J. Appl. Numer. Optim. 7 (2025), No. 2, pp. 233-252
Available online at http://jano.biemdas.com
https://doi.org/10.23952/jano.7.2025.2.06

ON APPROXIMATE EFFICIENT SOLUTIONS OF NONSMOOTH MINIMAX PROGRAMMING PROBLEMS INVOLVING INEQUALITY AND EQUALITY CONSTRAINTS WITH APPLICATIONS

THANH-HUNG PHAM

Faculty of Pedagogy and Faculty of Social Sciences & Humanities, Kien Giang University, Chau Thanh, Kien Giang, Vietnam

Abstract. This paper focuses on optimality conditions/duality theorems for ε-quasi solutions of nonsmooth minimax programming problems involving inequality and equality constraints. The main results obtained in this paper are the extensions of the results in Chuong and Kim [Ann. Oper. Res. 251 (2017), 73–87] and Bae, Hong and Kim [J. Nonlinear Var. Anal. 5 (2021), 709-720].

Keywords. Duality theorem; Efficient solution; Generalized L-invex-infine function; Minimax programming problem; Optimality condition.
2020 Mathematics Subject Classification. 90C25, 90C46.

1. Introduction

Let X denote the Asplund space (i.e., the Banach spaces whose separable subspaces have separable duals), and let Ω be a nonempty locally closed subset of X. Let $K := \{1,\ldots,m\}, I := \{1,\ldots,n\} \cup \emptyset$ and $J := \{1,\ldots,l\} \cup \emptyset$ be index sets. In what follows, Ω is always assumed to be SNC at the point under consideration. This assumption is automatically fulfilled when X is a finite dimensional space.

In this paper, we consider the following minimax programming problem involving inequality and equality constraints

$$\text{(MP)} \qquad \min_{x\in C} \max_{k\in K} f_k(x),$$

where

$$C := \{x \in \Omega \mid g_i(x) \leq 0, i \in I, h_j(x) = 0, j \in J\},$$

and the functions $f_k : X \to \mathbb{R}, k \in K, g_i : X \to \mathbb{R}, i \in I$ and $h_j : X \to \mathbb{R}, j \in J$ are locally Lipschitz functions. Suppose that $f := (f_1,\ldots,f_m), g := (g_1,\ldots,g_n), h := (h_1,\ldots,h_l)$.

Firstly, we introduce the concept of ε-quasi solutions for problem (MP) as follows.

Definition 1.1. (See [1, 2, 3]) Let $\varepsilon \geq 0$ and $\varphi(x) := \max_{k\in K} f_k(x), x \in X$. A point $\bar{x} \in C$ is said to be an ε-quasi solution to problem (MP) if $\varphi(x) + \varepsilon||x-\bar{x}|| \geq \varphi(\bar{x})$ for all $x \in C$.

E-mail address: thanhhungpham.math@gmail.com; pthung@vnkgu.edu.vn
Received 2 March 2025; Accepted 25 May 2025; Published online 30 July 2025.

Since the data of mathematical models of practical problems are obtained by measuring devices or statistical records, the models are also approximations, and hence their exact solutions are in fact also approximate solutions to the practical problems. Moreover, in numerical methods, we always have to find approximate solutions which converge to exact ones. Therefore, the study of approximate solutions for optimization problems is significant from both the theoretical aspect and computational applications. Among many other interesting research for approximate efficient solutions, optimality conditions and duality for approximate efficient solutions of multiobjective optimization problems were considered numerously by researchers; see, e.g., [4]-[20] and the references therein.

The minimax problem arises in decision-making, optimization, and game theory contexts, and minimax programming problems have attracted the attention of mathematicians in recent years. Optimality conditions/duality theorems/sensitivity analysis for minimax programming problems were investigated in [21]-[33] and the references therein. Recently, Bae et al. [1] obtained optimality conditions and duality for approximate efficient solutions of minimax programming problems with inequality constraints. Chuong and Kim [34] studied optimality conditions/duality theorems for efficient solutions of minimax programming problems involving inequality and equality constraints. To the best of our knowledge, there have been no papers studying the optimality conditions and dualilty for ε-quasi solutions of nonsmooth minimax programming problems involving inequality and equality constraints.

In this paper, we provide some new results on optimality conditions/duality theorems for ε-quasi solutions of nonsmooth minimax programming problems involving inequality and equality constraints. The rest of the paper is organized as follows. Section 1 and 2 present introduction, notations, and preliminaries. In Section 3, we investigate optimality conditions for ε-quasi solutions of nonsmooth minimax programming problems involving inequality and equality constraints. In Section 4, we investigate the Wolfe type dual problems/the Mond–Weir type dual problems. In Section 5, we provide applications to the nonsmooth multiobjective optimization problems involving inequality and equality constraints. Finally, conclusions are given in Section 6.

2. Preliminaries

Let us recall some concepts and calculus rules from Variational Analysis in [35]. Throughout the paper, all spaces under consideration are assumed to be Asplund (i.e., Banach spaces whose separable subspaces have separable duals). The canonical pairing between space X and its topological dual one X^* (equipped with the weak* topology w^*) is denoted by $\langle\cdot,\cdot\rangle$, while the symbol $||\cdot||$ stands for the norm in the considered space. The symbol $\mathbb{B}_X$ stands for the closed unit ball in space X. The closed unit ball in the dual space X^* is denoted by $\mathbb{B}_{X^*}$.The topological closure and the topological interior of a set $\Theta \subset X$ are denoted by $\mathrm{cl}\Theta$ and $\mathrm{int}\Theta$. As usual, the polar cone of Θ is the set $\Theta^\circ := \{x^* \in X^* \mid \langle x^*, x\rangle \le 0, \forall x \in \Theta\}$. Besides, the nonnegative (resp., nonpositive) orthant cone of Euclidean space $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+ = \{(x_1,\ldots,x_n) \mid x_i \ge 0, i = 1,\ldots,n\}$ (resp., $\mathbb{R}^n_-$) for $n \in \mathbb{N} := \{1,2,\ldots\}$, while $\mathrm{int}\mathbb{R}^n_+$ is used to indicate the topological interior of $\mathbb{R}^n_+$.

Given a set-valued map $F : X \rightrightarrows X^*$ between X and its dual X^*, we denote by

$$\mathop{\mathrm{Lim\,sup}}_{x\to\bar{x}} F(x) := \{x^* \in X \quad \mid \exists \text{ sequence } x_k \to \bar{x} \text{ and } x_k^* \xrightarrow{w^*} x^* \text{ with } x_k^* \in F(x_k), \forall k \in \mathbb{N}\}$$

the sequential Painlevé-Kuratowski upper/outer limit of F as $x \to \bar{x}$. Here the notation $\xrightarrow{w^*}$ indicates the convergence in the weak* topology of X^*.

A set $\Theta \subset X$ is called closed around $\bar{x} \in \Theta$ if there is a neighborhood U of $\bar{x}$ such that $\Theta \cap \mathrm{cl}U$ is closed. We say that Θ is locally closed if Θ is closed around x for every $x \in \Theta$.

Let $\Theta \subset X$ be closed around $\bar{x} \in \Theta$. The Fréchet/regular normal cone to Θ at $\bar{x}$ is defined by

$$\widehat{N}(\bar{x};\Theta) := \left\{ x^* \in X^* \mid \limsup_{x \xrightarrow{\Theta} \bar{x}} \frac{\langle x^*, x-\bar{x}\rangle}{\| x-\bar{x} \|} \leq 0 \right\},$$

where $x \xrightarrow{\Theta} \bar{x}$ means that $x \to \bar{x}$ with $x \in \Theta$. If $\bar{x} \notin \Theta$, we put $\widehat{N}(\bar{x};\Theta) := \emptyset$.

The Mordukhovich/limiting normal cone $N(\bar{x};\Theta)$ to Θ at $\bar{x} \in \Theta \subset X$ is obtained from Fréchet/regular normal cones by taking the sequential Painlevé Kuratowski upper limit as

$$N(\bar{x};\Theta) := \operatorname*{Lim\,sup}_{x \xrightarrow{\Theta} \bar{x}} \widehat{N}(x;\Theta).$$

If $\bar{x} \notin \Theta$, we put $N(\bar{x};\Theta) := \emptyset$. Specially, if Θ is convex set and $\bar{x} \in \Theta$, then

$$N(\bar{x};\Theta) = \{x^* \in X^* \mid \langle x^*, x-\bar{x}\rangle \leq 0, \forall x \in \Theta\}.$$

Let $f : X \to \bar{\mathbb{R}} := [-\infty, +\infty]$ be an extended real-valued function. The domain and epigraph of f are given by

$$\mathrm{dom} f := \{x \in X \mid |f(x)| < +\infty\},$$
$$\mathrm{epi} f := \{(x,\mu) \in X \times \mathbb{R} \mid \mu \geq f(x)\}.$$

Let $f : X \to \bar{\mathbb{R}}$ be finite at $\bar{x} \in \mathrm{dom} f$, then the Mordukhovich/limiting subdifferential of f at $\bar{x}$ is defined by

$$\partial f(\bar{x}) := \{x^* \in X^* \mid (x^*, -1) \in N((\bar{x}, f(\bar{x})); \mathrm{epi} f)\}$$

and

$$\widehat{\partial} f(\bar{x}) := \left\{x^* \in X^* \mid (x^*, -1) \in \widehat{N}((\bar{x}, f(\bar{x})); \mathrm{epi} f)\right\}.$$

If $|f(\bar{x})| = +\infty$, then one puts $\partial f(\bar{x}) = \widehat{\partial} f(\bar{x}) := \emptyset$.

A function $f : X \to \bar{\mathbb{R}}$ is said to be convex if, for all $x, y \in X$ and all $\lambda \in (0,1)$,

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda) f(y).$$

If f is a convex function and $\bar{x} \in X$, then

$$\partial f(\bar{x}) = \widehat{\partial} f(\bar{x}) = \{x^* \in X^* \mid f(x) - f(\bar{x}) \geq \langle x^*, x-\bar{x}\rangle, \forall x \in X\}.$$

Let $f : X \to \bar{\mathbb{R}}$ be finite at $\bar{x} \in \mathrm{dom} f$, then f is lower semi-continuous at $\bar{x} \in X$ if $\liminf_{x \to \bar{x}} f(x) \geq f(\bar{x})$. Given $\Theta \subset X$, consider the indicator function $\delta(\cdot;\Theta)$ defined by

$$\delta(x;\Theta) := \begin{cases} 0, & \text{if } x \in \Theta, \\ +\infty, & \text{otherwise.} \end{cases}$$

Furthermore, we have a relation between the Mordukhovich/limiting normal cone and the Mordukhovich/limiting subdifferential of the indicator function as follows

$$N(\bar{x};\Theta) = \partial \delta(\bar{x};\Theta), \forall \bar{x} \in \Theta.$$

We say that f is locally Lipschitz at $\bar{x} \in X$ if there exist a positive constant $L > 0$ and a neighbourhood U of $\bar{x}$ such that $|f(x_1) - f(x_2)| \leq L||x_1 - x_2||$ for all $x_1, x_2 \in U$.

Lemma 2.1. (See [35], Proposition 1.114) *Let $\psi : X \to \bar{\mathbb{R}}$ be finite at $\bar{x}$. If $\bar{x}$ is a local minimizer of ψ then $0 \in \partial\psi(\bar{x})$.*

Lemma 2.2. (See [35], Therorem 3.36) *Suppose that $\psi_k : X \to \bar{\mathbb{R}}, k = 1,\dots,m$ (with $m \geq 2$) are lower semi-continuous around $\bar{x} \in X$ and let all but one of these functions be Lipschitz continuous around $\bar{x}$. Then $\partial(\psi_1 + \cdots + \psi_m)(\bar{x}) \subset \partial\psi_1(\bar{x}) + \cdots + \partial\psi_m(\bar{x})$.*

A set $\Theta \subset X$ is sequentially normally compact (SNC) at $\bar{x} \in \Theta$ if, for any sequences

$$x_k \xrightarrow{\Theta} \bar{x} \text{ and } x_k^* \xrightarrow{w^*} 0 \text{ with } x_k^* \in \widehat{N}(x_k;\Theta),$$

$||x_k^*|| \to 0$ as $k \to +\infty$. Obviously, this SNC property is automatically satisfied in finite dimensional spaces.

