J. Appl. Numer. Optim. 8 (2026), No. 1, pp. 11-18
Available online at http://jano.biemdas.com
https://doi.org/10.23952/jano.8.2026.1.02

SOME RESULTS AND CONJECTURES RELATED TO FRANKL'S UNION CLOSED CONJECTURE

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Abstract. A union-closed family $\mathscr{F}$ is a finite collection of distinct subsets of a finite set such that the union of two subsets in $\mathscr{F}$ belongs to $\mathscr{F}$. Péter Frankl conjectured in 1979 that, for any such family, there exists an element that belongs to at least half of its sets. This conjecture is still unsolved. In this paper, we present some results and conjectures related to Frankl's conjecture. The new conjectures are based on evidence provided by some experimental results.
Keywords. Conjecture of Frankl; Optimization model; Power set; Union-closed sets.
2020 Mathematics Subject Classification. 05D05.

1. Introduction

In this paper, we use the following notations:

X	:	a finite set;
2^X	:	the family of all subsets of X;
$\mathscr{F}$	:	a family of distinct subsets of X;
$\mathscr{F}_x$	:	the subsets in $\mathscr{F}$ that contain $x \in X$.

A family $\mathscr{F}$ is called *union-closed* if $S \cup T \in \mathscr{F}$ for any two subsets $S \in \mathscr{F}$ and $T \in \mathscr{F}$. We then say that $\mathscr{F}$ is a *UC-family*. In this paper, we deal with the so-called *union-closed conjecture* (UCC):

Conjecture 1.1. If $\mathscr{F}$ is a UC-family, then at least one element $x \in X$ belongs to at least half of the sets in $\mathscr{F}$.

Without loss of generality, we assume that $\emptyset \in \mathscr{F}$, where $\emptyset$ denotes the empty set. For a simple proof, we refer to [6, Lemma 1]. When $\mathscr{F}$ contains only the empty set, the conjecture fails because then $|\mathscr{F}_x| = 0$ for every x and $|\mathscr{F}| = 1$. This case will always be excluded below.

The UCC was mentioned first in [3], by Péter Frankl, where he wrote 1979 as its year of origin. Since then, the conjecture became also known as the *Frankl conjecture*. Many solution attempts yielded several partial results; see [2, 6, 9, 10] for some early results from the previous century. Below we mention some of these results that are relevant for the current paper. For a

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Received 25 June 2025; Accepted 1 November 2025; Published online 1 April 2026.

quite complete historical survey up to 2015, we refer to [1]. But, after more than 45 years, the conjecture is still open.

Remark 1.1. It may be noted that, in [3], Frankl deals with intersection-closed (or *IC*-) families $\mathscr{G}$, for which $S \cap T \in \mathscr{G}$ holds for any two subsets $S \in \mathscr{G}$ and $T \in \mathscr{G}$. There he conjectured the existence of an element $x \in X$ that belongs to at most half of the subsets in $\mathscr{G}$. By taking for $\mathscr{G}$ the complements of the sets in a UC family $\mathscr{F}$ the two conjectures turn out to be equivalent. This is a consequence of the well-known theorem of De Morgan: the complement of the union of two sets is equal to the intersection of their complements.

Recently, the interest in the UCC was renewed due to a paper of Gilmer [4]. He showed the existence of an element $x \in X$ belonging to at least 1% of the subsets in $\mathscr{F}$. This was considered to be a breakthrough. It has led to some improvements up to slightly more than 38% instead of 1% [5, 11, 12]. Hopefully the current paper brings us closer to a clear yes or no answer to the UCC.

We call an element $x \in X$ as in the conjecture a *Frankl element*. With $\mathscr{F}_x$ as defined above, the UCC can be stated as follows:

$$\mathscr{F} \text{ is UC} \;\;\Rightarrow\;\; \exists x \in X \text{ such that } |\mathscr{F}_x| \geq \tfrac{1}{2}\,|\mathscr{F}|,$$

where we use $|.|$ to denote the cardinality of a set or family.

