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PARAMETRIC ANALYSIS OF OBJECTIVE FUNCTION COEFFICIENTS AND RIGHT-HAND-SIDE PARAMETERS OF LINEAR PROGRAMMING MODELS ACROSS THE ENTIRE FEASIBLE RANGE

TAMÁS KOLTAI[1,*], IMRE DIMÉNY[2]

[1]*Department of Management and Business Economics, Budapest University of Technology and Economics, Budapest, Hungary*

[2]*QSMS Research Group, Budapest University of Technology and Economics, Budapest, Hungary*

Abstract. Management decisions today can be supported by a large amount of data. To enable the effective use of the data, proper mathematical models are required, which can help one explore patterns that are useful for decision makers. If linear programming (LP) and related sensitivity analysis take advantage of increased computational power and the extended possibilities of informatics, then LP models might usefully serve as tools for data analytic. This paper demonstrates how parametric analysis for the entire feasible region of a right-hand side parameter or an objective function coefficient can be performed. Parameterised LPs are defined for the calculations, and techniques for speeding up the calculations are recommended. The proposed method is implemented in an AIMMS environment and illustrated with a production planning problem. The required computation time for the calculation is also analysed with the help of several size benchmark LP models. The extended LP sensitivity information presented in this paper clarifies the consequences of parameter changes and may lead to better management decisions whenever scarce resources must be allocated to alternatives and LP models are applied.

Keywords. Linear programming; Management decision; Parametric analyses; Production planning problem; Sensitivity analysis.

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1. INTRODUCTION

Business analytics (BA) helps managers to discover trends, patterns, and relationships by means of processing a large amount of collected data. However, mathematical models are required to explore and interpret information, relationships and patterns hidden in datasets. Proper mathematical models may lead to better management decisions and deliver a competitive edge, according to Davenport [1]. A problem that managers often face is how to allocate scarce resources to different operational possibilities, and many of the related problems can be formulated as linear programming (LP) models [2]. Practical applications of LP models date back to 1952 [3] and have been widely used since then [4, 5, 6, 7]. Recent applications of LP models to support management decision making include, for example, inventory routing problems under

*Corresponding author.
E-mail address: koltai.tamas@gtk.bme.hu (T. Koltai) dimeny.imre@gtk.bme.hu (I. Dimény).
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uncertainty [8], reservoir management [9] or optimizing the integration of hydroelectric and renewable energy sources [10].

Over the past decades, the computation time to solve an LP model has dramatically decreased due to the development of optimisation software [11] and algorithms to solve LP problems [12]. Today even large-scale practical problems can be solved easily.

Determining the optimal allocation of limited resources is just one step within the decision-making process. In practical cases, some values of the model parameters like transportation costs, material costs, or order volumes, capacities, etc. are based on approximations, expectations, forecasts and are thus subject to uncertainty [13]. Should there be deviation from the originally assumed value of some parameters, the optimality of the solution might be lost, or the solution could become unfeasible. Sensitivity analyses provide information about the effect of such changes.

LP sensitivity analysis is a post-optimisation process, and no insight is required about the nature of the uncertainty to perform it. Multiple approaches to LP sensitivity analysis exist, like ordinary sensitivity analysis [14], the tolerance approach [15], the global approach [16], and the global tolerance approach [17]. Recent advances of LP sensitivity analysis include a new computationally efficient geometric approach developed by Kaci and Radjef [18] and the global sensitivity analysis using a statistical tolerance approach developed by Curry et al. [19]. The application of LP sensitivity analysis is present in diverse industries. For example, Abbas and Ghayyib [20] demonstrate the use of sensitivity analysis in the oil transport sector, where changing parameters in LP models directly influence optimal solutions. Mollaeivaneghi et al. [21] apply parametric optimization in energy systems and demonstrate how the uncertainty of fuel costs impacts decision-making.

Many of the available tools to solve LP problems implement a simplex method and provide sensitivity information pertaining to the optimal solution. LP sensitivity analysis yields validity ranges for the objective value coefficients (OFC) and right-hand-side parameters (RHS). Even if a non-simplex-based algorithm is used, with some extra modelling effort the OFC and RHS sensitivity information can easily be obtained [22].

There are cases when the resulting sensitivity ranges are too tight to give the decision-maker confidence, and information about a wider range is required. Three similar LP problems presented in Dimény and Koltai [23] illustrate what can happen when sensitivity ranges are too tight. The problems presented have identical optimum, shadow price, and validity ranges, but the effects of increasing one of the RHS parameters beyond its validity interval are different: the problem becomes unfeasible; the slope of the objective value function remains almost unchanged; and the slope of the objective value function changes significantly.

In other cases, when the problem is degenerate and the optimal solution of the primal problem or the dual problem is not unique, the resulting sensitivity information can be misleading. The incorrect interpretation of sensitivity information may result in unfavourable management decisions (see, e.g., [24, 25]).

The objective of this paper is to complete traditional LP results with parametric analysis extended for the whole feasible range of any RHS and OFC parameter. The calculation framework and all the necessary LP models are defined, and techniques to speed up the calculation are recommended. The implementation of the calculation in AIMMS environment shows that these results can easily be obtained in practice. Test run on several models are also provided and the computational time of several size problems are provided and analysed.

The paper is structured as follows. Section 2 shows why sensitivity results for management decision-making is difficult to obtain in case of degenerate LP solutions, and how this problem is discussed in the literature. In Section 3, the analysis of the objective value function is recommended to cope with the problem described in Section 2, and a methodology is presented for mapping this function across the whole feasible range of the parameter examined. Section 4 presents a practical implementation for determining the objective value function and some examples illustrate the potential uses of this information. In Section 6, the computational performance of the proposed method is analysed. Finally, the conclusions of the presented research are summarized in Section 7.

2. Basic Concepts of LP Sensitivity Analyses under Degeneracy

All the notations used in this paper are listed in Table 1. Let us consider the max $\left(\mathbf{c}^T\mathbf{x}\right)$, $\mathbf{A}\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq 0$ form of a LP [26] where the elements of the $\mathbf{c}$ OFC vector are $c_1, c_2, \ldots, c_I$ and the elements of the $\mathbf{b}$ RHS parameters vector are $b_1, b_2, \ldots, b_J$.