A function $\varphi : X \to \mathbb{R}$ is said to be sequentially normally compact (SNC) at $\bar{x} \in X$ if gphφ is SNC at $(\bar{x},\varphi(\bar{x}))$. According to ([35], Corollary 1.69(i)), φ is SNC at $\bar{x} \in X$ if it is Lipschitz continuous around $\bar{x}$.

Lemma 2.3. (See [35], Corollary 3.5) *Let $\Theta_1,\Theta_2 \subset X$ be such that $\bar{x} \in \Theta_1 \cap \Theta_2$ and that at least one of $\{\Theta_1,\Theta_2\}$ is SNC at this point. If the normal qualification condition $N(\bar{x};\Theta_1) \cap (-N(\bar{x};\Theta_2)) = \{0\}$ is satisfied, then $N(\bar{x};\Theta_1 \cap \Theta_2) \subset N(\bar{x};\Theta_1) + N(\bar{x};\Theta_2)$.*

3. Optimality Conditions

In this section, we provide necessary and sufficient optimality conditions for ε-quasi solutions of nonsmooth minimax programming problems involving inequality and equality constraints.

For $\bar{x} \in \Omega$, we put

$$I(\bar{x}) := \{i \in I \mid g_i(\bar{x}) = 0\}, \qquad J(\bar{x}) := \{j \in J \mid h_j(\bar{x}) = 0\}.$$

Next, we recall the constraint qualification for problem (MP)which is due to the conditions in Chuong and Kim ([34]) as follows.

Definition 3.1. We say that condition (CQ) is satisfied at $\bar{x} \in \Omega$ if there do not exist $\mu_i \geq 0, i \in I(\bar{x})$ and $\gamma_j \geq 0, j \in J(\bar{x})$ such that $\sum\limits_{i\in I(\bar{x})} \mu_i + \sum\limits_{j\in J(\bar{x})} \gamma_j \neq 0$ and

$$0 \in \sum_{i\in I(\bar{x})} \mu_i\partial g_i(\bar{x}) + \sum_{j\in J(\bar{x})} \gamma_j \left(\partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x})\right) + N(\bar{x};\Omega).$$

Now, we propose a necessary optimality condition for an ε-quasi solution of problem (MP) under the condition (CQ).

Theorem 3.1. *Let $\bar{x} \in C$ be an ε-quasi solution to problem* (MP). *Suppose that the condition* (CQ) *at $\bar{x}$ holds. Then, there exist $\alpha := (\alpha_1,\dots,\alpha_m) \in \mathbb{R}^m_+$ with $\sum_{k\in K} \alpha_k = 1$, $\mu := (\mu_1,\dots,\mu_n) \in \mathbb{R}^n_+$ and $\gamma := (\gamma_1,\dots,\gamma_l) \in \mathbb{R}^l_+$ such that*

$$\begin{aligned} &0 \in \sum_{k\in K} \alpha_k\partial f_k(\bar{x}) + \sum_{i\in I} \mu_i\partial g_i(\bar{x}) + \sum_{j\in J} \gamma_j \left(\partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x})\right) + \varepsilon\mathbb{B}_{X^*} + N(\bar{x};\Omega),\\ &\alpha_k \left[f_k(\bar{x}) - \max_{k\in K} f_k(\bar{x})\right] = 0, k \in K, \\ &\mu_i g_i(\bar{x}) = 0, i \in I. \end{aligned} \tag{3.1}$$

Proof. Let $\bar{x} \in C$ be an ε-quasi solution ti problem (MP). Then $\bar{x}$ is a minimizer of $\min_{x\in C}\{\varphi(x)+\varepsilon||x-\bar{x}||\}$, with $\varphi(x) := \max_{k\in K} f_k(x)$. It follows easily that $\bar{x}$ is a minimizer of the following unconstrained scalar optimization problem

$$\min_{x\in X}\{\varphi(x)+\varepsilon||x-\bar{x}||+\delta(x;C)\}.$$

We deduce from Lemma 2.1 that

$$0 \in \partial\left(\varphi+\varepsilon||x-\bar{x}||+\delta(\cdot;C)\right)(\bar{x}). \tag{3.2}$$

Since function φ and and function $||\cdot-\bar{x}||$ are Lipschitz continuous around $\bar{x}$ and the function $\delta(\cdot;C)$ is lower semi-continuous around $\bar{x}$, we deduce from $\partial\delta(\bar{x};C) = N(\bar{x};C), \partial(||\cdot-\bar{x}||)(x) = \mathbb{B}_{X^*}$, Lemma 2.2 and (3.2) that

$$0 \in \partial\varphi(\bar{x})+\varepsilon\mathbb{B}_{X^*}+N(\bar{x};C). \tag{3.3}$$

According to Theorem 3.46(ii) (see [35] page 305-306), we have

$$\begin{aligned}\partial\varphi(\bar{x}) &= \partial\left(\max_{k\in K} f_k(\cdot)\right)(\bar{x})\\ &\subset \left\{\sum_{k\in K(\bar{x})}\alpha_k\partial f_k(\bar{x}) \mid \alpha_k\geq 0, k\in K(\bar{x}), \sum_{k\in K(\bar{x})}\alpha_k = 1\right\},\end{aligned} \tag{3.4}$$

where $K(\bar{x}) := \left\{k\in K \mid f_k(\bar{x}) = \max_{k\in K} f_k(\bar{x})\right\} \neq \emptyset$.

On the other hand, by letting $\widetilde{\Omega} := \{x\in X \mid g_i(x)\leq 0, i\in I, h_j(\bar{x}) = 0, j\in J\}$, we have, $C = \widetilde{\Omega}\cap\Omega$. The condition (CQ) being fulfilled at $\bar{x}$ entails that there do not exist $\mu_i\geq 0, i\in I(\bar{x})$ and $\gamma_j\geq 0, j\in J(\bar{x}) = J$ such that $\sum_{i\in I(\bar{x})}\mu_i+\sum_{j\in J}\gamma_j\neq 0$ and

$$0 \in \sum_{i\in I(\bar{x})}\mu_i\partial g_i(\bar{x})+\sum_{j\in J}\gamma_j\left(\partial h_j(\bar{x})\cup\partial(-h_j)(\bar{x})\right).$$

Therefore, by Corollary 4.36 (Mordukhovich, [35]), one has

$$\begin{aligned}N(\bar{x};\widetilde{\Omega}) \subset\ &\left\{\sum_{i\in I(\bar{x})}\mu_i\partial g_i(\bar{x})+\sum_{j\in J}\gamma_j\left(\partial h_j(\bar{x})\cup\partial(-h_j)(\bar{x})\right) \mid \mu_i\geq 0, i\in I(\bar{x}),\right.\\ &\left.\gamma_j\geq 0, j\in J\right\}.\end{aligned} \tag{3.5}$$

Since condition (CQ) is satisfied at $\bar{x}$, it follows from Corollary 3.37 (Mordukhovich, [35]) that

$$N(\bar{x};C) = N(\bar{x};\widetilde{\Omega}\cap\Omega) \subset N(\bar{x};\widetilde{\Omega})+N(\bar{x};\Omega). \tag{3.6}$$

This yields from (3.5)-(3.6) that

$$\begin{aligned}N(\bar{x};C) \subset\ &\left\{\sum_{i\in I(\bar{x})}\mu_i\partial g_i(\bar{x})+\sum_{j\in J}\gamma_j\left(\partial h_j(\bar{x})\cup\partial(-h_j)(\bar{x})\right) \mid \mu_i\geq 0, i\in I(\bar{x}),\right.\\ &\left.\gamma_j\geq 0, j\in J\right\}+N(\bar{x};\Omega).\end{aligned} \tag{3.7}$$

From (3.3), (3.4), and (3.7), we have

$$
\begin{aligned}
&0 \in \left\{ \sum_{k\in K(\bar{x})} \alpha_k \partial f_k(\bar{x}) \mid \alpha_k \geq 0, k \in K(\bar{x}), \sum_{k\in K(\bar{x})} \alpha_k = 1 \right\} + \varepsilon \mathbb{B}_{X^*} + \\
&\left\{ \sum_{i\in I(\bar{x})} \mu_i \partial g_i(\bar{x}) + \sum_{j\in J} \gamma_j \left(\partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x})\right) \mid \mu_i \geq 0, i \in I(\bar{x}), \gamma_j \geq 0, j \in J \right\} \\
&+N(\bar{x};\Omega),
\end{aligned} \tag{3.8}
$$

where $K(\bar{x}) := \left\{ k \in K \mid f_k(\bar{x}) = \max_{k\in K} f_k(\bar{x}) \right\} \neq \emptyset$. Now, we put $\alpha_k := 0$ for $k \in K \backslash K(\bar{x})$, $\mu_i := 0$ for $i \in I \backslash I(\bar{x})$. From (3.8), we can conclude that there exist $\alpha := (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m_+$ with $\sum\limits_{k\in K} \alpha_k = 1, \mu := (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n_+$ and $\gamma := (\gamma_1, \ldots, \gamma_l) \in \mathbb{R}^l_+$ such that

$$
\begin{aligned}
&0 \in \sum_{k\in K} \alpha_k \partial f_k(\bar{x}) + \sum_{i\in I} \mu_i \partial g_i(\bar{x}) + \sum_{j\in J} \gamma_j \left(\partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x})\right) + N(\bar{x};\Omega) + \varepsilon \mathbb{B}_{X^*}, \\
&\alpha_k \left[f_k(\bar{x}) - \max_{k\in K} f_k(\bar{x}) \right] = 0, k \in K, \\
&\mu_i g_i(\bar{x}) = 0, i \in I.
\end{aligned}
$$

The proof is complete. □

Remark 3.1. In Theorem 3.1, if we only consider inequality constraints, then we have [1, Theorem 3.1]. Besides, if $\varepsilon = 0$, then we obtain [34, Theorem 3.3].

The following example illustrates the conditions given in Theorem 3.1.

Example 3.1. Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f(x) = (f_1(x), f_2(x))$ with $f_1(x) = f_2(x) = -\frac{1}{4}|x|, x \in \mathbb{R}$, and let $g, h : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = -x^2, h(x) = x^2 + x, x \in \mathbb{R}$. We consider problem (MP) with $K = \{1,2\}, I = J = \{1\}$ and $\Omega = \mathbb{R}$. By simple computation, one has $C = \{0,1\}$. Now, we take $\bar{x} = 0 \in C, \varepsilon = 1$. Then, it is easy to show that $\bar{x}$ is an ε-quasi solution to problem (MP). Indeed, we have

$$
\max_{k\in K} f_k(x) + \varepsilon ||x - \bar{x}|| = -\frac{1}{4}|x| + |x| \geq 0 = \max_{k\in K} f_k(\bar{x}), \forall x \in C.
$$

On the other hand, take $\mathbb{B}_{X^*} = [-1,1], \bar{x} = 0, \varepsilon = 1, \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2_+$ with $\alpha_1 + \alpha_2 = 1$ and $\mu_i = 1, i \in I, \gamma_j = 1, j \in J$, we have $N(\bar{x};\Omega) = N(\bar{x};\mathbb{R}) = \{0\}$ and $\partial f_k(\bar{x}) = \left\{ -\frac{1}{4}, \frac{1}{4} \right\}, k \in K, \partial g(\bar{x}) = \{1\}, \partial h(\bar{x}) = \{1\}, \partial(-h)(\bar{x}) = \{-1\}$. It is easy to see that

$$
\begin{aligned}
&0 \in \left\{ -\frac{1}{4}, \frac{1}{4} \right\} + \{0\} + \{-1,1\} + \{0\} + [-1,1] \\
&= \sum_{k\in K} \alpha_k \partial f_k(\bar{x}) + \sum_{i\in I} \mu_i \partial g_i(\bar{x}) + \sum_{j\in J} \gamma_j \left(\partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x})\right) + N(\bar{x};\Omega) + \varepsilon \mathbb{B}_{X^*}, \\
&\alpha_k \left[f_k(\bar{x}) - \max_{k\in K} f_k(\bar{x}) \right] = 0, k \in K, \\
&\mu_i g_i(\bar{x}) = 0, i \in I.
\end{aligned}
$$

It is easy to see that the condition (CQ) is satisfied at $\bar{x}=0$. Indeed, there do not exist $\mu_i = 1 \geq 0, i \in I(\bar{x}) = \{1\}$ and $\gamma_j = 1 \geq 0, j \in J(\bar{x}) = \{1\}$ such that $\sum_{i\in I(\bar{x})} \mu_i + \sum_{j\in J(\bar{x})} \gamma_j = 2 \neq 0$ and

$$0 \in \{0\} + \{-1,1\} + \{0\} = \sum_{i\in I(\bar{x})} \mu_i \partial g_i(\bar{x}) + \sum_{j\in J(\bar{x})} \gamma_j \left(\partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x})\right) + N(\bar{x};\Omega).$$

Hence the condition (CQ) is satisfied at $\bar{x}=0$.

