v_1 in X and set $x_0 = S(v_0)$ and $x_1 = S(v_1)$. Note that $v_0 - Ax_0 \in N_C(x_0)$ and $v_1 - Ax_1 \in N_C(x_1)$. In particular, we derive $\langle v_0 - Ax_0, x_1 - x_0\rangle \leqslant 0$ and $\langle v_1 - Ax_1, x_0 - x_1\rangle \leqslant 0$. $\langle v_1 - Ax_1, x_0 - x_1\rangle \leqslant 0$ can also be written as $\langle v_1 - Ax_1, x_1 - x_0\rangle \geqslant 0$. Then, $\langle v_0 - Ax_0, x_1 - x_0\rangle \leqslant \langle v_1 - Ax_1, x_1 - x_0\rangle$, which is equivalent to $\langle Ax_1 - Ax_0, x_1 - x_0\rangle \leqslant \langle v_1 - v_0, x_1 - x_0\rangle$. Since A is strongly monotone, we have $\mu\|x_1 - x_0\|^2 \leqslant \langle Ax_1 - Ax_0, x_1 - x_0\rangle$, while

$$\langle v_1 - v_0, x_1 - x_0\rangle \leqslant \|v_1 - v_0\|\|x_1 - x_0\|.$$

Thus $\|x_1 - x_0\| \leqslant \mu^{-1}\|v_1 - v_0\|$, which implies that S is Lipschitz continuous in x with the constant μ^{-1}.

□

4. An Alternative Approach via Metric Regularity

In the previous section, we have established the equivalence

$$x = S(v) \iff 0 \in -v + Ax + N_C(x)$$

directly. Here, we present an alternative approach that uses the concept of metric regularity of set-valued mappings. This perspective provides a unified viewpoint connecting the three following parts:

- the existence and uniqueness of the solution,
- the equivalence of the generalized equation and the solution map,
- the Lipschitz stability of the solution map under perturbations in the parameter v.

The main idea is to interpret the solution map S as the inverse of $F(x) := Ax + N_C(x)$, a set-valued mapping. Inspiration by the classical results of [7, Theorem 9.43] and [5, Theorem 3E.6], we prove the equivalence of the metric regularity property of F, its monotonicity, and the Lipschitz continuity of its inverse. We would like to underline here that all of these properties are satisfied for our quadratic problem. By showing that F is strongly monotone in feasible directions (that is, C), one obtains metric regularity with modulus μ^{-1}. This immediately implies that the solution map $S(v)$ is globally Lipschitz continuous with constant μ^{-1} and that the generalized equation and the solution map are equivalent.

Definition 4.1 (Metric Regularity)**.** A set-valued mapping $F : X \rightrightarrows X$ is said to be metrically regular at $(\bar{x}, \bar{v}) \in \operatorname{gph} F$ with modulus $\kappa > 0$ if there exist neighborhoods U of $\bar{x}$ and V of $\bar{v}$ such that $d(x, F^{-1}(v)) \leq \kappa d(v, F(x))$ for all $x \in U$ and $v \in V$, where $d(x,A) := \inf\{\|x-a\| : a \in A\}$ is the distance from x to the set A.

In the following, we show the equivalence between the Lipschitz behavior of the solution map of our main quadratic problem, the metric regularity property of its inverse S^{-1} (i.e., F), and the strong monotonicity character of the map F. Each of them is demonstrated to be automatically linked to the other ones.

Theorem 4.1. *Let X be a Hilbert space, $C \subset X$ a nonempty, closed, and convex set, $A : X \to X$ a μ-strongly monotone operator on C with $\mu > 0$, and N_C the normal cone operator of C. Define $F := A + N_C$. Then, for every $v \in X$, $F^{-1}(v)$ is nonempty (in fact, a singleton), F is μ-strongly monotone on C, and F is metrically regular with modulus μ^{-1}, that is, for all $x \in C$ and $v \in X$, $d(x, F^{-1}(v)) \leq \frac{1}{\mu} d(v, F(x))$.*

Proof. For $x, y \in C$ and $u \in N_C(x)$, $v \in N_C(y)$, we have $A(x) + u \in F(x)$ and $A(y) + v \in F(y)$. Then

$$\langle (A(x)+u) - (A(y)+v), x-y \rangle = \langle A(x) - A(y), x-y \rangle + \langle u-v, x-y \rangle.$$

