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STOCHASTIC OPTIMIZATION APPROACH FOR PREDICTING HORSE RACING OUTCOMES

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Abstract. This paper uses the stochastic programming model by Peng and Uryasev (2025) to forecast horse racing outcomes. The model captures the nonlinear relationship between explanatory factors and horse running time. This random running time is characterized by quantile functions, allowing for considerable flexibility in shaping the conditional distribution as a function of factor data. The proposed model demonstrates good numerical efficiency and adaptability, making it well-suited for predicting various horse racing outcomes. The case study compares the proposed model with alternative approaches for predicting racing outcomes. The suggested model achieves comparable or superior performance while maintaining computational efficiency across various race scenarios.
Keywords. Factor model; Running time distribution; Ordering probability; Quantile mixture; Ranking model.
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1. INTRODUCTION

This paper adopts the novel framework proposed by [15] to predict horse racing outcomes. The framework effectively models the nonlinear dependence between explanatory factors and horse running time. The random running time is represented by a quantile function, enabling the prediction of its conditional distribution given the factors.

Accurate probabilistic predictions play a crucial role for bettors participating in the pari-mutuel betting system of horse racing. The most straightforward bet is the win-bet, where one must select the horse that finishes first. Traditionally, modeling the probabilities of winners is based on the knowledge of win-bet fraction. Empirical studies highlight a prevalent issue known as the favorite-longshot bias in the win-bet fraction. This bias arises because bettors tend to underbet favorites and overbet longshots due to the allure of higher potential rewards from longshot bets (see [1, 7, 10, 16], among others). Nevertheless, the win-bet fraction remains consistent with the true winning probability (see [12]), so we can reasonably assume that the win-bet fraction serves as a reliable estimate of the winning probability. In addition to adopting the win-bet fraction as an estimated winning probability, various common approaches for winning probability estimation include the application of logit or probit models, as demonstrated in studies such as [4, 6, 11, 14].

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Apart from win betting, there are other popular types of bets in horse racing. A place bet selects a horse to finish at least second (or third in the case of a show bet). More complex bets, such as exacta, trifecta, and quinella require accurately predicting the order of finish. These intricate bets add complexity to the betting experience, as they demand a precise estimation of ordering probabilities. Many ranking models rely on estimated winning probabilities, subsequently employing this knowledge to estimate the underlying distributions of horse running times. The most straightforward and convenient method for modeling ordering probabilities was proposed by [8]. The Harville model is essentially equivalent to assuming that the running times follow an exponential distribution. [3] enhanced the Harville model by introducing a logit model, offering a more refined approach. Alternatively, both [9] and [17] proposed ranking models for ordering probability, assuming normal and gamma distributions, respectively, for the running times. [3] and [12] showed that the Henery model fits better than the Harville model for particular racing data. Likewise, adopting the concept of fitting running time distributions, [2] evaluated the accuracy of the aforementioned distributions in estimating the probabilities of finishing second or third. These probabilities hold direct relevance for individuals placing place and show bets. However, the Henery and Stern models pose practical challenges in application due to their numerical inefficiency. Addressing this issue, [13] proposed a simple practical approximation for ordering probabilities for the Henery and Stern models but based on the simple assumption that the horses have the same abilities (i.e., equal mean running times).

Without fully relying on win-bet information or making naive assumptions, the model in this paper directly estimates the conditional distribution of running times. We formulate the convex optimization calibration problem. Notably, this calibration is race-invariant, as it does not rely exclusively on estimated winning probabilities from win-bet fraction data for a specific race. Therefore, this approach is both highly efficient and versatile, making it well-suited for a wide range of race scenarios while achieving comparable or superior performance (see Section 4).

The paper is structured as follows. Section 2 introduces the quantile mixture model with factors and its application to horse racing. Section 3 outlines the incorporation of assumed running time distributions into our model and explains the process of probability prediction. Empirical comparison of the proposed model with other models, referred to as benchmark models, for horse racing are presented in Section 4. The conclusion is provided in Section 5.

