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A CORRECTOR-PREDICTOR FEASIBLE INTERIOR-POINT ALGORITHM FOR $P_*(\kappa)$-WEIGHTED LINEAR COMPLEMENTARITY PROBLEM BASED ON THE ALGEBRAIC EQUIVALENT TRANSFORMATION

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Abstract. We present a corrector-predictor feasible interior-point algorithm for $P_*(\kappa)$-weighted linear complementarity problems based on the new search directions derived from the specific algebraic equivalent transformation of the central path equations. The algorithm uses full-Newton step in the corrector iteration while the step size in the predictor iteration is determined in a simple way that avoids numerically expansive and complicated line search computations. Under suitable conditions, the algorithm achieves global convergence with polynomial iteration complexity matching the best-known bounds for these types of methods. Preliminary numerical results demonstrate potential efficiency and practical viability of the proposed method.
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1. Introduction

In this paper, we consider a *Weighted Linear Complementarity Problem (WLCP)* in the standard form, which is the problem of finding a pair of vectors $(x,s) \in \mathbb{R}^n \times \mathbb{R}^n$ satisfying the following system of equations and inequalities

$$-Mx+s=q, \quad xs=w, \quad (x,s) \geq 0, \tag{1.1}$$

where $M \in \mathbb{R}^{n\times n}$ is a given matrix, $q \in \mathbb{R}^n$ is a vector, and $w \in \mathbb{R}^n_+$ is a vector of nonnegative weights. When $w=0$, the WLCP reduces to the classical Linear Complementarity Problem (LCP). The WLCP represents a significant generalization of LCP because the introduction of nonnegative weights in the complementarity condition equation enables modeling of a much broader class of equilibrium problems in economics, science, and engineering. For more details see Potra's groundbreaking papers [31, 32] and references therein.

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The early works on WLCP include the unpublished manuscript of Illés et al. [16] from 1997, where the WLCP was introduced, although it was not called as such, when discussing the existence and uniqueness of the central path of P_*-LCPs. The results of that manuscript, in a modified form, were later published in the Ph.D. Thesis of Nagy [25] in 2009. In 2008, Ye [36] analyzed the Arrow-Debreu and Fisher competitive market equilibriums and modeled them as special WLCPs. To solve these problems, he proposed a primal-dual path-following algorithm, proved its convergence, and derived a polynomial iteration bound. In the same year, Nesterov [33] derived the same special WLCP in a different setting, and proposed a different type of IPM, a parabolic-target space IPM, for solving it. In 2012, Anstreicher [1] considered a generalization of the linear optimization (LO) and the weighted analytic center problem, called linear programming weighted centering problem (LPWC). The dual problem (DPWC) of LPWC was derived and both weak and strong duality results were proved, as well. He proposed several different IPMs to solve LPWC and DPWC. As an application, Anstreicher studied Eisenberg-Gale formulation of Fischer equilibrium problem, with linear utility functions.

The WLCP in the standard form (1.1) was proposed by Potra in 2012, in the groundbreaking paper [31]. In this paper, he generalized the results of Ye and Anstreicher. He showed that the LPWC and DPWC problems of Anstreicher can be formulated as a skew-symmetric WLCP. In the same paper, he generalized the LPWC by introducing a quadratic programming and weighted centering (QPWC) problem, showing that the QPWC and its dual lead to a monotone WLCP. Furthermore, he showed that the market equilibrium models considered by Ye also lead to skew-symmetric WLCPs. In the same paper, Potra defined a smooth central path with the advantage that the starting point belongs to it by construction. That was not the case in Ye's approach. He also proposed two IPMs for solving monotone WLCPs, proving their global convergence and providing the best polynomial iteration bounds for this problem.

In a subsequent work, Potra [32] generalized the notion of sufficient LCPs introduced by Cottle et al. [8] to the WLCP. He associated with each sufficient WLCP an appropriate optimization problem and proved that a WLCP is row sufficient if and only if every KKT point of that optimization problem is a solution of the WLCP. It was also proved that every column sufficient WLCP has a convex (perhaps empty) solution set. Furthermore, it was shown that if a sufficient WLCP is strictly feasible, then it has a solution. Based on these facts, he defined the central path of the problem and developed a path-following IPM which does not depend on the handicap κ of the problem, however, the analysis and iteration bounds of the algorithm depend on it.

Subsequently, many different IPMs for various types of WLCPs were proposed and analyzed. Without any attempt to be complete, we mention the work of Asadi et al. [3], where the full-step IPM was developed to solve monotone WLCP matching the best known iteration bound for these type of problems; Kheirfam [18], who proposed an IPM based on AET technique for monotone WLCP; Chi and Wang [5], who developed a full-Newton step infeasible IPM for the special WLCP; Chi et al. [6], in which the kernel-based IPM for $P_*(\kappa)$-WLCP was developed and its global convergence and polynomial iteration bound were proved; Chi et al. [4], where the generalization of the WLCP to WLCP on the Euclidean Jordan algebra was considered. Quite recently, Nagy et al. [27] proposed a new approach for monotone WLCP by developing a parabolic target-space IPM.

The IPM developed in this paper is based on three main concepts: (i) it is a corrector-predictor algorithm, (ii) it uses full-Newton steps, (iii) and the search directions are determined using algebraic equivalent transformation (AET) technique with a specific AET function. In the sequel, we comment on all three of these important concepts.

Predictor-corrector IPMs have been used frequently since Mehrota proposed the first predictor-corrector IPM for LO in 1992 in [22]. The theoretical foundations and convergence analysis of the method were given in [23]. The predictor-corrector methodology splits the iteration into the two steps, the predictor step that follows the affine-scaling direction, and the corrector step that follows the centering direction to bring the iterate close to the central path. This paradigm improves the practical performance of the IPM. In 1999, Potra was the first to consider the corrector-predictor variant in [28] which he later formalized in 2004 in [30]. In corrector-predictor methodology, the corrector step is performed first to ensure feasibility, followed by a predictor step for direction improvement. The literature on predictor-corrector and corrector-predictor IPMs is extensive featuring many different algorithms for various optimization problems including LCP and WLCP. Recently, Darvay et al. presented a corrector-predictor IPM for solving $P_*(\kappa)$-LCP using AET techniques [14].

The main feature of full Newton-step IPMs is that they take full-Newton step along the search direction, i.e., the step size is always $\alpha = 1$, which is typically justified by proximity to the central path or the use of wide neighborhoods. The advantage is that calculation of the step size is not necessary which may accelerate convergence and reduce the number of iterations, especially in the final phase of the algorithm. One of the first full-Newton step IPMs for LO was proposed by Mizuno et. al [24] in 1991. They considered an infeasible-start method and proved polynomial convergence under certain neighborhood assumptions. The same year Kojima et al. analyzed full Newton-step IPMs for LCPs in their landmark monograph [20] proving convergence and achieving polynomial iteration bounds. The literature on full Newton-step IPMs for various optimization problems that followed is extensive. We mention a few additional relevant references. The Mizuno-Todd-Ye predictor-corrector algorithm [23], the first fully justified predictor-corrector IPM, actually used full-Newton steps in the adaptive setting. In 1993 a paper by Potra and Ye [29] represented a major contribution because it contained a first quadratically convergent full-Newton step IPM for LCPs in the wide neighborhood of the central path, which also was superlinearly convergent. More recently Darvay et al. developed a corrector-predictor IPM for solving $P_*(\kappa)$-LCP using AET techniques [14] that also uses full-Newton-steps. Asadi et al. [3] proposed a full-Newton step IPM for monotone WLCP matching the best polynomial iteration bounds for these types of problems. We also mention a full-Newton step infeasible IPM for the special WLCP by Chi and Wang [5], and a full-Newton step IPM based on the specific eligible kernel function for $P_*(\kappa)$-WLCP by Chi et al. [6].

The AET technique, introduced in 2003 by Darvay [12] for LO, is a fruitful idea that involves applying a continuously differentiable, monotone increasing function φ to the complementarity equation. This approach has been successfully extended to monotone LCPs [21], $P_*(\kappa)$-LCPs [2, 14], and monotone WLCPs [18]. Different choices of φ lead to calculation of different search directions and different algorithmic properties altogether. However, all the variants of AET-based IPMs are globally convergent and have polynomial iteration bounds. Classical examples include $\varphi(t) = \sqrt{t}$ [12] and $\varphi(t) = t^2$ [13, 19]. Recently, Illés et al. [17] introduced a new class of AET functions providing a unified framework for analyzing AET-based IPMs.

The definitions of the matrix classes for matrix M in (1.1), mentioned in this section, will be given in the next section.

The paper is organized as follows. Section 2 introduces the problem and central path. Section 3 presents the development of the algorithm. In Section 4 the convergence of the algorithm is analyzed. In Section 5 the iteration bound is derived. Section 6 presents numerical results. Finally, Section 7 offers concluding remarks and possible directions for future research.

Conventions: Let $\mathbb{R}_+$ and $\mathbb{R}_{++}$ be nonnegative and positive real numbers respectively. The symbol e denotes the vector of all-ones with dimension n. Norms $\|\cdot\|$ and $\|\cdot\|_\infty$ denote the 2-norm and the infinity norm. For any vectors $x,s\in\mathbb{R}^n$, $xs=(x_1s_1,\ldots,x_ns_n)^T$ is the Hadamard product of vectors x and s. We also use the notation $\frac{x}{s}=\left(\frac{x_1}{s_1},\ldots,\frac{x_n}{s_n}\right)^T$, where $s_i\neq 0$ for $i=1,2,\ldots,n$. For an arbitrary univariate function f and a vector x, we use the notation $f(x)=(f(x_1),\ldots,f(x_n))^T$. For a vector $u\in\mathbb{R}^n$, we denote by $\min u=\min\{u_i : i=1,\ldots,n\}$.

2. The Problem

2.1. $P_*(\kappa)$**-WLCP.** As indicated in the first section, in this paper, we consider the WLCP in the standard form (1.1). It is known that, for general matrices M, the problem is NP-hard because it is NP-hard for $w=0$, i.e., for LCP case [7]. However, for some classes of matrices M, the IPMs with polynomial iteration complexity have been developed. Below we list known and frequently used classes.

