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AN INERTIAL EXTRAGRADIENT ALGORITHM WITH NON-MONOTONE STEP SIZES FOR VARIATIONAL INEQUALITY PROBLEMS AND FIXED POINT PROBLEMS OF NONEXPANSIVE SEMIGROUPS

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Abstract. In this paper, we propose an inertial extragradient algorithm with non-monotone step sizes for approximating a common solution of a variational inequality problem of pseudomonotone and uniformly continuous operators and a fixed point problem of nonexpansive semigroups in the setting of real Hilbert spaces. By utilizing a self-adaptive technique that produces non-monotonic step sizes, our method ensures strong convergence without requiring Armijo-type line searches. Finally, to assess the performance of the proposed algorithm, we provide some numerical results.

Keywords. Inertial extragradient algorithm; Fixed point; Nonexpansive semigroup; Variational inequality.

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1. INTRODUCTION

Let C be a nonempty, convex, and closed subset of a real Hilbert space H, and let $B : H \to H$ be a mapping. The variational inequality problem investigated in this paper is to find a point $x^* \in C$ such that

$$\langle x - x^*, B(x^*)\rangle \geq 0, \quad \forall x \in C. \tag{1.1}$$

The solution of (1.1) is denoted by $VI(C,B)$. Problem (1.1) is recognized for its wide real applications, such as control problems, signal recovery problems, economics, and so on, and it includes complementarity, equilibrium, saddle point problems, and so on. In the past decases, a variety of efficient algorithms were proposed to solve (1.1); see, e.g., [5, 9, 11, 14, 19, 35] and the references therein.

With the aim of achieving faster convergence of algorithms, Polyak [17] introduced the inertial extrapolation process $x_{n+1} = x_n + \beta_1(x_n - x_{n-1}) - \beta_2 B(x_n)$, $n \geq 0$, where β_1 and β_2 are two real numbers. In recent years, incorporating inertial factors to accelerate convergence has gained popularity in algorithm development; see, e.g., [1, 7, 8, 24, 25]. By combination of the inertial type method, the viscosity method and the modified Tseng extra-gradient method, Thong et al. [31] introduced a Tseng's extra-gradient method. However, its applicability is constrained by the control parameter's dependency on the Lipschitz constant of B. In an effort to address this limitation, recent research investigated new step size rules for solving variational

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inequality problems. For example, Tian and Tong [30] proposed an algorithm with self-adaptive step sizes for finding a solution of the variational inequality problem involving monotone operators and the fixed point problem of a quasi-nonexpansive mapping in real Hilbert spaces. A detailed description of their algorithm is given as follows:

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = P_C[w_n - \lambda_n B(w_n)], \\ T_n = \{x \in H : \langle w_n - \lambda_n B(w_n) - y_n, x - y_n\rangle \leq 0\}, \\ z_n = P_{T_n}[w_n - \lambda_n B(y_n)], \\ x_{n+1} = (1-\beta_n)w_n + \beta_n T(z_n), \end{cases}$$

where the stepsize λ_n is generated by

$$\lambda_{n+1} := \begin{cases} \min\{\frac{\mu\|w_n - u_n\|}{\|B(w_n) - B(u_n)\|}, \lambda_n\} & \text{if } B(w_n) \neq B(u_n), \\ \lambda_n & \text{otherwise.} \end{cases} \tag{1.2}$$

In 2024, Wang et al. [33] proposed a projection and contraction method with a double inertial extrapolation step and self-adaptive step sizes to solve variational inequalities with quasi-monotonicity in real Hilbert spaces. For a given $x_0, x_1 \in C, \lambda_1 > 0, \mu \in (0,1), \alpha, \beta, \theta_n \in (0,1)$. Compute

$$\begin{cases} z_n = x_n + \beta(x_n - x_{n-1}), \\ w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C[w_n - \lambda_n B(w_n)], \end{cases}$$

where

$$\lambda_{n+1} := \begin{cases} \min\{\frac{\mu\|w_n - y_n\|}{\|B(w_n) - B(y_n)\|}, \lambda_n\} & \text{if } B(w_n) \neq B(y_n), \\ \lambda_n & \text{otherwise.} \end{cases}$$

If $w_n = y_n = x_n$, then stop. Otherwise, compute $v_n = w_n - y_n - \lambda_n(B(w_n) - B(y_n))$ and $x_{n+1} = (1-\alpha)z_n + \alpha(w_n - \gamma d_n v_n)$, where

$$d_n := \begin{cases} \frac{\langle w_n - y_n, v_n\rangle}{\|v_n\|^2}, & \text{if } v_n \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The step size criterion (1.2) used in [30] and [33] generates a non-increasing sequence of steps. This approach, however, may require computationally expensive. Seeking to improve upon this inefficiency, researchers proposed a modified version of criterion (1.2); see, e.g., [13, 28, 29, 33]. Recently, Tan and Cho [29] proposed two extragradient iterative algorithms by using inertial technique to solve (1.1). The specific steps of their algorithm are outlined below: for a given $x_0, x_1 \in C, \chi \in (0,1), \theta > 0, \tau_1 > 0$ and a sequence $\{\tau_n\}$ satisfying $\sum_{n=1}^{\infty} \xi_n < \infty$, $\{\alpha_n\} \subset (0,1)$ satisfies the following conditions: $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Compute $w_n = x_n + \theta_n(x_n - x_{n-1})$ with θ_n such that

$$\theta_n := \begin{cases} \min\{\frac{\varepsilon_n}{\|x_n - x_{n-1}\|}, \theta\} & \text{if } x_n \neq x_{n-1}, \\ \theta & \text{otherwise.} \end{cases}$$

where ε_n is satisfying the condition $\lim_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = 0$. Compute $y_n = P_C[w_n - \tau_n B(w_n)]$ and $z_n = P_{T_n}[w_n - \tau_n B(y_n)]$, where $T_n = \{x \in H : \langle w_n - \tau_n B(w_n) - y_n, x - y_n\rangle \leq 0\}$. Compute $x_{n+1} =$

$(1-\alpha_n-\beta_n)w_n+\beta_n z_n$ and update the step size τ_{n+1} by

$$\tau_{n+1} := \begin{cases} \min\{\frac{\mu\|w_n-y_n\|}{\|B(w_n)-B(y_n)\|}, \tau_n+\xi_n\} & \text{if } B(w_n)\neq B(y_n), \\ \tau_n+\xi_n & \text{otherwise.} \end{cases}$$

Here, we also consider the fixed point problem for a mapping S. Due to the wide applications of fixed point problems, various schemes were extensively investigated; see, e.g., [4, 6, 20, 21, 23].

A family $\Gamma_a := \{S(t) : 0 \le t < \infty\}$ of mappings from C into itself is called a nonexpansive semigroup on C if it satisfies the following conditions:

(i) $S(0)x = x$ for all $x \in C$;
(ii) $S(s+t) = S(s)S(t)$ for all $s,t \ge 0$;
(iii) $\|S(t)x - S(t)y\| \le \|x-y\|$ for all $x,y \in C$ and $t \ge 0$;
(iv) for all $x \in C$ and $t \ge 0$, $s \mapsto S(t)x$ is continuous.