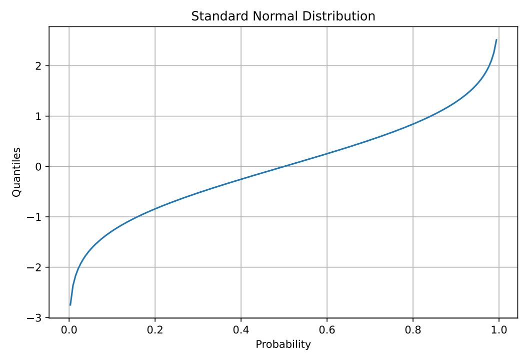
2. Model Description

Let $\xi_1, \cdots, \xi_n \in \Xi$ be the n labeled horse racing data, where each $\xi_j = (\boldsymbol{x}_j, y_j)$ consists of a vector of factors $\boldsymbol{x}_j = [x_{j1}, \ldots, x_{jK}] \in \mathbb{R}^K$ for horse j and a positive time y_j (in seconds) representing the horse's finish time. A quantile function, by definition, is an inverse of a cumulative distribution function (CDF). This paper assumes a continuous and strictly increasing CDF, ensuring a one-to-one correspondence between the single-valued CDF and its single-valued quantile function. Denote by $G(p, \boldsymbol{x}_j, \boldsymbol{\theta})$ a conditional quantile function of horse's running time given factor vector $\boldsymbol{x}_j \in \mathbb{R}^K$, a matrix of parameters, and a confidence level $p \in (0,1)$. We assume that $\{y_j\}_{j=1}^n$ are independent realizations from quantile function $G(p, \boldsymbol{x}_j, \boldsymbol{\theta})$.

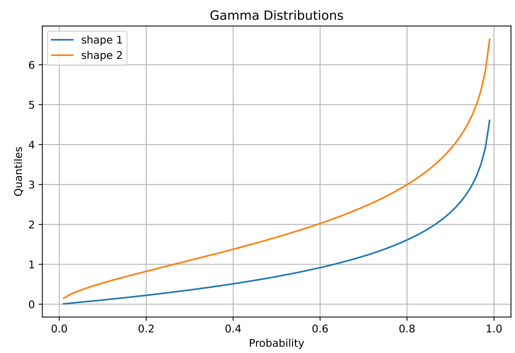
[15] proposed a factor model of mixtures for estimation of a quantile function. A quantile function is estimated as a linear combination of basis quantile functions:

$$G(p, \boldsymbol{x}, \boldsymbol{\theta}) = \sum_{i=0}^{I} f_i(\boldsymbol{x}, \boldsymbol{\theta}_i)\, Q_i(p), \tag{2.1}$$

where basis quantile functions, $Q_i(p)$, $i = 1,\dots,I$ depend on confidence level $p \in (0,1)$, coefficients, $f_i(\boldsymbol{x},\boldsymbol{\theta}_i)$, depend on the factor vector $\boldsymbol{x}$ and the vector of parameters, $\boldsymbol{\theta}_i$. Also, $Q_0(p) = 1$ and $f_0(\boldsymbol{x}) = 1$. Some example basis quantile functions are shown in Figure 1. These graphs were obtained by flipping axes of CDF graphs.



(A) Quantile of standard normal distribution.



(B) Quantiles of gamma distribution with shape parameters 1 and 2.

FIGURE 1. Graphs of quantile functions

This paper assumes that the functions $f_i(\boldsymbol{x},\boldsymbol{\theta}_i)$, $i = 0,\dots,I$ are linear combination of spline functions:

$$f_i(\boldsymbol{x},\boldsymbol{\theta}_i) = \sum_{k=0}^{K} \theta_{ik} B_k(x_k), \quad i = 1,\dots,I, \tag{2.2}$$

where K is the number of factors, $\boldsymbol{\theta}_i = [\theta_{i1},\dots,\theta_{iK}]$ is the vector of coefficients, $\boldsymbol{x} = [x_1,\dots,x_K]$ is the vector of factors, $B_k(x_k)$, $k = 1,\dots,I$ are so called basis spline functions and $B_0(x) = 1$.