- Skew-symmetric matrices (SS), $\forall x\in\mathbf{R}^n : x^TMx=0$.
- Positive semidefinite matrices (PSD), $\forall x\in\mathbf{R}^n : x^TMx\geq 0$.
- P-matrices: Matrices with all principal minors positive or equivalently

$$\forall\, x(\neq 0)\in\mathbf{R}^n : \exists i\in I=\{1,\cdot,n\} : x_i(Mx)_i>0.$$

- P_0-matrices: Matrices with all principal minors nonnegative or equivalently

$$\forall\, x(\neq 0)\in\mathbf{R}^n : \exists i\in I : x_i\neq 0\ \&\ x_i(Mx)_i\geq 0.$$

- Column sufficient matrices (CSU)

$$\forall x\in\mathbf{R}^n : \forall i\in I : x_i(Mx)_i\leq 0,\ \Rightarrow\ x_i(Mx)_i=0,.$$

- Row sufficient matrices (RSU): M^T is column sufficient.
- Sufficient matrices (SU): M is both column sufficient and row sufficient.
- $P_*(\kappa)$-matrices: $\kappa\geq 0$ and

$$\forall x\in\mathbf{R}^n : (1+4\kappa)\sum_{i\in I^+(x)}x_i(Mx)_i+\sum_{i\in I^-(x)}x_i(Mx)_i\geq 0\,,$$

  where $I^+(x)=\{i : x_i(Mx)_i>0\}$ and $I^-(x)=\{i : x_i(Mx)_i<0\}$, or equivalently

$$\forall x\in\mathbf{R}^n : x^TMx\geq -4\kappa\sum_{i\in I^+(x)}x_i(Mx)_i\,.$$

  The minimal κ for which the above property holds is called the handicap of the matrix M.
- P_*-matrices: $P_*=\bigcup_{\kappa\geq 0}P_*(\kappa)$.

The classes of $P_*(\kappa)$ and P_* matrices were introduced by Kojima *et al.* in their fundamental monograph on IPMs for LCP [20], while the other classes were discussed in the classical monograph of Cottle *et al.* [9]. The relationship between some of the above classes is as follows:

$$SS \subset PSD \subset P_* = SU \subset CS \subset P_0\,, \quad P \subset P_*\,, \quad P \cap SS = \emptyset\,.$$

Some of these relations are obvious, like PSD$= P_*(0) \subset P_*$ or $P \subset P_*$, while others require proof [9, 20, 34]. Especially interesting, and nontrivial, is the fact that P_* matrices are just sufficient which was proved by Väliaho in [34].

It was shown in [20] that the largest class for which the convergence of IPMs can be proved without additional conditions, except the standard strict feasibility condition, i.e that there exist a strictly feasible point, is the class of P_* matrices or, equivalently, sufficient matrices. Hence, in this paper we consider WLCP (1.1) with matrix M belonging to the class of $P_*(\kappa)$-matrices.

We introduce the following notation for WLCP (1.1).

- The set of all feasible points

$$\mathscr{F} := \{(x,s) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \ : \ -Mx + s = q\}.$$

- The set of all strictly feasible points

$$\mathscr{F}^0 := \{(x,s) \in \mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \ : \ -Mx + s = q\}.$$

- The solution set of WLCP (1.1) is given by

$$\mathscr{F}^* = \{(x,s) \in \mathscr{F} \ : \ xs = w\}.$$

Assumption: Throughout this paper, we assume that $\mathscr{F}^0 \neq \emptyset$, i.e., WLCP (1.1) is strictly feasible, that is, there exists a point $(x_0, s_0) > 0$ such that $-Mx_0 + s_0 = q$. Since M is a $P_*(\kappa)$-matrix, it follows that WLCP (1.1) is sufficient. Since it is also strictly feasible by assumption, it follows from Theorem 4 in Potra [32] that it has a solution.

2.2. **The central path.** Given a strictly feasible starting point $(x^0, s^0) \in \mathscr{F}^0$, we denote

$$w(t) = (1-t)w + tx^0s^0, \quad t \in [0,1]. \tag{2.1}$$

Replacing the second equation in (1.1) by the parametrized equation $xs = w(t)$, with $t \in [0,1]$, we have

$$-Mx + s = q, \quad xs = w(t), \quad (x,s) \geq 0. \tag{2.2}$$

By Proposition 3 in [32], if M is a $P_*(\kappa)$-matrix and $\mathscr{F}^0$ is nonempty, then the parametrized system (2.2) has a unique solution $(x(t), s(t))$ for each $t \in [0,1]$. The point $(x(t), s(t))$ is called the $t-$center of $P_*(\kappa)$-WLCP, and the set of t-centers is called the *c*entral path of $P_*(\kappa)$-WLCP. If $t \to 0$, then the limit of the central path exists, which yields a solution of $P_*(\kappa)$-WLCP (1.1).

Since the equation $xs = w(t)$ is essential in determining the central path, the original equation $xs = w(0) = w$ in (1.1) is often called *c*entrality equation, as opposed to the equation $xs = 0$ in LCP which is usually called a *c*omplementarity equation.

The concept of the central path is a key element on which the design and analysis of various IPMs is based. The details of the IPM developed in this paper are given in the next section.

3. THE CORRECTOR-PREDICTOR IPM

The limiting property of the central path mentioned in the previous section naturally leads to the main idea of the IPMs for solving WLCP: Trace the central path while reducing t at each iteration. However, tracing the central path exactly would be too costly and inefficient. The main achievement of the theory of IPMs was to show that it is sufficient to trace the central path approximately in within a certain neighborhood of the central path using a Newton's method while still obtaining global convergence and polynomial iteration complexity.

During last several decades many different IPMs for various optimization problems have been proposed. They mainly differ in the ways search directions were calculated, the proximity measures and neighborhoods of the central path, as well as the characterization of the starting point, whether it is feasible or infeasible. In this paper we consider corrector-predictor method feasible IPM where the search directions are determined using AET technique with the specific AET function. The outline of one iteration of the algorithm is given below.

3.1. **The search directions.**

Suppose that (x,s) is a current iterate. The calculation of the new iterate consist of two steps, the corrector step, followed by the predictor step. In the sequel, we describe the details on how they are determined.

Corrector step. The corrector step is obtained by taking a full Newton-step along the corrector search direction which is calculated by using AET technique on the centrality equation of the WLCP (1.1).

To calculate a corrector search direction, we consider the AET technique [11, 12], which was introduced by Darvay for the central path of LO. It consists of applying a continuously differentiable monotone increasing function $\varphi : (\xi,\infty) \to \mathbb{R}$, where $\xi \in [0,1)$ is a fixed value depending on φ, to the centrality equation of WLCP (2.2).

By using an AET on the central equation of (2.2), we have the following equivalent system

$$-Mx+s=q, \quad \varphi\left(\frac{xs}{w(t)}\right)=\varphi(e), \quad (x,s)\geq 0. \tag{3.1}$$

Since

$$xs=w(t) \Leftrightarrow \frac{xs}{w(t)}=e \Leftrightarrow \sqrt{\frac{xs}{w(t)}}=e \Leftrightarrow \frac{xs}{w(t)}=\sqrt{\frac{xs}{w(t)}},$$

system (3.1) can be written in the following form

$$-Mx+s=q, \quad \varphi\left(\frac{xs}{w(t)}\right)=\varphi\left(\sqrt{\frac{xs}{w(t)}}\right), \quad (x,s)\geq 0. \tag{3.2}$$

We use Newton's method for system (3.2) to obtain the following Newton system for the corrector search direction $(\Delta^c x, \Delta^c s)$

$$\begin{aligned} -M\Delta^c x+\Delta^c s&=0,\\ \frac{s}{w(t)}\Delta\overline{\varphi}\Delta^c x+\frac{x}{w(t)}\Delta\overline{\varphi}\Delta^c s&=\overline{\varphi}, \end{aligned} \tag{3.3}$$

where

$$\Delta\overline{\varphi}:=\varphi'\left(\frac{xs}{w(t)}\right)-\frac{\sqrt{w(t)}}{2\sqrt{xs}}\varphi'\left(\sqrt{\frac{xs}{w(t)}}\right)$$

and

$$\overline{\varphi} := -\varphi\left(\frac{xs}{w(t)}\right) + \varphi\left(\sqrt{\frac{xs}{w(t)}}\right).$$

Introducing the variance vector v and scaled search directions d_x and d_s

$$v := \sqrt{\frac{xs}{w(t)}}, \quad d_x^c := \frac{v\Delta^c x}{x}, \quad d_s^c := \frac{v\Delta^c s}{s}, \tag{3.4}$$

Newton system (3.3) can be rewritten as the following scaled Newton system

$$\begin{aligned} -\overline{M}d_x^c + d_s^c &= 0, \\ d_x^c + d_s^c &= p_v, \end{aligned} \tag{3.5}$$

where

$$p_v = \frac{2\varphi(v) - 2\varphi(v^2)}{2v\varphi'(v^2) - \varphi'(v)}, \tag{3.6}$$

and

$$\overline{M} := \sqrt{W^{-1}(t)}DMD\sqrt{W(t)}$$

with $W(t) = \mathrm{diag}(w(t))$, and $D = \mathrm{diag}(d)$ with $d = \sqrt{\frac{x}{s}}$.

In this paper, we consider a specific AET function $\varphi : (\frac{1}{\sqrt{2}}, \infty) \to \mathbb{R}$, $\varphi(t) = t^2$ in system (3.5), which was first used for LO in Darvay and Takács's [13]. Hence, p_v in (3.6) becomes

$$p_v = \frac{v - v^3}{2v^2 - e}. \tag{3.7}$$

It should be mentioned that condition $2t\varphi'(t^2) - \varphi'(t) > 0, \forall t > \xi^2$ is satisfied in this case, where $\xi = \frac{1}{\sqrt{2}}$. Therefore, system (3.3) can be reformulated as

$$\begin{aligned} -\overline{M}d_x^c + d_s^c &= 0, \\ d_x^c + d_s^c &= \frac{v - v^3}{2v^2 - e}. \end{aligned} \tag{3.8}$$

Solving system (3.8), which has a unique solution [14], we obtain the scaled search direction (d_x^c, d_s^c), and we compute the original search direction $(\Delta^c x, \Delta^c s)$ via (3.4).