The set of all the common fixed points of a family Γ_a is denoted by $\mathrm{Fix}(\Gamma_a)$, *i.e.*,

$$\mathrm{Fix}(\Gamma_a) := \{x \in C : S(t)x = x, t \ge 0\}.$$

The fixed point problem for a nonexpansive semigroup Γ_a is:

$$\text{find} \quad x \in C \quad \text{such that} \quad x \in \mathrm{Fix}(\Gamma_a). \tag{1.3}$$

A nonexpansive semigroup Γ_a on C is said to be uniformly asymptotically regular (u.a.r) on C if, for all $h > 0$ and any bounded subset E of C, $\lim_{s\to\infty}\sup_{x\in E}\|S(h)(S(s)x) - S(s)x\| = 0$.

The fixed point problem of nonexpansive semigroups has attracted much attention in recent years. Authors have extensively investigated various iterative algorithms; see, e.g., [3, 10, 12]. In this paper, we propose an inertial extragradient algorithm with non-monotone step sizes for a common solution of variational inequality problem (1.1) with pseudomonotone and uniformly continuous operators and fixed point (1.3) of nonexpansive semigroups in the setting of real Hilbert spaces. Unlike methods that rely on Armijo-type line searches, our algorithm achieves greater efficiency with the aid of a self-adaptive step size technique that generates non-monotonic sequence of step sizes. Our approach achieves strong convergence results without requiring the u.a.r condition, which was often assumed in previous works. Our step size properly includes those in [13, 18, 28, 29, 31]. Some numerical results are reported to confirm the efficacy of the proposed algorithm.

2. Preliminaries

Recall that a mapping $S : C \to H$ is said to be monotone if $\langle Sx - Sy, x-y\rangle \ge 0$ for all $x,y \in C$; pseudomonotone if $\langle Sx, y-x\rangle \ge 0 \Rightarrow \langle Sy, y-x\rangle \ge 0$ for all $x,y \in C$; nonexpansive if $\|Sx-Sy\| \le \|x-y\|$ for all $x,y \in C$; contraction if there exists a constant $0 \le k < 1$ such that $\|Sx-Sy\| \le k\|x-y\|$ for all $x,y \in C$; uniformly continuous if, for every $\varepsilon > 0$, there exists a $\delta > 0$, such that $\|Sx-Sy\| < \varepsilon$ whenever $\|x-y\| < \delta$.

From [2], one has the following celebrated results.

(i) $\langle u - P_C[u], v - P_C[u]\rangle \le 0$ for all $u \in H$ and $v \in C$;
(ii) $\|P_C[u] - P_C[v]\|^2 \le \langle P_C[u] - P_C[v], u - v\rangle$ for all $u,v \in H$;
(iii)] $\|P_C[u] - v\|^2 \le \|u-v\|^2 - \|u - P_C[u]\|^2$ for all $u,v \in H$.

Note that the following equality and inequality hold in Hilbert spaces

(i) $\|u+v\|^2 \leq \|u\|^2 + 2\langle v, u+v\rangle$ for all $u, v \in H$;
(ii) $2\langle u, v\rangle = \|u\|^2 + \|v\|^2 - \|u-v\|^2 = \|u+v\|^2 - \|u\|^2 - \|v\|^2$ for all $u, v \in H$.
In addition, the following lemmas are needed.

Lemma 2.1. [16] *Each Hilbert space H satisfies the Opial's conditions, i.e., for any sequence $\{u_n\}$ with $u_n \rightharpoonup u$, $\liminf_{n\to\infty} \|u_n - u\| < \liminf_{n\to\infty} \|u_n - v\|$ holds for every $v \in H$ with $v \neq u$.*

Lemma 2.2. [32] *A function F_1 defined on a convex domain is uniformly continuous if and only if, for every $\varepsilon_1 > 0$, there exists a $K_1 < \infty$ such that $\|F_1(u) - F_1(v)\| \leq K_1\|u - v\| + \varepsilon_1$.*

Lemma 2.3. [27] *Let $b_{n+1} \leq b_n + \vartheta_n$, where $\{\vartheta_n\}$ and $\{b_n\}$ two nonnegative sequences. If $\sum_{n=0}^{\infty} \vartheta_n < \infty$, then $\lim_{n\to\infty} b_n$ exists.*

Lemma 2.4. [22] *Let $a_{n+1} \leq (1-\kappa_n)a_n + \kappa_n\varphi_n$, where $\{a_n\}$ is a positive sequence, $\{\kappa_n\}$ is a sequence in $(0,1)$ with $\sum_{n=1}^{\infty} \kappa_n = \infty$ and φ_n is a real sequence. If $\limsup_{k\to\infty} \varphi_{n_k} \leq 0$ for all subsequences $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition $\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0$, then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.5. [26] *Let C be a nonempty, convex, bounded, and closed subset of a Hilbert space H, and let $\Gamma_a := \{S(t) : 0 \leq t < \infty\}$ be a nonexpansive semigroup defined on set C. For each $x \in C$ and $s > 0$ and for any $0 \leq h < \infty$, $\lim_{s\to\infty} \sup_{x\in C} \left\| \frac{1}{s}\int_0^s S(t)xdt - S(h)\left(\frac{1}{s}\int_0^s S(t)xdt\right)\right\| = 0$.*

Lemma 2.6. [34] *For each $u_1, \cdots, u_m \in H$ and $\eta_1, \cdots, \eta_m \in [0,1]$ with $\sum_{i=1}^{m} \eta_i = 1$, the following equality holds*

$$\|\eta_1 u_1 + \cdots + \eta_m u_m\|^2 = \sum_{i=1}^{m} \eta_i \|u_i\|^2 - \sum_{1\leq i\leq j\leq m} \eta_i\eta_j \|u_i - u_j\|^2.$$

3. The Proposed Method and Its Properties

Let H be real Hilbert spaces, let C be nonempty, convex, and closed subsets of H. Let $\Gamma_a := \{S(s) : 0 \leq s < \infty\}$ be one-parameter nonexpansive semigroup on H. Let $g : H \to H$ be a contraction mapping with constant $\varpi \in [0,1)$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$, $\{\varepsilon_n\}$, $\{s_n\}$, $\{\tau_n\}$, $\{\mu_n\}$, and $\{\rho_n\}$ be nonnegative sequences satisfying the following conditions:

(a) $\alpha_n + \beta_n + \delta_n = 1$ and $\liminf_{n\to\infty} \beta_n\delta_n > 0$;
(b) Let $\{\varepsilon_n\}$ be positive sequence such that $\lim_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = 0$;
(c) $0 < s_n < \infty$;
(d) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(e) $0 < \liminf_{n\to\infty} \mu_n < \limsup_{n\to\infty} \mu_n < 1$;
(f) $\sum_{n=1}^{\infty} \tau_n < \infty$ and $\lim_{n\to\infty} \rho_n = 0$.