Table 2 summarizes the basic LP problems used in the paper. The first row of Table 2 contains three LP problems. LP1 is the standard form of a primal linear programming problem, LP2 is a perturbed primal problem, where δ can take positive and negative values, and LP3 is the standard form of the dual linear programming problem. The optimal solution of the dual problem defines the marginal change of the objective function when a RHS parameter changes. The shadow price of a constraint is the change of the objective function when the RHS parameter of the constraint is increased with 1 unit.

LP sensitivity analysis provides an insight into how the optimal solution is altered when some parameters of the model are modified and defines validity ranges of the primal and dual optimum when OFC or RHS parameters of the model are changed. Within the validity range of an OFC parameter the primal optimal solution will be the same. Within the validity range of a RHS parameter the dual optimal solution will be the same.

Problems and possible solutions pertaining to the managerial interpretation of LP sensitivity analysis have an extensive literature. Evans and Baker [24] provided examples to demonstrate the possible effects of incorrect sensitivity information interpretation. Aucamp and Steinberg [27] showed that shadow prices are not necessarily equal to the dual variables except when the primal problem is non-degenerate. They provide an alternative definition for the shadow price. Akgül [28] introduced the positive and the negative shadow prices to differentiate the effect of the increase and the decrease of a parameter value. Gal [29] conducted an extensive survey on the managerial interpretation of shadow prices. Many other papers demonstrate that when sensitivity analysis results are misinterpreted, the related management decisions result in adverse financial results and/or improper operations (see, e.g., [25, 30]). The contradiction between the incorrect managerial decisions and the mathematically correct sensitivity information was resolved by differentiating between the mathematical and the managerial interpretation of sensitivity information by Koltai and Terlaky [31].

Sensitivity information can be classified in three types for which Hadigheh and Terlaky [32] gave descriptive names:

Type I (Basis Invariancy) determines those values of some RHS or OFC parameters for which a given optimal basis remains optimal. This is the traditional way of understanding sensitivity analysis.

Type II (Support Set Invariancy) sensitivity determines invariant support set ranges of some parameters so that variables with a zero value in the given optimal solution remain zero and variables with a positive value remains positive in the optimal solution of the perturbed problem.

Type III sensitivity (Optimal Partition Invariancy) returns ranges of some parameters for which the set of always-active constraints with respect to the primal optimal face and the dual optimal face is invariant.

Notation	Description
$\mathbf{A}$	Coefficient matrix of the LP problem
$\mathbf{b}$	Right-hand side parameters vector with elements b_j $(i = 1,\dots,J)$
$\mathbf{c}$	Objective function coefficient vector with elements c_i $(i = 1,\dots,I)$
$\mathbf{x}$	Decision variable vector of the primal problem with elements x_i $(i = 1,\dots,I)$
$\mathbf{x}^*$	Optimal solution of the primal problem with elements x_i^* $(i = 1,\dots,I)$
$\mathbf{y}$	Decision variable vector of the dual problem with elements y_j $(j = 1,\dots,J)$
$\mathbf{y}^*$	Optimal solution of the dual problem with elements y_j^* $(j = 1,\dots,J)$
OF^*	Optimal value of the objective function
$\mathbf{e}_i$	Unit vector with I elements, with $e_j = 1$ and $e_k = 0$ for all $k \neq j$
$\mathbf{e}_j$	Unit vector with J elements, with $e_i = 1$ and $e_k = 0$ for all $k \neq i$
δ	Perturbation parameter of an RHS parameter when Type III ranges are calculated
β	Decision variable when calculating the maximum feasible increase/decrease of an RHS parameter
β_j^+	Maximum feasible increase pertaining to the RHS parameter of constraint j
β_j^-	Maximum feasible decrease pertaining to the RHS parameter of constraint j
ξ_j	Change of RHS parameter b_i
λ	Binary direction parameter: 1 means increase, while -1 means decrease
$LP(\upsilon \leftarrow \nu)$	LP problem resulting from setting the value of parameter λ to ν
$OF^*(\upsilon \leftarrow \nu)$	Optimal value of the $LP(\lambda \leftarrow \nu)$ problem
I_s^k	Start point of the linearity interval k
I_e^k	End point of the linearity interval k
I_{rate}^k	Rate of change of the objective function in the linearity interval k
b_j^k	Value of b_j when calculating step k linearity interval
c_i^k	Value of c_i when calculating step k linearity interval
$SP_j^+\left(b_j^k\right)$	Right shadow prices of the modified $LP(b_j \leftarrow b_j^k)$ problem
$SP_j^-(b_j^k)$	Left shadow prices of the modified $LP(b_j \leftarrow b_j^k)$ problem
$\xi_j^+\left(b_j^k\right)$	Feasible increase of b_j^k when calculating Type III ranges in step k
$\xi_j^-(b_j^k)$	Feasible decrease of b_j^k when calculating Type III ranges in step k
$\gamma_i^+\left(c'\right)$	Maximal increase allowed for the c' objective function coefficient of variable i to remain within the Type III invariancy interval for the modified $LP\left(c_i \leftarrow c'\right)$ problem.
$\gamma_i^-\left(c'\right)$	Maximal decrease allowed for the c' objective function coefficient of variable i to remain within the Type III invariancy interval for the modified $LP\left(c_i \leftarrow c'\right)$ problem.
p_i	Rolled steel product i $(i = 1,2,3)$
m_j	Production line j $(j = 1,2,3)$
$x_{p_i m_j}$	Decision variable: produced quantity of product p_i on production line m_j

TABLE 1. Notations

Primal LP Problem (LP1)	**Perturbed Primal LP Problem (LP2)**	**Dual LP Problem (LP3)**
$Ax \leq b$	$Ax \leq b + \delta e_j$	$A^T y \geq c$
$x \geq 0$	$x \geq 0$	$y \geq 0$
$\max(c^T x)$	$\max(c^T x)$	$\min(b^T y)$
LP to calculate max. feasible change of RHS parameter (LP4)	**LP to calculate Type III intervals pertaining to RHS Parameter (LP5)**	**LP to calculate max. feasible Change of OFC Parameter (LP6)**
$Ax \leq b + \lambda \beta e_i$	$Ax \leq b + \delta e_j + \lambda \xi_j e_j$	$A^T y \geq c + \lambda \gamma_i e_i$
$\beta \geq 0$	$c^T x = OF^* + \lambda \xi_j y_i^*$	$b^T y = OF^* + \lambda \gamma_i x_i^*$
$x \geq 0$	$\xi_j \geq 0$	$\gamma_i \geq 0$
$\max(\beta)$	$\max(\xi_j)$	$\max(\gamma_i)$

TABLE 2. Summary of LP models used

When solving non-degenerate cases, the three types of sensitivities ranges are identical. However, in degenerate cases, LP solvers could provide different sensitivity information, depending on the optimal basis found. Table 3 contains an example of an LP problem to illustrate the different sensitivity ranges.