Remark 3.2. However, Example 3.1 does not satisfy conditions in [1, Theorem 3.1]. The reason is that the condition (CQ) in [1, Theorem 3.1] is not satisfied at $\bar{x}=0$. Indeed, one has

$$0 \in \{0\} + \{0\} = \sum_{i\in I(\bar{x})} \mu_i \partial g_i(\bar{x}) + N(\bar{x};\Omega),$$

for $\mu_i = 1 \geq 0, i \in I(\bar{x}) = \{1\}$. Hence condition (CQ) is not satisfied at $\bar{x}=0$.

On the other hand, Example 3.1 also does not satisfy the conditions in [34, Theorem 3.1]. It is easy to show that $\bar{x} = 0 \in C$ is not a solution to the problem (MP) in [34, Theorem 3.1]. In order to see this, let us take $\hat{x} = 1 \in C$,

$$\max_{k\in K} f_k(\hat{x}) = -\frac{1}{4}|\hat{x}| = -\frac{1}{4} < 0 = \max_{k\in K} f_k(\bar{x}).$$

It is easy to imply that $\bar{x} = 0 \in C$ is not satisfies condition

$$0 \in \sum_{k\in K} \alpha_k \partial f_k(\bar{x}) + \sum_{i\in I} \mu_i \partial g_i(\bar{x}) + \sum_{j\in J} \gamma_j \left(\partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x})\right) + N(\bar{x};\Omega).$$

Indeed, taking $\bar{x}=0, \alpha = (\alpha_1,\alpha_2) \in \mathbb{R}^2_+$ with $\alpha_1+\alpha_2 = 1$ and $\mu_i = 1, i \in I, \gamma_j = 1, j \in J$, we have $N(\bar{x};\Omega) = N(\bar{x};\mathbb{R}) = \{0\}$ and $\partial f_k(\bar{x}) = \left\{-\frac{1}{4}, \frac{1}{4}\right\}, k \in K, \partial g(\bar{x}) = \{1\}, \partial h(\bar{x}) = \{1\}, \partial(-h)(\bar{x}) = \{-1\}$. It is easy to see that

$$\begin{aligned} 0 &\notin \left\{-\frac{1}{4}, \frac{1}{4}\right\} + \{0\} + \{-1,1\} + \{0\} \\ &= \sum_{k\in K} \alpha_k \partial f_k(\bar{x}) + \sum_{i\in I} \mu_i \partial g_i(\bar{x}) + \sum_{j\in J} \gamma_j \left(\partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x})\right) + N(\bar{x};\Omega). \end{aligned}$$

The following definition is inspired by the Definition 3.5 in [34].

Definition 3.2. We say that (f,g,h) is generalized L-invex-infine on Ω at $\bar{x} \in \Omega$ if, for any $x \in \Omega$, $u_k^* \in \partial f_k(\bar{x}), k \in K$ and $x_i^* \in \partial g_i(\bar{x}), i \in I$ and $y_j^* \in \partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x}), j \in J$, there exists $v \in N(\bar{x};\Omega)^\circ$ such that

$$f_k(x) - f_k(\bar{x}) \geq \langle u_k^*, v\rangle, k \in K,$$
$$g_i(x) - g_i(\bar{x}) \geq \langle x_i^*, v\rangle, i \in I,$$
$$h_j(x) - h_j(\bar{x}) = w_j \left\langle y_j^*, v\right\rangle, j \in J,$$

where $w_j = 1$ (respectively, $w_j = -1$), where $y_j^* \in \partial h_j(\bar{x})$ (respectively, $y_j^* \in \partial(-h_j)(\bar{x})$), and

$$\langle b^*, v\rangle \leq ||x-\bar{x}||, \forall b^* \in \mathbb{B}_{X^*}.$$

Theorem 3.2. *Assume that $\bar{x} \in C$ satisfies condition (3.1). If (f,g,h) is generalized L-invex-infine on Ω at $\bar{x}$, then $\bar{x}$ is an ε-quasi solution to problem* (MP).

Proof. Let $\varphi(x) := \max_{k\in K} f_k(x)$ for $x \in X$. Since $\bar{x} \in C$ satisfies condition (3.1), there exist $\alpha_k \geq 0, k \in K$ with $\sum_{k\in K}\alpha_k = 1, \mu_i \geq 0, i \in I$ and $\gamma_j \geq 0, j \in J$ and $u_k^* \in \partial f_k(\bar{x}), k \in K, x_i^* \in \partial g_i(\bar{x}), i \in I$ and $y_j^* \in \partial h_j(\bar{x}) \cup (-h_j)(\bar{x}), j \in J$, as well as $b^* \in \mathbb{B}_{X^*}$ such that

$$-\left[\sum_{k\in K}\alpha_k u_k^* + \sum_{i\in I}\mu_i x_i^* + \sum_{j\in J}\gamma_j y_j^* + \varepsilon b^*\right] \in N(\bar{x};\Omega), \tag{3.9}$$

$$\alpha_k\left[f_k(\bar{x}) - \max_{k\in K} f_k(\bar{x})\right] = 0, k \in K, \tag{3.10}$$

$$\mu_i g_i(\bar{x}) = 0, i \in I.$$

Suppose on contrary that $\bar{x} \in C$ is not an ε-quasi solution to problem (MP). It then follows that there exists $\hat{x} \in C$ such that

$$\varphi(\hat{x}) + \varepsilon||\hat{x} - \bar{x}|| < \varphi(\bar{x}). \tag{3.11}$$

By the definition of polar cones and the generalized L-invex-infine property of (f, g, h) on Ω at $\bar{x}$, we deduce from (3.9) that, for such $\hat{x} \in \Omega$, there exists $v \in N(\bar{x};\Omega)^\circ$ such that

$$\begin{aligned} 0 &\leq \sum_{k\in K}\alpha_k\langle u_k^*, v\rangle + \sum_{i\in I}\mu_i\langle x_i^*, v\rangle + \sum_{j\in J}\gamma_j\left\langle y_j^*, v\right\rangle + \varepsilon\langle b^*, v\rangle \\ &\leq \sum_{k\in K}\alpha_k[f_k(\hat{x}) - f_k(\bar{x})] + \sum_{i\in I}\mu_i[g_i(\hat{x}) - g_i(\bar{x})] + \sum_{j\in J}\frac{1}{\omega_j}\gamma_j\left[h_j(\hat{x}) - h_j(\bar{x})\right] + \varepsilon||\hat{x} - \bar{x}||, \end{aligned}$$

where $\omega_j \in \{-1, 1\}$. Hence,

$$\sum_{k\in K}\alpha_k[f_k(\hat{x}) - f_k(\bar{x})] + \sum_{i\in I}\mu_i[g_i(\hat{x}) - g_i(\bar{x})] + \sum_{j\in J}\sigma_j\left[h_j(\hat{x}) - h_j(\bar{x})\right] + \varepsilon||\hat{x} - \bar{x}|| \geq 0, \tag{3.12}$$

where $\sigma_j := \dfrac{\gamma_j}{\omega_j} \in \mathbb{R}, j \in J$. Note that $\mu_i g_i(\bar{x}) = 0, \mu_i g_i(\hat{x}) \leq 0, i \in I$ and $h_j(\bar{x}) = 0, h_j(\hat{x}) = 0, j \in J$. Thus

$$\sum_{i\in I}\mu_i[g_i(\hat{x}) - g_i(\bar{x})] + \sum_{j\in J}\sigma_j\left[h_j(\hat{x}) - h_j(\bar{x})\right] \leq 0. \tag{3.13}$$

It follows from (3.12) and (3.13) that $\sum_{k\in K}\alpha_k[f_k(\hat{x}) - f_k(\bar{x})] + \varepsilon||\hat{x} - \bar{x}|| \geq 0$. Besides, by (3.10), it implies that

$$\begin{aligned} \sum_{k\in K}\alpha_k\varphi(\bar{x}) &= \sum_{k\in K}\alpha_k f_k(\bar{x}) \\ &\leq \sum_{k\in K}\alpha_k f_k(\hat{x}) + \varepsilon||\hat{x} - \bar{x}|| \\ &\leq \sum_{k\in K}\alpha_k \max_{k\in K} f_k(\hat{x}) + \varepsilon||\hat{x} - \bar{x}|| \\ &= \sum_{k\in K}\alpha_k\varphi(\hat{x}) + \varepsilon||\hat{x} - \bar{x}||. \end{aligned}$$

In view of $\alpha := (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}_+^m$ with $\sum_{k\in K}\alpha_k = 1$, one has $\varphi(\hat{x}) + \varepsilon||\hat{x} - \bar{x}|| \geq \varphi(\bar{x})$, which contradicts (3.11). Therefore, $\bar{x}$ is an ε-quasi efficient solution to problem (MP). □

Remark 3.3. In Theorem 3.2, if we only consider inequality constraints, then we [1, Theorem 3.2]. Besides, if $\varepsilon = 0$, then we obtain [34, Theorem 3.6].

Now, we present an example to show the importance of the generalized L-invex-infine property in Theorem 3.2.

Example 3.2. Let $f:\mathbb{R}\to\mathbb{R}^2$ be defined by $f(x)=(f_1(x),f_2(x))$ with

$$f_1(x)=f_2(x)=\begin{cases} x^2\cos\dfrac{1}{x}, & \text{if } x\neq 0,\\ 0, & \text{if } x=0,\end{cases}$$

and let $g,h:\mathbb{R}\to\mathbb{R}$ be given by $g(x)=x-1, h(x)=0$ for all $x\in\mathbb{R}$. We consider the problem (MP) with $K=\{1,2\}, I=J=\{1\}$ and $\Omega=\mathbb{R}$. By simple computation, one has

$$C:=\{x\in\Omega \mid g_i(x)\le 0, i\in I, h_j(x)=0, j\in J\}=(-\infty,1].$$

By choosing $\bar{x}=0\in C, \mu_i=0, i\in I, \varepsilon=\dfrac{1}{4\pi}, \mathbb{B}_{X^*}=[-1,1]$, it is easy to see that $N(\bar{x};\Omega)=N(\bar{x};\mathbb{R})=\{0\}$, $\partial f_1(\bar{x})=f_2(\bar{x})=[-1,1]$, and $\partial g(\bar{x})=\{1\}, \partial h(\bar{x})=\{0\}, \partial(-h)(\bar{x})=\{0\}, \alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^2_+$ with $\alpha_1+\alpha_2=1$. Therefore, we have

$$\begin{aligned}
&0\in[-1,1]+\{0\}+\{0\}+\left[-\frac{1}{4\pi},\frac{1}{4\pi}\right]\\
&=\sum_{k\in K}\alpha_k\partial f_k(\bar{x})+\sum_{i\in I}\mu_i\partial g_i(\bar{x})+\sum_{j\in J}\gamma_j\left(\partial h_j(\bar{x})\cup\partial(-h_j)(\bar{x})\right)+N(\bar{x};\Omega)+\varepsilon\mathbb{B}_{X^*},\\
&\alpha_k\left[f_k(\bar{x})-\max_{k\in K}f_k(\bar{x})\right]=0, k\in K,\\
&\mu_i g_i(\bar{x})=0, i\in I.
\end{aligned}$$