By the μ-strong monotonicity of A on C, the first term is at least $\mu\|x-y\|^2$. By the monotonicity of N_C, the second term is nonnegative. Hence F is μ-strongly monotone. Strong monotonicity implies injectivity: if $w \in F(x) \cap F(y)$, then

$$\langle w-w, x-y \rangle \geq \mu \|x-y\|^2 \implies x = y.$$

For $x, y \in C$ and $p \in F(x)$, $q \in F(y)$, strong monotonicity gives

$$\mu \|x-y\|^2 \leq \langle p-q, x-y \rangle \leq \|p-q\| \, \|x-y\|.$$

Hence, $\|x-y\| \leq \frac{1}{\mu}\|p-q\|$, so F^{-1} is single-valued and Lipschitz continuous with constant $1/\mu$. To see that $F^{-1}(v)$ is nonempty, we consider $f_v(x) = \frac{1}{2}\langle x, Ax \rangle - \langle v, x \rangle$ for all x in C. Since C is nonempty, closed, and convex, f_v has a minimizer $x_v \in C$, which satisfies the first-order optimality condition $0 \in A(x_v) - v + N_C(x_v)$. Hence $x_v \in F^{-1}(v)$ (see Section 3 above). Therefore, $F^{-1}(v)$ is nonempty. In fact, it is a singleton $\{x_v\}$.

Finally, the Lipschitz continuity of F^{-1} implies metric regularity: for all $x \in C$ and $v \in X$,

$$d(x, F^{-1}(v)) \leq \frac{1}{\mu} d(v, F(x)).$$

□

Proof of Theorem 1.2 (via metric regularity approach): From Theorem 4.1, we clearly have, for all $x \in C$ and $v \in X$, $d(x, F^{-1}(v)) \leq \mu^{-1} d(v, F(x))$. Note that the proof of Theorem 4.1 gives us the Lipschitz property of F^{-1} with the constant κ. Now, for the direct implication, if $x = S(v)$, then we have by definition $v \in F(x)$, which is equivalent to $0 \in -v + F(x)$. Conversely, if $0 \in -v + F(x)$, equivalently $v \in F(x)$, the metric regularity of F together with closeness property implies

$$d(x, F^{-1}(v)) \leq \mu^{-1} d(v, F(x)) = 0 \quad \implies \quad x \in F^{-1}(v) = S(v).$$

The uniqueness of the solution and Lipschitz continuity of the solution map are guaranteed by

$$\|S(v_1) - S(v_2)\| \leq \mu^{-1} \|v_1 - v_2\|, \quad \forall v_1, v_2 \in X.$$

Remark 4.1.
- This approach provides an alternative viewpoint that the metric regularity not only proves the equivalence of the generalized equation and the solution map, but also equivalently yields stability and Lipschitz continuity of $S(v)$ under perturbations in v.
- From the known characterization of global metric regularity (see, e.g., [1, 5]), in the global framework the metric regularity of a set-valued map is always equivalent to the Lipischitzness of its inverse.

As an important application of Theorem 1.2, we have the Hilbert version of the projection onto convex sets in $\mathbb{R}^n$.

Corollary 4.1. *(Projections onto Convex Sets). Let $C \subset X$ be nonempty, closed, and convex set, where X is a Hilbert space. For each $v \in X$, there exists a unique $x \in C$ such that $\|v-x\| = \inf_{y\in C}\|v-y\|$, denoted $x = P_C(v)$. The projection $P_C\colon X \to C$ is 1-Lipschitz, i.e.,*

$$\|P_C(u) - P_C(v)\| \leq \|u-v\|, \quad \forall u, v \in X.$$

Proof. The desired result is immediate from Theorem 1.2 and the well-known characterization of the metric projection onto closed and convex sets by the standard variational inequality. □

5. A Third Approach of the Stability of Problem (1.1) via Fixed Point Arguments

Stability properties of fixed points problems played a decisive role in establishing stable solutions to numerous mathematical models including evolution equations as in [3, 4] and the references therein. This gives us a further motivation to explore the fixed-point approach for problem (1.1). Thus we have to compare the stability results obtained via different approaches explored above. We first recall the elements of fixed point theory we need in the sequel. Consider a metric space (X,d). For any nonempty subset $A \subseteq X$ and $x \in X$, the distance from x to A is given by $d(x,A) := \inf\{d(x,y) \mid y \in A\}$, with the convention $d(x,\emptyset) = +\infty$. For nonempty subsets $A,B \subseteq X$, the excess of A over B is defined as $e(A,B) := \sup\{d(a,B) \mid a \in A\}$. By convention, $e(\emptyset,B) = 0$ for $B \neq \emptyset$, and $e(A,\emptyset) = +\infty$. The extended Hausdorff distance between $A,B \subseteq X$ is $h(A,B) := \max\{e(A,B), e(B,A)\}$. This extension accommodates infinite distances. The minimal distance between nonempty subsets A,B is $d(A,B) := \inf\{d(x,y) \mid x \in A, y \in B\}$. If either set is empty, we define $d(A,B) = h(A,B) = +\infty$. For a set-valued map $\Phi : X \rightrightarrows X$, the fixed-point set is denoted by $\mathrm{Fix}(\Phi) := \{x \in X \mid x \in \Phi(x)\}$. When Φ is a single-valued contraction, $\mathrm{Fix}(\Phi)$ retains this notation despite being a singleton.