The problem in Table 3 is degenerate, and multiple optimal solutions exist. The optimal basic solution found by CPLEX was $P_1(1,1,0)$, variables x_1 and x_2 are basic variables and constraints (1), (5), (6), and (7) are active.

$x_1 + x_2 + x_3 \leq 2$	(1)
$x_1 \leq 2$	(2)
$x_2 \leq 2$	(3)
$x_3 \leq 2$	(4)
$x_1 \geq 1$	(5)
$x_2 \geq 1$	(6)
$x_3 \geq 0$	(7)
$\max(x_1 + x_2 + x_3)$	

TABLE 3. Illustration LP problem (LP7)

Figure 1 shows the different types of sensitivity ranges pertaining to the RHS parameter of constraint 1 (b_1). The shadow price, that is the slope of the objective value function is 1.

The Type I range calculated by CPLEX is the [2,3] interval. The Type II range is the [2,4] interval and can be obtained intuitively. The maximum and minimum values of the RHS parameter must be determined by finding an optimal solution with $x_1 > 0$, $x_2 > 0$ and $x_3 = 0$. Decreasing the b_1 parameter value makes the problem unfeasible, while for $b_1 > 4$ the value of x_3 needs to be strictly positive.

The type III range is the [2,6] interval and can also be calculated intuitively since the value of b_1 also determines the value of the objective value function for $b_1 \leq 6$.

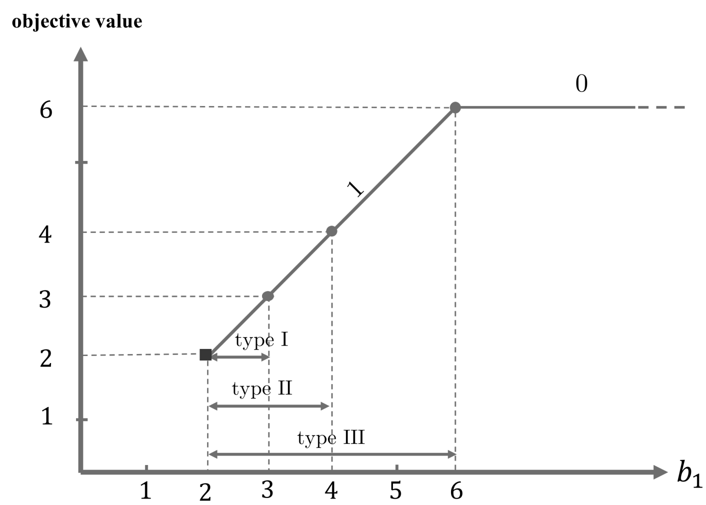

FIGURE 1. Objective value function pertaining to the b_1 RHS parameter of the $LP7$ problem

For this example, Cplex 20 and Gurobi 9.1 returns the [2,3] interval, while XA 16 returns the [2,4] interval as the RHS sensitivity range. It can also be seen from this example that commercial solvers provide different type I sensitivity information when the model is degenerate depending on the optimal solution found.

The information provided by solvers is of great value in many cases but type II and type III sensitivity information are far more significant for managerial decisions.

Type II sensitivity information is particularly important in case of assignment problems [33] and transportation problems [34]. The invariant support set preserves the shipping pattern but allows the change of loads on the transportation routes [35].

Type III sensitivity is required when the cost of switching between active variables is less significant as is often the case in production planning problems.

Adler and Monteiro [36] demonstrated that the rate of change of the objective value function remains unchanged within the Type III range and is either the interval between two consecutive breakpoints of the related objective value function or consists of a breakpoint itself. Consequently, type III range of some parameters depends only on the problem data and, from a practical point of view can be calculated using additional LPs [37]. The intervals pertaining to type III ranges are sometimes referred to as the linearity intervals, and the objective value function could be constructed by connecting the consecutive linearity intervals. A possible algorithm for this is presented by Adler and Monteiro [36] for the RHS parameters using the traditional way for calculating linearity intervals.

The additional LPs required to calculate Type III sensitivity intervals are presented in the second row of Table 2. LP5 is used to calculate type III intervals pertaining to a RHS parameter, and LP6 is used to calculate type III interval pertaining to an OFC parameter. Depending on the value of λ the result of LP5, LP6 will calculate the maximal increase (λ=1) or the maximal decrease (λ=-1). The optimal solution of the primal problem defines the optimal allocation of the limited resources.

Analysing the effect of changes of OFC and RHS parameters for the whole feasible/bounded region is now possible due to the easy accessibility of high-capacity computers and the development of computational methods. In this way, managers may obtain a comprehensive picture about the effects of the change of some critical parameters on the optimal objective value. However, algorithms are needed to perform the required calculations systematically and efficiently, and a proper user interface is also needed to facilitate management decision making based on these new results.

3. Algorithm to Calculate the Objective Value Function Pertaining to a RHS Parameter

Changing the RHS parameter in one of the directions will decrease the feasible region until the model becomes unfeasible. Therefore, as a first step, the maximal feasible increase and decrease pertaining to an RHS parameter must be calculated. One of these two values will be finite while the other will be infinite, creating this way a final infinite linearity interval.

For each constraint j, the maximal increases can be determined by solving LP4 problem presented in Table 2 with the value of λ=1.The maximal decrease can be calculated using the same model with λ =-1. The difference between these additional LPs and the original LP consists in using constraint j as an objective value function instead of being a constraint.