Assuming the usual conditions:

(C1) Feasible set C is nonempty, convex, and closed.
(C2) Operator $B : H \to H$ is pseudomonotone and uniformly continuous on H and sequentially weakly continuous on C.
(C3) Solution set $\Omega = \mathrm{Fix}(\Gamma_a) \bigcap VI(C,B)$ is nonnonempty.

Now, we present the following algorithm for finding the common solutions of (1.1) and (1.3).

Algorithm 3.1.

Step 0. The initial step:
Give $\chi \in (0,1), \gamma \in (0,2), \lambda_1 > 0, \theta > 0$, and let $x_0, x_1 \in H$ be arbitrary.
Give x_{n-1}, x_n.

Step 1. Choose θ_n such that

$$\theta_n := \begin{cases} \min\{\theta, \frac{\varepsilon_n}{\|x_n - x_{n-1}\|}\} & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 2. Set $a_n = x_n + \theta_n(x_n - x_{n-1})$, and compute $z_n = P_C[a_n - \lambda_n B(a_n]$. If $z_n = a_n$, then stop, and a_n is a solution to (1.1). Else, do Step 3.

Step 3. Compute $u_n = P_{Q_n}[a_n - \gamma\zeta_n d(a_n, z_n)]$, where

$$Q_n = \{x \in H : \langle a_n - \lambda_n B(a_n) - z_n, x - z_n\rangle \leq 0\},$$

$$\zeta_n := \begin{cases} \frac{\langle a_n - z_n, d(a_n,z_n)\rangle}{\|d(a_n,z_n)\|^2}, & \text{if } d(a_n,z_n) \neq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

$$d(a_n, z_n) = a_n - z_n - \lambda_n(B(a_n) - B(z_n)),$$

Step 4. Compute $v_n = \mu_n u_n + (1-\mu_n)\frac{1}{s_n}\int_0^{s_n} S(t)u_n dt$ and $x_{n+1} = \alpha_n g(x_n) + \beta_n x_n + \delta_n v_n$. Update

$$\lambda_{n+1} := \begin{cases} \min\{\frac{(\rho_n+\chi)\|a_n - z_n\|}{\|B(a_n) - B(z_n)\|}, \lambda_n + \tau_n\} & \text{if } B(a_n) \neq B(z_n), \\ \lambda_n + \tau_n & \text{otherwise.} \end{cases} \tag{3.3}$$

Set $n := n+1$ and go to Step 1.

Remark 3.1. By condition (b), we obtain from (3.1) that $\theta_n\|x_n - x_{n-1}\| \leq \varepsilon_n$. It follows that

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0. \tag{3.4}$$

Thus there exists $N_1 > 0$ such that

$$\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq N_1, \forall n \in \mathbf{N}. \tag{3.5}$$

Lemma 3.1. [15] *Assume that (C1)-(C3) hold and $\{a_n\}$ and $\{z_n\}$ are sequences generated by Algorithm 3.1. If there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ convergent weakly to a point $\tilde{x} \in H$ and $\lim_{k\to\infty}\|a_{n_k} - z_{n_k}\| = 0$, then $\tilde{x} \in VI(C,B)$.*

We now introduce the following lemma, which is essential for proving the global convergence of our proposed method.

Lemma 3.2. *Let $\{\lambda_n\}$ be a sequence defined by (3.3). Then*

$$\lim_{n\to\infty}\lambda_n = \lambda \in \Big[\min\{\frac{\chi}{M}, \lambda_1\}, \lambda_1 + \sum_{n=1}^{\infty}\tau_n\Big].$$

Proof. Since B is uniformly continuous, we obtain from Lemma 2.2 that, for any given $\varepsilon_1 > 0$, there exists a $K_1 < \infty$ such that $\|B(a_n) - B(z_n)\| \leq K_1\|a_n - z_n\| + \varepsilon_1$. If $B(a_n) - B(z_n) \neq 0$, then

$$\frac{(\rho_n+\chi)\|a_n - z_n\|}{\|B(a_n) - B(z_n)\|} \geq \frac{(\rho_n+\chi)\|a_n - z_n\|}{K_1\|a_n - z_n\| + \varepsilon_1} = \frac{(\rho_n+\chi)\|a_n - z_n\|}{(K_1+\varepsilon_2)\|a_n - z_n\|} \geq \frac{\chi}{M},$$

where $\varepsilon_1 = \varepsilon_2\|a_n - z_n\|$ for some $\varepsilon_2 \in (0,1)$ and $M = K_1 + \varepsilon_2$. Hence, from the definition of λ_{n+1}, $\{\lambda_{n+1}\}$ is bounded below by $\min\left(\frac{\chi}{M}, \lambda_1\right)$ and $\lambda_{n+1} \leq \lambda_n + \tau_n \leq \lambda_1 + \sum_{n=1}^{\infty} \tau_n$, which implies that $\min\{\frac{\chi}{M}, \lambda_1\} \leq \lambda_n \leq \lambda_1 + \sum_{n=1}^{\infty} \tau_n$. By Lemma 2.3, it follows that $\lim_{n\to\infty} \lambda_n$ denoted by $\lambda = \lim_{n\to\infty} \lambda_n$ exists. Clearly, we have $\lambda \in \left[\min\{\frac{\chi}{M}, \lambda_1\}, \lambda_1 + \sum_{n=1}^{\infty} \tau_n\right]$. □

Remark 3.2. It follows from Lemma 3.2 and condition (e) that

$$\lim_{n\to\infty}\left(1 - \frac{(\rho_n + \chi)\lambda_n}{\lambda_{n+1}}\right) = 1 - \chi > 0, \tag{3.6}$$

there exists $n_0 > 0$ such that, for all $n \geq n_0$, $1 - \frac{(\rho_n+\chi)\lambda_n}{\lambda_{n+1}} > \frac{1-\chi}{2} > 0$.