Theorem 5.1 ([6, Lemma 1])**.** *Let X be a complete metric space and let $T_1, T_2 : X \rightrightarrows X$ be two set-valued mappings such that $T_i(x)$ is nonempty and closed, for every $x \in X$, $i = 1,2$. Suppose that both T_1 and T_2 are Lipschitz continuous on X with the same Lipschitz constant $\kappa \in [0,1)$. Then*

$$h(\mathrm{Fix}(T_1), \mathrm{Fix}(T_2)) \leq \frac{1}{1-\kappa} \sup_{x\in X} h(T_1(x), T_2(x)).$$

An important result of Theorem 5.1 is as follows in Hilbert spaces.

Corollary 5.1. *Let X be a Hilbert space and let $T_1, T_2 : X \longrightarrow X$ be two single-valued mappings. Suppose that both T_1 and T_2 are contractions on X with the same Lipschitz constant $\kappa \in [0,1)$. Then $\|\mathrm{Fix}(T_1) - \mathrm{Fix}(T_2)\| \leq \frac{1}{1-\kappa} \sup_{x\in X} \|T_1(x) - T_2(x)\|$.*

For the completeness, we begin by justifying of the existence and uniqueness of the solution of problem (1.1) by using the famous Banach fixed point Theorem. As we have already seen, problem (1.1) is equivalent to the variational inequality

$$\langle Ax - v, y - x\rangle \geq 0, \quad \forall y \in C. \tag{5.1}$$

Proposition 5.1. *Let A be a self-adjoint, linear, and bounded operator over a Hilbert space X, with $dom(A) = X$, and let C be a closed, nonempty, and convex subset of X. Assume moreover that A is coercive over $C - C$ and fix an element $v \in C$. Then (5.1) has a unique solution.*

Proof. The proof is split into the following steps.

- Step 1. Let $\text{proj}_C : X \to C$ be the metric projectio. Given a vector $v \in C$, define the mapping $T_v : C \to C$ for every $y \in C$ by $T_v(y) = \text{proj}_C(y - \rho(Ay - v))$, where $\rho > 0$ is a parameter to be chosen.
- Step 2. Let us show that T_v is a contraction. For $y_1, y_2 \in C$, given that proj_C is nonexpansive, we clearly have $\|T_v(y_1) - T_v(y_2)\| \leq \|(y_1 - \rho(Ay_1 - v)) - (y_2 - \rho(Ay_2 - v))\|$. Thus $\|T_v(z)\| \leq \|z - \rho Az\|$, where $z = y_1 - y_2$.
- Step 3. Now, we choose ρ such that T_v is a contraction. Note that $\langle Az, z\rangle \geq \mu\|z\|^2$. Additionally, since A is bounded, then there exists $N > 0$ such that $N > \max\big(\|A\|, \mu\big)$. Expanding $\|z - \rho Az\|^2$, we obtain $\|z - \rho Az\|^2 = \|z\|^2 - 2\rho\langle Az, z\rangle + \rho^2\|Az\|^2$. Hence, $\|z - \rho Az\|^2 \leq (1 - 2\rho\mu + \rho^2N^2)\|z\|^2$. Thus, to see that T is a contraction, it suffices to require the following condition $0 \leq 1 - 2\rho\mu + \rho^2N^2 < 1$. Since $N > \max(\|A\|, \mu)$, we have $0 \leq 1 - 2\rho\mu + \rho^2N^2$. Moreover, to see $1 - 2\rho\mu + \rho^2N^2 < 1$, it suffices to take $\rho < \frac{2\mu}{N^2}$. In particular, let $\rho < \frac{\mu}{N^2}$. Then, for $\rho \in \left(0, \frac{\mu}{N^2}\right)$, mapping T_v is a contraction with contraction constant $1 - 2\rho\mu + \rho^2N^2 \in (0, 1)$.
- Step 4. We are in a position to apply the Banach Fixed-Point Theorem. Since C is closed and T maps C to itself, the Banach Fixed-Point Theorem guarantees the existence of a unique $x \in C$ such that
$$x = \text{proj}_C(x - \rho(Ax - v)). \tag{5.2}$$
- Step 5. Note that fixed point x satisfies (5.2). Thus, from the characterization of e projection $\text{proj}_C(z)$, for any $z \in X$,
$$\langle z - \text{proj}_C(z), y - \text{proj}_C(z)\rangle \leq 0, \quad \forall y \in C. \tag{5.3}$$
This means that vector $z - \text{proj}_C(z)$ forms a non-acute angle with all directions $y - \text{proj}_C(z)$ for $y \in C$. Setting $z = x - \rho(Ax - v)$ and $\text{proj}_C(z) = x$ in (5.3), one sees that, for all y in C, $\langle (x - \rho(Ax - v)) - x, y - x\rangle \leq 0$, which implies $\langle -\rho(Ax - v), y - x\rangle \leq 0$ for all $y \in C$. Since $\rho > 0$, we immediately obtain variational inequality (5.1) Thus The fixed-point equation $x = \text{proj}_C(x - \rho(Ax - v))$ is equivalent to variational inequality (5.1).