In this paper, the focus is on *tight* UC-families. We call a UC-family $\mathscr{F}$ *tight* if every element of X belongs to at most half of the sets in $\mathscr{F}$. More precisely, a UC-family $\mathscr{F}$ is tight if and only if

$$|\mathscr{F}_x| \leq \tfrac{1}{2}\,|\mathscr{F}|, \;\; \forall x \in X.$$

The relevance of this focus may be clear: if $\mathscr{F}$ is not tight, then the existence of a Frankl element is obvious, because then $|\mathscr{F}_x| > \frac{1}{2}\,|\mathscr{F}|$ for at least one $x \in X$. So, for proving the UCC, it suffices to consider tight UC-families. For such families, the UCC requires only that $|\mathscr{F}_x| = \frac{1}{2}\,|\mathscr{F}|$ for some $x \in X$. One surprising result of our research is that there is strong evidence that in the tight case $|\mathscr{F}_x| = \frac{1}{2}\,|\mathscr{F}|$ holds for every $x \in X$. One may easily understand that this certainly holds if $\mathscr{F}$ is the *power set* of X, i.e., if $\mathscr{F} = 2^X$ for some finite set X.

2. An Old Result

For future use, we present in this section one of the oldest results in [9, Theorem 1].

Theorem 2.1. *If the UCC holds for all UC-families $\mathscr{F}$ with $|\mathscr{F}|$ odd, then it also holds for all UC-families $\mathscr{F}$ with $|\mathscr{F}|$ even.*

Proof. Suppose that $\mathscr{F}$ is a UC-family with $|\mathscr{F}| = 2k$ for some $k > 1$. Let $U \in \mathscr{F}$ be such that $U \neq \emptyset$ and $|U|$ minimal. Then U is not the union of two distinct nonempty sets in $\mathscr{F}$. Therefore, the family $\mathscr{G}$ that arises from $\mathscr{F}$ by removing U will be a UC-family of size $2k-1$. Since $|\mathscr{G}|$ is odd, the assumption in the theorem implies that $\mathscr{G}$ satisfies the UCC. So, $|\mathscr{G}_x| \geq \frac{1}{2}|\mathscr{G}| = k - \frac{1}{2}$ for some x. Hence $|\mathscr{G}_x| \geq k = \frac{1}{2}|\mathscr{F}|$. Since $\mathscr{G} \subseteq \mathscr{F}$ implies $|\mathscr{G}_x| \leq |\mathscr{F}_x|$ for every x, we may conclude that $\mathscr{F}$ satisfies the UCC. □

A natural question is whether the converse also holds: if the UCC holds whenever $|\mathscr{F}|$ is even does it then also always hold when $|\mathscr{F}|$ is odd? If this were true, then a simple induction argument would yield that the UCC holds for every UC-family, since it obviously holds if $|\mathscr{F}| = 2$. But, as pointed out in [1], naive induction does not succeed in the UCC case.

3. Two Other Old Results

In this section, we present a new proof of two other old results, namely that if a UC-family $\mathscr{F}$ contains a singleton (or 1-set) $\{x\}$, then x is a Frankl element, and if $\mathscr{F}$ contains a 2-set $\{x,y\}$, then either x or y is a Frankl element. We first deal with the case where $\mathscr{F}$ contains a 2-set. After this the case where $\mathscr{F}$ contains a 1-set easily follows. The proof below uses only two elementary facts, namely the law of De Morgan and

$$|S\cup T| = |S| + |T| - |S\cap T|, \quad \forall S,T\subseteq X. \tag{3.1}$$

Theorem 3.1. *Let $\mathscr{F}$ be UC and $\{x,y\}\in\mathscr{F}$. Then either $|\mathscr{F}_x| \geq \frac{1}{2}|\mathscr{F}|$ or $\left|\mathscr{F}_y\right| \geq \frac{1}{2}|\mathscr{F}|$.*

