J. Appl. Numer. Optim. 8 (2026), No. 1, pp. 107-114
Available online at http://jano.biemdas.com
https://doi.org/10.23952/jano.8.2026.1.06

NECESSARY CONDITION OF WEAK NASH EQUILIBRIUM

A.V. ARUTYUNOV*, S.E. ZHUKOVSKIY

V.A. Trapeznikov Institute of Control Sciences of RAS, Moscow, Russia

Abstract. We study a N-person game in normal form. We introduce the concept of weak Nash equilibrium. Unlike the Nash equilibrium, a weak equilibrium always exists if only the strategy sets are compact and the loss functions are continuous. Under the assumption that the sets of pure strategies are defined by equality constraints, necessary conditions of the weak Nash equilibrium are obtained. The necessary conditions obtained remain informative even in the case when the first derivative of the mappings defining the constraints degenerates at the point of weak equilibrium.
Keywords. N-person game in normal form; Necessary optimality condition; Weak Nash equilibrium.
2020 Mathematics Subject Classification. 91A10, 91B50.

1. INTRODUCTION

Let N and n_i be given positive integers, and let $X_i \subset \mathbb{R}^{n_i}$ be given nonempty compact sets, $i = \overline{1,N}$. We denote the standard inner product in each space $\mathbb{R}^{n_i}$ by $\langle \cdot,\cdot \rangle$ and the standard norm by $|\cdot|$. Put $X := X_1 \times .... \times X_N$, where the symbol $\times$ stands for the Cartesian product. For every $i = \overline{1,N}$, we put $X_{-i} := \prod_{j \neq i} X_j$. We denote the elements of the space X_{-i} by $x_{-i} := (x_1,...,x_{i-1},x_{i+1},...,x_N)$. For a point $x = (x_1,...,x_N) \in X$, we use the notation $x = (x_i,x_{-i}) := (x_1,...,x_{i-1},x_i,x_{i+1},...,x_N)$. The notation (x_i,x_{-i}) does not mean that the component x_i is moved to the first place. It remains in its i-th place. In such a notation, this coordinate x_i is simply highlighted. The same notation is used below for other vectors in $\mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_N}$.

Let $\theta_i : \mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_N} \to \mathbb{R}$ be given continuous functions, $i = \overline{1,N}$. Consider the N-person game. In this game, the i-th player choose a strategy $x_i \in X_i$, $i \in \{1, ..., N\}$. His purpose is to minimize the value $\theta_i(x_i,x_{-i})$, where $x_{-i} \in X_{-i}$ is the collection of $N-1$ strategies x_j, $j \neq i$, choused by other players. A point $x^* = (x_1^*,...,x_N^*) \in X$ is said to be Nash equilibrium of the game if $\theta_i(x^*) = \theta_i(x_i^*,x_{-i}^*) = \min\limits_{x_i \in X_i} \theta_i(x_i,x_{-i}^*)$ for all $i \in \{1,...,N\}$. Define a function $V : \mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_N} \to \mathbb{R}$ by

$$V(x_1,...,x_N) = \sum_{i=1}^{N} \Big(\theta_i(x_1,...,x_N) - \min_{y_i \in X_i} \theta_i(y_i,x_{-i}) \Big), \quad (x_1,...,x_N) \in \mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_N}. \tag{1.1}$$

*Corresponding author.
E-mail address: arutyunov@cs.msu.ru (A.V. Arutyunov), s-e-zhuk@yandex.ru (S.E. Zhukovskiy).
Received 1 July 2025; Accepted 30 August 2025; Published online 1 April 2026.

This function was introduced in [1] for the investigation of the Nash equilibrium. It is known (see, for example, [2]) that function V is continuous, nonnegative, and $V(x) = 0$ if and only if the point $x \in X$ is Nash equilibrium.

By analogy with the concept of a generalized saddle point (see [3]), the following generalization of equilibrium can be introduced. We say that a point $x^* = (x_1^*, \dots, x_N^*) \in X$ is a weak Nash equilibrium if function V attains its minimum at x^*. It follows from this definition that, under the assumptions introduced, a weak Nash equilibrium always exists. At the same time, Nash equilibrium, as is known, may not exist. If the set of Nash equilibria for a given game is non-empty, then it coincides with the set of weak equilibria.