Therefore, we conclude the existence of a point $x^* \in C$ satisfying, for all $y \in C$, $\langle Ax^* - v, y - x^*\rangle \geq 0$. The uniqueness is clearly direct from the coercivity condition. □

Next, we state and prove our stability result of this section.

Theorem 5.2. *Let $C \subset X$ be a nonempty, closed, and convex subset of a Hilbert space X, and let $A : X \to X$ be a bounded, linear, and strongly monotone operator on C with constant $\mu > 0$. For* *any $v \in X$, let $x \in C$ denote the unique solution of $\langle Ax - v, y - x\rangle \geq 0$ for all $y \in C$. Then there exists $\lambda > 0$ with $\lambda < \mu^{-1}$ such that, for all $v_1, v_2 \in X$ with corresponding solutions $x_1, x_2 \in C$, $\|x_1 - x_2\| \leq \lambda \|v_1 - v_2\|$.*

Proof. The proof essentially relies on Corollary 5.1. For $i = 1, 2$, let us define the mappings:

$$T_{v_i}(y) = \text{proj}_C(y - \rho(Ay - v_i)).$$

From the proof of Proposition 5.1, by the same choice of ρ, both T_{v_1} and T_{v_2} are contractions with the same constant $\kappa = 1 - 2\rho\mu + \rho^2 N^2 \in [0, 1)$. Using Corollary 5.1, one sees the distance between $x_1 = \text{Fix}(T_{v_1})$ and $x_2 = \text{Fix}(T_{v_2})$

$$\|x_1 - x_2\| \leq \frac{1}{1-\kappa} \sup_{y \in X} \|T_{v_1}(y) - T_{v_2}(y)\|.$$

Using the nonexpansiveness of proj_C, we obtain $\|T_{v_1}(y) - T_{v_2}(y)\| \leq \rho\|v_1 - v_2\|$. Taking the supremum over $y \in X$ yields $\sup_{y \in X} \|T_{v_1}(y) - T_{v_2}(y)\| \leq \rho\|v_1 - v_2\|$. Accordingly, from Corollary 5.1, we conclude that

$$\|x_1 - x_2\| \leq \frac{\rho}{1-\kappa}\|v_1 - v_2\|.$$

Since $\rho < \frac{\mu}{N^2}$ and $\kappa = 1 - 2\rho\mu + \rho^2 N^2$, then $\frac{\rho}{1-\kappa} < \mu^{-1}$. Taking $\lambda = \frac{\rho}{1-\kappa}$ yields $\|x_1 - x_2\| \leq \lambda\|v_1 - v_2\| < \mu^{-1}\|v_1 - v_2\|$. Thus the solution map to our problem is Lipschitz continuous with respect to the perturbation in v. Moreover, solutions grow linearly with $\|v_1 - v_2\|$, scaled by λ with $\lambda < \mu^{-1}$. This completes the proof. □

Remark 5.1. In the first and second methods, we have $\|x_1 - x_2\| \leq \mu^{-1}\|v_1 - v_2\|$. However, in our third method, we obtain $\|x_1 - x_2\| \leq \lambda\|v_1 - v_2\|$ with $\lambda < \mu^{-1}$. This implies that the third method provides a potentially tighter bound on the solution mapping's Lipschitz continuity compared to the first and second methods.

