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SEQUENTIAL APPROXIMATE SOLUTIONS FOR ROBUST FRACTIONAL OPTIMIZATION PROBLEMS

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Abstract. The purpose of this paper is to establish sequential optimality conditions characterizing completely an approximate solution for a scalar constrained robust fractional optimization problem in the absence of constraint qualifications. Our approach is based essentially on the use of a combination of conjugate analysis and the concept of approximate subdifferential. Finally, we present an example to illustrate our main results.
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1. INTRODUCTION

Optimization problems in the face of data uncertainty have attracted the attention of researchers in the past decades and now play an important role in a variety of fields, such as economics, engineering, optimal control, mechanics, operation research, and so on. Robust optimization approach [1, 2] emerged as a powerful deterministic approach for studying optimization problems with data uncertainty. A robust fractional optimization problem is to optimize an objective fractional function over the constrained set defined by functions with data uncertainty.

In recent years, spotlight was focused on the optimality conditions and the duality results of the robust solution in uncertain fractional optimization problems by using robust optimization methodology; see, e.g., [3, 4, 5, 6, 7, 8]. In [4], Sun et al. investigated some characterizations of robust optimal solutions for uncertain fractional optimization and applications. In [5], Jeyakumar and Li established duality theorems for a fractional programming problem in the face of data uncertainty, and they also developed a strong duality for robust minimax fractional programming problems, when data in both of the objective and constraints are uncertainty [6]. In general, for establishing the approximate optimality conditions for robust convex optimization problems, we need a constraint qualifications; see, e.g., [9, 10, 11, 12, 13, 14, 15, 16]. Using a robust optimization approach, Lee and Lee [9] established optimality theorems and duality theorems for approximate solutions for the uncertain fractional optimization problem. In [10, 12],

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Lee and Lee also established optimality theorems and duality results of robust approximate optimal solutions for uncertain convex optimization problems under closed convex cone constraint qualifications. In [16], Zeng, Xu and Fu obtained some new results of robust approximate optimal solutions for fractional semi-infinite optimization problems with uncertain data under a robust type constraint qualification condition. However, we know that the constraint qualifications do not always hold even for finite-dimensional optimization problems and frequently fail for infinite-dimensional optimization problems. In order to eliminate these drawbacks, authors have paid their attention to characterize optimality conditions for convex optimization problems in the absence of data uncertainty without any constraint qualifications; see, e.g., [17, 18, 19, 20]. In [17], Boţ, Csetnek and Wanka obtained sequential characterizations of an optimal solution for composed convex optimization problems. In [18, 19], Jeyakumar et al. introduced the concept of sequential Lagrange multiplier rule for convex programs with convex constraints by using the concept of the epigraphs of conjugate functions in terms of approximate subdifferentials computed at an optimal solution without any constraint qualifications. In [20], Bai, Wu and Zhu obtained a sequential Lagrange multiplier condition of approximate solutions for a convex optimization problem without any constraint qualifications. Recently, Sun, Long and Chai [21] obtained a sequential Lagrange multiplier rule condition characterizing efficient solutions for a fractional optimization problem without any constraint qualification in terms of the approximate subdifferentials of the functions involved at the minimizer by using a technique that combines the conjugate analysis with the Brøndested-Rockafellar theorem. However, as far as we know, until now it seems that no enough results on sequential approximate optimality conditions for convex optimization problems with data uncertainty without any constraint qualifications. So, the goal of this paper is to establish sequential approximate optimality conditions for a scalar constrained robust fractional optimization problem in the absence of constraint qualification in terms of the approximate subdifferentials of the involved functions.

The rest of this paper is organized as follows. In Section 2, we recall some notions and definitions and give some preliminary results which are needed in the paper. In Section 3, without any constraint qualifications, we obtain some sequential optimality conditions characterizing an approximate solution for a fractional optimization problem with uncertainty data in terms of the approximate subdifferentials of the functions. In Section 4, we establish the approximate solutions under a constraint qualification and we present an example illustrating the main result of this paper. The last section, Section 5, ends this paper with some concluding remarks.