The linearity with respect to the coefficients $\boldsymbol{\theta} \in \mathbb{R}^{(I+1)\times(K+1)}$ not only facilitates an efficient formulation of convex optimization but also preserves an interpretable factor model structure. This structure allows for an analysis of an impact of each factor on the shape of the conditional quantile function. With (2.1) and (2.2), we define the conditional quantile of running time Y_j of horse j given $\boldsymbol{\theta}$ and factor values $\boldsymbol{x}_j$. If two horses have the same factor values, i.e., $\boldsymbol{x}_i = \boldsymbol{x}_j$, they have the same conditional quantile, $G(p,\boldsymbol{x}_i,\boldsymbol{\theta}) = G(p,\boldsymbol{x}_j,\boldsymbol{\theta})$, and consequently the same conditional CDFs of running times.

By utilizing available labeled horse race data, we can estimate $\boldsymbol{\theta}$ in the function G from (2.1). The quality of an estimate is assessed with the following loss function

$$\mathscr{L}(\boldsymbol{\theta}) := \frac{1}{n}\sum_{j=1}^{n} L(\boldsymbol{\theta};\boldsymbol{x}_j,y_j), \tag{2.3}$$

where n is the number of observations in the dataset and $L(\boldsymbol{\theta};\boldsymbol{x}_j,y_j)$ is the loss function for a horse j. The function L is defined as a weighted average error $\mathscr{E}_p$ with respect to $p \in (0,1)$,

$$L(\boldsymbol{\theta};\boldsymbol{x}_j,y_j) = \int_0^1 w(p)\mathscr{E}_p\Big(y_j - \sum_{i=0}^{I} Q_i(p)\sum_{k=0}^{K} \theta_{ik}B_k(x_{jk})\Big)dp, \tag{2.4}$$

where $w(p)$ is a nonnegative weight function and $\int_0^1 w(p)dp = 1$. In (2.4), the pinball loss is defined as follows

$$\mathscr{E}_p(Z) = pZ\mathbb{1}_{\{Z>0\}} - (1-p)Z\mathbb{1}_{\{Z\leq 0\}},$$

where $\mathbb{1}_{\{\cdot\}}$ is the indicator function. Note that if the weight function $w(p)$ is a point-measure, assigning mass 1 to a specific p, the minimization of (2.4) is transformed into quantile regression [5] at the probability level p.

Given the historical labeled horse data $\{\xi_j\}_{j=1}^n$ across multiple races, our problem of finding the optimal matrix of coefficients for horse racing finishing times is formulated from (2.3) and (2.4) as

$$\min_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \sum_{j=1}^{n} \int_0^1 w(p)\mathscr{E}_p\Big(y_j - \sum_{i=0}^{I} Q_i(p) \sum_{k=0}^{K} \theta_{ik} B_k(x_{jk})\Big)dp, \tag{2.5}$$

where $\boldsymbol{\Theta}$ is a convex feasible set. We have neglected the coefficient $\frac{1}{n}$ in (2.3) because it does not influence the optimal solution vector.

Furthermore, with the uniform weight function $w(p) = 1$, problem (2.5) is equivalent to minimization of the Continuous Ranked Probability Score (CRPS), see [15]. Similar to quantile regression, the minimization problem (2.5) is robust to outliers.

In practice, we discretize the problem (2.5) by taking a grid over $p \in (0,1)$. In the case study presented in Section 4, we set $\mathbb{P} = \{0.01, 0.03, \ldots, 0.99\}$, and the problem is reformulated as

$$\min_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \sum_{j=1}^{n} \sum_{p\in\mathbb{P}} \mathscr{E}_p\Big(y_j - \sum_{i=0}^{I} Q_i(p) \sum_{k=0}^{K} \theta_{ik} B_k(x_{jk})\Big)\Delta p, \tag{2.6}$$

where $\Delta p = 0.02$. An important advantage of this formulation is that it leads to a convex optimization problem in $\boldsymbol{\theta}$. Additionally, if factor values are always non-negative, we can impose an additional non-negativity constraint on $\boldsymbol{\theta}$ to directly ensure that the estimated quantile function $G(p, \boldsymbol{x}, \hat{\boldsymbol{\theta}})$ remains nondecreasing in p, conditioned on any value of factors.