Lemma 3.3. *For given $x_n \in H$, let ζ_n and λ_n satisfy (3.2) and (3.3). Then, for all $n \geq n_0$,*

$$\zeta_n \geq \frac{\left(1 - \frac{\lambda_n(\rho_n+\chi)}{\lambda_{n+1}}\right)\|a_n - z_n\|^2}{\|d(a_n, z_n)\|^2}. \tag{3.7}$$

Proof. Using the definition of $d(a_n, z_n)$ and (3.3), we have

$$\begin{aligned}\langle a_n - z_n, d(a_n, z_n)\rangle &\geq \|a_n - z_n\|^2 - \lambda_n\|a_n - z_n\|\|B(a_n) - B(z_n)\| \\ &\geq \left(1 - \frac{\lambda_n(\rho_n + \chi)}{\lambda_{n+1}}\right)\|a_n - z_n\|^2.\end{aligned}$$

From (3.2), we obtain, for all $n \geq n_0$,

$$\zeta_n \geq \frac{\left(1 - \frac{\lambda_n(\rho_n+\chi)}{\lambda_{n+1}}\right)\|a_n - z_n\|^2}{\|d(a_n, z_n)\|^2}.$$

□

Lemma 3.4. *Let $\{x_n\}$ be a sequence generated by the Algorithm 3.1 and $x^* \in \Omega$. Then, for all $n \geq n_0$,*

$$\|u_n - x^*\|^2 \leq \|a_n - x^*\|^2 - \|a_n - u_n - \gamma\zeta_n d(a_n, z_n)\|^2 - \gamma(2-\gamma)\left(\frac{1 - \frac{(\rho_n+\chi)\lambda_n}{\lambda_{n+1}}}{1 + \frac{(\rho_n+\chi)\lambda_n}{\lambda_{n+1}}}\right)^2\|a_n - z_n\|^2.$$

Proof. From (3.3), we have

$$\|d(a_n, z_n)\| \geq \|a_n - z_n\| - \lambda_n\|B(a_n) - B(z_n)\| \geq \left(1 - \frac{(\rho_n + \chi)\lambda_n}{\lambda_{n+1}}\right)\|a_n - z_n\|.$$

From Remark 3.2, for all $n \geq n_0$, we have

$$\|d(a_n, z_n)\| > \frac{(1-\chi)}{2}\|a_n - z_n\|. \tag{3.8}$$

Since $x^* \in C \subset Q_n$, we have

$$\begin{aligned}\|u_n - x^*\|^2 &\leq \langle u_n - x^*, a_n - \gamma\zeta_n d(a_n, z_n) - x^*\rangle \\ &= \frac{1}{2}(\|u_n - x^*\|^2 + \|a_n - x^*\|^2 - \|u_n - a_n\|^2 - 2\gamma\zeta_n\langle u_n - x^*, d(a_n, z_n)\rangle)\end{aligned}$$

which implies that $\|u_n - x^*\|^2 \leq \|a_n - x^*\|^2 - \|u_n - a_n\|^2 - 2\gamma\zeta_n\langle u_n - x^*, d(a_n, z_n)\rangle$. Since $z_n \in C, x^* \in VI(C, B)$, we obtain $\langle \lambda_n B(x^*), z_n - x^*\rangle \geq 0$. Using the pseudomonotonicity of B, we

have $\langle \lambda_n B(z_n), z_n - x^* \rangle \geq 0$. By using the property of the metriric projection, we obtain $\langle a_n - \lambda_n B(a_n) - z_n, z_n - x^* \rangle \geq 0$. Using the definition of $d(a_n, z_n)$, we obtain $\langle d(a_n, z_n), z_n - x^* \rangle \geq 0$, which implies that

$$\|u_n - x^*\|^2 \leq \|a_n - x^*\|^2 - \|u_n - a_n\|^2 - 2\gamma\zeta_n \langle u_n - z_n, d(a_n, z_n) \rangle. \tag{3.9}$$

From (3.8), we have $d(a_n, z_n) \neq 0$ for all $n \geq n_0$. This implies $\langle a_n - z_n, d(a_n, z_n) \rangle = \zeta_n \|d(a_n, z_n)\|^2$. Then, for all $n \geq n_0$,

$$\begin{aligned} &-2\gamma\zeta_n \langle d(a_n, z_n), u_n - z_n \rangle \\ &= -2\gamma\zeta_n^2 \|d(a_n, z_n)\|^2 + \|a_n - u_n\|^2 + \gamma^2\zeta_n^2 \|d(a_n, z_n)\|^2 - \|a_n - u_n - \gamma\zeta_n d(a_n, z_n)\|^2 \\ &= \|a_n - u_n\|^2 - \|a_n - u_n - \gamma\zeta_n d(a_n, z_n)\|^2 - \gamma(2-\gamma)\zeta_n^2 \|d(a_n, z_n)\|^2. \end{aligned} \tag{3.10}$$

Substituting (3.10) into (3.9), we find, for all $n \geq n_0$,

$$\|u_n - x^*\|^2 \leq \|a_n - x^*\|^2 - \|a_n - u_n - \gamma\zeta_n d(a_n, z_n)\|^2 - \gamma(2-\gamma)\zeta_n^2 \|d(a_n, z_n)\|^2. \tag{3.11}$$

It follows from (3.3) that

$$\|d(a_n, z_n)\| \leq \|a_n - z_n\| + \lambda_n \|B(u_n) - B(z_n)\| \leq \left(1 + \frac{\lambda_n(\rho_n + \chi)}{\lambda_{n+1}}\right) \|a_n - z_n\|. \tag{3.12}$$

Substituting (3.7) and (3.12) into (3.11), we obtain the desired conclusion immediately. □

Lemma 3.5. *Let $\{x_n\}$ be a sequence generated by the Algorithm 3.1 and $x^* \in \Omega$. Then $\{x_n\}$ is bounded.*

Proof. Observe that

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq \alpha_n \|g(x_n) - x^*\| + \beta_n \|x_n - x^*\| + \delta_n \|v_n - x^*\| \\ &\leq \alpha_n \|g(x_n) - g(x^*)\| + \alpha_n \|g(x^*) - x^*\| + \beta_n \|x_n - x^*\| + \delta_n \|v_n - x^*\| \\ &\leq \alpha_n \varpi \|x_n - x^*\| + \alpha_n \|g(x^*) - x^*\| + \beta_n \|x_n - x^*\| + \delta_n \|v_n - x^*\|. \end{aligned} \tag{3.13}$$

From Lemma 2.6 and Lemma 3.4, we have

$$\begin{aligned} \|v_n - x^*\|^2 &\leq \mu_n \|u_n - x^*\|^2 + (1-\mu_n) \left\| \frac{1}{s_n} \int_0^{s_n} S(t) u_n dt - \frac{1}{s_n} \int_0^{s_n} S(t) x^* dt \right\|^2 \\ &\quad - \mu_n(1-\mu_n) \left\| \frac{1}{s_n} \int_0^{s_n} S(t) u_n dt - u_n \right\|^2 \\ &\leq \|u_n - x^*\|^2 - \mu_n(1-\mu_n) \left\| \frac{1}{s_n} \int_0^{s_n} S(t) u_n dt - u_n \right\|^2 \\ &\leq \|a_n - x^*\|^2. \end{aligned} \tag{3.14}$$

In view of (3.5), we have

$$\|a_n - x^*\| \leq \|x_n - x^*\| + \alpha_n \left(\frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| \right) \leq \|x_n - x^*\| + \alpha_n N_1. \tag{3.15}$$

Substituting (3.14) and (3.15) into (3.13), we obtain

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq (1 - (1-\varpi)\alpha_n) \|x_n - x^*\| + \alpha_n \|g(x^*) - x^*\| + \delta_n \alpha_n N_1 \\ &\leq (1 - (1-\varpi)\alpha_n) \|x_n - x^*\| + \alpha_n (1-\varpi) \frac{\|g(x^*) - x^*\| + N_1}{1-\varpi} \\ &\leq \max \left\{ \|x_n - x^*\|, \frac{\|g(x^*) - x^*\| + N_1}{1-\varpi} \right\}. \end{aligned}$$

By induction on n, we obtain

$$\|x_n - x^*\| \leq \max\left\{\|x_{n_0} - x^*\|, \frac{\|g(x^*) - x^*\| + N_1}{1-\varpi}\right\}, \forall n \geq n_0.$$

Hence $\{x_n\}$ is bounded, so we deduce that $\{a_n\}, \{z_n\}, \{u_n\}$, and $\{g(x_n)\}$ are all bounded. □

4. Convergence Analysis

We now analyze the strong convergence of the proposed method. Our strong convergence avoids the two-case approach used in the literature. The following lemmas provide valuable tools for analyzing the convergence properties of the proposed method.