The linearity intervals defined by the type III sensitivity ranges can be calculated using the LP6 model presented in Table 2. These intervals can be used to create the objective value function pertaining to a RHS parameter by connecting the consecutive linearity intervals ($I^k, k = 1..K$).

Under conditions of degeneracy, the effect of the increase and decrease of the RHS elements can be different. The LP5 model of Table 2 with a properly set perturbation (δ) can be used to calculate the linearity intervals pertaining to the increase (δ>0) and decrease (δ<0) of any RHS parameter. An improperly set perturbation size could lead to numerical errors or an erroneous validity range [37].

Figure 2 shows the objective value function pertaining to an RHS parameter. If the perturbation (δ) is larger than the validity range pertaining to the shadow price at the original value of the RHS parameter (see b_2 in the figure), the calculated Type III interval is erroneous. In this case, the original RHS parameter value (b) is outside the validity range of the shadow price pertaining to the perturbed primal LP problem.

In any practical situation, decision makers can set the perturbation size by defining the minimum level of change that is practically acceptable, thereby avoiding the problem of overly small or overly large perturbation. However, to create a general solution that calculates Type III sensitivity information the process of setting the perturbation must be automatic. For that, first the value of δ must be set such way to prevent numerical errors. Next, to check whether the problem of setting an excessively large perturbation value exists, the type I validity range of the perturbed LP5 must be calculated. If the original value of the RHS parameter is inside the type I interval of the perturbed LP problem, then no further steps are required. If the original value of the RHS parameter is outside the type I interval a new smaller perturbation size must be calculated. These steps must be repeated until the original RHS parameter value is inside the type I validity interval of the shadow price of the perturbed LP [37].

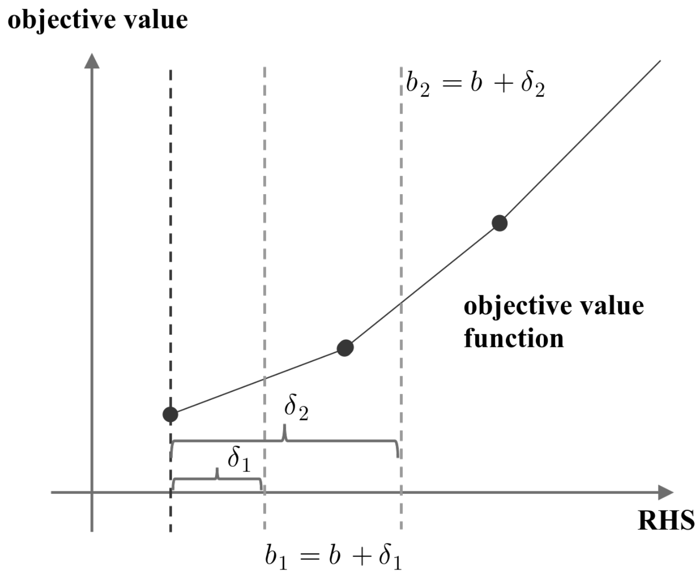

FIGURE 2. Objective value function pertaining to the b right-hand-side element

The $\delta_2 = (\delta_1 - \xi_1)/2$ formula can be used to calculate a new perturbation size, where δ_2 is the size of the new perturbation, δ_1 is the size of the previous perturbation and ξ_1 is the difference between the original perturbation and the edge of the left validity interval of the perturbed dual LP.

Figure 3 presents the situation when the original value of the RHS parameter (b_1) is outside the type I interval of the perturbed problem.

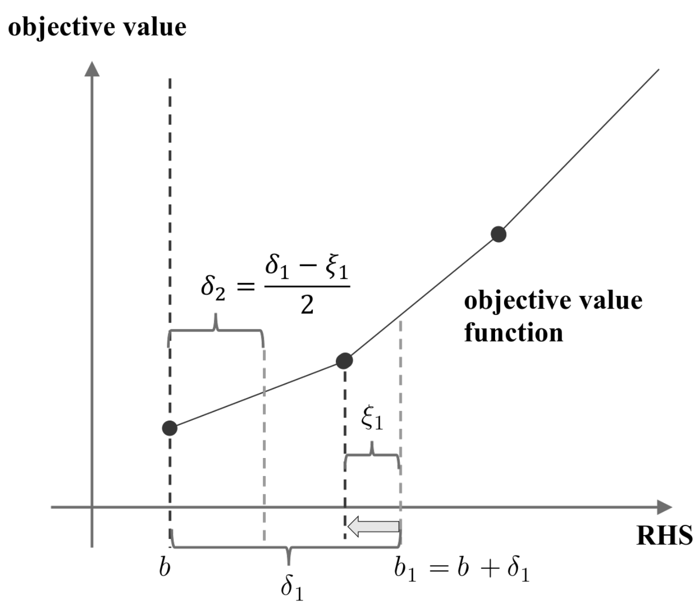

FIGURE 3. Calculation of proper perturbation size

When performing a parametric analysis of an LP problem, the calculation time also needs to be considered. The calculation time can be significantly decreased by performing the calculation only for relevant parameters, since many of the constraints and parameters of a real-life LP problem are just technical. Calculation time can be further decreased by taking advantage of initialising solver runs. LP5 model is used to calculate Type III sensitivity ranges of RHS parameters. Using solution $\mathbf{x}^*$ of the LP2 problem, the $(\mathbf{x}^*, 0)$ feasible solution of the LP5 model can be constructed [37]. When the LP5 solver run was initialised with the $(\mathbf{x}^*, 0)$ initial solution, an average 20 % decrease in the computational time was seen during the computational experiments presented in section 7 compared with running without the warm start.

To calculate the consecutive linearity intervals, the value of the objective function at the start (I_s^k) and end (I_e^k) points and the constant rate of change of the objective value function within the interval (I_{rate}^k) needs to be calculated for each interval I^k. When the original LP is feasible, the I^kintervals can be calculated starting from the original RHS parameter separately for both increasing and decreasing directions, until the maximum feasible change is reached.