6. A Further Stability under Global Variations

In this section, we extend the previous stability of quadratic problem (1.1) by considering the perturbation in all the data of the problem, including operator A, constraint set C, and parameter v. In this way, we aim at understanding much more the reaction and behavior of solutions to quadratic minimization problems with respect to total perturbations. Actually, we consider here that all of A, C, and v are subject to a perturbation by parameters of small size, say ε and ε', for which we present an important stability result of a Hölder type as follows: There exists a constant $L \geq 0$ such that $\|x_{\varepsilon'} - x_\varepsilon\| \leq L|\varepsilon' - \varepsilon|^{\frac{1}{2}}$, where x_ε is the unique solution to the perturbed problem:

$$\min_{x \in C_\varepsilon} \left(\frac{1}{2}\langle x, A_\varepsilon x\rangle - \langle v_\varepsilon, x\rangle \right), \quad \varepsilon \geq 0. \tag{6.1}$$

and $x_{\varepsilon'}$ is the unique solution to the problem:

$$\min_{x \in C_{\varepsilon'}} \left(\frac{1}{2}\langle x, A_{\varepsilon'} x\rangle - \langle v_{\varepsilon'}, x\rangle \right), \quad \varepsilon' \geq 0, \tag{6.2}$$

with $x_0 = x^*$, where x^* is the unique solution to our quadratic problem (1.1).

6.1. **Perturbation of the Problem.** We now consider the perturbed version of this problem, where operator A, constraint C, and vector v undergo small perturbations. Let x_ε be the solution to perturbed problem (6.1). For a given perturbation parameter $\varepsilon \geq 0$, the perturbed operator A_ε is defined by $A_\varepsilon = A + \varepsilon B$, where B is a linear operator with $\|B\| \leq 1$ for $\varepsilon \geq 0$. Moreover, perturbed operator A_ε retains the same properties as the original operator A and satisfies the condition: $\langle x, A_\varepsilon x\rangle \geq \mu\|x\|^2$ for all $x \in C_\varepsilon - C_\varepsilon$ and for a constant $\mu > 0$. Similarly, perturbed constraint set C_ε preserves the same properties as C and is defined as

$$C_\varepsilon = C + B(0, \varepsilon), \tag{6.3}$$

where $B(0,\varepsilon)$ represents the closed ball of radius ε centered at the origin. Additionally, the perturbed vector v_ε is given by: $v_\varepsilon = v + \varepsilon w$, where $w \in X$ still satisfies: $\|w\| \leq 1$.

6.2. **Existence and Uniqueness of the Solution under a small Perturbation.** Since the perturbations do not affect the fundamental properties of the problem, particularly the coercivity of A and the convexity of C, the existence and uniqueness of the solution x_ε remain guaranteed. The same holds for $x_{\varepsilon'}$, which is the unique solution to the problem (6.2).

Next, we state the following result.

Proposition 6.1. *(Hölder Continuity of Solutions with respect to Total Small Perturbation). Assume that conditions of Theorem 2.2 are satisfied. Then, given two positive real numbers ε and ε' close to an initial value $\varepsilon_0 \geq 0$, there exists a constant $L \geq 0$ such that $\|x_{\varepsilon'} - x_\varepsilon\| \leq L||\varepsilon' - \varepsilon|^{\frac{1}{2}}$, $\varepsilon, \varepsilon' \geq 0$, where x_ε (resp. $x_{\varepsilon'}$) is the unique solution to problem (6.1) (resp. (6.2)).*

Proof. Note that $x^* \in C$ is a solution to problem (1.1) if and only if x^* satisfies the variational inequality $\langle Ax^* - v, y - x^*\rangle \geq 0$, for all $y \in C$, which is the original problem corresponding to the following assumptions of Theorem 2.2

- A is a self-adjoint, linear, bounded operator over a Hilbert space X, with $\mathrm{dom}(A) = X$, and coercive over $C - C$.
- C is a closed, nonempty, and convex subset of X.
- $v \in X$.

The perturbed problem, subject to our treatment here, is formulated as follows: for each $\varepsilon \geq 0$, find $x_\varepsilon \in C_\varepsilon$ such that

$$\langle A_\varepsilon x_\varepsilon - v_\varepsilon, y - x_\varepsilon\rangle \geq 0, \quad \forall y \in C_\varepsilon. \tag{6.4}$$

To simplify our stability analysis of the solution, we act in the three following points

(1) In a first point, we perturb operator A, replacing it with A_ε, while constraints set C and vector v keep their initially fixed values, i.e., the corresponding problem to this step seeks to find $x_\varepsilon \in C_\varepsilon$ such that

$$\langle A_\varepsilon x_\varepsilon - v, y - x_\varepsilon\rangle \geq 0, \quad \forall y \in C. \tag{6.5}$$

(2) In the second one, we change constraint set C into the perturbed form C_ε given in (6.3).