3. Horse Racing Prediction

3.1. **Running Time Distribution.** Given the factor data for horses in a race, we can model the conditional running time distribution for all participating horses, characterized by quantile functions. Consider a labeled horse with factor data $\boldsymbol{x}$. Let Y be the random variable representing the running time of a horse. Here, we present conditional quantile functions for running time for normal and gamma basis functions. This selection of functions is motivated by [9], who assumed that running time follows a normal distribution with a fixed scale parameter of 1, and by [17], who considered a gamma distribution with a shape parameter of 1 or 2.

Example 3.1. Normal distribution

$$G(p, \boldsymbol{x}, \boldsymbol{\theta}) = f_0(\boldsymbol{x}, \boldsymbol{\theta}_0) + \Phi(p) f_1(\boldsymbol{x}, \boldsymbol{\theta}_1), \quad 0 < p < 1,$$

where $\Phi(p)$ is the quantile function of a standard normal random variable. In this case, Y follows a normal distribution with mean $f_0(\boldsymbol{x}, \boldsymbol{\theta}_0)$ and scale $f_1(\boldsymbol{x}, \boldsymbol{\theta}_1)$.

Example 3.2. gamma distribution

$$G(p, \boldsymbol{x}, \boldsymbol{\theta}) = f_0(\boldsymbol{x}, \boldsymbol{\theta}_0) + Q_r(p) f_1(\boldsymbol{x}, \boldsymbol{\theta}_1), \quad 0 < p < 1,$$

where $Q_r(p)$ is the quantile function of a gamma random variable with shape parameter r and scale parameter equal to 1. The random variable Y follows a gamma distribution with shape r, scale $f_1(\boldsymbol{x},\boldsymbol{\theta}_1)$, and a right shift of $f_0(\boldsymbol{x},\boldsymbol{\theta}_0)$.

Example 3.3. Mixture of normal and gamma distributions

$$G(p,\boldsymbol{x},\boldsymbol{\theta}) = f_0(\boldsymbol{x},\boldsymbol{\theta}_0) + \Phi(p) f_1(\boldsymbol{x},\boldsymbol{\theta}_1) + Q_r(p) f_2(\boldsymbol{x},\boldsymbol{\theta}_2), \quad 0 < p < 1,$$

where $\Phi(p)$ is the quantile function of a standard normal random variable, and $Q_r(p)$ is the quantile function of a gamma random variable with shape parameter r and scale parameter equal to 1.

In practice, the framework offers the flexibility to incorporate a wider range of distributions with various parameters to better fit the running time distribution. In this paper, we focus only on normal and gamma distributions with shape parameters 1 or 2, as well as their mixtures (see Section 4).

3.2. **Horse Racing Probability.** Let us consider a race with m horses and given factor data $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_m$. Denote by $Y_1,\ldots,Y_m$ the running time random variables for the m horses. We assume that these variables are independent given the observed factors. The model-based probability that horse i wins the race is given by

$$\begin{aligned} P_i^{(1)} &= P(Y_i < \min_{u \neq i}\{Y_u\}) \\ &= \int f_i(y_i) \prod_{u \neq i}^{m} [1 - F_u(y_i)] dy_i, \quad i = 1,2,\ldots,m, \end{aligned} \tag{3.1}$$

where $f_i(y)$ and $F_i(y)$ are the density and distribution functions of horse i. In addition to the win-bets, there are two common betting options in a race, known as place and show bets. The probabilities that a horse i finishes second or third are of direct interest to those making the place or show bets, denoted as $P_i^{(2)}$ and $P_i^{(3)}$. They are computed with the following formulation, respectively:

$$\begin{aligned} P_i^{(2)} &= P(Y_{(1)} < Y_i < Y_{(3)}) \\ &= \int f_i(y_i) \sum_{j \neq i}^{m} F_j(y_i) \prod_{u \neq i,j}^{m} [1 - F_u(y_i)] dy_i, \quad i = 1,2,\ldots,m, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} P_i^{(3)} &= P(Y_{(2)} < Y_i < Y_{(4)}) \\ &= \int f_i(y_i) \sum_{j_1,j_2 \neq i}^{m} F_{j_1}(y_i) F_{j_2}(y_i) \prod_{u \neq i,j_1,j_2}^{m} [1 - F_u(y_i)] dy_i, \quad i = 1,2,\ldots,m. \end{aligned} \tag{3.3}$$