Collecting increasing RHS intervals	**Collecting decreasing RHS intervals**
$k := 0$	$k := 0$
calculate β_j^+	calculate β_j^-
repeat	**repeat**
$b_j^k := \begin{cases} b_j, & k = 0 \\ I_e^{k-1}, & k \geq 1 \end{cases}$	$b_j^k := \begin{cases} b_j, & k = 0 \\ I_s^{k-1}, & k \geq 1 \end{cases}$
solve $LP(b_j \leftarrow b_j^k)$	solve $LP(b_j \leftarrow b_j^k)$
calculate $\xi_j^+(b_j^k)$	calculate $\xi_j^-(b_j^k)$
$I_s^k := b_j^k$	$I_e^k := b_j^k$
$I_e^k := I_s^k + \xi_j^+(b_j^k)$	$I_s^k := I_e^k - \xi_j^-(b_j^k)$
$I_{rate}^k := SP_j^+(b_j^k)$	$I_{rate}^k := SP_j^-(b_j^k)$
until $(I_e^k = \beta_j^+ \text{ or } \xi_j^+(b_j^k) = \infty)$	**until** $(I_e^k = \beta_j^- \text{ or } \xi_j^-(b_j^k) = \infty)$

TABLE 4. Algorithm for calculating consecutive RHS intervals

The pseudo-code for calculating the subsequent intervals is presented in Table 4. The first column presents the algorithm for collecting increasing RHS intervals, while the second column presents the algorithm for collecting decreasing intervals.

In the initial step the maximal feasible modification of the RHS parameter is calculated using LP5 for increasing intervals and LP6 for decreasing intervals.

The following steps are repeated until the calculated maximal feasible modification is reached, or the maximum increase/decrease is infinite:

- solve the modified $LP(b_j \leftarrow b_j^k)$ problem,
- calculate Type III ranges $\xi_j^+\left(b_j^k\right)$ and $\xi_j^-\left(b_j^k\right)$ using LP3,

- collect interval data: $\left(b_j^k, I_s^k + \xi_j^+\left(b_j^k\right)\right)$ and SP^+ for increasing intervals, $\left(I_e^k - \xi_j^-\left(b_j^k\right), b_j^k\right)$ and SP^- for decreasing intervals,
- set $b_j^{k+1} = I_e^{k-1}$ for increasing intervals and $b_j^{k+1} = I_s^{k-1}$ for decreasing intervals.

4. Algorithm for Calculating the Objective Value Function Pertaining to an OFC Parameter

The values of the OFC parameters do not influence the feasibility of the problem. Consequently, the modification of an OFC parameter does not influence the feasibility of the LP problem, either. However, after a certain increase or decrease in the OFC value the previously bounded LP problem may become unbounded. LP6 described in Table 2 can be used to calculate Type III sensitivity intervals where γ_i are the decision variables used to calculate the maximal decrease/increase allowed for the c_i OFC parameter.

Type III sensitivity analysis provides information about the invariance of the rate of change of the objective value function. The pseudo-code for calculating subsequent OFC intervals, is presented in Table 5.

Algorithm to collect increasing OFC intervals	**Algorithm to collect decreasing OFC intervals**
$k := 0$	$k := 0$
repeat	**repeat**
$c_i^k := \begin{cases} c_i, & k = 0 \\ I_e^{k-1}, & k \geq 1 \end{cases}$	$c_i^k := \begin{cases} c_i, & k = 0 \\ I_s^{k-1}, & k \geq 1 \end{cases}$
solve $LP(c_i \leftarrow c_i^k)$	solve $LP(c_i \leftarrow c_i^k)$
calculate $\gamma_i^+(c_i^k)$	calculate $\gamma_i^-(c_i^k)$
$I_s^k := c_i^k$	$I_s^k := I_e^k - \gamma_i^-(c_i^k)$
$I_e^k := I_s^k + \gamma_i^+(c_i^k)$	$I_e^k := c_i^k$
$I_{rate}^k := x_i$	$I_{rate}^k := x_i$
until $(\gamma_i^k = \infty)$	**until** $(\gamma_i^k = \infty)$

TABLE 5. Algorithm for calculating consecutive OFC intervals

The first column presents the algorithm for collecting increasing OFC intervals, while the second column presents the algorithm for collecting decreasing intervals.

When an OFC parameter (c_i) changes, the slope of the objective value function is equal to the value of the variable i in the optimal solution (x_i) of the modified $LP(c_i \leftarrow c_i^k)$ problem where c_i^k is the OFC parameter pertaining to variable i when calculating interval k.

When the maximum increase/decrease is finite, the following steps have to be followed:

- solve the modified $LP(c_i \leftarrow c_i^k)$ problem,
- calculate type III range for the required direction ($\gamma_i^+\left(c_i^k\right)$ and $\gamma_i^-\left(c_i^k\right)$ respectively),
- collect interval data and optimal value:

$\left(c_i^k, I_s^k + \gamma_i^+ \left(c_i^k\right)\right)$ for increasing intervals and $\left(I_e^k - \gamma_i^- \left(c_i^k\right), c_i^k\right)$ for decreasing intervals and $I_{rate}^k = x_i$,
- set $c_i^{k+1} = I_e^k$ for increasing intervals and $c_i^{k+1} = I_s^k$ for decreasing intervals.

In a manner analogous to the calculation of RHS sensitivity intervals, the computation time pertaining to the calculation of the OFC parameters can be decreased by taking advantage of the warm start functionality of the solver. The vector $\mathbf{y}' = (\mathbf{y}^*, 0)$ is a feasible solution for the additional LP problems pertaining to the calculation of the Type III sensitivity ranges of OFC parameters (4), where $\mathbf{y}^*$ is the optimal solution of the dual LP problem LP3. By instructing the solver to initialise the solver run using this information, the calculation of the additional LP problems can be accelerated.

5. Practical Implementation Objective Value Function Mapping using AIMMS

For the application of the suggested parametric analysis and to visualise the parametric objective value function of the LP models defined in the previous sections, the algorithm to connect the solver sessions needs to be implemented and a user interface needs to be created to enable the user to input information and visualise the results. The required computational platform will be implemented with AIMMS. AIMMS is a rapid model building and deployment platform that fulfils all three requirements with a solid mathematical modelling environment, a wide range of available solvers and easy-to-use user interface editor. AIMMS is often used for solving commercial optimisation problems in various industries [38].