We study a N-person game in normal form with the strategy sets X_i defined by equality-type constraints. The main result of the work is necessary conditions of weak Nash equilibrium. An important feature of this result is that the obtained necessary conditions remain informative even when the first derivative of the mappings defining the constraints degenerates at a weak equilibrium point.

2. Statement of the Problem and Example

Let us pass to the description of the game under consideration. We are given positive integers n_i and k_i, twice continuously differentiable mappings $F_i = (F_{i,1}, \dots, F_{i,k_i}) : \mathbb{R}^{n_i} \to \mathbb{R}^{k_i}$, nonempty compact sets $\overline{X}_i \subset \mathbb{R}^{n_i}$, points $x_i^* \in \mathrm{int}\overline{X}_i$ satisfying $F_i(x_i^*) = 0$, and continuously differentiable functions $\theta_i : \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_N} \to \mathbb{R}$, $i = \overline{1,N}$. Denote $x^* := (x_1^*, \dots, x_N^*)$ and

$$X_i := \{x_i \in \overline{X}_i : \ F_i(x_i) = 0\}, \quad i = \overline{1,N}.$$

The assumptions above imply that X_i is nonempty and compact and contains the point x_i^* for each $i = \overline{1,N}$. Define a function $v_i : \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_N} \to \mathbb{R}$ by

$$v_i(x) := \min_{y_i \in X_i} \theta_i(y_i, x_{-i}), \quad i = \overline{1,N}. \tag{2.1}$$

This function does not explicitly depend on the variable x_i and depends on the remaining variables x_{-i}. It follows from [5, Theorem 10.22] that the minimum function v_i is locally Lipschitz and differentiable in every direction.

Using the introduced notation, we obtain the following identity for the function V defined by the equality (1.1):

$$V(x_1, \dots, x_N) \equiv \sum_{i=1}^{N} \theta_i(x_1, \dots, x_N) - \sum_{i=1}^{N} v_i(x), \quad (x_1, \dots, x_N) \in X. \tag{2.2}$$

As noted above, the function V is non-negative, Lipschitz, and attains its minimum on the compact set X. The function V attains its minimum at the point x^* if and only if this point is a weak Nash equilibrium in the game with the loss functions θ_i and strategy sets X_i, $i = \overline{1,N}$. The equality $V(x^*) = 0$ holds if and only if the point x^* is Nash equilibrium.

In the specific case when $N = 2$ and $\theta_1 + \theta_2 = 0$, the weak Nash equilibrium coincides with the generalized saddle point of the function $f := \theta_1$ (see [3]).

A weak Nash equilibrium is a solution to the following constrained minimization problem:

$$\begin{array}{lll} \text{minimize} & V(x_1, \dots, x_N) & \text{over all} \quad x = (x_1, \dots, x_n) \\ \text{satisfying} & F_i(x_i) = 0, \quad x_i \in \overline{X}_i, & i = \overline{1,N}. \end{array} \tag{2.3}$$

In this case, the mappings F_i are smooth, while the function V may not be smooth. Moreover, at the weak equilibrium point x^*, the first derivatives of the constraints $\frac{\partial F_i}{\partial x_i}(x_i^*)$ may degenerate. Thus the problem is to minimize the function V satisfying the identities (2.2) and (2.1), subject to the constraints (2.3).

The following example shows the application of necessary optimality conditions from [4] to problem (2.3), and hence to the weak Nash equilibrium.

Example 2.1. Put

$$\overline{X}_1 = \overline{X}_2 := \{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_1| \leq 1, |\xi_1| \leq 1\}, \quad x^* = (x_1^*, x_2^*) = (0,0) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

Denote the coordinates of the vector $x_1 \in \overline{X}_1$ by $x_{1,1}$ and $x_{1,2}$. Analogously, for $x_2 \in \overline{X}_2$ we denote $x_2 = (x_{2,1}, x_{2,2})$. So, vectors x_1 and x_2 are two-dimensional, and the vector $x = (x_1, x_2)$ is four-dimensional. Let the mappings $F_1, F_2 : \mathbb{R}^2 \to \mathbb{R}$ defining the constraints be quadratic forms $F_1(x_1) = x_{1,1} \cdot x_{1,2}$ and $F_2(x_2) = x_{2,1} \cdot x_{2,2}$, $x_1, x_2 \in \mathbb{R}^2$. Then, the admissible set is $X = X_1 \times X_2$, where

$$X_1 := \{x_1 = (x_{1,1}, x_{1,2}) \in \overline{X}_1 : x_{1,1} \cdot x_{1,2} = 0\},$$

and

$$X_2 := \{x_2 = (x_{2,1}, x_{2,2}) \in \overline{X}_2 : x_{2,1} \cdot x_{2,2} = 0\}.$$