where $Y_{(k)}$ represents the k-th smallest random variable among $\{Y_s\}, s = 1,\ldots,m$. The model-based probability that horse i finishes 1st and horse j finishes 2nd (Exacta) is

$$\begin{aligned} P_{ij} &= P(Y_i < Y_j < \min_{u \neq i,j}\{Y_u\}) \\ &= \int f_j(y_j) F_i(y_j) \prod_{u \neq i,j}^{m} [1 - F_u(y_j)] dy_j, \quad i,j = 1,2,\ldots,m \;\&\; i \neq j. \end{aligned} \tag{3.4}$$

Similarly, the model-based probability that horses i, j, k finishes at the first, second, and third (Trifecta) is

$$\begin{aligned} P_{i,j,k} &= P(Y_i < Y_j < Y_k < \min_{u \neq i,j,k}\{Y_u\}) \\ &= \int \left(\int_{y_k}^{+\infty} f_j(y_j) F_i(y_j) dy_j \right) \prod_{u \neq i,j,k}^{m} [1 - F_u(y_k)] dy_k \end{aligned} \tag{3.5}$$

with $i, j, k = 1, 2, \ldots, m$ and $i \neq j \neq k$. Similar integrals can be computed for higher-order ranking probabilities. The common ranking models are summarized by [14]. With this approach, the winning probabilities of all m horses participating in a race can be estimated. The running time of each horse $i = 1, \ldots, m$ is modeled as a random variable following a distribution parameterized by β_i. Then, the parameters $\{\beta_i\}_{i=1}^m$ can be estimated by solving m equations derived from the winning probabilities in (3.1). However, this method is computationally intensive and requires separate estimation of $\{\beta_i\}$ for each race, as the parameters are race-specific.

In contrast, our model estimates the parameters $\boldsymbol{\theta}$ in (2.2) using data across multiple races, requiring a one-time estimation. Once the conditional distributions of horses in a race are determined, all density and distribution functions in (3.1) - (3.5) can be determined either analytically or numerically, and thus predicting race outcomes becomes straightforward. Furthermore, probabilities in (3.1)–(3.5) can also be estimated by simulation with a large sample size, once conditional distributions of horse running time are determined.

4. Case Study

The data utilized in this study were gathered from the Hong Kong Jockey Club covering the period from September 2008 to July 2018. The dataset includes races with varying distances and numbers of participants. We considered races with a distance of 1200 meters and exactly 14 participants. This selection criterion resulted in a dataset comprising 442 races and 6188 observations.

Earlier studies have shown that the win-bet fraction has high explanatory power. We have considered in our model only this single factor. The factor is transformed using Z-score normalization within each race. For the basis spline function in (2.2), we set the number of knots and the polynomial degree equal to 4 and 2, respectively. Following the approach of [9] and [17], we incorporate basis quantile functions of standard normal and standard gamma (with shape parameters 1 or 2) into our model. The model is then fitted using (2.6) with $\mathbb{P} = \{0.01, 0.03, \ldots, 0.99\}$. Results for our model are benchmarked with models by [1], [3], and [8], which are based on the single win-bet fraction factor. Let $\hat{\pi}_i$, $\hat{\pi}_{ij}$, and $\hat{\pi}_{ijk}$ denote probabilities of horse i finishing first, horses i and j finishing first and second, and horses i, j, and k finishing first, second, and third, respectively. In the model introduced by [3], referred to as the BLB model, the estimated probabilities for a given race are expressed as

$$\hat{\pi}_i = \frac{w_i^{\beta}}{\sum_r w_r^{\beta}}, \tag{4.1}$$

$$\hat{\pi}_{ij} = \hat{\pi}_i \hat{\pi}_{j|i} = \frac{w_i^{\beta}}{\sum_r w_r^{\beta}} \frac{w_{j|i}^{\mu}}{\sum_{s \neq i} w_{s|i}^{\mu}}, \tag{4.2}$$