Lemma 4.1. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1 and $\tilde{x} \in \Omega$. Then,*

$$\begin{aligned}\|x_{n+1} - \tilde{x}\|^2 \leq \|x_n - \tilde{x}\|^2 &+ \alpha_n\Big(2\|x_n - \tilde{x}\|\|g(\tilde{x}) - \tilde{x}\| + \|g(\tilde{x}) - \tilde{x}\|^2 + qq_n\Big) \\ &- \delta_n\|a_n - u_n - \gamma\zeta_n d(a_n, z_n)\|^2 - \gamma(2-\gamma)\delta_n\left(\frac{1 - \frac{(\rho_n+\chi)\lambda_n}{\lambda_{n+1}}}{1 + \frac{(\rho_n+\chi)\lambda_n}{\lambda_{n+1}}}\right)^2\|a_n - z_n\|^2 \\ &- \delta_n\mu_n(1-\mu_n)\left\|\frac{1}{s_n}\int_0^{s_n} S(t)u_n dt - u_n\right\|^2 - \beta_n\delta_n\|v_n - x_n\|^2.\end{aligned} \tag{4.1}$$

Proof. Let $\tilde{x} \in \Omega$. From the definition of a_n, we have

$$\begin{aligned}\|a_n - \tilde{x}\|^2 &\leq \|x_n - \tilde{x}\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 + 2\theta_n\|x_n - \tilde{x}\|\|x_n - x_{n-1}\| \\ &= \|x_n - \tilde{x}\|^2 + \alpha_n qq_n,\end{aligned} \tag{4.2}$$

where

$$qq_n = \theta_n\|x_n - x_{n-1}\|\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + 2\|x_n - \tilde{x}\|\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|.$$

From (3.4), one concludes $\lim_{n\to\infty} qq_n = 0$. By applying Lemma 2.6 and Lemma 3.4 together with (3.14) and (4.2), we have

$$\begin{aligned}\|x_{n+1} - \tilde{x}\|^2 &\leq \alpha_n\|g(x_n) - \tilde{x}\|^2 + \beta_n\|x_n - \tilde{x}\|^2 + \delta_n\|v_n - \tilde{x}\|^2 - \beta_n\delta_n\|v_n - x_n\|^2 \\ &\leq \alpha_n\Big(\varpi\|x_n - \tilde{x}\| + \|g(\tilde{x}) - \tilde{x}\|\Big)^2 + \beta_n\|x_n - \tilde{x}\|^2 + \delta_n\|v_n - \tilde{x}\|^2 - \beta_n\delta_n\|v_n - x_n\|^2 \\ &\leq (1-\delta_n)\|x_n - \tilde{x}\|^2 + \alpha_n\Big(2\|x_n - \tilde{x}\|\|g(\tilde{x}) - \tilde{x}\| + \|g(\tilde{x}) - \tilde{x}\|^2\Big) + \delta_n\|a_n - \tilde{x}\|^2 \\ &\quad - \delta_n\|a_n - u_n - \gamma\zeta_n d(a_n, z_n)\|^2 - \gamma(2-\gamma)\delta_n\left(\frac{1 - \frac{(\rho_n+\chi)\lambda_n}{\lambda_{n+1}}}{1 + \frac{(\rho_n+\chi)\lambda_n}{\lambda_{n+1}}}\right)^2\|a_n - z_n\|^2 \\ &\quad - \delta_n\mu_n(1-\mu_n)\left\|\frac{1}{s_n}\int_0^{s_n} S(t)u_n dt - u_n\right\|^2 - \beta_n\delta_n\|v_n - x_n\|^2\end{aligned}$$

$$\leq \|x_n-\tilde{x}\|^2+\alpha_n\Big(2\|x_n-\tilde{x}\|\|g(\tilde{x})-\tilde{x}\|+\|g(\tilde{x})-\tilde{x}\|^2+qq_n\Big)$$
$$-\delta_n\|a_n-u_n-\gamma\zeta_n d(a_n,z_n)\|^2-\gamma(2-\gamma)\delta_n\left(\frac{1-\frac{(\rho_n+\chi)\lambda_n}{\lambda_{n+1}}}{1+\frac{(\rho_n+\chi)\lambda_n}{\lambda_{n+1}}}\right)^2\|a_n-z_n\|^2$$
$$-\delta_n\mu_n(1-\mu_n)\Big\|\frac{1}{s_n}\int_0^{s_n}S(t)u_n dt-u_n\Big\|^2-\beta_n\delta_n\|v_n-x_n\|^2.$$

□

Lemma 4.2. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1 and $\tilde{x}\in\Omega$. If $\{\|x_{n_k}-\tilde{x}\|^2\}$ is a subsequence of $\{\|x_n-\tilde{x}\|^2\}$ satisfying*

$$\liminf_{k\to\infty}(\|x_{n_{k+1}}-\tilde{x}\|^2-\|x_{n_k}-\tilde{x}\|^2)\geq 0, \tag{4.3}$$

then

(a) $\lim_{k\to\infty}\|x_{n_{k+1}}-x_{n_k}\|=0$;

(b) The weak w-limit set $\omega_w(x_n)\subset\Omega$, where

$$\omega_w(x_n)=\Big\{x\in H: x_{n_i}\rightharpoonup x \text{ for some subsequences } \{x_{n_i}\} \text{ of } \{x_n\}\Big\}.$$