Put

$$f(x_1, x_2) := x_{1,1} \cdot x_{2,1} + \zeta_1(x_1) + \zeta_2(x_2),$$

and

$$\theta_1(x_1, x_2) := f(x_1, x_2), \quad \theta_2(x_1, x_2) := -f(x_1, x_2), \quad (x_1, x_2) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

Here, $\zeta_1 : \mathbb{R}^2 \to \mathbb{R}$ and $\zeta_2 : \mathbb{R}^2 \to \mathbb{R}$ are given continuously differentiable functions, ζ_1 satisfies the assumptions **(i)** and **(ii)** below, and ζ_2 satisfies the assumptions **(i′)** and **(ii′)** below. Note that the first term $x_{1,1} \cdot x_{2,1}$ in the definition of the function f depends on the first coordinates of both vectors x_1 and x_2, and therefore “mixes” these vectors.

In this example, we derive necessary conditions for the point $x^* := (0,0)$ to be a weak Nash equilibrium for the game with the loss functions θ_1 and θ_2 and the strategy sets X_1 and X_2. Let the function ζ_1 satisfy the following assumptions.

(i): The function $\zeta_1(\cdot, 0)$ over the segment $[-1,1]$ attains its maximum at the points -1 and 1 (and possibly at other points of the segment). Denote this maximal value by m_1.

(ii): The inequality $\zeta_1(0, x_{1,2}) \leq m_1$ holds for every $x_{1,2} \in [-1,1]$.

Let the function ζ_2 satisfy the following assumptions.

(i′): The function $\zeta_2(x_{2,1}, 0)$ over the segment $[-1,1]$ attains its maximum at the points -1 and 1 (but possibly at other points of the segment). Denote this maximal value by m_2.

(ii′): The inequality $\zeta_2(0, x_{2,2}) \leq m_2$ holds for every $x_{2,2} \in [-1,1]$.

Let us now compute the function $V : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ which is defined by formula (1.1). Denote

$$\varphi_1(x_1) := \max_{x_2 \in X_2} f(x_1, x_2) = \max_{x_2 \in X_2} \big(x_{1,1} \cdot x_{2,1} + \zeta_2(x_2)\big) + \zeta_1(x_1), \quad x_1 \in \mathbb{R}^2$$

and

$$\varphi_2(x_2) := \min_{x_1 \in X_1} f(x_1, x_2) = \min_{x_1 \in X_1} \big(x_{1,1} \cdot x_{2,1} + \zeta_1(x_1)\big) + \zeta_2(x_2), \quad x_2 \in \mathbb{R}^2.$$

We have

$$V(x_1, x_2) = \varphi_1(x_1) - \varphi_2(x_2).$$

Fix an arbitrary $x_1 \in \mathbb{R}^2$. Let us compute the value $\varphi_1(x_1)$. For every $x_2 \in X_2$, either $x_{2,1} = 0$ or $x_{2,2} = 0$. Consider the first case: $x_{2,1} = 0$. Then $x_{2,2}$ take all the values in the segment $[-1,1]$ by definition of X_2. Thus

$$\max_{x_2 \in X_2 \colon x_{2,1}=0} \big(x_{1,1} \cdot x_{2,1} + \zeta_2(x_2) \big) = \max_{x_{2,2} \in [-1,1]} \zeta_2(0, x_{2,2}).$$

Consider the second case: $x_{2,2} = 0$. Then $x_{2,1}$ takes all the values in the segment $[-1,1]$ by definition of X_2. Therefore, the assumption **(i′)** implies that

$$\max_{x_2 \in X_2 \colon x_{2,2}=0} \big(x_{1,1} \cdot x_{2,1} + \zeta_2(x_2) \big) = |x_{1,1}| + m_2.$$

Therefore,

$$\max_{x_2 \in X_2} \big(x_{1,1} \cdot x_{2,1} + \zeta_2(x_2) \big) = \max \Big\{ |x_{1,1}| + m_2, \max_{x_{2,2} \in [-1,1]} \zeta_2(0, x_{2,2}) \Big\}.$$