2. Preliminaries

Throughout this paper, let X and Y be two reflexive Banach spaces, X^* and Y^* be their respective topological duals paired in duality by $\langle .,. \rangle$. The dual spaces are endowed with the weak*-topologies $\omega(X^*,X)$ and $\omega(Y^*,Y)$, respectively. Let $Y_+ \subseteq Y$ be a nonempty and convex cone and its polar cone is given by $Y_+^* := \{y^* \in Y^* : \langle y^*, y\rangle \geq 0,\ \forall y \in Y_+\}$. The cone Y_+ induces on Y a partial order " $\leq_{Y_+}$ " defined by $y_1 \leq_{Y_+} y_2 \Longleftrightarrow y_2 - y_1 \in Y_+$ for all $y_1, y_2 \in Y$. Let C be a nonempty subset of X. The closure (resp. convex hull) is denoted by cl C, (resp. co C). If A and B are subsets of X, we have the relations cl $(\text{cl}\, A + \text{cl}\, B) = \text{cl}\,(A+B)$, and co $(\alpha A + \beta B) = \alpha$ co $(A) + \beta$ co (B), for any $\alpha, \beta \in \mathbb{R}$.

Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function. The effective domain and the epigraph of f are defined respectively by

$$\mathrm{dom} f := \{x \in X : \ f(x) \in \mathbb{R}\} \ \text{ and } \ \mathrm{epi} f := \{(x,r) \in X \times \mathbb{R} : \ f(x) \leq r\}.$$

The function f is said to be proper if its effective domain is nonempty. The conjugate function $f^* : X^* \to \mathbb{R} \cup \{+\infty\}$ is defined by $f^*(x^*) := \sup_{x \in X}\{\langle x^*, x\rangle - f(x)\}$ for all $x^* \in X^*$. The indicator function $\delta_C : X \to \mathbb{R} \cup \{+\infty\}$ of a nonempty subset $C \subseteq X$, is defined by

$$\delta_C(x) := \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise.} \end{cases}$$

The subdifferential of f at $\bar{x} \in \mathrm{dom} f$, denoted by $\partial_\varepsilon f(\bar{x})$, is defined

$$\partial_\varepsilon f(\bar{x}) := \{x^* \in X^* : \ \langle x^*, x - \bar{x}\rangle + f(\bar{x}) - \varepsilon \leq f(x), \ \forall x \in X\}.$$

When $\bar{x} \notin \mathrm{dom} f$, we set that $\partial_\varepsilon f(\bar{x}) = \emptyset$. If $\varepsilon = 0$, the set $\partial_0 f(\bar{x}) := \partial f(\bar{x})$ is the classical subdifferential of f at $\bar{x} \in \mathrm{dom} f$, i.e.,

$$\partial f(\bar{x}) := \{x^* \in X^* : \ \langle x^*, x - \bar{x}\rangle + f(\bar{x}) \leq f(x), \ \forall x \in X\}.$$

The set $N_\varepsilon(C, \bar{x})$ of ε-normal to C at $\bar{x}$ is defined as the ε-subdifferential of the indicator function δ_C at $\bar{x}$ i.e.,

$$N_\varepsilon(C, \bar{x}) := \partial_\varepsilon \delta_C(\bar{x}) = \{x^* \in X^* : \ \langle x^*, x - \bar{x}\rangle \leq \varepsilon, \ \forall x \in C\}.$$

If $\varepsilon = 0$, $N_0(C, \bar{x}) := \{x^* \in X^* : \ \langle x^*, x - \bar{x}\rangle \leq 0, \ \forall x \in C\}$ is said the normal cone and is denoted by $N(C, \bar{x})$. To Y, we attach an abstract maximal element with respect to " $\leq_{Y_+}$ ", denoted by $+\infty_Y$. We have $y \leq_{Y_+} +\infty_Y$, for any $y \in Y$. In the sequel we adopt the following convention

$$y + (+\infty_Y) = (+\infty_Y) + y = +\infty_Y, \quad \forall y \in Y \cup \{+\infty_Y\},$$

$$\alpha \cdot (+\infty_Y) = +\infty_Y, \quad \forall \alpha \geq 0.$$

Let $h : X \longrightarrow Y \cup \{+\infty\}$ be an extended vector-valued mapping, the effective domain, and the epigraph of h are defined respectively by

$$\mathrm{dom} h := \{x \in X : \ h(x) \in Y\} \text{ and } \mathrm{epi} h := \{(x,y) \in X \times Y : \ h(x) \leq_{Y_+} y\}.$$