Proof. Let $\mathscr{F}_x{}'$ denote the complement of $\mathscr{F}_x$ in $\mathscr{F}$ and similarly for y. We define the mapping $\varphi : \mathscr{F}_x{}'\cap\mathscr{F}_y{}' \to \mathscr{F}_x\cap\mathscr{F}_y$ as follows:

$$\varphi(U) = U\cup\{x,y\}\in\mathscr{F}_x\cap\mathscr{F}_y, \quad \forall U\in\mathscr{F}_x{}'\cap\mathscr{F}_y{}'.$$

One easily verifies that φ is injective. Hence

$$\left|\mathscr{F}_x\cap\mathscr{F}_y\right| \geq |\mathscr{F}_x{}'\cap\mathscr{F}_y{}'|. \tag{3.2}$$

According to De Morgan's theorem, the complement of $\mathscr{F}_x{}'\cap\mathscr{F}_y{}'$ in $\mathscr{F}$ is $\mathscr{F}_x\cup\mathscr{F}_y$. Hence, by adding the cardinality of $\mathscr{F}_x\cup\mathscr{F}_y$ to both sides of (3.2), we obtain

$$|\mathscr{F}_x\cap\mathscr{F}_y| + |\mathscr{F}_x\cup\mathscr{F}_y| \geq |\mathscr{F}|.$$

Due to (3.1), the sum at the left equals $|\mathscr{F}_x| + |\mathscr{F}_y|$. Hence $|\mathscr{F}_x| + |\mathscr{F}_y| \geq |\mathscr{F}|$, which implies the lemma. □

Corollary 3.1. *If a UC-family $\mathscr{F}$ contains a singleton, then it satisfies the UCC.*

Proof. Take $y = x$ in Lemma 3.1. □

We conclude from the above results that if a UC-family contains a subset S which is a 1-set or a 2-set, then S contains a Frankl element.

4. Some Results on 3-sets

Probably Sarvate and Renaud were the first to prove the results in the previous section, at least for 2-sets [9, Theorem 2]. In a subsequent paper, they showed that there is no such result for 3-sets [10]. They demonstrated this by presenting a UC family with 27 subsets of a set X with $|X| = 9$ whose smallest set is a 3-set which does not contain a Frankl element. In their example, the family does not contain the empty set. Later on, Poonen gave a different example in [6, page 267] with 28 subsets of the same set X. We found a smaller example $\mathscr{F}$, given in Figure 1 by its matrix representation $A_{\mathscr{F}}$. This 19×7 matrix is the so-called binary representation of a family $\mathscr{F}$ consisting of 19 subsets of a set with 7 elements. Its rows are the (binary) incidence vectors of the subsets in $\mathscr{F}$. So the first row represents the empty set, the second row the 3-set $Y = \{1, 2, 3\}$ and the third row the set $\{4, 5, 7\}$ and so on. We leave it to the reader to verify that $\mathscr{F}$ is a UC family. The column sums of $A_{\mathscr{F}}$ are respectively 9, 9, 9, 14, 14, 14, and 17. Since $\frac{1}{2}|\mathscr{F}| = 9.5$, it follows that no element of Y is a Frankl element and, moreover, each element outside Y is a Frankl element.

It may be worth mentioning that there is another proof for the fact that $\mathscr{F}$ has a Frankl element in the set $Z := \{4, 5, 6, 7\}$. The 4-set Z belongs to $\mathscr{F}$ and it contains three distinct 3-sets also in $\mathscr{F}$, namely $\{4, 5, 7\}$, $\{4, 6, 7\}$, and $\{5, 6, 7\}$. From this, we may conclude that Z contains

$$A_{\mathscr{F}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

FIGURE 1. UC family with a 3-set that does not contain a Frankl element.

a Frankl element of $\mathscr{F}$. This is due to one of the deepest results on the current topic, namely [6, Theorem 1]. Poonen used this result to prove that the UCC holds if $|X| \leq 7$ or $|\mathscr{F}| \leq 28$. According to [1], these bounds have been improved to respectively $|X| \leq 12$ and $|\mathscr{F}| \leq 50$.