So, it follows from assumption **(ii′)** and the obvious inequality $|x_{1,1}| \geq 0$ that

$$\max_{x_2 \in X_2} (x_{1,1} \cdot x_{2,1} + \zeta_2(x_2)) = |x_{1,1}| + m_2.$$

Finally, we have $\varphi_1(x_1) = |x_{1,1}| + m_2 + \zeta_1(x_1)$. Fix an arbitrary $x_2 \in \mathbb{R}^2$. In a similar way, from assumptions **(i)** and **(ii)**, we obtain that $-\varphi_2(x_2) = |x_{2,1}| - m_1 - \zeta_2(x_2)$. So, we have

$$V(x_1, x_2) = |x_{1,1}| + |x_{2,1}| + \zeta_1(x_1) - \zeta_2(x_2) + m_2 - m_1 \quad \forall x_1 \in \mathbb{R}^2, \quad x_2 \in \mathbb{R}^2.$$

Note that this function is only Lipschitz continuous. It has no partial derivatives at zero with respect to either the variable x_1 or the variable x_2.

Assume that the point $x^* = (0,0)$ is the weak Nash equilibrium (or, equivalently, x^* is a generalized saddle point of the function f on the set $X_1 \times X_2$). Then $(0,0)$ is the point of minimum of function V under the constraints $(x_1, x_2) \in \overline{X}_1 \times \overline{X}_2$,

$$F_1(x_1) = 0, \quad F_2(x_2) = 0.$$

In other words, $(0,0)$ is a solution to the problem (2.3).

The latter is equivalent to the fact that zero is a solution to the problem with an abnormal constraint

$$\varphi_1(x_1) \to \min, \quad x_1 \in \overline{X}_1, \quad F_1(x_1) = 0,$$

and zero is the solution to the problem with abnormal constraint

$$-\varphi_2(x_2) \to \min, \quad x_2 \in \overline{X}_2, \quad F_2(x_2) = 0.$$

Let us apply [4, Theorem 4.1] to the first of these problems. Fix an arbitrary nonzero vector $\bar{\xi} \in \mathbb{R}^2$ such that

$$\langle \bar{Q}\bar{\xi}, \bar{\xi} \rangle = 0 \quad \text{and} \quad \bar{Q}\bar{\xi} \neq 0. \tag{2.4}$$

Here,

$$\bar{Q} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The part (a) of Theorem 4.1 from [4] implies that $\varphi_1^-(0, \bar{\xi}) \geq 0$ and $\varphi_1^-(0, -\bar{\xi}) \geq 0$. Here,

$$\varphi_1^-(0, \bar{\xi}) := \liminf_{t \to 0+} \frac{\varphi_1(t\bar{\xi}) - \varphi_1(0)}{t}$$

is the lower directional derivative of the function φ_1 at zero in the direction $\bar{\xi}$. The vector $\bar{\xi} = (1,0)$ satisfies condition (2.4). Therefore, directly computing $\varphi_1^-(0,\bar{\xi})$ and $\varphi_1^-(0,-\bar{\xi})$ and using the inequalities $\varphi_1^-(0,\bar{\xi}) \geq 0$ and $\varphi_1^-(0,-\bar{\xi}) \geq 0$, we obtain

$$1+\frac{\partial \zeta_1}{\partial x_{1,1}}(0) \geq 0, \quad 1-\frac{\partial \zeta_1}{\partial x_{1,1}}(0) \geq 0.$$

Hence, $\frac{\partial \zeta_1}{\partial x_{1,1}}(0) \in [-1,1]$. Note that $\bar{\xi} = (0,1)$ also satisfies conditions (2.4). Therefore, directly computing $\varphi_1^-(0,\bar{\xi})$ and $\varphi_1^-(0,-\bar{\xi})$ and using the inequalities $\varphi_1^-(0,\bar{\xi}) \geq 0$ and $\varphi_1^-(0,-\bar{\xi}) \geq 0$, we obtain

$$\frac{\partial \zeta_1}{\partial x_{1,2}}(0) \geq 0, \quad -\frac{\partial \zeta_1}{\partial x_{1,2}}(0) \geq 0.$$

Hence, $\frac{\partial \zeta_1}{\partial x_{1,2}}(0) = 0$.