The mapping h is said to be proper if $\mathrm{dom} h \neq \emptyset$. For any $x_1, x_2 \in X$ and any $\lambda \in [0,1]$, h is said Y_+-convex if

$$h(\lambda x_1 + (1-\lambda)x_2) \leq_{Y_+} \lambda h(x_1) + (1-\lambda)h(x_2).$$

For $y^* \in Y_+^*$, let us consider the following composite function $y^* \circ h : X \to \mathbb{R} \cup \{+\infty\}$ defined by

$$(y^* \circ h)(x) := \begin{cases} \langle y^*, h(x)\rangle, & \text{if } x \in \mathrm{dom} h, \\ +\infty, & \text{otherwise.} \end{cases}$$

It is easy to see that h is Y_+-convex if and only if $y^* \circ h$ is convex, for each $y^* \in Y_+^*$. The mapping h is said to be star Y_+-lower semicontinuous if $y^* \circ h$ is lower semicontinuous for all $y^* \in Y_+^*$. In what follows, we endow $X^* \times \mathbb{R}$ with the product topology of $\omega(X^*, X)$ and the usual Euclidean topology.

The following lemma characterizes the epigraph of the conjugate of the sums of two functions.

Lemma 2.1. [22] *Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be to proper convex functions with* $\mathrm{dom} f \cap \mathrm{dom} g \neq \emptyset$.

i) If f and g are lower semicontinuous, then $\mathrm{epi}(f+g)^* = \mathrm{cl}\ (\mathrm{epi} f^* + \mathrm{epi} g^*)$.
ii) If f or g is continuous at some $\bar{x} \in \mathrm{dom} f \cap \mathrm{dom} g$, then $\mathrm{epi}(f+g)^* = \mathrm{epi} f^* + \mathrm{epi} g^*$.

The following proposition describes the epigraph of a conjugate function in terms of the approximate subdifferential (see [18]).

Proposition 2.1. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper convex lower semicontinuous function and $\bar{x} \in \mathrm{dom} f$. Then* $\mathrm{epi} f^* = \bigcup_{\varepsilon \geq 0} \{(x^*, \langle x^*, \bar{x}\rangle + \varepsilon - f(\bar{x})) \ : \ x^* \in \partial_\varepsilon f(\bar{x})\}$.

We also recall the following proposition describing the epigraph of the conjugate of the upper envelope of a family of convex proper lower semicontinuous functions.

Proposition 2.2. [22] *Let $f_i : X \to \mathbb{R} \cup \{+\infty\}$, $i \in I$ (where I is an arbitrary index set) be a family of convex proper lower semicontinuous functions. Suppose that there exists $x_0 \in X$ such that* $\sup_{i\in I} f_i(x_0) < +\infty$. *Then* $\mathrm{epi}(\sup_{i\in I} f_i)^* = \mathrm{cl}\big[\mathrm{co} \bigcup_{i\in I} \mathrm{epi} f_i^*\big]$, *where* $\sup_{i\in I} f_i : X \to \mathbb{R} \cup \{+\infty\}$ *is defined by* $\sup_{i\in I} f_i(x) = \sup_{i\in I} f_i(x)$ *for all $x \in X$.*

3. Sequential Approximate Optimality Conditions

In this section, we present, without any constraint qualification, some sequential necessary and sufficient optimality conditions characterizing completely an approximate solution for a robust fractional optimization problem. For this, let us consider the following fractional programming problem

$$(\mathscr{P}) \qquad \begin{cases} \min \ \dfrac{f(x)}{g(x)} \\ \text{s.t. } h(x) \in -Y_+, \\ \qquad x \in C, \end{cases}$$

where X and Y are two reflexive Banach spaces, C is a nonempty, convex, and closed subset of X, Y_+ is a nonempty, convex, and closed cone of Y, f, $-g : X \to \mathbb{R}$ are convex lower semicontinuous functions and $h : X \to Y \cup \{+\infty_Y\}$ is a proper and Y_+-convex mapping such that $f(x) \geq 0$ and $g(x) > 0$ for all $x \in C \cap h^{-1}(-Y_+)$.