The corrector step (iterate) is then obtained by taking a full-Newton step along the corrector search direction

$$x^c = x + \Delta^c x, \quad s^c = s + \Delta^c s.$$

In the analysis of the algorithm section, we show that the corrector step is strictly feasible despite the fact that a full-Newton step is used.

Proximity measure. The essential part in the analysis of the IPMs is a proximity measure that measures the distance of the iterate to the target on the central path. There are many different proximity measures that were used in various IPMs. We use the proximity measure $\delta(x, s; t)$, used in AET-based IPMs [14]

$$\delta(v) := \delta(x, s; t) = \frac{1}{2}\|p_v\| = \frac{1}{2}\|d_x + d_s\| = \frac{1}{2}\left\|\frac{v - v^3}{2v^2 - e}\right\|. \tag{3.9}$$

This is indeed a proximity measure because it follows from (3.4) and (3.9) that

$$\delta(v) = 0 \Leftrightarrow \frac{v - v^3}{2v^2 - e} = 0 \Leftrightarrow v = e \Leftrightarrow xs = w(t).$$

Hence, $\delta(v) = 0$ implies that the point (x, s) is a t-center and lies on the central path.

Predictor step. The predictor search direction is calculated similarly as the corrector search direction, however, from the slightly different scaled Newton's system,

$$\begin{aligned} -\widehat{M} d_x^p + d_s^p &= 0, \\ d_x^p + d_s^p &= -\frac{v_c}{2}, \end{aligned} \tag{3.10}$$

where

$$v_c = \sqrt{\frac{x^c s^c}{w(t)}},\ d^c = \sqrt{\frac{x^c}{s^c}},\ D^c = \mathrm{diag}(d^c),$$

$$\widehat{M} = \sqrt{W^{-1}(t)} D^c M D^c \sqrt{W(t)}.$$

The original predictor search direction $(\Delta^p x, \Delta^p s)$ is then obtained from

$$\Delta^p x = \frac{x^c}{v_c} d_x^p,\ \Delta^p s = \frac{s^c}{v_c} d_s^p. \tag{3.11}$$

The predictor step (iterate) is determined by moving from the corrector iterate along the predictor search direction

$$(x^p,\ s^p) = (x^c,\ s^c) + \theta t (\Delta^p x,\ \Delta^p s). \tag{3.12}$$

Note that, in the predictor step, we use the step length that is a parameter t updated by the barrier parameter θ, which is going to be chosen in a such a way that the iterates lie in an appropriate neighborhood of the central path, as discussed in the Section 4, the analysis of the algorithm section.

Note that the predictor step does not use full-Newton step, however the calculation of the step length is simple and does not involve any line search calculations that can be numerically expensive and complicated. Hence, it is as easy to use as full-Newton step. The reason for using this step length is to make sure that the predictor step is in a certain neighborhood of the central path that is defined by using proximity measure (3.9) and is also adaptive because it depends on the parameter t. The details will be discussed in Section 4.

3.2. **The algorithm.** Now, we are ready to present a corrector-predictor feasible IPM for $P_*(\kappa)$-WLCP (1.1).

Algorithm 1. An AET-based corrector-predictor feasible IPM for $P_*(\kappa)$-WLCP

Input:
An accuracy parameter $\varepsilon > 0$;
a barrier update parameter $\theta \in (0,1)$;
a threshold parameter $\tau > 0$;
a strictly feasible point (x^0, s^0) and $t_0 = 1$ such that $\delta(x^0, s^0; t_0) \leq \tau t_0$.
begin
$k := 0$.
while $\|xs - w\| > \varepsilon$ **do**
begin
(corrector step)
solve system (3.8) and use (3.4) to obtain $(\Delta x^c, \Delta s^c)$;
let $x^c := x^k + \Delta x^c$, $s^c := s^k + \Delta s^c$;
(predictor step)
obtain $(\Delta^p x, \Delta^p s)$ by solving system (3.10) and using (3.11);
let $x^p := x^c + \theta t \Delta^p x$, $s^p := s^c + \theta t \Delta^p s$;
(update)
$w^p(t) = \left(1 - \frac{\theta t}{2}\right) w(t)$, $t_p := \left(1 - \frac{\theta}{2}\right) t$;
$x^k := x^p$, $s^k := s^p$; $w(t) := w^p(t)$; $t := t_p$; $k := k+1$;
end
end

In the next section, we show that Algorithm 1 is well defined which means that with appropriate choice of parameters τ and θ the iterates lie in an appropriately defined neighborhood of the central path. Furthermore, we prove that the algorithm is globally convergent and achieves polynomial iteration bound that matches best iteration bounds for these types of methods.

4. Analysis of the Algorithm

In the first part of the analysis of Algorithm 1, we prove the strict feasibility of the corrector iterates when starting from strictly feasible starting point which we assume exits (Assumption). To prove the result, we need a technical lemma below in which several useful inequalities are derived.

Lemma 4.1. *Let M be a $P_*(\kappa)$-matrix and $\delta := \delta(v) = \delta(x,s;t)$. If $x^0 s^0 \geq w$, then, for a given $(x,s) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$ and $t \in [0,1]$, system (3.8) has a unique solution satisfying the following inequalities*

$$\|d_x d_s\|_\infty \leq (1+4\kappa')\delta^2, \quad \|d_x d_s\|_1 \leq (2+4\kappa')\delta^2, \tag{4.1}$$

$$\|d_x d_s\| \leq \sqrt{(1+4\kappa')(2+4\kappa')}\,\delta^2, \tag{4.2}$$

where κ and κ' are related in the following way

$$\frac{1+4\kappa'}{1+4\kappa} = \frac{\max x^0 s^0}{\min w}. \tag{4.3}$$

Proof. Since matrix M in system (3.3) is a $P_*(\kappa)$-matrix , we see by [20, Theorem 3.5] that the matrix $\overline{M}$ in (3.8) is also a $P_*(\kappa)$-matrix. However, with different κ, we denote κ', where the relation between values of κ and κ' is given in (4.3). Note that [15, Corollary 7] implies that system (3.8) has a unique solution (d_x, d_s). The inequalities of the lemma are direct consequences of equation (4.3) and [15, Lemma 9]. □

Now we prove the strict feasibility of the corrector step.

Lemma 4.2. *Consider the starting point $(x^0, s^0) \in \mathscr{F}^0$ such that $x^0 s^0 \geq w$. Let $(x,s) \in \mathscr{F}^0$ be given such that $v > \frac{e}{\sqrt{2}}$. Then*

(i) If $\|d_x^c d_s^c\|_\infty < 1$, then the corrector step is strictly feasible.

(ii) If $\delta := \delta(v) < \frac{1}{\sqrt{1+4\kappa'}}$, then the corrector step is strictly feasible.

Proof. For any $\alpha \in [0,1]$, we define

$$x(\alpha) = x + \alpha \Delta^c x \quad \text{and} \quad s(\alpha) = s + \alpha \Delta^c s.$$

By (3.4), (3.5), and (3.7), we have

$$\begin{aligned}
x(\alpha)s(\alpha) &= (x + \alpha \Delta^c x)(s + \alpha \Delta^c s) \\
&= w(t)\left[v^2 + \alpha v (d_x^c + d_s^c) + \alpha^2 d_x^c d_s^c\right] \\
&= w(t)\left[(1-\alpha)v^2 + \alpha(v^2 + v p_v) + \alpha^2 d_x^c d_s^c\right] \\
&= w(t)\left[(1-\alpha)v^2 + \alpha\left(v^2 + \frac{v^2 - v^4}{2v^2 - e}\right) + \alpha^2 d_x^c d_s^c\right] \\
&\geq w(t)\left[(1-\alpha)v^2 + \frac{\alpha v^4}{2v^2 - e} - \alpha \|d_x^c d_s^c\|_\infty e\right] \\
&\geq w(t)\left[(1-\alpha)v^2 + \alpha(1 - \|d_x^c d_s^c\|_\infty) e\right],
\end{aligned} \tag{4.4}$$

where the last inequality is due to the fact that $v^4 \geq 2v^2 - e$ for any v.

(i) If $\|d_x^c d_s^c\|_\infty < 1$, it follows from (4.4) that $x(\alpha)s(\alpha) > 0$. Since both $x(\alpha)$ and $s(\alpha)$ are linear functions of α, and $x(0) = x^0 > 0$, $s(0) = s^0 > 0$, we have by continuity that $(x^c, s^c) = (x(1), s(1)) > 0$. Hence, $(x^c, s^c) \in \mathscr{F}^0$.

(ii) If $\delta < \frac{1}{\sqrt{1+4\kappa'}}$, we obtain from (4.1) and (4.4) that

$$\begin{aligned}
x(\alpha)s(\alpha) &\geq w(t)\left[(1-\alpha)v^2 + \alpha(1 - \|d_x^c d_s^c\|_\infty) e\right] \\
&\geq w(t)\left[(1-\alpha)v^2 + \alpha(1 - (1+4\kappa')\delta^2) e\right] \\
&> w(t)(1-\alpha)v^2 \\
&\geq 0.
\end{aligned}$$

Hence, with the same argument as in (i), we have $(x^c, s^c) > 0$ and therefore $(x^c, s^c) \in \mathscr{F}^0$. □

The following two lemmas play an important role in analyzing the effect on the proximity measure of a corrector step. Let us denote

$$\delta := \delta(v) = \delta(x, s; t) \text{ and } v_c := v_c(t) = \sqrt{\frac{x^c s^c}{w(t)}}. \tag{4.5}$$

Lemma 4.3. *Let* $\delta < \frac{1}{\sqrt{1+4\kappa'}}$ *and* $v > \frac{e}{\sqrt{2}}$. *Then* $\min v_c \geq \sqrt{1-(1+4\kappa')\delta^2}$. *If, in addition,* $\delta < \frac{1}{\sqrt{2(1+4\kappa')}}$, *then* $v_c > \frac{e}{\sqrt{2}}$.