Let us apply Theorem 4.1 from [4] to the second problem. Repeating the above reasoning, we obtain that

$$\frac{\partial \zeta_2}{\partial x_{2,1}}(0) \in [-1,1], \quad \frac{\partial \zeta_2}{\partial x_{2,2}}(0) = 0.$$

As a result, we obtain the following assertion. If zero is a weak Nash equilibrium in the game under consideration, then V attains its minimum at zero, and therefore, the functions ζ_i satisfies the following relations

$$\frac{\partial \zeta_1}{\partial x_{1,1}}(0) \in [-1,1], \quad \frac{\partial \zeta_1}{\partial x_{1,2}}(0) = 0, \quad \frac{\partial \zeta_2}{\partial x_{2,1}}(0) \in [-1,1], \quad \frac{\partial \zeta_2}{\partial x_{2,2}}(0) = 0. \tag{2.5}$$

If at least one of the relations in (2.5) is violated, then zero is not a weak Nash equilibrium in this game. But, of course, a weak Nash equilibrium does exist in this example, and it is simply not zero.

In this example, the constraints in problem (2.3) degenerate at zero, i.e.,

$$\frac{\partial F_1}{\partial x_1}(0) = \frac{\partial F_2}{\partial x_2}(0) = 0.$$

Therefore, the first-order necessary conditions (see, for example, [5, Section 10.4]) are not informative in it. Let us now proceed to the formulation of the general result.

3. Necessary Conditions of Weak Nash Equilibrium

For every $i = \overline{1,N}$, denote by $Q_i(x^*_{-i})$ the set of all points of minimum $x_i \in X_i$ of the function $\theta_i(\cdot, x^*_{-i})$ over the set X_i. Let $\ker A$ stand for the kernel of a linear mapping A and $\operatorname{im} A$ stand for its image. Assume also that $x_i^* \in \operatorname{int}\overline{X}_i$ for all i.

Assume that $x^* = (x_1^*, \ldots, x_N^*)$ is a solution to problem (2.3). The classical first-order regularity conditions for this problem are as follows: The linear mappings $\frac{\partial F_i}{\partial x_i}(x_i^*)$, $i \in \overline{1,N}$, are surjective. If these assumptions are violated, then more general second-order regularity conditions can be used. They consist in the existence of a vector ξ, that satisfies the relations

$$\frac{\partial F_i}{\partial x_i}(x_i^*)\xi_i = 0, \quad \frac{\partial^2 F_i}{\partial x_i^2}(x_i^*)[\xi_i, \xi_i] \in \ker \frac{\partial F_i}{\partial x_i}(x_i^*), \tag{3.1}$$

$$\operatorname{im}\frac{\partial F_i}{\partial x_i}'(x_i^*) + \frac{\partial^2 F_i}{\partial x_i^2}[\xi_i, \ker \frac{\partial F_i}{\partial x_i}(x_i^*)] = \mathbb{R}^{k_i}, \qquad i = \overline{1,N}. \tag{3.2}$$

In this case, if the first-order condition is satisfied for some i, then the corresponding second-order condition is satisfied with $\xi_i = 0$. Here, $\frac{\partial^2 F_i}{\partial x_i^2}(x_i^*)[\xi_i, \cdot]$ is the $k_i \times n_i$ matrix such that its j-th row equals to the vector $\frac{\partial^2 F_{i,j}}{\partial x_i^2}(x_i^*)\xi_i$.

Theorem 3.1. *Assume that the point $x^* \in \mathrm{int}(\overline{X}_1 \times \ldots \times \overline{X}_N)$ is a weak Nash equilibrium for the N-person game. Let the vector $\xi = (\xi_1, \ldots, \xi_N)$ satisfy equalities (3.1) and (3.2). Then, for this vector ξ, the following inequalities hold*

$$\sum_{i=1}^{N} \min_{y_i \in Q_i(x_{-i}^*)} \left\langle \frac{\partial \theta_i}{\partial x_{-i}}(y_i, x_{-i}^*), \xi_{-i} \right\rangle$$
$$\leq \sum_{i=1}^{N} \left\langle \frac{\partial \theta_i}{\partial x}(x^*), \xi \right\rangle \leq -\sum_{i=1}^{N} \max_{y_i \in Q_i(x_{-i}^*)} \left\langle \frac{\partial \theta_i}{\partial x_{-i}}(y_i, x_{-i}^*), \xi_{-i} \right\rangle. \tag{3.3}$$