For the practical implementation, AIMMS version 4.42 was used to create the required mathematical models, implement the algorithms, and create the necessary user interface. CPLEX version 12.7.1 was applied to solve the generated LP models and obtain type I sensitivity information. The warm start functionality of CPLEX can be instructed to search for an adequate initial solution defined in Section 3 and 4. Using this option, the calculation time of type III sensitivity ranges can be significantly decreased.

AIMMS own structural language helps in creation the required procedures to calculate the type III ranges for all the parameters.

A part of the constraints and variables used in an LP model are auxiliary and the objective value function pertaining to the RHS parameter of these constraint must be ignored in the calculation. Similarly, the OFC pertaining to an auxiliary variable doesn't contains relevant managerial information. The related sensitivity calculations can also be spared. The separation between the parameters which needs detailed investigation cannot be done automatically. With the use of the built-in user interface editor, a user-friendly platform that can help the decision-makers in selecting parameters of the model that have practical relevance.

The following parameterized models are implemented in AIMMS:

- general parameterised linear program to solve the $LP(c_i \leftarrow c_i^k)$ and the $LP(b_j \leftarrow b_j^k)$ problems,
- modified parameterised linear program to calculate the maximal feasible increase/decrease of the RHS parameters,
- parameterised linear program to calculate the type III ranges of the RHS parameters, and OFC parameters.

The algorithms presented in Table 4 and 5 are implemented to collect all type III intervals. The algorithm determines the data pertaining to the intervals, which then are visualised in various AIMMS pages using both tables and charts with the help of the build-in user interface editor. One page is created for defining the input data and the solution of the LP problem. Two pages are created to present the RHS and OFC interval data in table and chart format.

6. Illustration of the Calculation and Visualisation of the Results

For illustrative purposes, a simple LP model taken from Schrage [39] is presented. In this problem the weekly production plan of rolled steel manufacturing must be determined. In the example, rolled steel is produced in three different thicknesses (p_1, p_2, p_3) using three production lines (m_1, m_2, m_3) with the following restrictions:

- product p_1 can be produced only on production line m_1,
- product p_2 can be produced on any production line,
- product p_3 can be produced only on production lines m_2 *and* m_3,
- production lines have a working capacity set to 35 hours/week,
- the three production lines also differ in speed and production costs,
- the contracted demand must be satisfied for each product,
- the conveyor system which transports the rolls has a capacity 600 tons for the given period.

Let $x_{p_i m_j}$ denote the produced quantity of product p_i on production line m_j.

The objective is to maximize revenue. The objective function coefficients are calculated based on the contribution margin of the product and the speed and cost of the production line.

Variable	$x_{p_1m_1}$	$x_{p_2m_1}$	$x_{p_2m_2}$	$x_{p_2m_3}$	$x_{p_3m_2}$	$x_{p_3m_3}$	**RHS**	**notation**
Constraint								
Production ($m1$)	0.111	0.111					≤ 35	b_1
Production ($m2$)			0.1667				≤ 35	b_2
Production ($m3$)				0.222	0.222		≤ 35	b_3
Conveyor	1	1	1	1	1	1	≤ 600	b_4
Demand ($p1$)	1						≥ 218	b_5
Demand ($p2$)		1					≥ 114	b_6
Demand ($p3$)					1	1	≥ 111	b_7
OFC	15.889	17.889	16.5	15.222	17.5	16.222		
Notation	c_1	c_2	c_3	c_4	c_5	c_6		

TABLE 6. LP formulation of the rolled steel production problem

Table 6 summarizes the parameters and the variables of the LP problem. The LP formulated to solve the steel production problem has multiple optimal solutions, hence the problem is degenerate and type I sensitivity ranges calculated by LP solvers depend on what solution is found. At the optimal solution the maximum revenue is 10074.47 USD and can be achieved with multiple production quantities. Table 8 contains two optimal solutions, but there are many optimums. Since p_2 can be produced alternatively on m_2 *and* m_3 with the same cost consequences any part of the production quantity of $x_{p_2m_2}$ can be transferred from m_2 to m_3, but in this case the other production quantities of m_2 *and* m_3 must be modified.

Two extreme solutions pertaining to $x_{p_2m_2} = 0$ and to $x_{p_2m_2} = 17$ are summarised in Table 7.

Variable	$x_{p_1m_1}$	$x_{p_2m_1}$	$x_{p_2m_2}$	$x_{p_2m_3}$	$x_{p_3m_2}$	$x_{p_3m_3}$
Solution 1	218	97	17	0	193	75
Solution 2	218	97	0	17	210	58

TABLE 7. Possible optimal solutions of the steel production problem

Since the problem has several optimal solutions, the problem is dual degenerate. However, the difficulties pertaining to degeneracy are not relevant, since the shadow price is determined for the whole feasible range.