The fractional programming problem $(\mathscr{P})$ in the face of data uncertainty in the constraints can be captured by

$$(\mathscr{P}') \qquad \begin{cases} \min \ \max\limits_{(u,v)\in U\times V} \dfrac{f(x,u)}{g(x,v)} \\ \text{s.t. } h(x,w) \in -Y_+, \\ \qquad x \in C, \end{cases}$$

where U, V, and W be convex and compact subsets of reflexive Banach space Z, $f, g : X \times Z \to \mathbb{R}$ are functions such that, for any $(u,v) \in U \times V$, $f(.,u)$, $-g(.,v)$ are convex lower semicontinuous functions and, for any $x \in X$, $f(x,.)$, $g(x,.)$ are continuous functions. Let $h(.,w) : X \to Y \cup \{+\infty_Y\}$, $w \in W$ be a proper, Y_+-convex and star Y_+-lower semicontinuous on X. Following [2], the robust counterpart of the problem $(\mathscr{P}')$, namely (P), is formulated as follows

$$(\mathrm{P}) \qquad \begin{cases} \min \ \max\limits_{(u,v)\in U\times V} \dfrac{f(x,u)}{g(x,v)} \\ \text{s.t. } h(x,w) \in -Y_+, \ \forall w \in W, \\ \qquad x \in C. \end{cases}$$

We suppose that $f(x,u) \geq 0$ and $g(x,v) > 0$ for any $x \in \mathbf{A} := \{x \in C : h(x,w) \in -Y_+,\ \forall w \in W\}$, $u \in U$, $v \in V$. Let $\bar{x} \in \mathbf{A}$, $\varepsilon \geq 0$. The following notations are need in the sequel

$$\bar{r} := \max_{(u,v)\in U\times V} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \varepsilon,$$
$$\bar{\varepsilon} := \varepsilon \min_{v\in V} g(\bar{x},v)$$

and we suppose that $\bar{r} \geq 0$.

Remark 3.1. Since U and V are supposed compact subsets, the continuity of the functions $f(x,.)$, $g(x,.)$ ensures that the expressions $\max_{(u,v)\in U\times V} \frac{f(x,u)}{g(x,v)}$, $\max_{u\in U} f(x,u)$ and $\min_{v\in V} g(x,v)$ are finite numbers for all $x \in X$.

Let us recall the following definition of the approximate solution for problem (P).

Definition 3.1. Let $\varepsilon \geq 0$. A point $\bar{x} \in \mathbf{A}$ is said to be an ε-solution to (P) if, for any $x \in \mathbf{A}$,

$$\max_{(u,v)\in U\times V} \frac{f(x,u)}{g(x,v)} \geq \max_{(u,v)\in U\times V} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \varepsilon.$$

When $\varepsilon = 0$, the ε-solution of (P) becomes the exact solution.

Now, we illustrate the above definition with the following example.

Example 3.1. Let $U = V = W := [1,2]$, $C := \mathbb{R}_+$ and $f,g,h : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be defined respectively by $f(x,u) := ux+1$, $g(x,v) := vx+2$, and $h(x,w) := 2wx-3$, for any $x \in \mathbb{R}$, $u \in U$, $v \in V$,and $w \in W$. Then, problem (P) becomes

$$(\mathrm{P}_1) \qquad \begin{cases} \min \displaystyle\max_{(u,v)\in U\times V} \frac{ux+1}{vx+2} \\ \text{s.t. } 2wx-3 \leq 0,\ w \in [1,2], \\ \qquad x \in \mathbb{R}_+. \end{cases}$$

It is easy to check that $\mathbf{A} = \{x \in \mathbb{R} \ :\ 0 \leq x \leq \frac{3}{2}\}$. By taking $\varepsilon = \frac{1}{3}$, it follows that $\bar{x} = 0$ is an ε-solution to (P_1).

We present an example of a fractional programming problem with uncertainty proving that the approximate solution occurs but the exact solution does not occur.

Example 3.2. Let $U = V = W := [1,2]$, $C := [-1,1]$ and $f,g,h : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be defined respectively by $f(x,u) := u(\frac{x}{4}+\frac{1}{4})$, $g(x,v) := v(-\frac{x}{4}+\frac{1}{2})$, and $h(x,w) := \max\{0,wx\}$ for any $x \in \mathbb{R}$, $u \in U$, $v \in V$ and $w \in W$. Then, problem (P) becomes

$$(\mathrm{P}_2) \qquad \begin{cases} \min \displaystyle\max_{(u,v)\in U\times V} \frac{u(\frac{x}{4}+\frac{1}{4})}{v(-\frac{x}{4}+\frac{1}{2})} \\ \text{s.t. } \max\{0,wx\} \leq 0,\ w \in [1,2], \\ \qquad x \in [-1,1], \end{cases}$$

Obviously, $\mathbf{A} = [-1,0]$. By taking $\varepsilon = 1$ and $\bar{x} = 0$, it follows that $\bar{x}$ is an approximate solution to (P_2), but not an exact solution.