Here,

$$\left\langle \frac{\partial \theta_1}{\partial x_{-1}}(y_1, x_{-1}^*), \xi_{-1} \right\rangle = \sum_{j \neq 1} \left\langle \frac{\partial \theta_i}{\partial x_j}(y_1, x_2^*, \ldots, x_N^*), \xi_j \right\rangle$$

and the analogous notation is used for other numbers $i = 2, \ldots, N$. If at least one of these inequalities is satisfied as equality, then there exist vectors $\lambda_1 = (\lambda_{1,1}, \ldots, \lambda_{1,N})$ and $\lambda_2 = (\lambda_{2,1}, \ldots, \lambda_{2,N})$, $\lambda_{1,i}$, $\lambda_{2,i} \in \mathbb{R}^{k_i}$, $i = \overline{1,N}$ and a vector

$$\eta \in \sum_{i=1}^{N} \frac{\partial \theta_i}{\partial x}(x^*) + \partial_{Cl}\left(\sum_{i=1}^{N} v_i(x^*)\right), \tag{3.4}$$

such that

$$\lambda_{2,i} \frac{\partial F_i}{\partial x_i}(x_i^*) = 0 \quad \forall i = \overline{1,N}, \tag{3.5}$$

and for the vector

$$z_i = \lambda_{1,i} \frac{\partial F_i}{\partial x_i}(x_i^*) + \lambda_{2,i} \frac{\partial^2 F_i}{\partial x_i^2}(x_i^*)\xi_i \in \mathbb{R}^{n_i}, \qquad z = (z_1, \ldots, z_N), \quad i = \overline{1,N} \tag{3.6}$$

the equality

$$\eta + z = 0 \tag{3.7}$$

takes place. Here, $\partial_{Cl}\left(\sum_{i=1}^{N} v_i(x^*)\right)$ is the generalized gradient (Clarke's gradient) of the function $\sum_{i=1}^{N} v_i$ at the point x^*, and the multiplication in (3.6) is the multiplication of the rows $\lambda_{1,i}$ and $\lambda_{2,i}$ by the corresponding matrices.

Note that if ξ is zero, then condition (3.3) is satisfied automatically. Note also that by Theorem 10.13 in [5] we have $\partial_{Cl}\left(\sum_{i=1}^{N} v_i\right) \subseteq \sum_{i=1}^{N} \partial_{Cl} v_i$. Therefore, the formula (3.4) of Theorem 3.1 implies that

$$\eta \in \sum_{i=1}^{N} \frac{\partial \theta_i}{\partial x}(x^*) + \sum_{i=1}^{N} \partial_{Cl} v_i(x^*).$$

Proof. Let x^* be a weak Nash equilibrium in the N-player game. Then x^* is a solution to problem (2.3). The definition of the function V implies that x^* is a solution to the following constrained optimization problem

$$\sum_{i=1}^{N} \theta_i(x_1,...,x_N) - \sum_{i=1}^{N} v_i(x) \to \min,$$

$$(x_1,...,x_N) \in \overline{X}_1 \times ... \times \overline{X}_N, \qquad F_i(x_i) = 0, \quad i = \overline{1,N}.$$

Here, the function $v_i : \mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_N} \to \mathbb{R}$ is defined by the formula

$$v_i(x) := \min_{y_i \in X_i} \theta_i(y_i, x_{-i}), \quad i = \overline{1,N}.$$

The assumptions of the theorem imply that $\xi = (\xi_1,...,\xi_N)$ satisfies relations (3.1) and (3.2). Therefore, ξ satisfies the assumption (a) of Theorem 4.1 in [4]. Applying this theorem, in view of $x^* \in \mathrm{int}(\overline{X}_1 \times ... \times \overline{X}_N)$ and the first inequality from (4.2) in [4], we have

$$\sum_{i=1}^{N} \left\langle \frac{\partial \theta_i}{\partial x}(x^*), \xi \right\rangle - \sum_{i=1}^{N} v_i^-(x^*, \xi) \geq 0.$$

Here,

$$v_i^-(x^*, \xi) := \liminf_{t \to 0+} \frac{v_i(x^* + t\xi) - v_i(x^*)}{t}$$

is the lower directional derivative of the function v_i at the point x^* in the direction ξ. Applying Danskin's formula (see [5, Theorem 10.22]) o lower directional derivative of the minimum function v_i we obtain

$$v_i^-(x^*, \xi) = \min_{y_i \in Q_i(x^*_{-i})} \left\langle \frac{\partial \theta_i}{\partial x_{-i}}(y_i, x^*_{-i}), \xi_{-i} \right\rangle \forall i.$$