Proof. Using (4.1), we obtain

$$\begin{aligned}
&\delta_{n_k}\|a_{n_k}-u_{n_k}-\gamma\zeta_{n_k}d(a_{n_k},z_{n_k})\|^2+\gamma(2-\gamma)\delta_{n_k}\left(\frac{1-\frac{(\rho_{n_k}+\chi)\lambda_{n_k}}{\lambda_{n_k+1}}}{1+\frac{(\rho_{n_k}+\chi)\lambda_{n_k}}{\lambda_{n_k+1}}}\right)^2\|a_{n_k}-z_{n_k}\|^2\\
&+\delta_{n_k}\mu_{n_k}(1-\mu_{n_k})\Big\|\frac{1}{s_{n_k}}\int_0^{s_{n_k}}S(t)u_{n_k}dt-u_{n_k}\Big\|^2+\beta_{n_k}\delta_{n_k}\|v_{n_k}-x_{n_k}\|^2\\
\leq\ &\|x_{n_k}-\tilde{x}\|^2-\|x_{n_{k+1}}-\tilde{x}\|^2+\alpha_{n_k}\Big(2\|x_{n_k}-\tilde{x}\|\|g(\tilde{x})-\tilde{x}\|+\|g(\tilde{x})-\tilde{x}\|^2+qq_{n_k}\Big).
\end{aligned}$$

From (4.3) and $\lim_{n\to\infty}\alpha_n=0$, we have

$$\begin{aligned}
&\limsup_{k\to\infty}\Big[\delta_{n_k}\|a_{n_k}-u_{n_k}-\gamma\zeta_{n_k}d(a_{n_k},z_{n_k})\|^2+\gamma(2-\gamma)\delta_{n_k}\left(\frac{1-\frac{(\rho_{n_k}+\chi)\lambda_{n_k}}{\lambda_{n_k+1}}}{1+\frac{(\rho_{n_k}+\chi)\lambda_{n_k}}{\lambda_{n_k+1}}}\right)^2\|a_{n_k}-z_{n_k}\|^2\\
&+\delta_{n_k}\mu_{n_k}(1-\mu_{n_k})\Big\|\frac{1}{s_{n_k}}\int_0^{s_{n_k}}S(t)u_{n_k}dt-u_{n_k}\Big\|^2+\beta_{n_k}\delta_{n_k}\|v_{n_k}-x_{n_k}\|^2\Big]\\
\leq\ &\limsup_{k\to\infty}\Big\{\|x_{n_k}-\tilde{x}\|^2-\|x_{n_{k+1}}-\tilde{x}\|^2+\alpha_{n_k}\Big(2\|x_{n_k}-\tilde{x}\|\|g(\tilde{x})-\tilde{x}\|+\|g(\tilde{x})-\tilde{x}\|^2+qq_{n_k}\Big)\Big\}\\
=\ &-\liminf_{k\to\infty}(\|x_{n_{k+1}}-\tilde{x}\|^2-\|x_{n_k}-\tilde{x}\|^2)\\
\leq\ &0.
\end{aligned}$$

In view of (3.6) and conditions (a) and (e), we have

$$\lim_{k\to\infty}\|a_{n_k}-u_{n_k}-\gamma\zeta_{n_k}d(a_{n_k},z_{n_k})\|=\lim_{k\to\infty}\|a_{n_k}-z_{n_k}\|=0,$$
$$\lim_{k\to\infty}\|v_{n_k}-x_{n_k}\|=\lim_{k\to\infty}\Big\|\frac{1}{s_{n_k}}\int_0^{s_{n_k}}S(t)u_{n_k}dt-u_{n_k}\Big\|=0. \tag{4.4}$$

From (3.2), we have $\zeta_{n_k}\|d(a_{n_k},z_{n_k})\|\le\|a_{n_k}-z_{n_k}\|$. It follows that

$$\|a_{n_k}-u_{n_k}\|\le\|a_{n_k}-u_{n_k}-\gamma\zeta_{n_k}d(a_{n_k},z_{n_k})\|+\gamma\|a_{n_k}-z_{n_k}\|,$$

which implies that $\lim_{k\to\infty}\|a_{n_k}-u_{n_k}\|=0$.

On the other hand, from (3.5), we have

$$\|a_{n_k}-x_{n_k}\|=\alpha_{n_k}\Big(\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|\Big)\le\alpha_{n_k}N_1.$$

Thus $\lim_{k\to\infty}\|a_{n_k}-x_{n_k}\|=0$. Observe that $\|u_{n_k}-x_{n_k}\|\le\|u_{n_k}-a_{n_k}\|+\|a_{n_k}-x_{n_k}\|$. It follows that $\lim_{k\to\infty}\|u_{n_k}-x_{n_k}\|=0$. Further, for all $h\ge0$, we see that

$$\|u_{n_k}-S(h)u_{n_k}\|\le\Big\|u_{n_k}-\frac{1}{s_{n_k}}\int_0^{s_{n_k}}S(t)u_{n_k}dt\Big\|+\Big\|\frac{1}{s_{n_k}}\int_0^{s_{n_k}}S(t)u_{n_k}dt-S(h)\Big(\frac{1}{s_{n_k}}\int_0^{s_{n_k}}S(t)u_{n_k}dt\Big)\Big\|$$
$$+\Big\|S(h)\Big(\frac{1}{s_{n_k}}\int_0^{s_{n_k}}S(t)u_{n_k}dt\Big)-S(h)u_{n_k}\Big\|.$$

Using (4.4) and Lemma 2.5, we obtain $\lim_{k\to\infty}\|u_{n_k}-S(h)u_{n_k}\|=0$. From the definition of $x_{n_{k+1}}$, we have $\|x_{n_{k+1}}-x_{n_k}\|\le\alpha_{n_k}\|g(x_{n_k})-x_{n_k}\|+\delta_{n_k}\|v_{n_k}-x_{n_k}\|$. It follows from (4.4) that $\lim_{k\to\infty}\|x_{n_{k+1}}-x_{n_k}\|=0$.

Now, we prove that $\omega_w(x_n)\subset\Omega$. Since $\{x_n\}$ is bounded, we see that $\omega_w(x_n)$ is nonempty. Let $\tilde{x}\in\omega_w(x_n)$. Thus there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup\tilde{x}$ as $k\to\infty$. Since $\lim_{k\to\infty}\|a_{n_k}-x_{n_k}\|=0$, we have that $a_{n_k}\rightharpoonup\tilde{x}$ as $k\to\infty$. In view of $\lim_{k\to\infty}\|a_{n_k}-z_{n_k}\|=0$, it follows from Lemma 3.1 that $\tilde{x}\in VI(C,B)$.