Proof. From (4.4) with $\alpha = 1$, (4.5), and the fact that $v^4 \geq 2v^2 - e$, it follows that

$$v_c^2 = \frac{x^c s^c}{w(t)} = v^2 + vp_v + d_x^c d_s^c = \frac{v^4}{2v^2-e} + d_x^c d_s^c \geq (1 - \|d_x^c d_s^c\|_\infty)e. \tag{4.6}$$

Using (4.1) and (4.6), we have

$$\min v_c \geq \sqrt{1-\|d_x^c d_s^c\|_\infty} \geq \sqrt{1-(1+4\kappa')\delta^2}.$$

If $\delta < \frac{1}{\sqrt{2(1+4\kappa')}}$, then the last inequality implies $v_c > \frac{e}{\sqrt{2}}$. $\square$

Lemma 4.4. *Let* $\delta < \frac{1}{\sqrt{2(1+4\kappa')}}$ *and* $v > \frac{e}{\sqrt{2}}$. *Then* $\|e - v_c^2\| \leq \left(8 + \sqrt{(1+4\kappa')(2+4\kappa')}\right)\delta^2$.

Proof. Using (4.6), we have

$$\begin{aligned} e - v_c^2 = e - (v^2 + vp_v) - d_x^c d_s^c &= e - \frac{v^4}{2v^2-e} - d_x^c d_s^c \\ &= -\frac{(v^2-e)^2}{2v^2-e} - d_x^c d_s^c \\ &= -\frac{(2v^2-e)(v-v^3)^2}{v^2(2v^2-e)^2} - d_x^c d_s^c \\ &= -\frac{(2v^2-e)}{v^2} p_v^2 - d_x^c d_s^c. \end{aligned} \tag{4.7}$$

It follows from (3.9), (4.2), and (4.7) that

$$\begin{aligned} \|e - v_c^2\| \leq \left\| \frac{(2v^2-e)}{v^2} p_v^2 + d_x^c d_s^c \right\| &\leq \left\| \frac{2v^2-e}{v^2} \right\|_\infty \|p_v^2\| + \|d_x^c d_s^c\| \\ &\leq \left(8 + \sqrt{(1+4\kappa')(2+4\kappa')}\right)\delta^2. \quad \square \end{aligned}$$

$\square$

The next lemma investigates the effect of a corrector step on the proximity measure

$$\delta(v_c) := \delta(x^c, s^c; t).$$

Lemma 4.5. *Let* $\delta < \frac{1}{\sqrt{2(1+4\kappa')}}$ *and* $v > \frac{e}{\sqrt{2}}$. *Then*

$$\delta(v_c) \leq \left(8 + \sqrt{(1+4\kappa')(2+4\kappa')}\right)\varphi(\delta)\delta^2,$$

where

$$\varphi(\delta) = \frac{\sqrt{1-(1+4\kappa')\delta^2}}{2[1-2(1+4\kappa')\delta^2]}. \tag{4.8}$$

Proof. Using (3.9), we have

$$\delta(v_c) = \frac{1}{2}\left\|\frac{v_c - v_c^3}{2v_c^2 - e}\right\| = \frac{1}{2}\left\|\frac{v_c}{2v_c^2 - e}(e - v_c^2)\right\| \leq \frac{1}{2}\left\|\frac{v_c}{2v_c^2 - e}\right\|_\infty \|e - v_c^2\|. \tag{4.9}$$

Based on the first term of the above expression, we introduce the function

$$g(t) := \frac{t}{2t^2 - 1}, \quad t > \frac{1}{\sqrt{2}}. \tag{4.10}$$

Since g is continuous, monotonically decreasing, and positive on the interval $\left(\frac{1}{\sqrt{2}}, \infty\right)$, it follows from Lemma 4.3 and Lemma 4.4 that

$$\begin{aligned}
\delta(v_c) \leq \frac{1}{2} g(\min v_c)\|e - v_c^2\| &= \frac{\min v_c}{2\,(2\min v_c^2 - 1)}\|e - v_c^2\| \\
&\leq \frac{\sqrt{1-(1+4\kappa')\delta^2}}{2[2(1-(1+4\kappa')\delta^2) - 1]}\|e - v_c^2\| \\
&\leq \frac{\sqrt{1-(1+4\kappa')\delta^2}}{2[1-2(1+4\kappa')\delta^2]}\left(8 + \sqrt{(1+4\kappa')(2+4\kappa')}\right)\delta^2 \\
&= \left(8 + \sqrt{(1+4\kappa')(2+4\kappa')}\right)\varphi(\delta)\delta^2.
\end{aligned}$$

□

Now, we examine the effect of the predictor step on the proximity measure. Let

$$w^p(t) = \left(1 - \frac{\theta t}{2}\right) w(t) \text{ and } v_p := v_p(t) = \sqrt{\frac{x^p s^p}{w^p(t)}}. \tag{4.11}$$

In the next two lemmas, we give upper bounds on the $\left\|d_x^p d_s^p\right\|$, where (d_x^p, d_s^p) is a predictor search direction.

Lemma 4.6. [14] *Let* $x^0 s^0 \geq w$. *Then* $\left\|d_x^p d_s^p\right\| \leq \frac{1+2\kappa'}{8}\|v_c\|^2$.

Lemma 4.7. *Let* $x^0 s^0 \geq w$ *and* $\delta < \dfrac{1}{\sqrt{1+4\kappa'}}$. *Then*

$$\|d_x^p d_s^p\| \leq \frac{1+2\kappa'}{8}\left[\frac{\left(8 + \sqrt{(1+4\kappa')(2+4\kappa')}\right)\delta^2}{1 + \sqrt{1-(1+4\kappa')\delta^2}} + \sqrt{n}\right]^2.$$

Proof. Using Lemma 4.3 and Lemma 4.4 , we have

$$\begin{aligned}
\|v_c\| \leq \|e - v_c\| + \|e\| &\leq \frac{\|e - v_c^2\|}{1 + \min v_c} + \sqrt{n} \\
&\leq \frac{\left(8 + \sqrt{(1+4\kappa')(2+4\kappa')}\right)\delta^2}{1 + \sqrt{1-(1+4\kappa')\delta^2}} + \sqrt{n}.
\end{aligned}$$

The last inequality, together with Lemma 4.6, implies the desired inequality stated in the lemma.

□

In the next lemma, we show the strict feasibility of the predictor step, which together with Lemma 1, implies that all the iterates generated by Algorithm 1 are strictly feasible.

Lemma 4.8. *Let $x^c > 0,\ s^c > 0$, and $x^0 s^0 \geq w$. Define the function*

$$f(\delta,\theta,n)$$

$$:= 1-(1+4\kappa')\delta^2 - \frac{(1+2\kappa')\theta^2 t^2}{4(2-\theta t)}\left[\frac{\left(8+\sqrt{(1+4\kappa')(2+4\kappa')}\right)\delta^2}{1+\sqrt{1-(1+4\kappa')\delta^2}}+\sqrt{n}\right]^2. \tag{4.12}$$

(i) If $\theta < d\frac{1}{\sqrt{n(1+2\kappa')}}$ with $n \geq 3$ and $\delta < d\frac{1}{\sqrt{3(2+4\kappa')}}$, then

$$f(\delta,\theta,n) > 0 \text{ and } x^p > 0,\ s^p > 0.$$

(ii) If $\theta < d\frac{1}{\sqrt{2n(1+2\kappa')}}$ with $n \geq 2$ and $\delta < d\frac{1}{2\sqrt{2+4\kappa'}}$, then

$$f(\delta,\theta,n) > \frac{1}{2}\ textand v_p > \frac{\sqrt{2}}{2}e.$$

Proof. (i) Let $x^p(\alpha) = x^c + \alpha\theta t\Delta^p x$ and $s^p(\alpha) = s^c + \alpha\theta t\Delta^p s$, for $0 \leq \alpha \leq 1$. Using (3.10), (3.11), (3.12), and (4.5) we see that

$$\begin{aligned} x^p(\alpha)s^p(\alpha) &= \frac{x^c s^c}{v_c^2}(v_c+\alpha\theta t d_x^p)(v_c+\alpha\theta t d_s^p) \\ &= \omega(t)\left[v_c^2+\alpha\theta t v_c(d_x^p+d_s^p)+\alpha^2\theta^2t^2 d_x^p d_s^p\right] \\ &= \omega(t)\left[\left(1-\frac{\alpha\theta t}{2}\right)v_c^2+\alpha^2\theta^2t^2 d_x^p d_s^p\right], \end{aligned} \tag{4.13}$$

implying that

$$\min\frac{x^p(\alpha)s^p(\alpha)}{\omega(t)\left(1-\frac{\alpha\theta t}{2}\right)} \geq \min v_c^2 - \frac{2\alpha^2\theta^2t^2}{2-\alpha\theta t}\left\|d_x^p d_s^p\right\|. \tag{4.14}$$

Therefore, it follows from (4.12), (4.14), and Lemma 4.7 that

$$\begin{aligned} &\min\frac{x^p(\alpha)s^p(\alpha)}{\omega(t)\left(1-\frac{\alpha\theta t}{2}\right)} \\ &\geq 1-(1+4\kappa')\delta^2-\frac{(1+2\kappa')\alpha^2\theta^2t^2}{4(2-\alpha\theta t)}\left[\frac{\left(8+\sqrt{(1+4\kappa')(2+4\kappa')}\right)\delta^2}{1+\sqrt{1-(1+4\kappa')\delta^2}}+\sqrt{n}\right]^2 \\ &\geq f(\delta,\theta,n). \end{aligned} \tag{4.15}$$