(3) In the third and final step, we take into account simultaneous perturbations of parameter v.

(1) Let x_ε and $x_{\varepsilon'}$ be solutions of the corresponding perturbed variational inequalities:

$$\begin{cases} \langle (A+\varepsilon B)x_\varepsilon - v, y - x_\varepsilon\rangle \geq 0, & \forall y \in C, \\ \langle (A+\varepsilon' B)x_{\varepsilon'} - v, y - x_{\varepsilon'}\rangle \geq 0, & \forall y \in C. \end{cases}$$

- By adding these variational inequalities when we make $y = x_{\varepsilon'}$ in the first inequality and $y = x_\varepsilon$ in the second one, we obtain $\langle (A+\varepsilon B)x_\varepsilon - (A+\varepsilon' B)x_{\varepsilon'}, x_{\varepsilon'} - x_\varepsilon\rangle \geq 0$.
- We expand and simplify the last inequality as follows:

$$\langle A(x_\varepsilon - x_{\varepsilon'}), x_{\varepsilon'} - x_\varepsilon\rangle + \langle B(\varepsilon x_\varepsilon - \varepsilon' x_{\varepsilon'}), x_{\varepsilon'} - x_\varepsilon\rangle \geq 0.$$

  Using coercivity of A ($\langle Az, z\rangle \geq \mu\|z\|^2$) and Cauchy-Schwarz inequality, we see that

$$-\mu\|x_\varepsilon - x_{\varepsilon'}\|^2 + \|\varepsilon x_\varepsilon - \varepsilon' x_{\varepsilon'}\| \cdot \|x_\varepsilon - x_{\varepsilon'}\| \geq 0.$$

  Accordingly, we have

$$\mu\|x_\varepsilon - x_{\varepsilon'}\| \leq \|\varepsilon x_\varepsilon - \varepsilon' x_{\varepsilon'}\|. \tag{6.6}$$

- We now look for a bound of $\|\varepsilon x_\varepsilon - \varepsilon' x_{\varepsilon'}\|$ as follows. Rewrite and apply the triangle inequality

$$\|\varepsilon x_\varepsilon - \varepsilon' x_{\varepsilon'}\| \leq |\varepsilon - \varepsilon'|\|x_\varepsilon\| + \varepsilon'\|x_\varepsilon - x_{\varepsilon'}\|. \tag{6.7}$$

- In this step, we have to find a bound of $\|x_\varepsilon\|$ that should be uniform over ε. Let x_0 be the element of C minimum in norm. Setting $y = x_0$ in variational inequality (6.5) yields

$$\langle (A+\varepsilon B)x_\varepsilon - v, x_0 - x_\varepsilon\rangle \geq 0.$$

  Expanding and using the coercivity condition, we derive $\|x_\varepsilon\| \leq M$ for all $\varepsilon \leq \varepsilon_0 < \mu$, where $M = \frac{\|Ax_0 - v\| + \varepsilon_0\|x_0\|}{\mu - \varepsilon_0} + \|x_0\|$.
- Taking into account $\|x_\varepsilon\| \leq M$ in (6.7) and combining with (6.6), we see that

$$\mu\|x_\varepsilon - x_{\varepsilon'}\| \leq |\varepsilon - \varepsilon'|M + \varepsilon'\|x_\varepsilon - x_{\varepsilon'}\|.$$

  Rearranging for $\|x_\varepsilon - x_{\varepsilon'}\|$, $\|x_\varepsilon - x_{\varepsilon'}\| \leq \frac{M}{\mu - \varepsilon_0}|\varepsilon - \varepsilon'|$.
- The solutions satisfy the following Lipschitz stability bound

$$\|x_\varepsilon - x_{\varepsilon'}\| \leq \frac{M}{\mu - \varepsilon_0}|\varepsilon - \varepsilon'|, \tag{6.8}$$

  where M and $\varepsilon_0 < \mu$ are constants, independently of $\varepsilon, \varepsilon'$.