and

$$\hat{\pi}_{ijk} = \hat{\pi}_i \hat{\pi}_{j|i} \hat{\pi}_{k|i,j} = \frac{w_i^{\beta}}{\sum_r w_r^{\beta}} \frac{w_{j|i}^{\mu}}{\sum_{s \neq i} w_{s|i}^{\mu}} \frac{w_{k|i,j}^{\omega}}{\sum_{t \neq i,j} w_{t|i,j}^{\omega}}, \tag{4.3}$$

where w_i = win-bet fraction of the horse i, $w_{j|i} = w_j/(1-w_i)$, $\hat{\pi}_{j|i} = \hat{\pi}_j/(1-\hat{\pi}_i)$, $w_{k|i,j} = w_k/(1-w_i-w_j)$, and $\hat{\pi}_{k|i,j} = \hat{\pi}_k/(1-\hat{\pi}_i-\hat{\pi}_j)$. The parameters β, (β,μ), and (β,μ,ω) in (4.1) - (4.3) are estimated separately using the maximum likelihood method. The likelihood is computed as the product of the estimated probabilities of the realized outcomes across all races in the dataset. Note that setting all parameters in (4.1) - (4.3) equal to 1 precisely restores the Ali and Harville models. In particular, we extend the formula in (4.1) to account for cases where horse i finishes second or third. The parameter β is then estimated separately for each case by maximizing the likelihood function, where the realized outcome corresponds to finishing in 2nd or 3rd place, respectively, as in the BLB model.

TABLE 1. Log-likelihoods of Models (442 Races in Hong Kong)

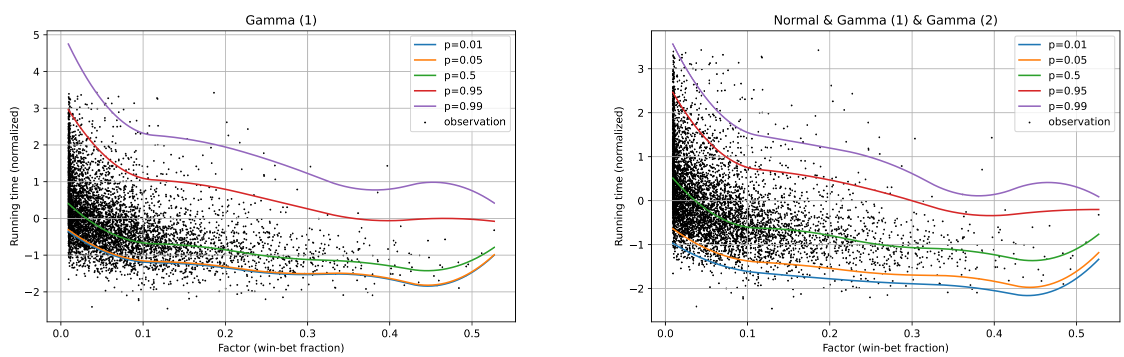
	First (Win)	**Second**	**Third**	**Exacta**	**Trifecta**
Ali	−874.53	−1010.11	−1119.24	———	———
Harville	———	———	———	−1797.40	−2745.86
BLB	−862.85[a]	−1006.26	−1084.39	−1785.50[a]	−2731.34[a]
Normal	−892.90	−1006.36	−1079.47	−1823.14	−2774.54
gamma $(r=1)$	−978.79	−1089.99	−1131.63	−1997.27	−3012.46
gamma $(r=2)$	−901.59	−1020.81	−1082.59	−1844.35	−2819.54
Normal & gamma $(r=1)$	−865.82[c]	−998.35[a]	−1075.15[b]	−1786.56[c]	−2739.62[c]
Normal & gamma $(r=2)$	−866.37	−998.44[b]	−1074.99[a]	−1789.06	−2747.17
Normal & gamma $(r=1)$ **& gamma** $(r=2)$	−865.57[b]	−998.80[c]	−1075.24[c]	−1785.95[b]	−2735.61[b]

Index ***a*** = largest value (top 1), index ***b*** = second largest value (top 2), index ***c*** = third largest value (top 3).