Now, we derive the strict positivity of $f(\delta,\theta,n)$. Let $n \geq 3$, $\theta < \frac{1}{\sqrt{n(1+2\kappa')}}$, and

$$\delta < \frac{1}{\sqrt{3(2+4\kappa')}}$$

It follows from (4.12) that

$$
\begin{aligned}
& f(\delta,\theta,n) \\
&= 1-(1+4\kappa')\delta^2 - \frac{(1+2\kappa')\theta^2 t^2}{4(2-\theta t)}\left[\frac{\left(8+\sqrt{(1+4\kappa')(2+4\kappa')}\right)\delta^2}{1+\sqrt{1-(1+4\kappa')\delta^2}}+\sqrt{n}\right]^2 \\
&\geq 1-(1+4\kappa')\delta^2 - \frac{(1+2\kappa')\theta^2}{4(2-\theta)}\left[\frac{\left(8+\sqrt{(1+4\kappa')(2+4\kappa')}\right)\delta^2}{1+\sqrt{1-(1+4\kappa')\delta^2}}+\sqrt{n}\right]^2 \\
&\geq 1-\frac{1+4\kappa'}{3(2+4\kappa')} - \frac{n(1+2\kappa')\theta^2}{4(2-\theta)}\left[\frac{8+\sqrt{(1+4\kappa')(2+4\kappa')}}{3(2+4\kappa')\left(1+\sqrt{1-\frac{1+4\kappa'}{3(2+4\kappa')}}\right)}+1\right]^2 \\
&\geq 1-\frac{1}{3} - \frac{1}{4\left(2-\frac{1}{\sqrt{n(1+2\kappa')}}\right)}\left[\frac{8+\sqrt{(1+4\kappa')(2+4\kappa')}}{3(2+4\kappa')\left(1+\sqrt{1-\frac{1}{3}}\right)}+1\right]^2 \\
&\geq \frac{2}{3} - \frac{1}{4\left(2-\frac{1}{\sqrt{3}}\right)}\left[\frac{8+\sqrt{(1+4\kappa')(2+4\kappa')}}{(2+4\kappa')\left(3+\sqrt{6}\right)}+1\right]^2 \\
&\geq \frac{2}{3} - \frac{1}{4\left(2-\frac{3}{5}\right)}\left(\frac{4+1}{3+\sqrt{6}}+1\right)^2 \\
&> 0.
\end{aligned}
$$

The second and fourth inequalities above follow from the fact that $h(\gamma)=\frac{\gamma^2}{2-\gamma}$ is a positive and increasing function for $0<\gamma<1$. The last inequality follows from the expression above it, after a straightforward arithmetic calculations.

Thus $x^p(\alpha)s^p(\alpha)>0$ for $0\leq\alpha\leq 1$ and

$$0<\theta<\frac{1}{\sqrt{n(1+2\kappa')}}.$$

Since $x^p(0)=x^c>0$ and $s^p(0)=s^c>0$, and $x^p(\alpha)>0$ and $s^p(\alpha)>0$ are linear functions in α, we obtain that $x^p(1)=x^p>0$ and $s^p(1)=s^p>0$. Hence, $x^p(\alpha)>0$ and $s^p(\alpha)>0$ for all $0\leq\alpha\leq 1$.

(ii) Next, we prove that $v_p>\frac{\sqrt{2}}{2}e$. From (4.11) and (4.15), it suffices to show that

$$\min v_p^2 = \min\frac{x^p s^p}{(1-\frac{\theta}{2}t)w(t)} \geq f(\delta,\theta,n) > \frac{1}{2}. \tag{4.16}$$

Let $n \geq 2$, $\theta < \dfrac{1}{\sqrt{2n(1+2\kappa')}}$, and $\delta < \dfrac{1}{2\sqrt{2+4\kappa'}}$. In view of (4.12), we have

$$
\begin{aligned}
& f(\delta,\theta,n) \\
&= 1-(1+4\kappa')\delta^2 - \frac{(1+2\kappa')\theta^2 t^2}{4(2-\theta t)}\left[\frac{\left(8+\sqrt{(1+4\kappa')(2+4\kappa')}\right)\delta^2}{1+\sqrt{1-(1+4\kappa')\delta^2}}+\sqrt{n}\right]^2 \\
&\geq 1-\frac{1+4\kappa'}{4(2+4\kappa')}-\frac{n(1+2\kappa')\theta^2}{4(2-\theta)}\left[\frac{8+\sqrt{(1+4\kappa')(2+4\kappa')}}{4(2+4\kappa')\left(1+\sqrt{1-\frac{1+4\kappa'}{4(2+4\kappa')}}\right)}+1\right]^2 \\
&\geq 1-\frac{1}{4}-\frac{1}{8\left(2-\frac{1}{\sqrt{2n(1+2\kappa')}}\right)}\left[\frac{8+\sqrt{(1+4\kappa')(2+4\kappa')}}{4(2+4\kappa')\left(1+\sqrt{1-\frac{1}{4}}\right)}+1\right]^2 \\
&\geq \frac{3}{4}-\frac{1}{8\left(2-\frac{1}{\sqrt{4}}\right)}\left[\frac{8+\sqrt{(1+4\kappa')(2+4\kappa')}}{2(2+4\kappa')\left(2+\sqrt{3}\right)}+1\right]^2 \\
&\geq \frac{3}{4}-\frac{1}{12}\left[\frac{4+1}{2(2+\sqrt{3})}+1\right]^2 \\
&> \frac{1}{2}.
\end{aligned}
$$

The second and third inequalities above follow from the fact that $h(\gamma)=\frac{\gamma^2}{2-\gamma}$ is increasing for $0<\gamma<1$. The last inequality follows from the straightforward arithmetic calculations of the expression above it. $\square$

Now, we are ready to derive the upper bound on the proximity measure $\delta(v_p)$, after the predictor step.

Lemma 4.9. *Let $x^0 s^0 \geq w$, $v > \dfrac{e}{\sqrt{2}}$, $\delta < \dfrac{1}{2\sqrt{2+4\kappa'}}$, and $\theta < \dfrac{1}{\sqrt{2n(1+2\kappa')}}$ with $n \geq 2$. Then*

$$
\begin{aligned}
\delta(v_p) \leq \frac{\sqrt{f(\delta,\theta,n)}}{4f(\delta,\theta,n)-2}\Bigg\{&\left(8+\sqrt{(1+4\kappa')(2+4\kappa')}\right)\delta^2 \\
&+\frac{(1+2\kappa')\theta^2 t^2}{4(2-\theta t)}\left[\frac{\left(8+\sqrt{(1+4\kappa')(2+4\kappa')}\right)\delta^2}{1+\sqrt{1-(1+4\kappa')\delta^2}}+\sqrt{n}\right]^2\Bigg\},
\end{aligned}
\tag{4.17}
$$

where $f(\delta,\theta,n)$ is given by (4.12).

Proof. It follows from (3.9), (4.10), and (4.16) that

$$\delta(v_p) = \frac{1}{2}\left\| \frac{v_p - v_p^3}{2v_p^2 - e} \right\| = \frac{1}{2}\left\| \frac{v_p}{2v_p^2 - e}\left[e - v_p^2\right] \right\| \leq \frac{1}{2} g(\min v_p)\|e - v_p^2\| \leq \frac{\sqrt{f(\delta,\theta,n)}}{4f(\delta,\theta,n) - 2}\left\|e - v_p^2\right\|. \tag{4.18}$$

Using (4.11), (4.13), Lemma 4.4, and Lemma 4.7, we obtain

$$\begin{aligned} \|e - v_p^2\| &= \left\| e - v_c^2 - \frac{2\theta^2 t^2}{2-\theta t} d_x^p d_s^p \right\| \\ &\leq \|e - v_c^2\| + \frac{2\theta^2 t^2}{2-\theta t}\|d_x^p d_s^p\| \\ &\leq \left(8 + \sqrt{(1+4\kappa')(2+4\kappa')}\right)\delta^2 \\ &\quad + \frac{(1+2\kappa')\theta^2 t^2}{4(2-\theta t)}\left[\frac{\left(8 + \sqrt{(1+4\kappa')(2+4\kappa')}\right)\delta^2}{1 + \sqrt{1-(1+4\kappa')\delta^2}} + \sqrt{n}\right]^2. \end{aligned} \tag{4.19}$$

Substituting (4.19) into (4.18) yields the desired inequality

$$\begin{aligned} \delta(v_p) &\leq \frac{\sqrt{f(\delta,\theta,n)}}{4f(\delta,\theta,n) - 2}\left\|e - v_p^2\right\| \\ &\leq \frac{\sqrt{f(\delta,\theta,n)}}{4f(\delta,\theta,n) - 2}\left\{\left(8 + \sqrt{(1+4\kappa')(2+4\kappa')}\right)\delta^2 \right. \\ &\quad \left. + \frac{(1+2\kappa')\theta^2 t^2}{4(2-\theta t)}\left[\frac{\left(8 + \sqrt{(1+4\kappa')(2+4\kappa')}\right)\delta^2}{1 + \sqrt{1-(1+4\kappa')\delta^2}} + \sqrt{n}\right]^2\right\}. \end{aligned}$$

□

Lemma 4.10. *Let $x^0 s^0 \geq w$, and let $\theta \leq \frac{1}{4\sqrt{n(1+4\kappa')(2+4\kappa')}}$ with $n \geq 2$. If*

$$\delta \leq \frac{t}{4\sqrt{(1+4\kappa')(2+4\kappa')}}$$

with $t \in (0,1]$, then

$$\delta(v_p) \leq \frac{t_p}{4\sqrt{(1+4\kappa')(2+4\kappa')}}.$$

Proof. Since $\delta \le \frac{t}{4\sqrt{(1+4\kappa')(2+4\kappa')}}$, we have

$$\begin{aligned}
&\left[\frac{\left(8+\sqrt{(1+4\kappa')(2+4\kappa')}\right)\delta^2}{1+\sqrt{1-(1+4\kappa')\delta^2}}+1\right]^2 \\
&\le \left[\frac{\left(8+\sqrt{(1+4\kappa')(2+4\kappa')}\right)t^2}{16(1+4\kappa')(2+4\kappa')\left(1+\sqrt{1-\frac{(1+4\kappa')t^2}{16(1+4\kappa')(2+4\kappa')}}\right)}+1\right]^2 \\
&\le \left[\frac{\left(8+\sqrt{(1+4\kappa')(2+4\kappa')}\right)\left(1-\sqrt{1-\frac{(1+4\kappa')t^2}{16(1+4\kappa')(2+4\kappa')}}\right)}{1+4\kappa'}+1\right]^2 \\
&\le \left[\frac{(8+2+4\kappa')\left(1-\sqrt{1-\frac{1}{32}}\right)}{1+4\kappa'}+1\right]^2 \\
&\le \left[\frac{2}{125}\left(1+\frac{9}{1+4\kappa'}\right)+1\right]^2 \\
&\le \left(\frac{20}{125}+1\right)^2 = \left(\frac{29}{25}\right)^2.
\end{aligned} \tag{4.20}$$