(2) At this stage, we are in a position to consider the variation on the constraints set while we keep the vector v fixed, so, let C_ε and $C_{\varepsilon'}$ be the perturbed sets defined as:

$$C_\varepsilon = C + B(0,\varepsilon),\ C_{\varepsilon'} = C + B(0,\varepsilon').$$

The corresponding problem to this step is as follows. For each $\varepsilon \geq 0$, find $x_\varepsilon \in C_\varepsilon$ such that $\langle A_\varepsilon x_\varepsilon - v, y - x_\varepsilon\rangle \geq 0$, for all $y \in C_\varepsilon$. It is useful to observe that the Hausdorff distance between C_ε and $C_{\varepsilon'}$ is given in our case by: $d_H(C_\varepsilon, C_{\varepsilon'}) = |\varepsilon - \varepsilon'|$. Without loss of generality, we assume that $\varepsilon' > \varepsilon$ and see that $C_\varepsilon \subset C_{\varepsilon'}$. Then, for $x_\varepsilon \in C_\varepsilon$ and $x_{\varepsilon'} \in C_{\varepsilon'}$, we immediately see that, for all $y \in C_\varepsilon$, $\langle Ax_\varepsilon - v, y - x_\varepsilon\rangle \geq 0$, and for all $z \in C_{\varepsilon'}$, $\langle Ax_{\varepsilon'} - v, z - x_{\varepsilon'}\rangle \geq 0$. Since that $C_\varepsilon \subset C_{\varepsilon'}$, we can assume that $x_{\varepsilon'} \notin C_\varepsilon$ (otherwise, if $x_{\varepsilon'} \in C_\varepsilon$, then $x_\varepsilon = x_{\varepsilon'}$ due to uniqueness of the solution to each perturbed quadratic problem, and in this case we have nothing to prove else). Thus, we obtain $\mu\|x_\varepsilon - x_{\varepsilon'}\|^2 \leq \langle Ax_\varepsilon - Ax_{\varepsilon'}, x_\varepsilon - x_{\varepsilon'}\rangle$. For all y $\in C_\varepsilon$ and all $z \in C_{\varepsilon'}$, $\mu\|x_\varepsilon - x_{\varepsilon'}\|^2 \leq \langle Ax_\varepsilon - Ax_{\varepsilon'}, x_\varepsilon - x_{\varepsilon'}\rangle + \langle Ax_\varepsilon - v, y - x_\varepsilon\rangle + \langle Ax_{\varepsilon'} - v, z - x_{\varepsilon'}\rangle$. Then, a simplification in the previous inequality leads to

$$\mu\|x_\varepsilon - x_{\varepsilon'}\|^2 \leq \langle Ax_\varepsilon - v, y - x_{\varepsilon'}\rangle + \langle Ax_{\varepsilon'} - v, z - x_\varepsilon\rangle \quad \forall y \in C_\varepsilon \quad \forall z \in C_{\varepsilon'}. \tag{6.9}$$

Therefore, taking $z = x_\varepsilon$ and $y = \text{proj}_{C_\varepsilon}(x'_\varepsilon)$ in (6.9), we see that

$$\mu \|x_{\varepsilon'} - x_\varepsilon\|^2 \leq (\|A\|\|x_\varepsilon\| + \|v\|)\|\text{proj}_{C_\varepsilon}(x'_\varepsilon) - x'_\varepsilon\|$$

and

$$\mu \|x_{\varepsilon'} - x_\varepsilon\|^2 \leq (\|A\|\|x_\varepsilon\| + \|v\|) d(x_{\varepsilon'}, C_\varepsilon),$$

with $d(x_{\varepsilon'}, C_\varepsilon)$ being the distance between $x_{\varepsilon'}$ and the set C_ε. Note that $d(x_{\varepsilon'}, C_\varepsilon) \leq d_H(C_{\varepsilon'}, C_\varepsilon) \leq |\varepsilon' - \varepsilon|$. Hence

$$\mu \|x_{\varepsilon'} - x_\varepsilon\|^2 \leq (\|A\|\|x_\varepsilon\| + \|v\|)|\varepsilon' - \varepsilon|, \tag{6.10}$$

which implies by the triangular inequality $\|x_\varepsilon\| \leq \|x_\varepsilon - x^*\| + \|x^*\|$. Since $\mu\|x_\varepsilon - x^*\|^2 \leq (\|A\|\|x^*\| + \|v\|)\varepsilon$ and $x_{\varepsilon'} = x^*$ when $\varepsilon' = 0$, it results that

$$\|x_\varepsilon\| \leq \varepsilon^{\frac{1}{2}} \mu^{-\frac{1}{2}} (\|A\|\|x^*\| + \|v\|)^{\frac{1}{2}} + \|x^*\|. \tag{6.11}$$