In the next section, we present the result of some computational experiments that inspired the results presented in the rest of the paper, including the focus on the tight case.

5. NUMERICAL EXPERIMENTS

As a reason for the lack of progress in proving the UCC has been mentioned the lack of knowledge on the minimal maximum frequency among all UC-families of a given size [1, page 2056]. In order to gain some knowledge of this type, we deal in this section with the following optimization problem:

$$\min \left\{ \max_{x \in X} |\mathscr{F}_x| \;:\; \mathscr{F} \subseteq 2^X, |\mathscr{F}_x| \geq 1, \forall x \in X, \mathscr{F} \text{ is } UC, |\mathscr{F}| = m \right\}, \quad 2 \leq m \leq 2^n. \tag{5.1}$$

In this problem $n = |X|$, and we look for the smallest possible value of the maximal value of $|\mathscr{F}_x|$, where $\mathscr{F}$ is a UC-family consisting of m subsets of X, and x runs through X, with $|\mathscr{F}_x| \geq 1$ for each x.

Problem (5.1) can be put in the form of a linear optimization problem with binary variables. For a detailed description of this binary linear optimization problem we refer to [8]. By using the optimization package MOSEK under the CVX routine in Matlab, we solved this problem for $m = 2$ to 2^n for some not too large values of n. Table 1 shows the outcome for $n = 4$, where the elements $x \in X$ are permuted such that $|\mathscr{F}_x|$ is nonincreasing.

This table suggests that Frankl's conjecture is *tight* if and only if $|\mathscr{F}|$ is a power of 2. Similar experiments for $n = 2$ to 7 confirmed this behavior. As a result we became rather convinced that

$\lvert\mathscr{F}\rvert = m$	$\lvert\mathscr{F}_1\rvert$	$\lvert\mathscr{F}_2\rvert$	$\lvert\mathscr{F}_3\rvert$	$\lvert\mathscr{F}_4\rvert$	$\max_i \lvert\mathscr{F}_i\rvert/\lvert\mathscr{F}\rvert$
2	1	1	1	1	0.5000
3	2	1	1	1	0.6667
4	2	2	2	2	0.5000
5	3	3	3	1	0.6000
6	4	4	3	1	0.6667
7	4	4	4	3	0.5714
8	4	4	4	4	0.5000
9	5	5	5	1	0.5556
10	6	6	5	4	0.6000
11	7	6	6	5	0.6364
12	7	7	6	6	0.5833
13	8	7	7	6	0.6154
14	8	8	7	7	0.5714
15	8	8	8	8	0.5333
16	8	8	8	8	0.5000

TABLE 1. Optimal solutions for problem (5.1) if $n = 4$ and $2 \le m \le 2^n$.

four related conjectures are also true. These conjectures are presented in the next section. As we will explain, each of these conjectures implies the UCC. We conclude this section by showing

X	1	2	3	4	5	6	7
1	0	0	0	0	0	0	0
2	0	0	1	0	1	0	1
3	0	1	0	1	0	1	0
4	0	1	1	1	1	1	1
5	1	0	0	0	0	0	0
6	1	0	1	0	1	0	1
7	1	1	0	1	0	1	0
8	1	1	1	1	1	1	1

TABLE 2. Solution of problem (5.1) for $n = 7$ and $m = 8$.

a particular solution of problem (5.1), as shown in Table 2. In this table the first row shows the elements of $X = \{1, 2, \dots, 7\}$. The remaining rows are the incidence vectors in $\{0,1\}^n$ of the m subsets in $\mathscr{F}$, whereas each column shows the incidence vector in $\{0,1\}^m$ of the subfamily $\mathscr{F}_x$, $1 \le x \le 7$. It is clear from Table 2 that each $x \in X$ occurs in 4 sets, so we have $\lvert\mathscr{F}_x\rvert = 4$ for each x.