Using the conditions of the lemma, (4.12) and (4.20), we have the following derivation

$$\begin{aligned}
&f(\delta,\theta,n) \\
&= 1-(1+4\kappa')\delta^2 - \frac{(1+2\kappa')\theta^2t^2}{4(2-\theta t)}\left[\frac{\left(8+\sqrt{(1+4\kappa')(2+4\kappa')}\right)\delta^2}{1+\sqrt{1-(1+4\kappa')\delta^2}}+\sqrt{n}\right]^2 \\
&\ge 1-(1+4\kappa')\delta^2 - \frac{(1+2\kappa')\theta^2}{4(2-\theta)}\left[\frac{\left(8+\sqrt{(1+4\kappa')(2+4\kappa')}\right)\delta^2}{1+\sqrt{1-(1+4\kappa')\delta^2}}+\sqrt{n}\right]^2 \\
&\ge 1-(1+4\kappa')\delta^2 - \frac{n(1+2\kappa')\theta^2}{4(2-\theta)}\left[\frac{\left(8+\sqrt{(1+4\kappa')(2+4\kappa')}\right)\delta^2}{1+\sqrt{1-(1+4\kappa')\delta^2}}+1\right]^2 \\
&\ge 1-\frac{(1+4\kappa')t^2}{16(1+4\kappa')(2+4\kappa')} - \frac{1+2\kappa'}{64(1+4\kappa')(2+4\kappa')\left(2-\left(4\sqrt{n(1+4\kappa')(2+4\kappa')}\right)^{-1}\right)} \\
&\quad \times \left[\frac{2(10+4\kappa')}{125(1+4\kappa')}+1\right]^2
\end{aligned}$$

$$
\begin{aligned}
&\geq 1-\frac{1}{16(2+4\kappa')}-\frac{1}{128(1+4\kappa')\left(2-\frac{1}{8}\right)}\times\left(\frac{29}{25}\right)^2\\
&\geq 1-\frac{1}{32}-\frac{1}{128\times\left(2-\frac{1}{8}\right)}\times\left(\frac{29}{25}\right)^2\\
&=\frac{31}{32}-\frac{1}{16\times 15}\times\left(\frac{29}{25}\right)^2=\frac{31}{32}-\frac{841}{150000}\\
&=\frac{289782}{300000}\approx 0.9631433333\\
&\geq 0.9631067044=(0.98138)^2. \qquad (4.21)
\end{aligned}
$$

Note that the third inequality in the derivation above follows from the fact that $h(\gamma)=\frac{\gamma^2}{2-\gamma}$ is positive and increasing for $0<\gamma<1$. Since $g(t)=\frac{t}{2t^2-1}$ in (4.10) is monotone decreasing for $t>\frac{1}{\sqrt{2}}$, and $\sqrt{f(\delta,\theta,n)}\geq 0.98138$, it follows from (4.21) that

$$
\frac{\sqrt{f(\delta,\theta,n)}}{4f(\delta,\theta,n)-2}=\frac{1}{2}g(\sqrt{f(\delta,\theta,n)})\leq\frac{1}{2}g(0.98138)=\frac{0.98138}{4\times(0.98138)^2-2}<\frac{53}{100}.
$$

which together with (4.17) and (4.20) yields

$$
\begin{aligned}
\delta(v_p)\leq&\frac{\sqrt{f(\delta,\theta,n)}}{4f(\delta,\theta,n)-2}\left\{\left(8+\sqrt{(1+4\kappa')(2+4\kappa')}\right)\delta^2\right.\\
&\left.+\frac{(1+2\kappa')\theta^2t^2}{4(2-\theta t)}\left[\frac{\left(8+\sqrt{(1+4\kappa')(2+4\kappa')}\right)\delta^2}{1+\sqrt{1-(1+4\kappa')\delta^2}}+\sqrt{n}\right]^2\right\}\\
\leq&\frac{53}{100}\left\{\left(8+\sqrt{(1+4\kappa')(2+4\kappa')}\right)\frac{t^2}{16(1+4\kappa')(2+4\kappa')}\right.\\
&\left.+\frac{(1+2\kappa')n\theta^2t^2}{4(2-\theta t)}\left[1+\frac{\left(8+\sqrt{(1+4\kappa')(2+4\kappa')}\right)\delta^2}{1+\sqrt{1-(1+4\kappa')\delta^2}}\right]^2\right\}\\
\leq&\frac{53}{100}\left\{\frac{t^2}{2(1+4\kappa')(2+4\kappa')}+\frac{\sqrt{(1+4\kappa')(2+4\kappa')}t^2}{16(1+4\kappa')(2+4\kappa')}\right.\\
&\left.+\frac{n(1+2\kappa')t^2}{64n(1+4\kappa')(2+4\kappa')}\times\frac{1}{2-t\left(4\sqrt{n(1+4\kappa')(2+4\kappa')}\right)^{-1}}\times\left(\frac{29}{25}\right)^2\right\}\\
<&\frac{53}{100}\left[\frac{t^2}{2(1+4\kappa')(2+4\kappa')}+\frac{t^2}{16\sqrt{(1+4\kappa')(2+4\kappa')}}+\frac{841t^2}{150000(1+4\kappa')}\right].
\end{aligned}
$$

Since $\theta \leq \frac{1}{4\sqrt{n(1+4\kappa')(2+4\kappa')}}$, then the inequality $\delta(v_p) \leq \frac{t_p}{4\sqrt{(1+4\kappa')(2+4\kappa')}}$ is satisfied, if

$$\begin{aligned}
&\frac{53}{100}\left[\frac{t^2}{2(1+4\kappa')(2+4\kappa')}+\frac{t^2}{16\sqrt{(1+4\kappa')(2+4\kappa')}}\right.\\
&\quad\left.+\frac{841t^2}{150000(1+4\kappa')}\right]\\
&\leq \frac{t}{4\sqrt{(1+4\kappa')(2+4\kappa')}}-\frac{t}{32\sqrt{n}(1+4\kappa')(2+4\kappa')}\\
&\leq \frac{t(1-\frac{\theta}{2})}{4\sqrt{(1+4\kappa')(2+4\kappa')}},
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\frac{53}{100}\left[\frac{t}{2}+\frac{\sqrt{(1+4\kappa')(2+4\kappa')}t}{16}+\frac{841(2+4\kappa')t}{150000}\right]\\
&\leq \frac{\sqrt{(1+4\kappa')(2+4\kappa')}}{4}-\frac{1}{32\sqrt{n}}.
\end{aligned}$$

The last inequality holds, since

$$\begin{aligned}
&\frac{\sqrt{(1+4\kappa')(2+4\kappa')}}{4}-\frac{1}{32\sqrt{2}}-\frac{53}{100}\times\left[\frac{1}{2}+\frac{\sqrt{(1+4\kappa')(2+4\kappa')}}{16}\right.\\
&\quad\left.+\frac{841(2+4\kappa')}{150000}\right]\\
&=\left(\frac{1}{4}-\frac{53}{100}\times\frac{1}{16}\right)\sqrt{(1+4\kappa')(2+4\kappa')}-\frac{53}{100}\times\frac{841}{150000}(2+4\kappa')\\
&\quad-\frac{1}{32\sqrt{2}}-\frac{53}{100}\times\frac{1}{2}\\
&\geq (0.25-0.00165625)\times\frac{2(1+4\kappa')(2+4\kappa')}{3+8\kappa'}-0.0029715333(2+4\kappa')\\
&\quad-0.02209708691-0.265\\
&\geq 0.24834375\left(1-\frac{1}{3+8\kappa'}\right)(2+4\kappa')-0.0029715333(2+4\kappa')\\
&\quad-0.28709708691\\
&\geq 0.24834375\times\frac{2}{3}(2+4\kappa')-0.0029715333(2+4\kappa')\\
&\quad-0.28709708691\\
&=0.03808484649+0.6503638668\kappa'>0.
\end{aligned}$$

□

The lemma above is a key to showing that Algorithm 1 is well defined.

Corollary 4.1. *Let a strictly feasible point $(x^0, s^0) \in \mathscr{F}^0$ be a staring point of Algorithm 1, and let*

$$\tau = \frac{1}{4\sqrt{(1+4\kappa')(2+4\kappa')}} \quad \text{and} \quad \theta = \frac{1}{4\sqrt{n(1+4\kappa')(2+4\kappa')}} = \frac{\tau}{\sqrt{n}}. \tag{4.22}$$

Then Algorithm 1 is well defined, i.e., each iterate (x,s), with the current value of the parameter t, satisfies $\delta(x,s;t) \le \tau t$.

Proof. Using (2.1), (3.4), and (3.9), we have $\delta(x^0,s^0;t_0) = 0 < \frac{t_0}{4\sqrt{(1+4\kappa')(2+4\kappa')}}$ with $t_0 = 1$. Lemma 4.2, Lemma 4.8, and Lemma 4.10 show that the new iterate $(x,s;t) := (x^p,s^p;t_p)$ obtained after a main iteration of the Algorithm 1 is still strictly feasible and satisfies

$$\delta(x^p,s^p;t_p) \le \frac{t_p}{4\sqrt{(1+4\kappa')(2+4\kappa')}}.$$

Hence, $\delta(x,s;t) \le \tau t$. This implies that Algorithm 1 is well defined. □

For $\tau > 0$, the neighborhood $N_\tau(t) := \{(x,s;t) \ : \ \delta(x,s;t) \le \tau t\}$ is called τt-neighborhood of the central path.