Next, we show that $\tilde{x}\in\mathrm{Fix}(\Gamma_a)$. Since $\lim_{k\to\infty}\|u_{n_k}-x_{n_k}\|=0$, we have $u_{n_k}\rightharpoonup\tilde{x}$ as $k\to\infty$. Now, for all $r\ge0$, we have

$$\|u_{n_k}-S(r)\tilde{x}\|\le\|u_{n_k}-S(r)u_{n_k}\|+\|S(r)u_{n_k}-S(r)\tilde{x}\|\le\|u_{n_k}-S(r)u_{n_k}\|+\|u_{n_k}-\tilde{x}\|.$$

It follows that $\liminf_{k\to\infty}\|u_{n_k}-S(r)\tilde{x}\|\le\liminf_{k\to\infty}\|u_{n_k}-\tilde{x}\|$. By the Opial's property of space H (Lemma 2.1), we obtain that $S(r)\tilde{x}=\tilde{x}$ for all $r\ge0$, which implies that $\tilde{x}\in\mathrm{Fix}(\Gamma_a)$. Since $\tilde{x}\in\omega_w(x_n)$, it follows that $\omega_w(x_n)\subset\Omega$. Therefore, the claim (b) in lemma is proved. □

Theorem 4.1. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1. Then, $\{x_n\}$ converges strongly to $\tilde{x}\in\Omega$, where $\tilde{x}=P_\Omega[g(\tilde{x})]$.*

Proof. Let $\tilde{x}\in\Omega$. First, we show that $\limsup_{k\to\infty}\langle g(\tilde{x})-\tilde{x},x_{n_{k+1}}-\tilde{x}\rangle\le0$. Let $\{x_{n_{k_j}}\}$ be a subsequence of $\{x_{n_k}\}$ which converges weakly to some $\hat{x}\in\Omega$, and

$$\lim_{j\to\infty}\langle g(\tilde{x})-\tilde{x},x_{n_{k_j}}-\tilde{x}\rangle=\limsup_{k\to\infty}\langle g(\tilde{x})-\tilde{x},x_{n_k}-\tilde{x}\rangle.$$

Since $\{x_{n_{k_j}}\}$ converges weakly to $\hat{x}\in\Omega$ and $\tilde{x}=P_\Omega[g(\tilde{x})]$, it follows that

$$\limsup_{k\to\infty}\langle g(\tilde{x})-\tilde{x},x_{n_{k+1}}-\tilde{x}\rangle=\limsup_{k\to\infty}\langle g(\tilde{x})-\tilde{x},x_{n_k}-\tilde{x}\rangle=\langle g(\tilde{x})-\tilde{x},\hat{x}-\tilde{x}\rangle\le0. \tag{4.5}$$

It follows that

$$\begin{aligned}
&\|x_{n_k+1}-\tilde{x}\|^2\\
&\leq \left\|\alpha_{n_k}(g(x_{n_k})-g(\tilde{x}))+\beta_{n_k}(x_{n_k}-\tilde{x})+\delta_{n_k}(v_{n_k}-\tilde{x})\right\|^2+2\alpha_{n_k}\langle g(\tilde{x})-\tilde{x},x_{n_k+1}-\tilde{x}\rangle\\
&\leq \alpha_{n_k}\|g(x_{n_k})-g(\tilde{x})\|^2+\beta_{n_k}\|x_{n_k}-\tilde{x}\|^2+\delta_{n_k}\|v_{n_k}-\tilde{x}\|^2+2\alpha_{n_k}\langle g(\tilde{x})-\tilde{x},x_{n_k+1}-\tilde{x}\rangle\\
&\leq \alpha_{n_k}\varpi\|x_{n_k}-\tilde{x}\|^2+\beta_{n_k}\|x_{n_k}-\tilde{x}\|^2+\delta_{n_k}\|v_{n_k}-\tilde{x}\|^2+2\alpha_{n_k}\langle g(\tilde{x})-\tilde{x},x_{n_k+1}-\tilde{x}\rangle.
\end{aligned}$$

From (4.2), (3.14), and Lemma 3.4, we have

$$\begin{aligned}
\|x_{n_k+1}-\tilde{x}\|^2 &\leq \Big(1-(1-\varpi)\alpha_{n_k}\Big)\|x_{n_k}-\tilde{x}\|^2+\delta_{n_k}\alpha_{n_k}qq_{n_k}+2\alpha_{n_k}\langle g(\tilde{x})-\tilde{x},x_{n_k+1}-\tilde{x}\rangle\\
&\leq (1-\sigma_{n_k})\|x_{n_k}-\tilde{x}\|^2+\sigma_{n_k}\left(\frac{qq_{n_k}}{1-\varpi}+\frac{2\langle g(\tilde{x})-\tilde{x},x_{n_k+1}-\tilde{x}\rangle}{1-\varpi}\right),
\end{aligned}\tag{4.6}$$

where $\sigma_{n_k}=(1-\varpi)\alpha_{n_k}$. Let $\varphi_{n_k}=\frac{qq_{n_k}}{1-\varpi}+\frac{2\langle g(\tilde{x})-\tilde{x},x_{n_{n_k+1}}-\tilde{x}\rangle}{1-\varpi}$. In view of $\sum_{n_k=1}^{\infty}\alpha_{n_k}=\infty$ and $\lim_{k\to\infty}\alpha_{n_k}=0$, we find that $\sum_{n_k=1}^{\infty}\sigma_{n_k}=\infty$ and $\lim_{k\to\infty}\sigma_{n_k}=0$. It follows from (4.5) that $\limsup_{k\to\infty}\varphi_{n_k}\leq 0$. Hence, $\lim_{n\to\infty}\|x_n-\tilde{x}\|^2=0$. Consequently, $\lim_{n\to\infty}\|x_n-\tilde{x}\|=0$, and x_n converges strongly to $\tilde{x}$. This completes the proof. □

Remark 4.1. The rate of convergence depends on the specific properties of φ_{n_k} and α_{n_k}. Let $d_k^2=\|x_{n_k}-\tilde{x}\|^2$. Inequality (4.6) becomes $d_{k+1}^2\leq(1-\sigma_{n_k})d_k^2+\sigma_{n_k}\varphi_{n_k}$. If $\limsup_{k\to\infty}\varphi_{n_k}=0$, for any $\varepsilon>0$, then there exists k_0 such that, for all $k\geq k_0$, $\varphi_{n_k}\leq\varepsilon$. Therefore, for all $k\geq k_0$, we have $d_{k+1}^2\leq(1-\sigma_{n_k})d_k^2+\sigma_{n_k}\varepsilon$. To analyze the rate, we can consider specific choices for α_n

Case 1. $\alpha_n=\frac{1}{n+1}$. We can obtain $O(\frac{1}{n})$.

Case 2. $\alpha_n=\frac{c}{n}$ for sufficient large n $\big(c>0$ such that $\alpha_n\in(0,1)\big)$. The convergence rate in this case is typically logarithmic. More precisely, we can expect $\|x_{n_k}-\tilde{x}\|^2=O\big(\frac{1}{\log(n_k)}\big)$ or a related logarithmic term.

Consider the case where $\varphi_{n_k}\leq 0$ for large k. Then $d_{k+1}^2\leq\Big(1-\frac{(1-\varpi)c}{n_k}\Big)d_k^2$ and

$$\log d_{k+1}^2\leq\log d_k^2+\log\left(1-\frac{(1-\varpi)c}{n_k}\right)\approx\log d_k^2-\frac{(1-\varpi)c}{n_k},$$

which implies

$$\log d_k^2\leq\log d_{k_0}^2-(1-\varpi)c\sum_{i=k_0}^{k-1}\frac{1}{n_i}\approx\log d_{k_0}^2-c(1-\varpi)\log\left(\frac{k}{k_0}\right).$$

It follows that $d_k^2\leq d_{k_0}^2\left(\frac{k_0}{k}\right)^{(1-\varpi)c}$, which is a polynomial rate.