Hence, condition

$$\begin{aligned}
&0\in\sum_{k\in K}\alpha_k\partial f_k(\bar{x})+\sum_{i\in I}\mu_i\partial g_i(\bar{x})+\sum_{j\in J}\gamma_j\left(\partial h_j(\bar{x})\cup\partial(-h_j)(\bar{x})\right)+N(\bar{x};\Omega)+\varepsilon\mathbb{B}_{X^*},\\
&\alpha_k\left[f_k(\bar{x})-\max_{k\in K}f_k(\bar{x})\right]=0, k\in K,\\
&\mu_i g_i(\bar{x})=0, i\in I
\end{aligned}$$

holds. However $\bar{x}=0$ is not an ε-quasi solution to problem (MP). Indeed, we can choose $x=\dfrac{1}{3\pi}\in C$. Clearly,

$$\max_{k\in K}f_k(x)+\varepsilon||x-\bar{x}||=-\frac{1}{9\pi^2}+\frac{1}{12\pi^2}<0=\max_{k\in K}f_k(\bar{x}).$$

The reason is that f is (f,g,h) is not generalized L-invex-infine on Ω at $\bar{x}=0$. Indeed, take $\hat{x}=\dfrac{1}{\pi}\in\Omega, u_k^*=0\in\partial f_k(\bar{x})=[-1,1], k\in K$. Then, it is easy to see that $N(\bar{x};\Omega)^\circ=N(\bar{x};\mathbb{R})^\circ=\mathbb{R}$ and

$$f_k(\hat{x})-f_k(\bar{x})=-\frac{1}{\pi}<0=\langle u_k^*,v\rangle, \forall v\in N(\bar{y};\Omega)^\circ, k\in K.$$

4. Duality Theorems

4.1. **Wolfe type duality.** In this section, we consider the dual problem of the problem (MP) in the Wolfe type.

For $x\in X, := (\alpha_k), \alpha_k\ge 0, k\in K, \mu:=(\mu_i), \mu_i\ge 0, i\in I, \gamma:=(\gamma_j), \gamma_j\ge 0, j\in J$ and $f:=(f_k), k\in K, g_i:=(g_i), i\in I, h:=(h_j), j\in J$, Ω is a nonempty locally closed subset of X. Let

$$L(x,\alpha,\mu,\gamma):=\varphi(x)+\sum_{i\in I}\mu_i g_i(x)+\sum_{j\in J}\gamma_j h_j(x),$$

where $\varphi(x) = \max_{k\in K} f_k(x)$. For $\varepsilon \geq 0$, we consider the Wolfe type dual problem (WMD) with respect to the primal problem (MP) as follows:

$$
\text{(WMD)} \quad \begin{cases} \max & L(y,\alpha,\mu,\gamma) \\ \text{s.t.} & 0 \in \sum_{k\in K} \alpha_k \partial f_k(y) + \sum_{i\in I} \mu_i \partial g_i(y) + \sum_{j\in J} \gamma_j \left[\partial h_j(y) \cup \partial(-h_j)(y)\right] \\ & +N(y;\Omega) + \varepsilon \mathbb{B}_{X^*}, \\ & \alpha_k\left(f_k(y) - \varphi(y)\right) = 0, k \in K, \\ & \sum_{j\in J} \gamma_j h_i(y) - \sum_{j\in J} \sigma_j h_j(y) \leq 0, \forall \sigma := (\sigma_1,\ldots,\sigma_l) \in \mathbb{S}\left(0, ||\gamma||\right), \\ & y \in \Omega, \alpha := (\alpha_1,\ldots,\alpha_m) \in \mathbb{R}^m_+, \sum_{k\in K} \alpha_k = 1, \\ & \mu := (\mu_1,\ldots,\mu_n) \in \mathbb{R}^n_+, \gamma := (\gamma_1,\ldots,\gamma_l) \in \mathbb{R}^l_+, \end{cases}
$$

where $\mathbb{S}\left(0, ||\gamma||\right) := \left\{\sigma := (\sigma_1,\ldots,\sigma_l) \in \mathbb{R}^l \mid ||\sigma|| = ||\gamma||\right\}$.

The feasible set of problem (WMD) is defined by

$$
\begin{aligned} C_{\text{WMD}} := \Bigg\{ & (y,\alpha,\mu,\gamma) \in \Omega \times \mathbb{R}^m_+ \times \mathbb{R}^n_+ \times \mathbb{R}^l_+ \mid 0 \in \sum_{k\in K} \alpha_k \partial f_k(y) \\ & + \sum_{i\in I} \mu_i \partial g_i(y) + \sum_{j\in J} \gamma_j \left[\partial h_j(y) \cup \partial(-h_j)(y)\right] + N(y;\Omega) + \varepsilon \mathbb{B}_{X^*}, \\ & \alpha_k\left(f_k(y) - \varphi(y)\right) = 0, k \in K, \sum_{k\in K} \alpha_k = 1, \\ & \sum_{j\in J} \gamma_j h_j(y) - \sum_{j\in J} \sigma_j h_j(y) \leq 0, \forall \sigma := (\sigma_1,\ldots,\sigma_l) \in \mathbb{S}\left(0, ||\gamma||\right) \Bigg\}, \end{aligned}
$$

where $\mathbb{S}\left(0, ||\gamma||\right) := \left\{\sigma := (\sigma_1,\ldots,\sigma_l) \in \mathbb{R}^l \mid ||\sigma|| = ||\gamma||\right\}$.

Definition 4.1. Let $\varepsilon \geq 0$. A point $(\bar{y},\bar{\alpha},\bar{\mu},\bar{\gamma}) \in C_{\text{WMD}}$ is said to be an ε-quasi solution to problem (WMD) if

$$
L(\bar{y},\bar{\alpha},\bar{\mu},\bar{\gamma}) + \varepsilon||(\bar{y},\bar{\alpha},\bar{\mu},\bar{\gamma}) - (y,\alpha,\mu,\gamma)|| \geq L(y,\alpha,\mu,\gamma), \forall (y,\alpha,\mu,\gamma) \in C_{\text{WMD}}.
$$

Theorem 4.1. (ε-quasi weak duality) *Suppose that $x \in C$ and $(y,\alpha,\mu,\gamma) \in C_{\text{WMD}}$. If (f,g,h) is generalized $L-$invex–infine on Ω at y, then $\varphi(x) + \varepsilon||x-y|| \geq L(y,\alpha,\mu,\gamma)$, where $\varphi(x) = \max_{k\in K} f_k(x)$.*

Proof. Since $(y,\alpha,\mu,\gamma) \in C_{\text{WMD}}$, there exist $\alpha := (\alpha_k)_{k\in K}, \alpha_k \geq 0$ with $\sum_{k\in K} \alpha_k = 1, u^*_k \in \partial f_k(y), k \in K, \mu := (\mu_i)_{i\in I}, \mu_i \geq 0, x^*_i \in \partial g_i(y), i \in I$ and $\gamma := (\gamma_j)_{j\in J}, \gamma_j \geq 0, y^*_j \in \partial h_j(y) \cup \partial(-h_j)(y)$, as well as $b^* \in \mathbb{B}_{X^*}$ such that

$$
-\left(\sum_{k\in K} \alpha_k u^*_k + \sum_{i\in I} \mu_i x^*_i + \sum_{j\in J} \gamma_j y^*_j + \varepsilon b^*\right) \in N(y;\Omega), \tag{4.1}
$$

$$
\alpha_k\left(f_k(y) - \varphi(y)\right) = 0, k \in K, \tag{4.2}
$$

and

$$
\sum_{j\in J} \gamma_j h_j(y) - \sum_{j\in J} \sigma_j h_j(y) \leq 0 \text{ for all } \sigma \in \mathbb{R}^l \text{ with } ||\sigma|| = ||\gamma||. \tag{4.3}
$$

By the definition of polar cones, the generalized L-invex-infine property of (f,g,h) on Ω at y, and (4.1), for such $x \in C$, there exists $v \in N(y;\Omega)^{\circ}$ such that

$$\begin{aligned} 0 &\leq \sum_{k\in K} \alpha_k \langle u_k^*, v\rangle + \sum_{i\in I} \mu_i \langle x_i^*, v\rangle + \sum_{i\in I} \gamma_j \langle y_j^*, v\rangle + \varepsilon \langle b^*, v\rangle \\ &\leq \sum_{k\in K} \alpha_k [f_k(x) - f_k(y)] + \sum_{i\in I} \mu_i [g_i(x) - g_i(y)] + \sum_{j\in J} \frac{1}{\omega_j} \gamma_j \left[h_j(x) - h_j(y)\right] + \varepsilon ||x-y||, \end{aligned}$$

where $\omega_j \in \{-1,1\}, j \in J$. Thus

$$\sum_{k\in K} \alpha_k [f_k(x) - f_k(y)] + \sum_{i\in I} \mu_i [g_i(x) - g_i(y)] + \sum_{j\in J} \sigma_j \left[h_j(x) - h_j(y)\right] + \varepsilon ||x-y|| \geq 0, \tag{4.4}$$

where $\sigma_j := \frac{\gamma_j}{\omega_j} \in \mathbb{R}, j \in J$. From $x \in C$, it is obvious that $\sum_{i\in I} \mu_i g_i(x) \leq 0$ and $\sum_{j\in J} \sigma_j h_j(x) = 0$. Therefore, one implies from (4.4) that

$$\sum_{k\in K} \alpha_k [f_k(x) - f_k(y)] - \sum_{i\in I} \mu_i g_i(y) - \sum_{j\in J} \sigma_j h_j(y) + \varepsilon ||x-y|| \geq 0,$$

where $\sigma := (\sigma_1, \ldots, \sigma_l) \in \mathbb{R}^l$. By (4.2), we see that

$$\sum_{k\in K} \alpha_k f_k(x) - \varphi(y) - \sum_{i\in I} \mu_i g_i(y) - \sum_{j\in J} \sigma_j h_j(y) + \varepsilon ||x-y|| \geq 0,$$

or equivalently,

$$\sum_{k\in K} \alpha_k f_k(x) - \left(L(y,\alpha,\mu,\gamma) - \sum_{j\in J} \gamma_j h_j(y) \right) - \sum_{j\in J} \sigma_j h_j(y) + \varepsilon ||x-y|| \geq 0.$$

Since $||\sigma|| = ||\gamma||$, it follows from (4.3) that $\sum_{k\in K} \alpha_k f_k(x) - L(y,\alpha,\mu,\gamma) + \varepsilon ||x-y|| \geq 0$, which implies that (Since $(\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m_+$ such that $\sum_{k\in K} \alpha_k = 1$)

$$\begin{aligned} L(y,\alpha,\mu,\gamma) &\leq \sum_{k\in K} \alpha_k f_k(x) + \varepsilon ||x-y|| \\ &\leq \sum_{k\in K} \alpha_k \max_{k\in K} f_k(x) + \varepsilon ||x-y|| \\ &= \sum_{k\in K} \alpha_k \varphi(x) + \varepsilon ||x-y|| \\ &= \varphi(x) + \varepsilon ||x-y||. \end{aligned}$$

The proof is complete. □

Remark 4.1. In Theorem 4.1, if we only consider inequality constraints, then we have [1, Theorem 4.1]. Besides, if $\varepsilon = 0$, then we obtain [34, Theorem 4.1].