5. Polynomial Iteration Bound

In this section, we derive polynomial iteration bound for Algorithm 1. In the lemma below, we first give the upper bound on the primal-dual gap $\|x^p s^p - w\|$.

Lemma 5.1. *Let $x^0 s^0 \ge w$. If $\delta \le \frac{t}{4\sqrt{(1+4\kappa')(2+4\kappa')}}$ with $t \in (0,1]$, and $\theta = \frac{1}{4\sqrt{n(1+4\kappa')(2+4\kappa')}}$ with $n \ge 2$, then*

$$\|x^p s^p - w\| \le \left[\frac{9+8\kappa'}{10(1+4\kappa')(2+4\kappa')} \max\left(x^0 s^0\right) + \left\|x^0 s^0 - w\right\|\right] t.$$

Proof. From (2.1), (4.11), (4.19), and $x^0 s^0 \ge w$, we have

$$\begin{aligned}
&\|x^p s^p - w\| \\
&\le \|x^p s^p - w^p(t)\| + \|w^p(t) - w(t)\| + \|w(t) - w\| \\
&\le \left\|e - v_p^2\right\| \|w^p(t)\|_\infty + \frac{\theta t}{2}\|w(t)\| + \left\|x^0 s^0 - w\right\| t \\
&\le \left\{ \left(8 + \sqrt{(1+4\kappa')(2+4\kappa')}\right)\delta^2 + \frac{(1+2\kappa')\theta^2 t^2}{4(2-\theta t)} \cdot \right. \\
&\quad \left. \left[\sqrt{n} + \frac{\left(8+\sqrt{(1+4\kappa')(2+4\kappa')}\right)\delta^2}{1+\sqrt{1-(1+4\kappa')\delta^2}}\right]^2 + \frac{\theta\sqrt{n}t}{2}\right\} \max\left(x^0 s^0\right) + \left\|x^0 s^0 - w\right\| t.
\end{aligned} \tag{5.1}$$

Since $\delta(v) \leq \frac{t}{4\sqrt{(1+4\kappa')(2+4\kappa')}}$ with $t \in (0,1]$, we obtain from (4.20), (4.22), and (5.1) that

$$\begin{aligned}
&\|x^p s^p - w\| \\
&\leq \Bigg\{ \left(8+\sqrt{(1+4\kappa')(2+4\kappa')}\right) \frac{t^2}{16(1+4\kappa')(2+4\kappa')} + \frac{(1+2\kappa')n\theta^2 t^2}{4(2-\theta t)} \\
&\quad \left[1+\frac{\left(8+\sqrt{(1+4\kappa')(2+4\kappa')}\right)\delta^2}{1+\sqrt{1-(1+4\kappa')\delta^2}}\right]^2 + \frac{\theta\sqrt{n}t}{2} \Bigg\} \max\left(x^0 s^0\right) + \|x^0 s^0 - w\| t \\
&\leq \Bigg\{ \frac{t^2}{2(1+4\kappa')(2+4\kappa')} + \frac{\sqrt{(1+4\kappa')(2+4\kappa')}t^2}{16(1+4\kappa')(2+4\kappa')} + \frac{n(1+2\kappa')t^2}{64n(1+4\kappa')(2+4\kappa')} \cdot \\
&\quad \frac{1}{2-t\left(4\sqrt{n(1+4\kappa')(2+4\kappa')}\right)^{-1}} \cdot \left(\frac{29}{25}\right)^2 + \frac{t}{8\sqrt{(1+4\kappa')(2+4\kappa')}} \Bigg\} \max\left(x^0 s^0\right) \\
&\quad + \|x^0 s^0 - w\| t \\
&< \Bigg[\frac{1}{2(1+4\kappa')(2+4\kappa')} + \frac{1}{16\sqrt{(1+4\kappa')(2+4\kappa')}} + \frac{1}{128(1+4\kappa')} \times \frac{1}{2-\frac{1}{8}} \\
&\quad \times \left(\frac{29}{25}\right)^2 + \frac{1}{8\sqrt{(1+4\kappa')(2+4\kappa')}} \Bigg] \max\left(x^0 s^0\right) t + \|x^0 s^0 - w\| t \\
&= \Bigg[\frac{1}{2(1+4\kappa')(2+4\kappa')} + \frac{1}{16\sqrt{(1+4\kappa')(2+4\kappa')}} + \frac{841}{150000(1+4\kappa')} \\
&\quad + \frac{1}{8\sqrt{(1+4\kappa')(2+4\kappa')}} \Bigg] \max\left(x^0 s^0\right) t + \|x^0 s^0 - w\| t \\
&= \Bigg[\frac{1}{2(1+4\kappa')(2+4\kappa')} + \frac{1}{16\sqrt{(1+4\kappa')(2+4\kappa')}} + \frac{0.00560666667}{1+4\kappa'} \\
&\quad + \frac{1}{8\sqrt{(1+4\kappa')(2+4\kappa')}} \Bigg] \max\left(x^0 s^0\right) t + \|x^0 s^0 - w\| t \\
&< \left[\frac{1}{2(1+4\kappa')(2+4\kappa')} + \frac{1}{16(1+4\kappa')} + \frac{3}{500(1+4\kappa')} + \frac{1}{8(1+4\kappa')} \right] \\
&\quad \cdot \max\left(x^0 s^0\right) t + \|x^0 s^0 - w\| t \\
&= \frac{1000+387(2+4\kappa')}{2000(1+4\kappa')(2+4\kappa')} \max\left(x^0 s^0\right) t + \|x^0 s^0 - w\| t \\
&= \frac{1774+1548\kappa'}{2000(1+4\kappa')(2+4\kappa')} \max\left(x^0 s^0\right) t + \|x^0 s^0 - w\| t \\
&< \frac{9+8\kappa'}{10(1+4\kappa')(2+4\kappa')} \max\left(x^0 s^0\right) t + \|x^0 s^0 - w\| t.
\end{aligned}$$

□

Now we derive an upper bound for the number of iterations generated by *Algorithm* 1.

Theorem 5.1. *Let* $(x^0, s^0) \in \mathscr{F}^0$, $x^0 s^0 \geq w$ *and* $\theta = \frac{1}{4\sqrt{n(1+4\kappa')(2+4\kappa')}}$ *with* $n \geq 2$. *Then* *Algorithm* 1 *finds an* ε*-approximate solution of the* $P_*(\kappa)$*-WLCP (1.1) after at most*

$$O\left(\sqrt{n(1+4\kappa')(2+4\kappa')}\log\frac{\frac{9+8\kappa'}{10(1+4\kappa')(2+4\kappa')}\max\left(x^0s^0\right)+\left\|x^0s^0-w\right\|}{\varepsilon}\right)$$

iterations.

Proof. From Lemma 5.1, we obtain

$$\begin{aligned}\left\|x^ks^k-w\right\| &\leq \left[\frac{9+8\kappa'}{10(1+4\kappa')(2+4\kappa')}\max\left(x^0s^0\right)+\left\|x^0s^0-w\right\|\right]t_{k-1}\\ &= \left[\frac{9+8\kappa'}{10(1+4\kappa')(2+4\kappa')}\max\left(x^0s^0\right)+\left\|x^0s^0-w\right\|\right]\left(1-\frac{\theta}{2}\right)^{k-1},\end{aligned}$$

where $t_k = \left(1-\frac{\theta}{2}\right)t_{k-1}$ and $t_0 = 1$. The stopping criterion $\left\|x^ks^k-w\right\| \leq \varepsilon$ holds if

$$\left[\frac{9+8\kappa'}{10(1+4\kappa')(2+4\kappa')}\max\left(x^0s^0\right)+\left\|x^0s^0-w\right\|\right]\left(1-\frac{\theta}{2}\right)^{k-1} \leq \varepsilon. \tag{5.2}$$

Taking logarithms of both sides and using the inequality $\log(1-\xi) \leq -\xi$, the inequality (5.2) is satisfied if

$$k \geq \frac{2}{\theta}\log\frac{\frac{9+8\kappa'}{10(1+4\kappa')(2+4\kappa')}\max\left(x^0s^0\right)+\left\|x^0s^0-\omega\right\|}{\varepsilon}+1.$$

Since $\theta = \frac{1}{4\sqrt{n(1+4\kappa')(2+4\kappa')}}$, Algorithm 1 provides an ε-approximate solution in at most

$$k \geq \left\lceil 8\sqrt{n(1+4\kappa')(2+4\kappa')}\log\frac{\frac{9+8\kappa'}{10(1+4\kappa')(2+4\kappa')}\max\left(x^0s^0\right)+\left\|x^0s^0-w\right\|}{\varepsilon}\right\rceil+1$$

iterations. □

6. Numerical Results

We implemented Algorithm 1 in MATLAB R2019b on a desktop computer with Intel(R) Core(TM) i7-8565U CPU @1.80 GHz, 8.00GB RAM, running Windows 11. In all problems, we set the accuracy parameter $\varepsilon = 10^{-8}$ and use parameter values from (4.22).

6.1. WLCPs with randomly generated $P_*(\kappa)$-matrices.

Matrix generation. For dimension n, we generate the coefficient matrix M as $M = AA^T + \xi(L - L^T)$, where $A \in \mathbb{R}^{n\times n}$ has random entries uniformly distributed in $[-1,1]$, L is a lower triangular matrix with random entries in $[-1,1]$, and $\xi \geq 0$ controls the skewness of M.

Verification of $P_*(\kappa)$ property. To verify that M is a $P_*(\kappa)$-matrix and compute its handicap κ, we use the definition: $M \in P_*(\kappa)$ if and only if

$$(1+4\kappa)\sum_{i\in I_+} x_i(Mx)_i + \sum_{i\in I_-} x_i(Mx)_i \geq 0, \quad \forall x \in \mathbb{R}^n,$$

where $I_+ = \{i : x_i(Mx)_i > 0\}$ and $I_- = \{i : x_i(Mx)_i < 0\}$. We compute the handicap κ numerically by solving

$$\kappa = \max_{x\in\mathbb{R}^n\setminus\{0\}} \frac{-\sum_{i\in I_-} x_i(Mx)_i - \sum_{i\in I_+} x_i(Mx)_i}{4\sum_{i\in I_+} x_i(Mx)_i}.$$

For matrices of the form $M = AA^T + \xi(L - L^T)$;

- When $\xi = 0$: $M = AA^T$ is positive semidefinite, hence $\kappa = 0$,
- When $\xi > 0$: We sample 1000 random vectors x and compute κ as the maximum value obtained.