Substituting (6.11) into (6.10) yields

$$\begin{aligned} \mu \|x_{\varepsilon'} - x_\varepsilon\|^2 &\leq (\|A\|\|x_\varepsilon\| + \|v\|)(|\varepsilon' - \varepsilon|) \\ &\leq \left(\|A\|\varepsilon^{\frac{1}{2}}\mu^{-\frac{1}{2}}(\|A\|\|x^*\| + \|v\|)^{\frac{1}{2}} + \|x^*\|\| + \|v\|\right)|\varepsilon' - \varepsilon| \end{aligned}$$

and $\|x_{\varepsilon'} - x_\varepsilon\| \leq k\mu^{-\frac{1}{2}}(|\varepsilon' - \varepsilon|)^{\frac{1}{2}}$, where $k = \left(\|A\|\varepsilon^{\frac{1}{2}}\mu^{-\frac{1}{2}}(\|A\|\|x^*\| + \|v\|)^{\frac{1}{2}} + \|x^*\| + \|v\|\right)^{\frac{1}{2}}$.

(3) Given an initially fixed value of v, the corresponding perturbed form of v is actually given by $v_\varepsilon = v + \varepsilon w$, and $v_{\varepsilon'} = v + \varepsilon' w$, where $w \in X$ still satisfies: $\|w\| \leq 1$. We still keep the notation x_ε as the solution to totally perturbed problem (6.4). Then, from Theorem 5.2, we obtain $\|x_{\varepsilon'} - x_\varepsilon\| \leq \mu^{-1}\|v_{\varepsilon'} - v_\varepsilon\|$. Clearly, we have $\|v_{\varepsilon'} - v_\varepsilon\| \leq |\varepsilon - \varepsilon'|$. Then, $\|x_{\varepsilon'} - x_\varepsilon\| \leq \mu^{-1}|\varepsilon' - \varepsilon|$. Therefore, by keeping in mind the previous steps, including estimate (6.8), we are able to conclude

$$\|x_{\varepsilon'} - x_\varepsilon\| \leq \mu^{-1}|\varepsilon' - \varepsilon| + \frac{M}{\mu - \varepsilon_0}|\varepsilon - \varepsilon'| + \mu^{-\frac{1}{2}}k(|\varepsilon' - \varepsilon|)^{\frac{1}{2}}.$$

For $|\varepsilon - \varepsilon'|$ in the neighborhood of zero (by assumption), $|\varepsilon - \varepsilon'| \leq |\varepsilon' - \varepsilon|^{\frac{1}{2}}$ holds. Thus

$$\|x_{\varepsilon'} - x_\varepsilon\| \leq \left(\mu^{-1} + \frac{M}{\mu - \varepsilon_0} + \mu^{-\frac{1}{2}}k\right)|\varepsilon' - \varepsilon|^{\frac{1}{2}}.$$

Define the constant $L = \left(\mu^{-1} + \frac{M}{\mu - \varepsilon_0} + \mu^{-\frac{1}{2}}k\right) \geq 0$ and observe that the last inequality simplifies to $\|x_{\varepsilon'} - x_\varepsilon\| \leq L|\varepsilon' - \varepsilon|^{\frac{1}{2}}$. This completes the proof. □

7. Conclusion

The present study focuses on the stability of quadratic minimization problem (1.1) from several complementary perspectives. First, by reformulating the problem as an abstract equilibrium problem, we demonstrated that the coercivity of the main operator together with the boundedness of the constraint set guarantees the uniqueness of the solution as well as the required quantitative stability estimates. Second, an extremely detailed proof of the result briefly justified in [5, Theorem 6H.1] was presented, establishing existence, uniqueness, and Lipschitz continuity of the solution mapping via variational inequality arguments. Furthermore, we proved the equivalence of the metric regularity property of the inverse of solution map S, its monotonicity, and the Lipschitz continuity of S. We would like to underline here that all of these properties are satisfied for our quadratic problem. Third, the fixed-point method was employed, yielding a sharper Lipschitz bound $\|x_1 - x_2\| \leq \lambda \|v_1 - v_2\|$, $\lambda < \mu^{-1}$, which improves upon the estimates obtained in the first two approaches. Finally, Hölder continuity was obtained in our paper under simultaneous perturbations of operator A, constraint set C, and parameter v, leading to explicit error bounds that quantify the deviation of solutions in terms of the perturbation size. Finally, these results refine the stability theory of quadratic minimization problems in Hilbert spaces and strengthen the mathematical understanding of their sensitivity to data variations.

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