Theorem 4.2. (ε-quasi strong duality) *Let $\bar{x} \in C$ be an ε-quasi solution to problem* (MP). *Suppose that the condition* (CQ) *for problem* (MP) *is satisfied at $\bar{x} \in C$. Then there exists $(\bar{\alpha}, \bar{\mu}, \bar{\gamma}) \in \mathbb{R}^m_+ \times \mathbb{R}^n_+ \times \mathbb{R}^l_+$ such that $(\bar{x}, \bar{\alpha}, \bar{\mu}, \bar{\gamma}) \in C_{\mathrm{WMD}}$ and $\varphi(\bar{x}) = L(\bar{x}, \bar{\alpha}, \bar{\mu}, \bar{\gamma})$, where $\varphi(\bar{x}) = \max_{k\in K} f_k(\bar{x})$. If (f,g,h) is generalized L-invex-infine on Ω at $y \in \Omega$, then $(\bar{x}, \bar{\alpha}, \bar{\mu}, \bar{\gamma})$ is an ε-quasi solution to problem* (WMD).

Proof. Let $\bar{x} \in C$ be an ε-quasi solution to problem (MP). According to Theorem 3.1, then there exist $\alpha_k \geq 0, k \in K$ with $\sum_{k\in K} \alpha_k = 1, \mu_i \geq 0, i \in I$ and $\gamma_j \geq 0, j \in J$ such that

$$\begin{aligned} &0 \in \sum_{k\in K} \alpha_k f_k(\bar{x}) + \sum_{i\in I} \mu_i \partial g_i(\bar{x}) + \sum_{j\in J} \gamma_j \left(\partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x})\right) + \varepsilon \mathbb{B}_{X^*} + N(\bar{x};\Omega), \\ &\alpha_k \left(f_k(\bar{x}) - \varphi(\bar{x})\right) = 0, k \in K, \\ &\quad \mu_i g_i(\bar{x}) = 0, \qquad i \in I. \end{aligned} \tag{4.5}$$

Putting

$$\bar{\alpha}_k := \frac{\alpha_k}{\sum\limits_{k\in K} \alpha_k}, k \in K, \bar{\mu}_i := \frac{\mu_i}{\sum\limits_{k\in K} \alpha_k}, i \in I, \bar{\gamma}_j := \frac{\gamma_j}{\sum\limits_{k\in K} \alpha_k}, j \in J,$$

one has $\bar{\alpha} := (\bar{\alpha}_1, \ldots, \bar{\alpha}_m) \in \mathbb{R}^m_+$ with $\sum_{k\in K} \bar{\alpha}_k = 1$, $\bar{\mu} := (\bar{\mu}_i)_{i\in I}, \bar{\mu}_i \geq 0, i \in I, \bar{\gamma} := (\bar{\gamma}_j)_{j\in J}, \bar{\gamma}_j \geq 0, j \in J$. Furthermore, the assertion in (4.5) is also valid when α_k's, μ_i's and γ_j's are replaced by $\bar{\alpha}_k$'s, $\bar{\mu}_i$'s and $\bar{\gamma}_j$'s, respectively.

On the other hand, from $\bar{x} \in C$, one has $h_j(\bar{x}) = 0, j \in J$. It is easy to imply that $\sum_{j\in J} \bar{\gamma} h_j(\bar{x}) = 0$ and $\sum_{j\in J} \sigma_j h_j(\bar{x}) = 0$, for all $\sigma \in \mathbb{R}^l$ with $||\sigma|| = ||\bar{\gamma}||$. This implies that

$$\sum_{j\in J} \bar{\gamma}_j h_j(\bar{x}) - \sum_{j\in J} \sigma_j h_j(\bar{x}) \leq 0 \text{ for all } \sigma \in \mathbb{R}^l \text{ with } ||\sigma|| = ||\bar{\gamma}||.$$

So, $(\bar{x}, \bar{\alpha}, \bar{\mu}, \bar{\gamma}) \in C_{\mathrm{WMD}}$. Besides, one has $\sum_{i\in I} \bar{\mu}_i g_i(\bar{x}) = \sum_{j\in J} \bar{\gamma}_j h_j(\bar{x}) = 0$. Therefore, one has

$$\varphi(\bar{x}) = \varphi(\bar{x}) + \sum_{i\in I} \bar{\mu}_i g_i(\bar{x}) + \sum_{j\in J} \bar{\gamma}_j h_j(\bar{x}) = L(\bar{x}, \bar{\alpha}, \bar{\mu}, \bar{\gamma}).$$

If (f,g,h) is generalized L-invex-infine on Ω at any $y \in \Omega$, then one finds by Theorem 4.1 that

$$L(\bar{x}, \bar{\alpha}, \bar{\mu}, \bar{\gamma}) + \varepsilon ||(\bar{x}, \bar{\alpha}, \bar{\mu}, \bar{\gamma}) - (y, \alpha, \mu, \gamma)|| = \varphi(\bar{x}) + \varepsilon ||(\bar{x}, \bar{\alpha}, \bar{\mu}, \bar{\gamma}) - (y, \alpha, \mu, \gamma)|| \geq L(y, \alpha, \mu, \gamma),$$

for any $(y, \alpha, \mu, \gamma) \in C_{\mathrm{WMD}}$. This means that $(\bar{x}, \bar{\alpha}, \bar{\mu}, \bar{\gamma})$ is an ε-quasi solution to problem (WMD). The proof is complete. □

Remark 4.2. In Theorem 4.2, if we only consider inequality constraints, then we have [1, Theorem 4.2]. Besides, if $\varepsilon = 0$, then we obtain [34, Theorem 4.3].

If $f_k, k \in K$ and $g_i, i \in I, h_j, j \in J$ are continuously differentiable and convex, then we have the following corollary.

Corollary 4.1. *Suppose that Ω is convex and $f_k, k \in K, g_i, i \in I, h_j, j \in J$ are continuously differentiable and convex on Ω. Let $y \in \Omega$ and $\bar{x} \in C$ be an ε-quasi solution of problem* (MP). *Assume that the condition* (CQ) *for problem* (MP) *is satisfied at $\bar{x} \in C$. Then there exists $(\bar{\alpha}, \bar{\mu}, \bar{\gamma}) \in \mathbb{R}^m_+ \times \mathbb{R}^n_+ \times \mathbb{R}^l_+$ such that $(\bar{x}, \bar{\alpha}, \bar{\mu}, \bar{\gamma}) \in C_{\mathrm{WMD}}$ and $\varphi(\bar{x}) = L(\bar{x}, \bar{\alpha}, \bar{\mu}, \bar{\gamma})$, where $\varphi(\bar{x}) = \max_{k\in K} f_k(\bar{x})$. Moreover, $(\bar{x}, \bar{\alpha}, \bar{\mu}, \bar{\gamma})$ is an ε-quasi solution to problem* (WMD).

Theorem 4.3. (ε-quasi converse duality) *Let $(\bar{x}, \bar{\alpha}, \bar{\mu}, \bar{\gamma}) \in C_{\mathrm{WMD}}$ such that $\varphi(\bar{x}) = L(\bar{x}, \bar{\alpha}, \bar{\mu}, \bar{\gamma})$. If $\bar{x} \in C$ and (f,g,h) is generalized L-invex-infine on Ω at $\bar{x}$, then $\bar{x}$ is an ε-quasi solution to problem* (MP).

Proof. Since $(\bar{x}, \bar{\alpha}, \bar{\mu}, \bar{\gamma}) \in C_{\mathrm{WMD}}$, there exist $\bar{\alpha} := (\bar{\alpha}_k)_{k\in K}, \bar{\alpha}_k \geq 0$ with $\sum_{k\in K} \bar{\alpha}_k = 1, u^*_k \in \partial f_k(\bar{x}), k \in K, \bar{\mu} := (\bar{\mu}_i)_{i\in I}, \bar{\mu}_i \geq 0, x^*_i \in \partial g_i(\bar{x}), i \in I$ and $\bar{\gamma} := (\bar{\gamma}_j)_{j\in J}, \bar{\gamma}_j \geq 0, y^*_j \in \partial h_j(\bar{x}) \cup$

$\partial(-h_j)(\bar{x})$, as well as $b^* \in \mathbb{B}_{X^*}$ such that

$$-\left(\sum_{k\in K}\bar{\alpha}_k u_k^* + \sum_{i\in I}\bar{\mu}_i x_i^* + \sum_{j\in J}\bar{\gamma}_j y_j^* + \varepsilon b^*\right) \in N(\bar{x};\Omega), \tag{4.6}$$

$$\bar{\alpha}_k\left(f_k(\bar{x}) - \varphi(\bar{x})\right) = 0, k \in K, \tag{4.7}$$

and

$$\sum_{j\in J}\bar{\gamma}_j h_j(\bar{x}) - \sum_{j\in J}\bar{\sigma}_j h_j(\bar{x}) \le 0 \text{ for all } \bar{\sigma} \in \mathbb{R}^l \text{ with } ||\bar{\sigma}|| = ||\bar{\gamma}||. \tag{4.8}$$

Suppose on contrary that $\bar{x} \in C$ is not an ε-quasi solution to problem (MP). It then follows that there exists $\hat{x} \in C$ satisfying

$$\varphi(\hat{x}) + \varepsilon||\hat{x} - \bar{x}|| < \varphi(\bar{x}). \tag{4.9}$$

By the generalized L-invex-infine property of (f, g, h) on Ω at $\bar{x}$, the definition of polar cones and (4.6), for such $\hat{x} \in C$, there exists $v \in N(\bar{x};\Omega)^\circ$ such that

$$\begin{aligned} 0 &\le \sum_{k\in K}\bar{\alpha}_k\langle u_k^*, v\rangle + \sum_{i\in I}\bar{\mu}_i\langle x_i^*, v\rangle + \sum_{i\in I}\bar{\gamma}_j\left\langle y_j^*, v\right\rangle + \varepsilon\langle b^*, v\rangle \\ &\le \sum_{k\in K}\bar{\alpha}_k\left[f_k(\hat{x}) - f_k(\bar{x})\right] + \sum_{i\in I}\bar{\mu}_i\left[g_i(\hat{x}) - g_i(\bar{x})\right] + \sum_{j\in J}\frac{1}{\omega_j}\bar{\gamma}_j\left[h_j(\hat{x}) - h_j(\bar{x})\right] + \varepsilon||\hat{x} - \bar{x}||, \end{aligned}$$

where $\omega_j \in \{-1, 1\}, j \in J$. Thus

$$\sum_{k\in K}\bar{\alpha}_k\left[f_k(\hat{x}) - f_k(\bar{x})\right] + \sum_{i\in I}\bar{\mu}_i\left[g_i(\hat{x}) - g_i(\bar{x})\right] + \sum_{j\in J}\bar{\sigma}_j\left[h_j(\hat{x}) - h_j(\bar{x})\right] + \varepsilon||\hat{x} - x|| \ge 0, \tag{4.10}$$

where $\bar{\sigma}_j := \dfrac{\bar{\gamma}_j}{\omega_j} \in \mathbb{R}, j \in J$. Besides, from $\hat{x} \in C$, it is obvious that $\sum_{i\in I}\bar{\mu}_i g_i(\hat{x}) \le 0$ and $\sum_{j\in J}\bar{\sigma}_j h_j(\hat{x}) = 0$. Therefore, one implies from (4.10) that

$$\sum_{k\in K}\bar{\alpha}_k\left[f_k(\hat{x}) - f_k(\bar{x})\right] - \sum_{i\in I}\bar{\mu}_i g_i(\bar{x}) - \sum_{j\in J}\bar{\sigma}_j h_j(\bar{x}) + \varepsilon||\hat{x} - \bar{x}|| \ge 0,$$

where $\bar{\sigma} := (\bar{\sigma}_1, \ldots, \bar{\sigma}_l) \in \mathbb{R}^l$. Since $||\bar{\sigma}|| = ||\bar{\gamma}||$, it follows from (4.8) that

$$\sum_{k\in K}\bar{\alpha}_k\left[f_k(\hat{x}) - f_k(\bar{x})\right] - \sum_{i\in I}\bar{\mu}_i g_i(\bar{x}) - \sum_{j\in J}\bar{\gamma}_j h_j(\bar{x}) + \varepsilon||\hat{x} - \bar{x}|| \ge 0. \tag{4.11}$$

Combining (4.7) and (4.11), one has

$$\sum_{k\in K}\bar{\alpha}_k f_k(\hat{x}) - \varphi(\bar{x}) - \sum_{i\in I}\bar{\mu}_i g_i(\bar{x}) - \sum_{j\in J}\bar{\gamma}_j h_j(\bar{x}) + \varepsilon||\hat{x} - \bar{x}|| \ge 0. \tag{4.12}$$

In addition, since

$$\varphi(\bar{x}) = L(\bar{x}, \bar{\alpha}, \bar{\mu}, \bar{\gamma}) = \varphi(\bar{x}) + \sum_{i\in I}\bar{\mu}_i g_i(\bar{x}) + \sum_{j\in J}\bar{\gamma}_j h_j(\bar{x}),$$

it follows that $\sum_{i\in I}\bar{\mu}_i g_i(\bar{x}) + \sum_{j\in J}\bar{\gamma}_j h_j(\bar{x}) = 0$. This together with (4.12) implies that (Since $(\bar{\alpha}_1, \ldots, \bar{\alpha}_m) \in \mathbb{R}^m_+$ such that $\sum_{k\in K}\bar{\alpha}_k = 1$)

$$\varphi(\bar{x}) \le \sum_{k\in K}\bar{\alpha}_k f_k(\hat{x}) + \varepsilon||\hat{x} - \bar{x}|| \le \sum_{k\in K}\bar{\alpha}_k \max_{k\in K} f_k(\hat{x}) + \varepsilon||\hat{x} - \bar{x}|| = \varphi(\hat{x}) + \varepsilon||\hat{x} - \bar{x}||,$$

which contradicts (4.9). The proof is complete. $\square$

Remark 4.3. In Theorem 4.3, if we only consider inequality constraints, then we have [1, Theorem 4.3]. Besides, if $\varepsilon = 0$, then we obtain [34, Theorem 4.4].