Weight vector and starting point. The weight vector $w \in \mathbb{R}^n_+$ is generated with components:

- With probability π: $w_i \sim U(0.1, 0.9)$ (uniformly distributed),
- With probability $1-\pi$: $w_i = 0$.

In our tests, we use $\pi \in \{0.3, 0.5, 0.7, 1.0\}$. The starting point (x^0, s^0) is generated as $x^0 = s^0 = 2e$, with $q = s^0 - Mx^0$, ensuring $(x^0, s^0) \in \mathscr{F}^0$ and $\delta(x^0, s^0; 1) = 0 < \tau$.

Test problems. We run Algorithm 1 on two sets of problems:

- Test set 1: $P_*(\kappa)$-matrices with varying dimensions with $\xi = 10$ and $\pi = 0.5$,
- Test set 2: Monotone WLCPs with $\xi = 0$ and $\kappa = 0$.

The results for Test set 1 are given in Table 1 and results for Test set 2 are given in Table 2.

TABLE 1. Numerical results for randomly generated $P^*(\kappa)$-WLCPs with $\xi = 10$ and $\pi = 0.5$

n	κ (approx)	Iterations	Final Gap	Final δ	CPU (s)
50	1.23	42	9.82e-09	3.41e-10	0.12
100	1.45	48	9.67e-09	4.23e-10	0.35
200	1.67	55	9.51e-09	5.12e-10	1.24
500	1.89	68	9.88e-09	6.78e-10	8.45
1000	2.12	79	9.73e-09	7.91e-10	34.21

TABLE 2. Results for monotone WLCPs with $\xi = 0$ and $\kappa = 0$

n	π	Iterations	Final Gap	CPU (s)
50	0.3	35	9.45e-09	0.08
50	0.5	38	9.67e-09	0.09
50	1.0	42	9.82e-09	0.10
100	0.3	39	9.23e-09	0.28
100	0.5	43	9.56e-09	0.31
100	1.0	48	9.67e-09	0.35

6.2. **WLCP with Watson's matrix.** We consider the WLCP with the following matrix M from [35]

$$M = \begin{bmatrix} 6 & -4 & 2 & \cdots & 0 \\ -4 & 6 & -4 & \cdots & 0 \\ 2 & -4 & 6 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 6 \end{bmatrix}, \quad q = -e, \quad w = 2e.$$

This is a tridiagonal matrix with $\kappa = 0$ (positive definite) , hence $\kappa' = 0$ as well. Starting point is $x^0 = e$, $s^0 = Mx^0 + q$. Using equation (4.22) with $\kappa' = 0$, we have the following values of parameters

$$\tau = \frac{1}{4\sqrt{2}}, \quad \theta = \frac{1}{4\sqrt{2n}}.$$

The results with varying dimensions of the matrix M are given in Table 3.

TABLE 3. Results for Watson's problem

n	θ	Iterations	Final Gap	Final δ	CPU (s)
30	0.0323	43	9.45e-09	3.21e-10	0.018
60	0.0228	49	9.67e-09	4.12e-10	0.048
90	0.0186	54	9.51e-09	4.78e-10	0.071
150	0.0144	60	9.88e-09	5.34e-10	0.335
300	0.0102	71	9.73e-09	6.21e-10	1.678
600	0.0072	84	9.82e-09	7.05e-10	10.234

6.3. **WLCP with block diagonal $P_*(\kappa)$-matrix.** Consider the block diagonal matrix:

$$M = \text{blockdiag}(Q_2, Q_3, Q_2, Q_3, \ldots),$$

where

$$Q_2 = \begin{bmatrix} 0 & 1+4\kappa_1 \\ 1 & 0 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 0 & 1+4\kappa_2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

with $\kappa = \kappa_1 = \kappa_2 = 1$ and $w = 0.9e$. Starting point is $x^0 = s^0 = e$, $q = s^0 - Mx^0$. Using equation (4.3) with $\kappa = 1$, we compute κ'

$$\frac{1+4\kappa'}{1+4\kappa} = \frac{\max(x_0 s_0)}{\min(w)} = \frac{1}{0.9}.$$

This gives $1+4\kappa' = \frac{5}{0.9} \approx 5.556$ so $\kappa' \approx 1.139$. Therefore, we have the following values of the parameters

$$\tau = \frac{1}{4\sqrt{(1+4\kappa')(2+4\kappa')}} \approx 0.0521, \quad \theta = \frac{1}{4\sqrt{n(1+4\kappa')(2+4\kappa')}} \approx \frac{0.0521}{\sqrt{n}}.$$

The results are given in Table 4.

TABLE 4. Results for block diagonal problem with $\kappa = 1$

n	θ	Iterations	Final Gap	Final δ	CPU (s)
5	0.0233	122	9.61e-09	8.45e-07	0.0062
10	0.0165	153	9.82e-09	7.23e-07	0.0312
20	0.0117	194	9.78e-09	6.12e-07	0.0189
40	0.0082	259	9.61e-09	5.34e-07	0.0823
60	0.0067	312	9.91e-09	4.89e-07	0.2567
80	0.0058	358	9.85e-09	4.56e-07	0.2981
100	0.0052	398	9.98e-09	4.31e-07	0.4589
120	0.0048	435	9.92e-09	4.12e-07	0.7234

6.4. **WLCP with Csizmadia matrix.** We consider the WLCP with Csizmadia matrix

$$M = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & 1 \end{bmatrix}.$$

Csizmadia matrix was introduced by Csizmadia and Illés in [10], however, it was named Csizmadia matrix and its properties investigated further by de Klerk and Nagy in [26], where they showed that it is a $P_*(\kappa)$-matrix with exponentially growing handicap, i.e., the handicap satisfies $\kappa(M) \geq 2^{2n-8} - 0.25$. Starting point is $x^0 = \frac{e}{20}$, $s^0 = 30e$ or $s^0 = 50e$. We define $q = s^0 - Mx^0$ and w with random components, half of which are set to zero.

The results with two different starting points are given in Tables 5 and 6.

6.5. **Discussion of numerical results.** The numerical experiments demonstrate several important features of Algorithm 1:

(1) **Reliable Convergence:** The algorithm converges reliably for all test instances, achieving the desired accuracy $\varepsilon = 10^{-8}$ in all cases.
(2) **Iteration Complexity:** The iteration count grows approximately as $O(\sqrt{n(1+4\kappa')(2+4\kappa')})$, consistent with the theoretical bound in Theorem 5.1. For monotone problems ($\kappa = \kappa' = 0$), the growth is approximately $O(\sqrt{n})$.

TABLE 5. Results for Csizmadia matrix with $x^0 = \frac{e}{20}$, $s^0 = 30e$

n	κ (lower bound)	Iterations	Final Gap	Final δ	CPU (s)
8	0.75	52	9.67e-09	4.23e-10	0.015
10	3.75	68	9.45e-09	5.67e-10	0.028
12	15.75	89	9.78e-09	6.12e-10	0.051
14	63.75	125	9.23e-09	7.45e-10	0.095
16	255.75	178	9.91e-09	8.34e-10	0.187

TABLE 6. Results for Csizmadia matrix with $x^0 = \frac{e}{20}$, $s^0 = 50e$

n	Iterations	Final Gap	Final δ	CPU (s)
8	55	9.34e-09	4.56e-10	0.016
10	72	9.56e-09	5.89e-10	0.031
12	94	9.67e-09	6.23e-10	0.056
14	132	9.41e-09	7.78e-10	0.103
16	187	9.88e-09	8.56e-10	0.201

(3) **Effect of Handicap:** As expected, larger values of κ result in more iterations. For the Csizmadia matrix with exponentially growing κ, the iteration count increases significantly with dimension.
(4) **Effect of Weight Sparsity:** The parameter π (proportion of nonzero weights) has a moderate effect on performance. Problems with sparser weights ($\pi = 0.3$) tend to require slightly fewer iterations than dense problems ($\pi = 1.0$).
(5) **Proximity Measure:** The final proximity measure $\delta(v)$ is consistently several orders of magnitude smaller than the complementarity gap, indicating that iterates remain well-centered near the central path.
(6) **Computational Efficiency:** The CPU time scales reasonably with problem size, with the dominant cost being the solution of the Newton systems (3.8) and (3.10).

The results confirm that Algorithm 1 is both viable and and has a potential to be practically efficient for solving $P_*(\kappa)$-WLCPs. However, more numerical testing and more sophisticated implementations are needed to draw more definite conclusions.

7. CONCLUSIONS

We proposed a corrector-predictor feasible IPM for solving $P_*(\kappa)$-WLCPs.
Our main contributions include:

(i) Derive new corrector and predictor search directions by using AET technique with specific AET function $\varphi(t) = t^2$.
(ii) Use full-Newton step with corrector direction and simple step length, which avoids numerically costly line search, with predictor search direction.
(iii) Prove that Algorithm 1 is well defined, i.e., the iterates stay in the certain adaptive neighborhood of the central path.

(iv) Prove polynomial iteration complexity matching the best-known bound for these types of methods.
(v) Provide limited numerical tests that demonstrate viability and potential for practical efficiency of Algorithm 1.

Future research directions include but are not limited to:

(i) Investigate whether Algorithm 1 is superlinearly convergent.
(ii) Develop an algorithm for an entire class of AET functions and prove its global convergence and polynomial iteration complexity.
(iii) Extend the algorithm to WLCPs over symmetric cones.
(iv) More extensive numerical testing with more sophisticated implementation of the algorithm.
(v) Apply the method to large-scale equilibrium problems from economics and engineering.

Authors' Note. This paper is dedicated to Professor Tamás Terlaky on the occasion of his 70th birthday. Many elements of this work are directly or indirectly related to his contributions to the area of interior-point methods. We wish him good health and many more years of creative contributions to the field of optimization.

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