Another important observation is that the columns for the elements 3, 5 and 7 are equal: $\mathscr{F}_3 = \mathscr{F}_5 = \mathscr{F}_7$. By removing two of these columns the eight rows remain different and the UC property is maintained. Finally, also $\mathscr{F}_2 = \mathscr{F}_4 = \mathscr{F}_6$. After removing also two of these columns, three mutually different columns are left. The rows of the corresponding submatrix are all possible binary words of length 3. So the resulting family is the power set of the set $X = \{1,2,3\}$.

Summarizing, we concluded in this way that tightness occurs only if $m = 2^k$ for some k and in that case, after removing identical columns, $\mathscr{F}$ turns out to be the power set of a k-set $Y \subseteq X$.

6. Four Related Conjectures

We made clear in Section 1 that it suffices to prove the UCC for the tight case.

As we saw that, in Table 2, it may happen that $\mathscr{F}_y = \mathscr{F}_x$ for $x, y \in X$. We then say that x and y are *equivalent*, denoting this as $x \simeq y$. Indeed, $\simeq$ is an equivalence relation in the formal sense. Following Poonen [6], its equivalence classes are called *blocks*. If x and y are equivalent elements then one of these elements can be neglected, because this hurts neither the UC-property nor the UCC. After doing this the blocks becomes singletons. A family will be called *clean* if its blocks are singletons and there are no 'idle' elements, i.e., elements x with $|\mathscr{F}_x| = 0$. Our ultimate goal is to show that if a tight UC-family is clean then it is a power set.

We use U' to denote the complement of any subset U of X. If $\mathscr{F}$ is a clean UC-family, it contains the set X. As mentioned before, we assume that $\mathscr{F}$ also contains its complement X', which is the empty set. Hence, we always have $|\mathscr{F}| \geq 2$ and $|\mathscr{F}_x| \geq 1$ for every $x \in X$.

We proceed with a simple lemma.

Lemma 6.1. *If $\mathscr{F}$ is closed under taking complements, then $|\mathscr{F}_x| = \frac{1}{2}|\mathscr{F}|$ for each $x \in X$.*

Proof. For each subset U in $\mathscr{F}$ and for each element x in X we have either $x \in U$ or $x \in U'$. Since U' also belongs to $\mathscr{F}$, x belongs to exactly half of the number of sets in $\mathscr{F}$. This implies the lemma. □

Taking into account our assumption that $\mathscr{F}$ is clean we have the following four conjectures.

Conjecture 6.1. If $\mathscr{F}$ is UC and tight, then $\mathscr{F} = 2^X$.

Conjecture 6.2. If $\mathscr{F}$ is UC and tight, then $U \in \mathscr{F}$ implies $U' \in \mathscr{F}$.

Conjecture 6.3. If $\mathscr{F}$ is UC and tight, then $|\mathscr{F}_x| = \frac{1}{2}|\mathscr{F}|$ for every $x \in X$.

Conjecture 6.4. If $\mathscr{F}$ is UC and tight, then $|\mathscr{F}|$ is even.

Proposition 6.1. *One has the following implications:*

Conjecture 6.1 $\Rightarrow$ Conjecture 6.2 $\Rightarrow$ Conjecture 6.3 $\Rightarrow$ Conjecture 6.4 $\Rightarrow$ UCC.