4.2. **Mond–Weir Type duality.** In this section, we consider the dual problem of problem (MP) in the Mond–Weir type. Note that results related to the Mond–Weir type were not studied in [1, 34]. Therefore, the obtained results in this subsection are new.

For $x \in X, \alpha := (\alpha_k), \alpha_k \geq 0, k \in K, \mu := (\mu_i), \mu_i \geq 0, i \in I, \gamma := (\gamma_j), \gamma_j \geq 0, j \in J$ and $f := (f_k), k \in K, g_i := (g_i), i \in I, h := (h_j), j \in J$, Ω is a nonempty locally closed subset of X. Let $L(x, \alpha, \mu, \gamma) := \varphi(x)$, where $\varphi(x) = \max_{k \in K} f_k(x)$. For $\varepsilon \geq 0$. We consider the Mond–Weir type dual problem (MWMD) with respect to the primal problem (MP) as follows:

$$\text{(MWMD)} \quad \begin{cases} \max & L(y, \alpha, \mu, \gamma) \\ \text{s.t.} & 0 \in \sum_{k \in K} \alpha_k \partial f_k(y) + \sum_{i \in I} \mu_i \partial g_i(y) + \sum_{j \in J} \gamma_j \left[\partial h_j(y) \cup \partial(-h_j)(y)\right] \\ & +N(y; \Omega) + \varepsilon \mathbb{B}_{X^*}, \alpha_k \left(f_k(y) - \varphi(y)\right) = 0, k \in K, \\ & \sum_{i \in I} \mu_i g_i(y) + \sum_{j \in J} \sigma_j h_j(y) \geq 0, \forall \sigma := (\sigma_1, \ldots, \sigma_l) \in \mathbb{S}\left(0, ||\gamma||\right), \\ & y \in \Omega, \alpha := (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m_+, \sum_{k \in K} \alpha_k = 1, \\ & \mu := (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n_+, \gamma := (\gamma_1, \ldots, \gamma_l) \in \mathbb{R}^l_+, \end{cases}$$

where $\mathbb{S}\left(0, ||\gamma||\right) := \left\{\sigma := (\sigma_1, \ldots, \sigma_l) \in \mathbb{R}^l \mid ||\sigma|| = ||\gamma||\right\}$.

The feasible set of problem (MWMD) is defined by

$$C_{\text{MWMD}} := \Bigg\{ (y, \alpha, \mu, \gamma) \in \Omega \times \mathbb{R}^m_+ \times \mathbb{R}^n_+ \times \mathbb{R}^l_+ \mid 0 \in \sum_{k \in K} \alpha_k \partial f_k(y) + \sum_{i \in I} \mu_i \partial g_i(y)$$
$$+ \sum_{j \in J} \gamma_j \left[\partial h_j(y) \cup \partial(-h_j)(y)\right] + N(y; \Omega) + \varepsilon \mathbb{B}_{X^*}, \alpha_k \left(f_k(y) - \varphi(y)\right) = 0, k \in K, \sum_{k \in K} \alpha_k = 1,$$
$$\sum_{i \in I} \mu_i g_i(y) + \sum_{j \in J} \sigma_j h_j(y) \geq 0, \forall \sigma := (\sigma_1, \ldots, \sigma_l) \in \mathbb{S}\left(0, ||\gamma||\right) \Bigg\},$$

where $\mathbb{S}\left(0, ||\gamma||\right) := \left\{\sigma := (\sigma_1, \ldots, \sigma_l) \in \mathbb{R}^l \mid ||\sigma|| = ||\gamma||\right\}$.

Definition 4.2. Let $\varepsilon \geq 0$. A point $(\bar{y}, \bar{\alpha}, \bar{\mu}, \bar{\gamma}) \in C_{\text{MWMD}}$ is said to be an ε-quasi solution to problem (MWMD) if

$$L(\bar{y}, \bar{\alpha}, \bar{\mu}, \bar{\gamma}) + \varepsilon ||(\bar{y}, \bar{\alpha}, \bar{\mu}, \bar{\gamma}) - (y, \alpha, \mu, \gamma)|| \geq L(y, \alpha, \mu, \gamma), \forall (y, \alpha, \mu, \gamma) \in C_{\text{MWMD}}.$$

Theorem 4.4. (ε-quasi weak duality) *Suppose that $x \in C$ and $(y, \alpha, \mu, \gamma) \in C_{\text{MWMD}}$. If (f, g, h) is generalized L-invex-infine on Ω at y, then $\varphi(x) + \varepsilon ||x - y|| \geq L(y, \alpha, \mu, \gamma)$, where $\varphi(x) = \max_{k \in K} f_k(x)$.*

Proof. Since $(y, \alpha, \mu, \gamma) \in C_{\text{MWMD}}$, there exist $\alpha := (\alpha_k)_{k \in K}, \alpha_k \geq 0$ with $\sum_{k \in K} \alpha_k = 1, u^*_k \in \partial f_k(y), k \in K, \mu := (\mu_i)_{i \in I}, \mu_i \geq 0, x^*_i \in \partial g_i(y), i \in I$ and $\gamma := (\gamma_j)_{j \in J}, \gamma_j \geq 0, y^*_j \in \partial h_j(y) \cup \partial(-h_j)(y)$, as well as $b^* \in \mathbb{B}_{X^*}$ such that

$$-\left(\sum_{k \in K} \alpha_k u^*_k + \sum_{i \in I} \mu_i x^*_i + \sum_{j \in J} \gamma_j y^*_j + \varepsilon b^*\right) \in N(y; \Omega), \tag{4.13}$$

$$\alpha_k \left(f_k(y) - \varphi(y)\right) = 0, k \in K, \tag{4.14}$$

and

$$\sum_{i \in I} \mu_i g_i(y) + \sum_{j \in J} \sigma_j h_j(y) \geq 0 \text{ for all } \sigma \in \mathbb{R}^l \text{ with } ||\sigma|| = ||\gamma||. \tag{4.15}$$

By the definition of the polar cone, the generalized L-invex-infine property of (f,g,h) on Ω at y and (4.13), for such $x \in C$, there exists $v \in N(y;\Omega)^\circ$ such that

$$\begin{aligned} 0 &\leq \sum_{k\in K} \alpha_k \langle u_k^*, v\rangle + \sum_{i\in I} \mu_i \langle x_i^*, v\rangle + \sum_{i\in I} \gamma_j \langle y_j^*, v\rangle + \varepsilon \langle b^*, v\rangle \\ &\leq \sum_{k\in K} \alpha_k [f_k(x) - f_k(y)] + \sum_{i\in I} \mu_i [g_i(x) - g_i(y)] + \sum_{j\in J} \frac{1}{\omega_j} \gamma_j \left[h_j(x) - h_j(y)\right] + \varepsilon ||x-y||, \end{aligned}$$

where $\omega_j \in \{-1,1\}, j \in J$. Thus

$$\sum_{k\in K} \alpha_k [f_k(x) - f_k(y)] + \sum_{i\in I} \mu_i [g_i(x) - g_i(y)] + \sum_{j\in J} \sigma_j \left[h_j(x) - h_j(y)\right] + \varepsilon ||x-y|| \geq 0, \quad (4.16)$$

where $\sigma_j := \frac{\gamma_j}{\omega_j} \in \mathbb{R}, j \in J$. From $x \in C$, it is obvious that $\sum_{i\in I} \mu_i g_i(x) \leq 0$ and $\sum_{j\in J} \sigma_j h_j(x) = 0$. Therefore, one implies from (4.16) that

$$\sum_{k\in K} \alpha_k [f_k(x) - f_k(y)] - \sum_{i\in I} \mu_i g_i(y) - \sum_{j\in J} \sigma_j h_j(y) + \varepsilon ||x-y|| \geq 0,$$

where $\sigma := (\sigma_1, \ldots, \sigma_l) \in \mathbb{R}^l$. Since $||\sigma|| = ||\gamma||$, (4.15) yields that $\sum_{k\in K} \alpha_k [f_k(x) - f_k(y)] + \varepsilon ||x-y|| \geq 0$. It follows by (4.14) that $\sum_{k\in K} \alpha_k f_k(x) - \varphi(y) + \varepsilon ||x-y|| \geq 0$, which implies that (Since $(\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m_+$ such that $\sum\limits_{k\in K} \alpha_k = 1$)

$$\begin{aligned} L(y,\alpha,\mu,\gamma) = \varphi(y) &\leq \sum_{k\in K} \alpha_k f_k(x) + \varepsilon ||x-y|| \\ &\leq \sum_{k\in K} \alpha_k \max_{k\in K} f_k(x) + \varepsilon ||x-y|| \\ &= \sum_{k\in K} \alpha_k \varphi(x) + \varepsilon ||x-y|| \\ &= \varphi(x) + \varepsilon ||x-y||. \end{aligned}$$

The proof is complete. □

Theorem 4.5. (ε-quasi strong duality) *Let $\bar{x} \in C$ be an ε-quasi solution to problem* (MP). *Suppose that the condition* (CQ) *for problem* (MP) *holds at $\bar{x} \in C$. Then there exists $(\bar{\alpha}, \bar{\mu}, \bar{\gamma}) \in \mathbb{R}^m_+ \times \mathbb{R}^n_+ \times \mathbb{R}^l_+$ such that $(\bar{x}, \bar{\alpha}, \bar{\mu}, \bar{\gamma}) \in C_{\text{MWMD}}$ and $f(\bar{x}) = L(\bar{x}, \bar{\alpha}, \bar{\mu}, \bar{\gamma})$. If (f,g,h) is generalized L-invex-infine on Ω at $y \in \Omega$, then $(\bar{x}, \bar{\alpha}, \bar{\mu}, \bar{\gamma})$ is an ε-quasi solution to problem* (MWMD).