Using the parametric approach of Dinkelbach [24], we transform problem (P) into the robust non-fractional convex optimizaton problem $(\mathrm{P_r})$ with a parametric $r \in \mathbb{R}_+$.

$$(\mathrm{P_r}) \qquad \begin{cases} \min \ \max\limits_{u \in U} f(x,u) - r \min\limits_{v \in V} g(x,v) \\ \text{ s.t. } h(x,w) \in -Y_+, \ \forall w \in W, \\ \qquad x \in C. \end{cases}$$

Definition 3.2. Let $\varepsilon \geq 0$. A point $\bar{x} \in \mathbf{A}$ is said an ε-solution to $(\mathrm{P_r})$ if, for any $x \in \mathbf{A}$,

$$\max_{u \in U} f(x,u) - r \min_{v \in V} g(x,v) \geq \max_{u \in U} f(\bar{x},u) - r \min_{v \in V} g(\bar{x},v) - \varepsilon.$$

The following lemma describes the relationship between the solution of problem (P) and problem $(\mathrm{P_{\bar{r}}})$.

Lemma 3.1. [9, Lemma 2.1] *A point $\bar{x} \in \mathbf{A}$ is an ε-solution to*(P) *if and only if $\bar{x}$ is an $\bar{\varepsilon}$-solution to* $(\mathrm{P_{\bar{r}}})$.

It is easy to obtain the following lemma.

Lemma 3.2. *Let C be a closed and convex subset of X and $Y_+ \subseteq Y$ be a closed convex cone. Let $h(.,w) : X \to Y \cup \{+\infty_Y\}$, $w \in W$ be a proper, Y_+-convex and star Y_+-lower semicontinuous on X. If $\mathbf{A} \neq \emptyset$, then*

$$\operatorname{epi}\delta^*_{\mathbf{A}} = \operatorname{cl}\operatorname{co}\Big(\bigcup_{y^* \in Y^*_+, w \in W} \operatorname{epi}(y^* \circ h(.,w))^* + \operatorname{epi}\delta^*_C \Big).$$

In what follows, we establish sequential optimality conditions for approximate solutions for the robust fractional optimization problem in terms of epigraphs of conjugate functions without any constraint qualification. Note that the weak* convergence of the sequence $\{x^*_n\}$ of X^* to x^* is designated by $x^*_n \to_* x^*$.

Theorem 3.1. *For* (P)*, let $\bar{x} \in \mathbf{A}$, $\varepsilon \geq 0$, and $\bar{r} = \max_{(u,v) \in U \times V} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \varepsilon \geq 0$. Then, $\bar{x}$ is an ε-solution to problem* (P) *if and only if there exist $p \geq 1$, $\{(u^*_n, r_n)\}$, $\{(v^*_n, s_n)\}$, $\{(w^*_{i,n}, t_{i,n})\}$, $\{(q^*_n, \gamma_n)\} \subseteq X^* \times \mathbb{R}$ $(i = 1,...,p)$, $y^*_n \in Y^*_+$, $w_n \in W$ and $\alpha_{i,n} \geq 0$ $(\sum_{i=1}^p \alpha_{i,n} = 1)$, such that*

$$\begin{cases} (u^*_n, r_n) \in \operatorname{epi}(\max\limits_{u \in U} f(.,u))^*, \ (v^*_n, s_n) \in \operatorname{epi}(\bar{r}(-\min\limits_{v \in V} g(.,v)))^*, \ (q^*_n, \gamma_n) \in \operatorname{epi}\delta^*_C, \\ (w^*_{i,n}, t_{i,n}) \in \operatorname{epi}(y^*_n \circ h(.,w_n))^*, \ (i = 1, \dots, p), \\ u^*_n + v^*_n + \sum\limits_{i=1}^p \alpha_{i,n} w^*_{i,n} + q^*_n \to_* 0, \\ r_n + s_n + \sum\limits_{i=1}^p \alpha_{i,n} t_{i,n} + \gamma_n \to 0. \end{cases}$$