This equality and the previous inequality imply

$$\sum_{i=1}^{N} \langle \frac{\partial \theta_i}{\partial x}(x^*), \xi \rangle - \sum_{i=1}^{N} \min_{y_i \in Q_i(x^*_{-i})} \left\langle \frac{\partial \theta_i}{\partial x_{-i}}(y_i, x^*_{-i}), \xi_{-i} \right\rangle \geq 0.$$

Thus the first inequality in (3.3) is proved.

The second inequality in Theorem 4.1 in [4] implies that the following relations hold

$$\sum_{i=1}^{N} \left\langle \frac{\partial \theta_i}{\partial x}(x^*), \xi \right\rangle - \sum_{i=1}^{N} v_i(x^*, -\xi) \geq 0.$$

Applying Danskin's formula (see [5, Theorem 10.22]) again, we compute the lower directional derivative of the minimum function v_i in the direction $-\xi$. We obtain

$$v_i^-(x^*, -\xi) = \min_{y_i \in Q_i(x^*_{-i})} \left\langle \frac{\partial \theta_i}{\partial x_{-i}}(y_i, x^*_{-i}), -\xi_{-i} \right\rangle = - \max_{y_i \in Q_i(x^*_{-i})} \left\langle \frac{\partial \theta_i}{\partial x_{-i}}(y_i, x^*_{-i}), \xi_{-i} \right\rangle$$

for all $i = \overline{1,N}$.

This equality and the previous inequality imply

$$\sum_{i=1}^{N} \left\langle \frac{\partial \theta_i}{\partial x}(x^*), \xi \right\rangle + \sum_{i=1}^{N} \max_{y_i \in Q_i(x^*_{-i})} \left\langle \frac{\partial \theta_i}{\partial x_{-i}}(y_i, x^*_{-i}), \xi_{-i} \right\rangle \geq 0.$$

Thus the second inequality in (3.3) is also proved.

Let us prove the second part of the theorem. Assume that at least one of the inequalities in (3.3) holds as an equality. Then the part (b) of Theorem 4.1 in [4] implies that there exist vectors $\lambda_1 = (\lambda_{1,1}, ..., \lambda_{1,N})$, $\lambda_2 = (\lambda_{2,1}, ..., \lambda_{2,N})$, $\lambda_{1,i}$, $\lambda_{2,i} \in \mathbb{R}^{k_i}$, and a vector $\eta \in \partial_{Cl} V(x^*)$ such that (3.5) takes place and the equality $\eta + z = 0$, i.e., (3.7), holds for the vectors z_i defined by the formula (3.6). The function V satisfies identity (2.2). It is known that (see, for example, [5, Excercise 10.16]) the generalized gradient of the sum of a smooth function and a locally Lipschitz function equals to the sum of the gradient of the smooth function and generalized gradient of the locally Lipschitz function. Therefore, identity (2.2) and inclusion $\eta \in \partial_{Cl} V(x^*)$ imply that

$$\eta \in \partial_{Cl} V(x^*) = \sum_{i=1}^{N} \frac{\partial \theta_i}{\partial x}(x^*) + \partial_{Cl}\left(\sum_{i=1}^{N} v_i(x^*)\right).$$

Therefore, inclusion (3.4) is proved. □

Acknowledgments

This research was supported by the Russian Science Foundation (Project no. 25-21-00525).

References

[1] H. Nikaido, K. Isoda, Note on noncooperative convex games, Pacific J. 5 (1955), 807–815.

[2] F. Facchinei, C. Kanzow, Generalized Nash equilibrium problems, Ann. Oper. Res. 175 (2010), 177–211.

[3] A.V. Arutyunov, S.E. Zhukovskiy, On stability of the generalized saddle points, Optimization 73 (2024), 3519–3536.

[4] A.V. Arutyunova, A.F. Izmailov, I. Shvartsman, Optimality conditions for 2-regular problems with nonsmooth objective functions, Nonlinear Anal. 90 (2013), 37–45.

[5] F.H. Clarke, Functional Analysis, Calculus of Variations and Optimal Control, Springer, New York, 2013.