Proof. Let $\bar{x} \in C$ be an ε-quasi solution to problem (MP). From Theorem 3.1, one sees that there exist $\alpha_k \geq 0, k \in K$ with $\sum_{k\in K} \alpha_k = 1, \mu_i \geq 0, i \in I$ and $\gamma_j \geq 0, j \in J$ such that

$$\begin{aligned} &0 \in \sum_{k\in K} \alpha_k f_k(\bar{x}) + \sum_{i\in I} \mu_i \partial g_i(\bar{x}) + \sum_{j\in J} \gamma_j \left(\partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x})\right) + \varepsilon \mathbb{B}_{X^*} + N(\bar{x};\Omega), \\ &\alpha_k \left(f_k(\bar{x}) - \varphi(\bar{x})\right) = 0, k \in K, \\ &\mu_i g_i(\bar{x}) = 0, i \in I. \end{aligned} \quad (4.17)$$

Putting

$$\bar{\alpha}_k := \frac{\alpha_k}{\sum\limits_{k\in K} \alpha_k}, k \in K, \bar{\mu}_i := \frac{\mu_i}{\sum\limits_{k\in K} \alpha_k}, i \in I, \bar{\gamma}_j := \frac{\gamma_j}{\sum\limits_{k\in K} \alpha_k}, j \in J,$$

one has $\bar{\alpha} := (\bar{\alpha}_1, \ldots, \bar{\alpha}_m) \in \mathbb{R}^m_+$ with $\sum_{k\in K} \bar{\alpha}_k = 1$, $\bar{\mu} := (\bar{\mu}_i)_{i\in I}, \bar{\mu}_i \geq 0, i \in I, \bar{\gamma} := (\bar{\gamma}_j)_{j\in J}, \bar{\gamma}_j \geq 0, j \in J$. Furthermore, the assertion in (4.17) is also valid when α_k's, μ_i's and γ_j's are replaced by $\bar{\alpha}_k$'s, $\bar{\mu}_i$'s and $\bar{\gamma}_j$'s, respectively. On the other hand, from $\bar{x} \in C$, one has $h_j(\bar{x}) = 0, j \in J$. It is easy to see that $\sum_{j\in J}\sigma_j h_j(\bar{x})=0$. From $\bar{\mu}_i g_i(\bar{x})=0, i\in I$, one has $\sum_{i\in I}\bar{\mu}_i g(\bar{x})+\sum_{j\in J}\sigma_j h_j(\bar{x})=0$ for all $\sigma:=(\sigma_1,\cdots,\sigma_l)\in\mathbb{S}(0,||\bar{\gamma}||)$ and $||\sigma||=||\bar{\gamma}||$. So, $(\bar{x},\bar{\alpha},\bar{\mu},\bar{\gamma})\in C_{\text{MWMD}}$. Besides, $\varphi(\bar{x})=L(\bar{x},\bar{\alpha},\bar{\mu},\bar{\gamma})$. If (f,g,h) is generalized L-invex-infine on Ω at any $y\in\Omega$, by Theorem 4.4 yields

$$L(\bar{x},\bar{\alpha},\bar{\mu},\bar{\gamma})+\varepsilon||(\bar{x},\bar{\alpha},\bar{\mu},\bar{\gamma})-(y,\alpha,\mu,\gamma)||=\varphi(\bar{x})+\varepsilon||(\bar{x},\bar{\alpha},\bar{\mu},\bar{\gamma})-(y,\alpha,\mu,\gamma)||\geq L(y,\alpha,\mu,\gamma),$$

for any $(y,\alpha,\mu,\gamma)\in C_{\text{MWMD}}$. This means that $(\bar{x},\bar{\alpha},\bar{\mu},\bar{\gamma})$ is an ε-quasi solution to problem (MWMD). The proof is complete. □

Theorem 4.6. (ε-quasi converse duality) *Suppose that $(\bar{x},\bar{\alpha},\bar{\mu},\bar{\gamma})\in C_{\text{MWMD}}$. If $\bar{x}\in C$ and (f,g,h) is generalized L-invex-infine on Ω at $\bar{x}$, then $\bar{x}$ is an ε-quasi solution to problem* (MP).

Proof. Since $(\bar{x},\bar{\alpha},\bar{\mu},\bar{\gamma})\in C_{\text{MWMD}}$, then there exist $\bar{\alpha}:=(\bar{\alpha}_k)_{k\in K},\bar{\alpha}_k\geq 0$ with $\sum_{k\in K}\bar{\alpha}_k=1, u_k^*\in\partial f_k(\bar{x}), k\in K, \bar{\mu}:=(\bar{\mu}_i)_{i\in I},\bar{\mu}_i\geq 0, x_i^*\in\partial g_i(\bar{x}), i\in I$ and $\bar{\gamma}:=(\bar{\gamma}_j)_{j\in J},\bar{\gamma}_j\geq 0, y_j^*\in\partial h_j(\bar{x})\cup\partial(-h_j)(\bar{x})$, as well as $b^*\in\mathbb{B}_{X^*}$ such that

$$-\left(\sum_{k\in K}\bar{\alpha}_k u_k^*+\sum_{i\in I}\bar{\mu}_i x_i^*+\sum_{j\in J}\bar{\gamma}_j y_j^*+\varepsilon b^*\right)\in N(\bar{x};\Omega), \tag{4.18}$$

$$\bar{\alpha}_k\left(f_k(\bar{x})-\varphi(\bar{x})\right)=0, k\in K, \tag{4.19}$$

and

$$\sum_{i\in I}\bar{\mu}_i g_i(\bar{x})+\sum_{j\in J}\bar{\sigma}_j h_j(\bar{x})\geq 0 \text{ for all } \bar{\sigma}\in\mathbb{R}^l \text{ with } ||\bar{\sigma}||=||\bar{\gamma}||. \tag{4.20}$$

Suppose on contrary that $\bar{x}\in C$ is not an ε-quasi solution of problem (MP). It then follows that there exists $\hat{x}\in C$ satisfying

$$\varphi(\hat{x})+\varepsilon||\hat{x}-\bar{x}||<\varphi(\bar{x}). \tag{4.21}$$

By the generalized L-invex-infine property of (f,g,h) on Ω at $\bar{x}$, the definition of the polar cone and (4.18), for such $\hat{x}\in C$, there exists $v\in N(\bar{x};\Omega)^\circ$ such that

$$\begin{aligned}0 &\leq \sum_{k\in K}\bar{\alpha}_k\langle u_k^*,v\rangle+\sum_{i\in I}\bar{\mu}_i\langle x_i^*,v\rangle+\sum_{i\in I}\bar{\gamma}_j\left\langle y_j^*,v\right\rangle+\varepsilon\langle b^*,v\rangle\\ &\leq \sum_{k\in K}\bar{\alpha}_k\left[f_k(\hat{x})-f_k(\bar{x})\right]+\sum_{i\in I}\bar{\mu}_i\left[g_i(\hat{x})-g_i(\bar{x})\right]+\sum_{j\in J}\frac{1}{\omega_j}\bar{\gamma}_j\left[h_j(\hat{x})-h_j(\bar{x})\right]\varepsilon||\hat{x}-\bar{x}||,\end{aligned}$$

where $\omega_j\in\{-1,1\}, j\in J$. Thus

$$\sum_{k\in K}\bar{\alpha}_k\left[f_k(\hat{x})-f_k(\bar{x})\right]+\sum_{i\in I}\bar{\mu}_i\left[g_i(\hat{x})-g_i(\bar{x})\right]+\sum_{j\in J}\bar{\sigma}_j\left[h_j(\hat{x})-h_j(\bar{x})\right]+\varepsilon||\hat{x}-x||\geq 0,$$

where $\bar{\sigma}_j:=\frac{\bar{\gamma}_j}{\omega_j}\in\mathbb{R}, j\in J$. From $\hat{x}\in C$, it is obvious that $\sum_{i\in I}\bar{\mu}_i g_i(\hat{x})\leq 0$ and $\sum_{j\in J}\bar{\sigma}_j h_j(\hat{x})=0$. Thus

$$\sum_{k\in K}\bar{\alpha}_k\left[f_k(\hat{x})-f_k(\bar{x})\right]-\sum_{i\in I}\bar{\mu}_i g_i(\bar{x})-\sum_{j\in J}\bar{\sigma}_j h_j(\bar{x})+\varepsilon||\hat{x}-\bar{x}||\geq 0,$$

where $\bar{\sigma}:=(\bar{\sigma}_1,\ldots,\bar{\sigma}_l)\in\mathbb{R}^l$. Since $||\bar{\sigma}||=||\bar{\gamma}||$, (4.20) yields that $\sum_{k\in K}\bar{\alpha}_k\left[f_k(\hat{x})-f_k(\bar{x})\right]+\varepsilon||\hat{x}-\bar{x}||\geq 0$, which together with (4.19) implies that (Since $(\bar{\alpha}_1,\ldots,\bar{\alpha}_m)\in\mathbb{R}^m_+$ such that $\sum_{k\in K}\bar{\alpha}_k=1$)

$$\varphi(\bar{x})\leq\sum_{k\in K}\bar{\alpha}_k f_k(\hat{x})+\varepsilon||\hat{x}-\bar{x}||\leq\sum_{k\in K}\bar{\alpha}_k\max_{k\in K}f_k(\hat{x})+\varepsilon||\hat{x}-\bar{x}||=\varphi(\hat{x})+\varepsilon||\hat{x}-\bar{x}||,$$

which contradicts (4.21). The proof is complete. □

5. Application to the Multiobjective Optimization Problem

In this section, we consider the following multiobjective optimization problem involving inequality and equality constraints:

$$\text{(OP)} \quad \begin{array}{l} \min f(x) := (f_1(x), \ldots, f_m(x)), \\ \text{s.t. } x \in C := \{x \in \Omega \mid g_i(x) \leq 0, i \in I, h_j(x) = 0, j \in J\}, \end{array}$$

and the functions $f_k : X \to \mathbb{R}, k \in K, g_i : X \to \mathbb{R}, i \in I$ and $h_j : X \to \mathbb{R}, j \in J$ are locally Lipschitz functions. Suppose that Ω is always assumed to be SNC at the point under consideration.

Definition 5.1. Let $\varepsilon := (\varepsilon_1, \ldots, \varepsilon_m) \in \mathbb{R}^m_+$. A point $\bar{x} \in C$ is said to be an ε-quasi weakly solution to problem (OP) if $f(x) + \varepsilon||x - \bar{x}|| - f(\bar{x}) \notin -\text{int}\mathbb{R}^m_+$ for all $x \in C$.

Theorem 5.1. *Let $\varepsilon := (\varepsilon_1, \ldots, \varepsilon_m) \in \mathbb{R}^m_+ \backslash \{0\}$ and $\bar{x} \in C$ be an ε-quasi weakly solution to problem* (OP). *Suppose that the qualification condition* (CQ) *for problem* (OP) *holds at $\bar{x}$. Then, there exist $\alpha := (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m_+$ with $\sum_{k \in K} \alpha_k = 1, \mu := (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n_+$ and $\gamma := (\gamma_1, \ldots, \gamma_l) \in \mathbb{R}^l_+$ such that*

$$\begin{aligned} &0 \in \sum_{k \in K} \alpha_k f_k(\bar{x}) + \sum_{i \in I} \mu_i \partial g_i(\bar{x}) + \sum_{j \in J} \gamma_j \left(\partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x})\right) + N(\bar{x}; \Omega) + \bar{\varepsilon}\mathbb{B}_{X^*}, \\ &\mu_i g_i(\bar{x}) = 0, i \in I, \end{aligned} \tag{5.1}$$

where $\bar{\varepsilon} := \max_{k \in K}\{\varepsilon_k\}$.