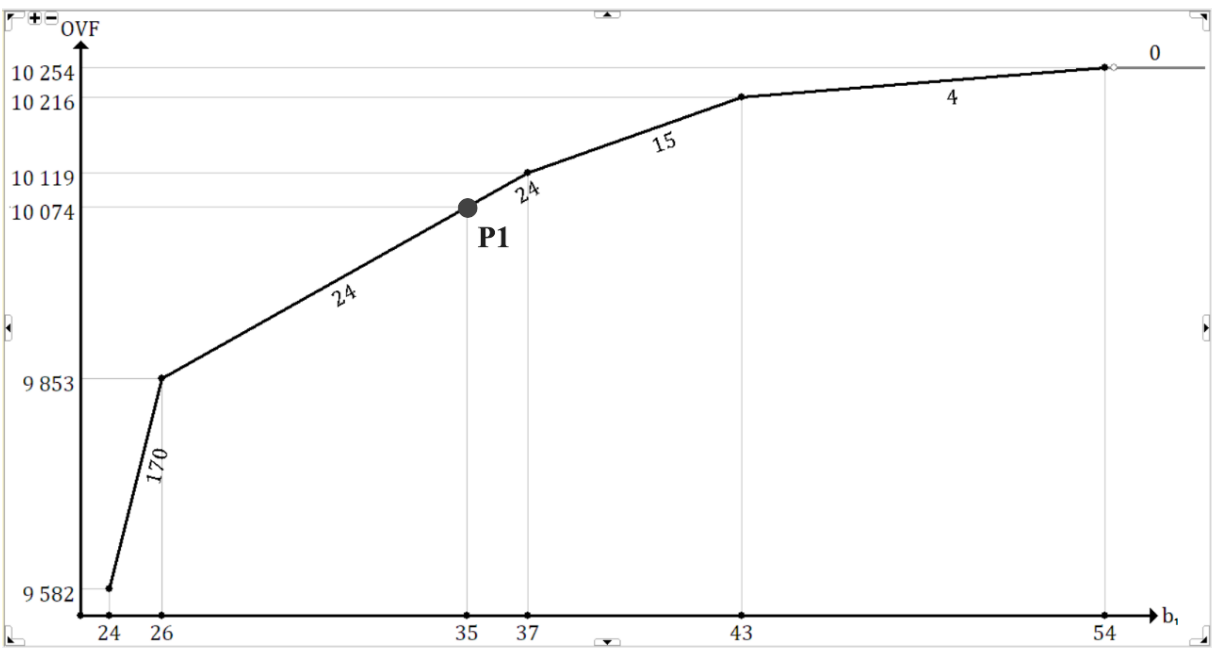

FIGURE 4. Objective value function pertaining to the production capacity of production line m_1

Figure 4 contains the objective value function pertaining to the production capacity of production line m_1. Point P1 marks the initial value of the related RHS parameter. The connected lines show the linearity intervals of the objective value function. The figure indicates that if capacity drops below 24 tons, the problem becomes unfeasible and increasing the available capacity from 54 tons upward has no effect on the value of the objective function.

Figure 5 contains the objective value function pertaining to the RHS parameter of the capacity of the conveyor system. Besides the shadow price, the chart also shows that if capacity drops below 443 tons, the LP problem becomes unfeasible. This is the minimum level of capacity required to satisfy the orders. The point where further increase in the capacity has no more effect on the objective value is also plotted.

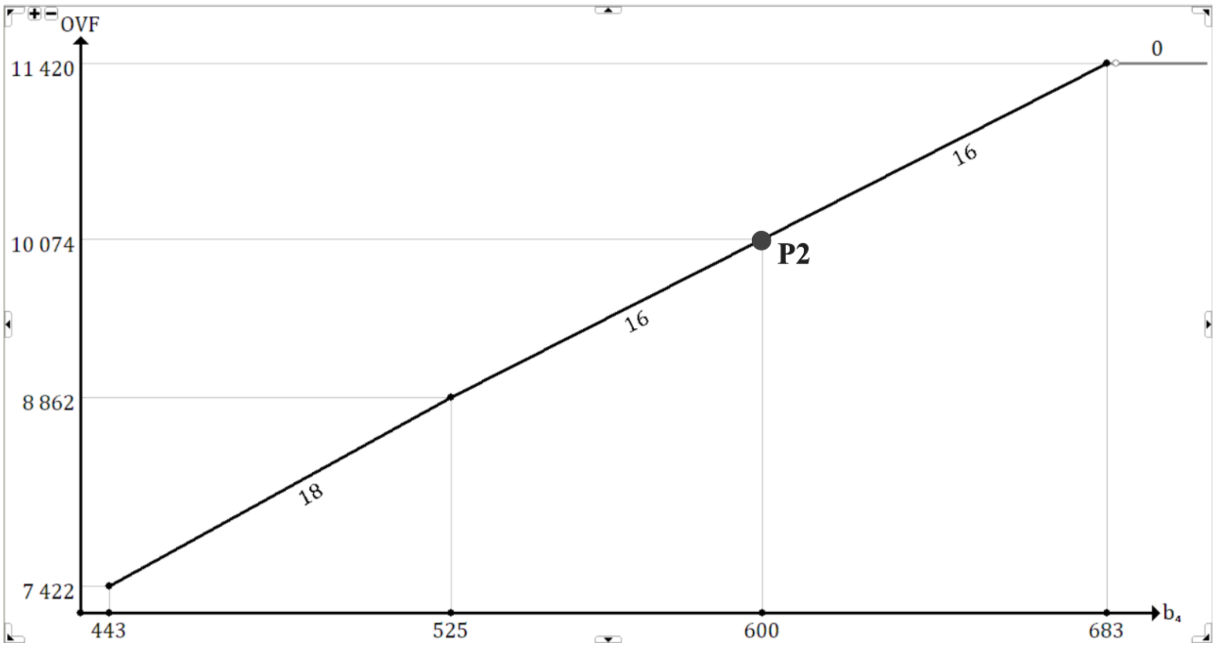

FIGURE 5. Objective value function pertaining to the capacity of the conveyor system