Table 1 presents the log-likelihood results for different horse race outcomes in Hong Kong, encompassing horses finishing first (win), second, third, Exacta, and Trifecta. Benchmark models, including Ali, Harville, and BLB models, are also reported. Our quantile mixture models with factors incorporate various combinations of basis quantile functions, utilizing standard normal and standard gamma distributions with shape parameters 1 or 2. The likelihoods for our models are computed using estimated probabilities obtained via simulation for each race, with a sample size of 10^5. For each race outcome in the table, the top three largest log-likelihood values are indexed with ***a***, ***b***, and ***c***, respectively. The models corresponding to these top three values are regarded as the best-performing models for the respective race outcome.

Within the benchmark models group, the BLB model consistently outperform both Ali and Harville models across all types of outcomes. Among our models that incorporate a single distribution, the one with a normal quantile demonstrates the best performance, closely followed by the model with a gamma quantile $(r = 2)$. This finding aligns with the results of [3], [12], and [2], which suggest that the normal ranking model fits the data better than the gamma ranking model. Our primary focus is on comparing the BLB model with our quantile mixture model, which incorporates multiple distributions. For Win, Exacta, and Trifecta outcomes, our models, while not surpassing the BLB model, exhibit very close performance, with log-likelihood differences ranging from 0.5 to 5.

This is reasonable, given that parameter estimation in the BLB model directly maximizes likelihoods, whereas our model calibrates by fitting running-time distributions. However, we observe that our models incorporating multiple quantiles outperform the BLB model for outcomes involving horses finishing second and third. Overall, our model that integrates the mixture of three quantile functions demonstrates excellent performance. Furthermore, our model can be significantly improved by incorporating additional factors, such as weather conditions, track conditions, weight carried, recent performance, and other horse- and race-related variables.



(A) Model with a gamma $(r = 1)$ quantile function (B) Model with standard normal, gamma $(r = 1)$, and gamma $(r = 2)$ quantile functions

FIGURE 2. The figure displays quantile values from models incorporating various basis quantile functions alongside true observations from the data. The probability levels p correspond to $0.01, 0.05, 0.5, 0.95$, and 0.99.

Figure 2 presents data points from our dataset alongside the quantile values estimated by our models at different probability levels across varying factor values. Two model configurations are compared: one using only a single gamma $(r = 1)$ quantile function and another incorporating all the previously mentioned quantile functions. As expected, the quantile value curves corresponding to different probability levels p do not intersect and exhibit a nonlinear dependence on the factor.

The figure highlights the model's flexibility in fitting the data by expanding the set of quantile functions. When restricted to only a gamma $(r = 1)$ quantile, the model provides a poor fit to the data. In particular, the curves at different probability levels fail to adequately separate the data, with those for $p = 0.01$ and $p = 0.05$ being nearly indistinguishable. By incorporating additional normal and gamma ($r = 2$) quantiles, the model achieves a significantly improved fit, effectively distinguishing data points across different probability levels. This comparison between the two configurations in the figure aligns with the poor and good performance shown in Table 1.

5. Conclusion

Case study in Section 4 demonstrated strong capability and flexibility of our model in estimating probabilities for various horse racing outcomes when incorporating a sufficient number of quantile functions. Our model achieved similar or superior performance compared to benchmark models by [1], [2], [3], and [4]. We recommend using a mixture of standard normal, gamma ($r = 1$), and gamma ($r = 2$) quantile functions for fitting the running time distribution. The model can be further improved by incorporating additional explanatory factors and increasing the number of quantile functions.

Our model is capable of predicting a broad range of horse racing outcomes, including higher-order ranking probabilities. The ranking models by [9] and [17] for exotic betting similarly rely on fitting the distribution of running time. However, these models are race-specific and, in their first step, are based solely on the estimated winning probabilities. Notably, they do not account for observed running times when solving for distribution parameters. Hence, our approach provides a significant logical advancement in fitting distributions, offering accommodation of various distributions and their parameters, while also incorporating observations of running time. Our methodology is numerically efficient since we only conduct calibration by minimizing (2.6). Unlike race-specific models, our approach remains race-invariant, enabling us to efficiently compute probabilities of outcomes.

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