Proof. Suppose that $\bar{x} \in C$ is an ε-quasi weakly solution to problem (OP), and let $\widehat{f}(x) := f_k(x) - f_k(\bar{x}), k \in K, x \in X$. We prove that $\bar{x}$ is an ε-quasi solution to the minimax programming problem

$$\min_{x \in C} \max_{k \in K} \widehat{f}_k(x). \tag{5.2}$$

To do this, let us put $\widehat{\varphi}(x) := \max\limits_{k \in K} \widehat{f}_k(x)$ and we need prove that

$$\widehat{\varphi}(x) + \bar{\varepsilon}||x - \bar{x}|| \geq \widehat{\varphi}(\bar{x}), \forall x \in C. \tag{5.3}$$

Indeed, if (5.3) does not hold, then there exists $x_0 \in C$ such that $\widehat{\varphi}(x_0) + \bar{\varepsilon}||x_0 - \bar{x}|| < \widehat{\varphi}(\bar{x})$. Since $\widehat{\varphi}(\bar{x}) = \max_{k \in K} \widehat{f}_k(\bar{x}) = \max\limits_{k \in K} \left\{ f_k(\bar{x}) - f_k(\bar{x}) \right\} = 0$, one has

$$\widehat{\varphi}(x_0) + \bar{\varepsilon}||x_0 - \bar{x}|| = \max_{k \in K} \left\{ f_k(x_0) - f_k(\bar{x}) \right\} + \bar{\varepsilon}||x_0 - \bar{x}|| < 0.$$

Therefore, $f(x_0) - f(\bar{x}) + \varepsilon||x_0 - \bar{x}|| \in -\text{int}\mathbb{R}^m_+$, which contradicts the fact that $\bar{x}$ is an ε-quasi weakly solution of problem (OP). So, by employ the condition (3.1) in Theorem 3.1, but applied to problem (5.2), we find $\alpha := (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m_+$ with $\sum_{k \in K} \alpha_k = 1, \mu := (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n_+$ and $\gamma := (\gamma_1, \ldots, \gamma_l) \in \mathbb{R}^l_+$ such that

$$\begin{aligned} &0 \in \sum_{k \in K} \alpha_k \partial \widehat{f}_k(\bar{x}) + \sum_{i \in I} \mu_i \partial g_i(\bar{x}) + \sum_{j \in J} \gamma_j \left(\partial h_j(\bar{x}) \cup \partial(-h_j)(\bar{x})\right) + N(\bar{x}; \Omega) + \bar{\varepsilon}\mathbb{B}_{X^*}, \\ &\alpha_k \left[\widehat{f}_k(\bar{x}) - \max_{k \in K} \widehat{f}_k(\bar{x}) \right] = 0, k \in K, \\ &\mu_i g_i(\bar{x}) = 0, i \in I. \end{aligned} \tag{5.4}$$

It is easy to see that (5.4) implies (5.1). Thus the proof of Theorem 5.1 is complete. □

Remark 5.1. In Theorem 5.1, if we only consider inequality constraints, then we have [1, Theorem 5.1]. Besides, if $\varepsilon = 0$, then we obtain in [34, Theorem 5.1].

Theorem 5.2. *Let $\bar{\varepsilon} := (\bar{\varepsilon}, \ldots, \bar{\varepsilon}) \in \mathbb{R}^m_+$ and $p := (p_1, \ldots, p_m), q := (q_1, \ldots, q_m), g := (g_1, \ldots, g_n)$, $h := (h_1, \ldots, h_l)$. Assume that $\bar{x}$ is in C such that (5.1) holds. If (f, g, h) is generalized L-invex-infine on Ω at $\bar{x}$, then $\bar{x}$ is an ε-quasi weakly solution to problem* (OP).

Proof. Similar to the proof of Theorem 5.1, we set $\widehat{f}_k(x) := f_k(x) - f_k(\bar{x})$, $k \in K$, $x \in X$. Now, it is easy to see that $\bar{x}$ satisfies condition (5.1). Let $\widehat{f} := (\widehat{f}_1, \ldots, \widehat{f}_m)$. Because (f, g, h) is generalized $L-$invex-infine on Ω at $\bar{x}$, one sees that $(\widehat{f}, g, h)$ is generalized L-invex-infine on Ω at $\bar{x}$. We apply Theorem 3.2 to follow that $\bar{x}$ is an $\bar{\varepsilon}$-quasi solution to the minimax programming problem $\min_{x \in C} \max_{k \in K} \widehat{f}_k(x)$, which means that $\widehat{\phi}(\bar{x}) \leq \widehat{\phi}(x) + \bar{\varepsilon}||x - \bar{x}||$ for all $x \in C$, where $\widehat{\phi}(x) := \max\limits_{k \in K} \widehat{f}_k(x)$. On the other hand, one has

$$\max_{k \in K} \left\{ f_k(x) - f_k(\bar{x}) \right\} + \bar{\varepsilon}||x - \bar{x}|| \geq 0, \forall x \in C,$$

which deduces that $f(x) - f(\bar{x}) + \bar{\varepsilon}||x - \bar{x}|| \notin -\mathrm{int}\mathbb{R}^m_+$ for all x in C. Thus $\bar{x}$ is an ε-quasi weakly solution to problem (OP). The proof is complete. $\square$

Remark 5.2. In Theorem 5.2, if we only consider inequality constraints, then we have [1, Theorem 5.2]. Besides, if $\varepsilon = 0$, then we obtain [34, Theorem 5.2].

6. Conclusion

In this article, we first obtained necessary optimality conditions for ε-quasi solutions of problem (MP) in terms of the Mordukhovich/limiting subdifferentials. Next, we introduced the new definition of generalized L-invex-infine property via the Mordukhovich/limiting subdifferential of locally Lipschitz functions. Using generalized L-invex-infine property, we established sufficient optimality condition and duality theorems for ε-quasi solutions of the considered problem. Besides, some of the obtained results were applied to optimality conditions for ε-quasi solution of nonsmooth multiobjective optimization problems involving inequality and equality constraints. The obtained results in this paper are new and different from the results in [1, 34]. A further research for studying ε-quasi solutions of nonsmooth minimax fractional optimization problems involving inequality and equality constraints can be investigated in the future research.

Acknowledgments

The author is grateful to the Reviewer for useful suggestions which improved the contents of this article.

References

[1] K. D. Bae, Z. Hong, D. S. Kim, A minimax approach to characterize quasi ε-Pareto solutions in multiobjective optimization problems, J. Nonlinear Var. Anal. 5 (2021), 709-720.

[2] J. H. Lee, L. G. Jiao, On quasi $\varepsilon-$solution for robust convex optimization problems, Optim. Lett. 11 (2017), 1609–1622.

[3] L.G. Jiao, D.S. Kim, Y.Y. Zhou, Quasi $\varepsilon-$solution in a semi-infinite programming problem with locally Lipschitz data, Optim. Lett. 15 (2021), 1759–1772.

[4] T. D. Chuong, D. S. Kim, Approximate solutions of multiobjective optimization problems, Positivity 20 (2016), 187-207.

[5] M. Fakhara, M.R. Mahyarinia, J. Zafarani, On approximate solutions for nonsmooth robust multiobjective optimization problems, Optimization 68 (2019), 1653-1683.

[6] N.H. Hugn, H.N. Tuan, N.V. Tuyen, On approximate quasi pareto solutions in nonsmooth semi–infinite interval–valued vector optimization problems, Appl. Anal. 102 (2023), 2432–2448.

[7] D.S. Kim, T.Q. Son, An approach to $\varepsilon-$duality theorems for nonconvex semi-infinite multiobjective optimization problems, Taiwanese J. Math. 22 (2018), 1261-1287.

[8] G.S. Kim, M.H. Kim, G.M. Lee, Optimality theorems for linear fractional optimization problems involving integral functions defined on $C^n[0,1]$, J. Nonlinear Var. Anal. 7 (2023), 279-290.

[9] T. H. Pham, Approximate proper efficiencies in nonsmooth semi-infinite multiobjective optimization problems, RAIRO Oper. Res. 58 (2024), 1163–1186.

[10] T. H. Pham, Approximate optimal solutions for multiobjective optimization problems with infinite constraints, Bull. Malays. Math. Sci. Soc. 47 (2024), 70.

[11] T. H. Pham, On approximate positively properly efficient solutions in nonsmooth semi-infinite multiobjective optimization problems with data uncertainty, J. Appl. Numer. Optim. 6 (2024), 271-289.

[12] T. H. Pham, Optimality and duality for approximate solutions of multiobjective optimization problems involving intersection of closed sets, Optimization (2025). https://doi.org/10.1080/02331934.2025.2466043

[13] T.V. Su, C.V. Tiep, Optimality conditions for nonconvex mathematical programming problems using weak subdifferentials and augmented normal cones, Appl. Set-Valued Anal. Optim. 6 (2024), 245-260.

[14] T. Shitkovskaya, Z. Hong, D. S. Kim, G. R. Piao, Approximate necessary optimality in fractional semi-infinite multiobjective optimization, J. Nonlinear Convex Anal. 21 (2020), 195-204.

[15] T. Shitkovskaya, L. Jiao, D. S. Kim, Multi-criteria optimization problems with fractional objectives: approximate optimality and approximate duality, J. Nonlinear Convex Anal. 22 (2021), 1117-1131.

[16] T. Q. Son, D. S. Kim, $\varepsilon-$Mixed duality for nonconvex multiobjective programs with an infinite number of constraints, J. Glob. Optim. 57 (2013), 447-465.

[17] T. Q. Son, N. V. Tuyen, C. F. Wen, Optimality conditions for approximate Pareto solutions of a nonsmooth vector optimization problem with an infinite number of constraints, Acta Math. Vietnam. 45 (2020), 435-448.

[18] X. K. Sun, L. P. Tang, J. Zeng, Characterizations of approximate duality and saddle point theorems for nonsmooth robust vector optimization, Numer. Funct. Anal. Optim. 41 (2020), 462–482.

[19] X. K. Sun, K. L. Teo, X. J. Long, Characterizations of robust ε-quasi optimal solutions for nonsmooth optimization problems with uncertain data, Optimization 70 (2021), 847–870.

[20] J. C. Liu, $\varepsilon-$Properly efficient solutions to nondifferentiable multi-objective programming problems, Appl. Math. Lett. 12 (1999), 109–113.

[21] D. T. V. An, D.T. Ngoan, N. V. Tuyen, Optimality conditions and sensitivity analysis in parametric convex minimax programming, Appl. Anal. 103 (2024), 2997-3016.

[22] D. T. V. An, N. H. Hung, D. T. Ngoan, N. V. Tuyen, Optimality conditions and sensitivity analysis in parametric nonconvex minimax programming, J. Global Optim. 90 (2024), 53-72.

[23] T. Antczak, Nonsmooth minimax programming under locally Lipschitz $(\phi,\rho)-$invexity, Appl. Math. Comput. 217 (2011), 9606-9624.

[24] T. Antczak, N. Abdulaleem, $E-$differentiable minimax programming under $E-$convexity, Ann. Oper. Res. 300 (2021), 1-22.

[25] K. D. Bae, G. R. Piao, Z. Hong, D. S. Kim, On minimax fractional semi-infinite programming problems with applications, Numer. Func. Anal. Opt. 42 (2021), 1522-1538.

[26] K. D. Bae, T. Shitkovskaya, D. S. Kim, A note on minimax optimization problems with an infinite number of constraints, J. Appl. Numer. Optim. 3 (2021), 521-531.

[27] Z. Hong, K. D. Bae, D. S. Kim, Minimax programming as a tool for studying robust multi-objective optimization problems, Ann. Oper. Res. 319 (2022), 1589-1606.

[28] D. T. K. Huyen, D. S. Kim, N. D. Yen, Optimality conditions for nondifferentiable minimax programs and vector optimization problems. J. Optim. Theory Appl. 200 (2024), 703-723.

[29] A. Jayswal, K. Kummari, V. Singh, Optimality and parametric duality for nonsmooth minimax fractional programming problems involving $L-$invex-infine functions, Pesqui. Oper. 36 (2016), 227-239.

[30] H. Y. Li, D. S. Kim, K. D Bae, Optimality conditions for nonsmooth minimax fractional optimization problems with an infinite number of constraints, J. Nonlinear Convex Anal. 25 (2024), 3019-3031.
[31] S. K. Mishra, J. S. Rautela, On nondifferentiable minimax fractional programming under generalized $\alpha-$type I invexity, J. Appl. Math. Comput. 31 (2009), 317-334.
[32] T. V. Su, Optimality and duality in nondifferentiable minimax programming problems involving ε-quasi solutions, Positivity 28 (2024), 72.
[33] T. Antczak, Parametric optimality and duality results for nondifferentiable L-univex multiobjective fractional programming problems, Commun. Optim. Theory 2024 (2024), 21.
[34] T. D. Chuong, D. S. Kim, Nondifferentiable minimax programming problems with applications, Ann. Oper. Res. 251 (2017), 73–87.
[35] B. S. Mordukhovich, Variational Analysis and Generalized Differentiation. I: Basic Theory, Springer, Berlin, 2006.