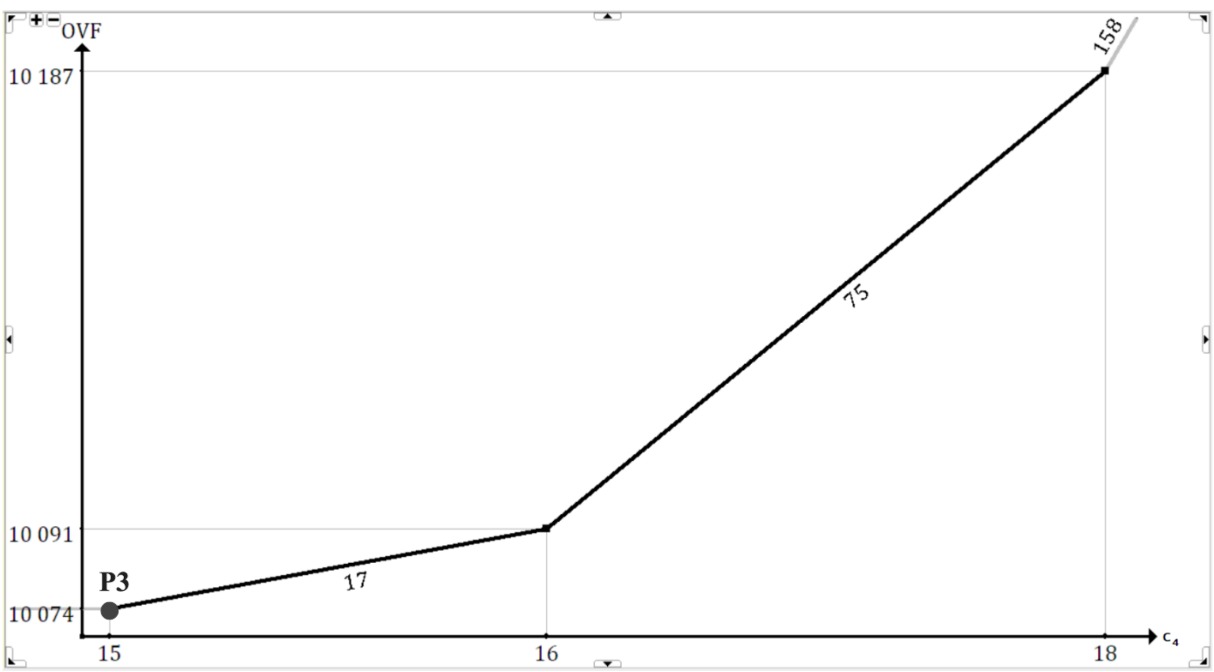

FIGURE 6. Objective value function pertaining to the $x_{p_2 m_3}$ OFC parameter

Figure 6 contains the objective value function pertaining to the $x_{p_2m_3}$OFC parameter. The chart shows how the effect of a possible increase in the contribution margin of product p_2 would influence the objective value.

The three objective value functions presented in Figures 4, 5 and 6 are different topologically as well. The change of the slope of the objective value when leaving the closest linearity interval to the optimal solution may differ and, different changes of the slope require different managerial decisions. The slope of the objective value pertaining to the capacity of the conveyor system is almost the same when it has a non-zero shadow price. Managers may expect identical changes of profit when the capacity of the conveyor system changes for some reason. However, the slope of the objective value function pertaining to the $x_{p_2m_3}$ OFC parameter changes significantly from one linearity interval to another. Large changes influence the profit more significantly, which is something that must be exploited if the change is positive or avoided if the change is negative.

In practice, this type of OVF chart may help managers to evaluate the expected changes of capacities, prices, demand, etc. and may lead to better management decisions.

7. Computational analysis

The computational performance of the presented method for mapping the objective value function in the whole feasible range of a parameter was examined using the Netlib suite of linear optimisation problems [40]. This set of LP problems is widely used to compare computational results [41]. The Netlib library [42] contains various size LP problems, ranging from 32 variables and 27 constraints up to 15695 variables and 16675 constraints.

Dataset name	**Variables**	**Constraints**	**LP solve time (s)**	**Avg. time per RHS (s)**	**Avg. time per OFC (s)**
KB2	41	52	0.27	2.78	2.84
TUFF	587	364	0.21	7.37	4.09
DEGEN3	1818	1503	0.36	0.84	3.38
BNL1	1175	643	0.21	22.33	37.32
WOOD1P	2594	244	0.36	1.55	6.55
BLEND	83	74	0.08	0.96	1.41
QAP8	1632	912	0.31	0.51	85.77
AFIRO	32	27	0.07	0.49	0.30
AGG2	302	516	0.27	0.79	1.16
BEACONF	262	173	0.22	0.37	0.33
BRANDY	249	220	0.27	8.42	7.83
PILOT	3652	1441	0.69	2.49	4.76
MAROS_R7	9408	3136	1.05	11.65	5.89

TABLE 8. Summary of the computational results

Table 8 contains the name of the dataset, the number of variables and constraints, the computation time to obtain the optimal solution of the original LP problem and finally the average computation time of the OVF pertaining to a RHS and an OFC parameter in seconds.

Computational time was evaluated using a laptop computer with a 1.8 GHz Intel i7 processor and 16GB of RAM.

The results show that any of the problems can be solved very quickly even on a standard laptop computer. The computational time required to obtain the optimal solution is less than one second in all but one case. When parametric analysis is performed for the whole feasible range of a single parameter, the computation time is larger than the time to get the optimal solution, but in most cases less than 10 seconds is required. Even in the case of the largest problem (MAROS_R7), the computation time is only 11.65 seconds.

The table also shows that computation time for generating the objective value function might be very different for similar size problems. This difference is a consequence of the model structure. The short time taken to obtain the objective value function in each of the cases shows that the objective value function for the whole feasible range of a parameter can be obtained easily in any real-life decision-making environment.

8. Conclusion

Sensitivity analysis provides information about the behaviour of an objective function when some critical parameters of the model change in the close neighbourhood of the original value. Parametric analysis extends sensitivity analysis for a wider interval. The change of the optimal value of the objective function is analysed further away from the original parameter value providing this way an overall view of the change of an important parameter.

In this paper, the implementation of parametric analysis of LP models in the whole feasible range of a parameter is presented. The required algorithms for efficient performance of the calculations are defined and the results are illustrated with help of a production planning sample problem. The input interface, the organisation of all the calculation and the visualisation of the results are implemented with the AIMMS mathematical programming environment. The information provided by the presented method generates all the possible optimal objective function values in the complete feasible range of any OFC and RHS parameters.

There are two main benefits of this information:

1. The problem of degeneracy, the calculation difficulties of left and right shadow prices and the misinterpretation of sensitivity analysis results in case of degeneracy are avoided by mapping the objective value function for all values.
2. Furthermore, by studying the patterns of the objective value function in a wider range, managers may recognise possibilities that are hidden when only parameter changes within a small interval are available. In this way, the presented results extend the scope of traditional sensitivity analysis provided by most of the commercially available LP solvers. Thanks to the provision of additional information about the parameter changes, the effect of the parameter change is known, and this is true not solely in the close neighbourhood of the original value, but also across the whole feasible region.

The presented method can be used to support OM decisions whenever a limited quantity of resources must be allocated to alternatives and a linear programming model can be used to describe the decision context. In practice, the solution of production planning, transportation logistics and capital allocation problems may benefit from the application of the suggested method. The created AIMMS platform presents the results in table format as well as graphically to facilitate the recognition of typical patterns of changes. Overall, this extended sensitivity information may provide a more comprehensive picture about the consequences of parameter changes for managers and may lead to better managerial decisions.

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