Proof. Since the power set 2^X of X is closed under taking complements the implication from Conjecture 6.1 to Conjecture 6.2 is trivial. The implication from Conjecture 6.2 to Conjecture 6.3 is an immediate consequence of Lemma 6.1. Furthermore, if Conjecture 6.3 holds then Conjecture 6.4 also holds, because $|\mathscr{F}| \geq 2$ and $|\mathscr{F}|/2$ is integral. Finally we deal with the implication from Conjecture 6.4 to the UCC. Let $\mathscr{F}$ be a UC-family with $|\mathscr{F}|$ odd. Then Conjecture 6.4 implies that $\mathscr{F}$ is not tight. But this means that $\mathscr{F}$ satisfies the UCC. This proves that all odd UC-families satisfy the UCC. By Theorem 2.1 this implies that also all UC-families with $|\mathscr{F}|$ even satisfy the UUC. Hence, Conjecture 6.4 implies that the UCC always holds. This completes the proof. □

We conclude this section by proving the inverse implications of two of the four implications in Proposition 6.1.

Proposition 6.2. *The UCC implies Conjecture 6.4.*

Proof. Assuming the UCC, we let $\mathscr{F}$ be a tight UC-family. The UC property implies that $|\mathscr{F}_x| \geq \frac{1}{2}|\mathscr{F}|$ for some $x \in X$. The tightness of $\mathscr{F}$ implies $|\mathscr{F}_x| \leq \frac{1}{2}|\mathscr{F}|$. Hence we obtain $|\mathscr{F}_x| = \frac{1}{2}|\mathscr{F}|$. Since $|\mathscr{F}_x|$ is integral, it follows that $|\mathscr{F}|$ is even. □

Apparently, due to the last two propositions we may conclude that Conjecture 6.4 is equivalent with the UCC. We do not yet know if this also holds true for the conjectures 6.1 to 6.3.

For the proof of the second inverse implication we need one more lemma.

Lemma 6.2. *If $\mathscr{F}$ is UC and closed under taking complements, then $\mathscr{F}$ is closed under taking intersections.*

Proof. Let $\mathscr{F}$ be UC and closed under taking complements and $U,V \in \mathscr{F}$. Then also $U',V' \in \mathscr{F}$ and due to the UC-property also $U' \cup V' \in \mathscr{F}$. Using De Morgan's law, we may now write

$$U \cap V = \big(U' \cup V'\big)' \in \mathscr{F}.$$

This proves the lemma. □

Proposition 6.3. *Conjecture 6.2 implies Conjecture 6.1.*

Proof. Suppose that Conjecture 6.2 holds and that $\mathscr{F}$ satisfies its hypothesis. Then $\mathscr{F}$ is UC and tight, and therefore by Conjecture 6.2 also closed under taking complements. By Lemma 6.2, this implies that $\mathscr{F}$ is IC. Now let $x \in X$ and

$$W := \bigcap_{U \in \mathscr{F}_x} U.$$

Due to the IC-property one has $W \in \mathscr{F}$. Now, we suppose that also $y \in W$. This means $y \in U$ for every $U \in \mathscr{F}_x$, whence $U \in \mathscr{F}_y$ for any such U. Therefore, $\mathscr{F}_y \supseteq \mathscr{F}_x$. By Lemma 6.1, we also have $|\mathscr{F}_y| = |\mathscr{F}_x|$. Thus we obtain $\mathscr{F}_y = \mathscr{F}_x$. Our assumption that $\mathscr{F}$ is clean now implies that $y = x$. Thus we may conclude that $W = \{x\} \in \mathscr{F}$. Since this holds for every $x \in X$, we have $\{x\} \in \mathscr{F}$ for every $x \in X$. Since $\mathscr{F}$ is UC, this implies $\mathscr{F} = 2^X$. □

7. Conclusion

It is almost incomprehensible that a basic problem as Frankl's conjecture has remained unsolved for so long. Every mathematician should feel challenged by this fundamental problem. As should be clear, this short note leaves the conjecture open. But hopefully it brings us somehow closer to the solution.

On the other hand, it may well be one of those problems that are beyond the capacity of the human mind, just to remind us that there are things we will never understand [7].

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