Case 3. $\alpha_n=\frac{c}{n^p}$ for $0<p<1$. The sum $\Sigma\frac{1}{n_k^p}$ diverges. The convergence rate here is typically sublinear. We can expect a rate of the form $\|x_{n_k}-\tilde{x}\|^2=O\big(\frac{1}{n_k^q}\big)$ for some $q>0$.

5. Numerical Examples

We illustrate the efficacy of our algorithm through two numerical examples in this section, and provide a comparison with the methods proposed by Tian and Tong [30], and Tan and Cho [29].

Example 5.1. Let $B:\mathbb{R}^n\to\mathbb{R}^n$ be defined by $B(x)=Mx+q$, where $q\in\mathbb{R}^n$, $M=N^TN+U+D$, D is a $n\times n$ diagonal matrix whose diagonal terms are nonnegative (hence M is positive symmetric definite), $N$is a $n\times n$ matrix, and U is a $n\times n$ skew-symmetric matrix. The feasible set C is given by $C=\{x\in\mathbb{R}^n:-3\leq x_i\leq 3, i=1,\cdots,n\}$. It is clear that B is pseudomonotone and uniformly continuous. For $q=0$, the solution set is $VI(C,B)=\{0\}$. We define the mapping $S(r):\mathbb{R}^n\to\mathbb{R}^n$ by $S(r)x=10^{-r}x$. Clearly, $S(r)$ is a nonexpansive semigroup and $\mathrm{Fix}(\Gamma_a)=\{0\}$. In all the tests, the parameters are taken as follows:

- In our method, we choose $\lambda_1=0.9,\chi=0.4,\theta=0.5,\tau_n=\frac{1}{(n+1)^{1.1}},t_n=4.5,r=2,\gamma=0.0001,\varepsilon_n=\frac{1}{(2n+1)^3},\rho_n=\frac{1}{(n+1)},\alpha_n=\frac{1}{2(n+10)},\beta_n=\frac{n}{2(n+10)},\delta_n=\frac{(n+19)}{2(n+10)}$, and $\varpi=0.5$.
- In the method proposed by Tan and Cho [29], $\tau_1=0.9,\mu=0.8,\theta=0.7$, $\xi_n=\frac{1}{(n+1)^{1.1}}$, $\varepsilon_n=\frac{1}{(n+1)^2}$, $\alpha_n=\frac{1}{n+1}$, and $\beta_n=0.9(1-\alpha_n)$.
- In the method proposed by Tian and Tong [30], $\lambda_0=0.15,\mu=0.4,\alpha_n=0.25$, and $\beta_n=0.5$.

The convergence of the algorithm in Tong [30] was established under the assumption that mapping B is monotone and L-Lipschitz continuous, and mapping T is quasinonexpansive. However, in [29], convergence was proven under the assumption that mapping B is pseudomonotone and L-Lipschitz continuous. It is easy to see that mapping B is monotone and L-Lipschitz continuous with $L=\|M\|$. On the other hand, let $Tx=\frac{1}{2}xsin\|x\|$. Clearly, the unique fixed point of T is 0, i.e., $B(T)=\{0\}$ and mapping T is quasinonexpansive. For experiments, q is equal to zero vector, all the entries of N,U,D are generated randomly. The initial values $x_0=x_1$ are chosen at random using $randn(n,1)$ in MATLAB and take the stopping criterion as $\|x_{n+1}-x_n\|<10^{-8}$. The iteration numbers, denoted by No. It., and the computational time with different dimensions are given in tables 1-2. The result of this experiment is reported in the Table 1 with comparison of the proposed method to the methods in [29] and [30].

Table 1: Numerical results for Example 5.1

	The proposed method		The method in [29]		The method in [30]	
	No. It.	CPU(Sec.)	No. It.	CPU(Sec.)	No. It.	CPU(Sec.)
$n=100$	18	0.019	32	0.057	25	0.054
$n=300$	19	0.024	35	0.062	27	0.059
$n=500$	19	0.034	36	0.107	27	0.095
$n=800$	19	0.0108	37	0.137	28	0.177

Example 5.2. Let $H=\{x=(x_1,x_2,\cdots,x_i,\cdots):\sum_{n=1}^{\infty}|x_i|^2<\infty\}$. Let t_1 and t_2 be two positive real numbers such that $\frac{t_2}{k_1+1}<\frac{t_1}{k_1}<t_1<t_2$ for some $k_1>1$. Let the feasible set be given by $C=\{x\in H:\|x\|\leq t_1\}$ and the operator $B:H\longrightarrow H$ be defined by $B(x)=(t_2-\|x\|)x$ for all $x\in H$. Now, we show that B is L-Lipschitz continuous on H and pseudomonotone on C. Indeed, for any $x,y\in H$,

$$\|B(x)-B(y)\|\leq t_2\|x-y\|+\|x\|\|x-y\|+|\|x\|-\|y\||\|y\|\leq(t_2+2t_1)\|x-y\|.$$

Thus B is Lipschitz continuous with $L=t_2+2t_1$. Next, we show that B is pseudomonotone. Indeed, if $\langle B(x),y-x\rangle\geq 0$ for all $x,y\in C$, that is, $\langle(t_2-\|x\|)x,y-x\rangle\geq 0$. Since $\|x\|\leq t_2$, we have $\langle x,y-x\rangle\geq 0$. Therefore,

$$\langle B(y),y-x\rangle=\langle(t_2-\|y\|)y,y-x\rangle\geq(t_2-\|y\|)(\langle y,y-x\rangle-\langle x,y-x\rangle)=(t_2-\|y\|)\|x-y\|^2\geq 0.$$

For the experiment, we choose $t_1 = 3, t_2 = 5$, and $k_1 = 1.1$. We take the stopping criterion as $\|x_{n+1} - x_n\| < 10^{-8}$. In addition, we continue to use the same control parameters and the same mapping T for [30] as in Example 5.1. The result of this experiment is reported in the Table 2.

Table 2: Numerical results for Example 5.2

	The proposed method		The method in [29]		The method in [30]	
	No. It.	CPU(Sec.)	No. It.	CPU(Sec.)	No. It.	CPU(Sec.)
$n = 100$	24	0.013	51	0.057	34	0.043
$n = 1000$	24	0.028	71	0.065	73	0.079
$n = 10000$	26	0.037	78	0.118	132	0.172
$n = 100000$	28	0.281	102	0.825	309	2.73

6. Conclusions

In this paper, we presented an alternated inertial iterative algorithm for finding a common solution to problem (1.1) and problem (1.3) in real Hilbert space. Our stepsize can be computed easily by a simple closed formula and does not need neither linesearch computation nor the information of the Lipschitz constant of B. We obtained a strong convergence theorem of solutions under some mild conditions. Numerical results indicate that our proposed method compared with other associated ones in the literature is efficient.

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