Proof. $(\Rightarrow)$. Assume that $\bar{x}$ is an ε-solution to (P). By Lemma 3.1, $\bar{x}$ is an $\bar{\varepsilon}$-solution to $(\mathrm{P_{\bar{r}}})$, i.e., for any $x \in \mathbf{A}$,

$$\max_{u \in U} f(x,u) - \bar{r} \min_{v \in V} g(x,v) \geq \max_{u \in U} f(\bar{x},u) - \bar{r} \min_{v \in V} g(\bar{x},v) - \varepsilon \min_{v \in V} g(\bar{x},v).$$

By using the scalar indicator function $\delta_{\mathbf{A}}$, we transform problem $(\mathrm{P_{\bar{r}}})$ into an unconstrained minimization problem

$$(Q_{\bar{r}}) \qquad \min_{x \in X} \, (\Phi_{\bar{x}} + \delta_{\mathbf{A}})(x),$$

where $\Phi_{\bar{x}} : X \to \mathbb{R} \cup \{+\infty\}$ is defined, for any $x \in X$, by

$$\Phi_{\bar{x}}(x) := \max_{u \in U} f(x,u) - \bar{r} \min_{v \in V} g(x,v).$$

It is easy to see that $\Phi_{\bar{x}}$ is $\mathbb{R}_+$-convex. The convexity of $\delta_{\mathbf{A}} : X \to \mathbb{R} \cup \{+\infty\}$ is immediate since $\mathbf{A}$ is convex. Therefore $(Q_{\bar{r}})$ is a convex programming problem and hence we obtain that $\bar{x} \in \mathbf{A}$ is an ε-solution of (P) if and only if $\bar{x}$ is an $\bar{\varepsilon}$-solution to $(\mathrm{Q}_{\bar{r}})$, i.e., $0 \in \partial_{\bar{\varepsilon}}(\Phi_{\bar{x}} + \delta_{\mathbf{A}})(\bar{x})$, which yields

$$(0,0) = (0, -\Phi_{\bar{x}}(\bar{x}) + \bar{\varepsilon}) \in \mathrm{epi}(\Phi_{\bar{x}} + \delta_{\mathbf{A}})^*. \tag{3.1}$$

Moreover, by Lemma 2.1, we obtain

$$\begin{aligned}
\mathrm{epi}(\Phi_{\bar{x}} + \delta_{\mathbf{A}})^* &= \mathrm{cl}\left[\mathrm{epi}\Phi_{\bar{x}}^* + \delta_{\mathbf{A}}^*\right] \\
&= \mathrm{cl}\left[\mathrm{epi}\Big(\max_{u \in U} f(.,u) + \bar{r}(-\min_{v \in V} g(.,v))\Big)^* + \mathrm{epi}\delta_{\mathbf{A}}^*\right] \\
&= \mathrm{cl}\left[\mathrm{cl}\left[\mathrm{epi}(\max_{u \in U} f(.,u))^* + \mathrm{epi}\big(\bar{r}(-\min_{v \in V} g(.,v))\big)^*\right] + \mathrm{epi}\delta_{\mathbf{A}}^*\right] \\
&= \mathrm{cl}\left[\mathrm{epi}(\max_{u \in U} f(.,u))^* + \mathrm{epi}\big(\bar{r}(-\min_{v \in V} g(.,v))\big)^* + \mathrm{epi}\delta_{\mathbf{A}}^*\right].
\end{aligned}$$

By Lemma 3.2, we have

$$\begin{aligned}
\mathrm{epi}\ \delta_{\mathbf{A}}^* &= \mathrm{cl\ co}\left(\bigcup_{y^* \in Y_+^*, w \in W} \mathrm{epi}(y^* \circ h(.,w))^* + \mathrm{epi}\delta_C^*\right) \\
&= \mathrm{cl}\left[\mathrm{co}\left(\bigcup_{y^* \in Y_+^*, w \in W} \mathrm{epi}(y^* \circ h(.,w))^*\right) + \mathrm{epi}\delta_C^*\right].
\end{aligned}$$

Then,

$$\begin{aligned}
\mathrm{epi}\big(\Phi_{\bar{x}} + \delta_{\mathbf{A}}\big)^* = \mathrm{cl}\Bigg[&\mathrm{epi}(\max_{u \in U} f(.,u))^* + \mathrm{epi}\big(\bar{r}(-\min_{v \in V} g(.,v))\big)^* \\
&+ \mathrm{co}\Big(\bigcup_{y^* \in Y_+^*, w \in W} \mathrm{epi}(y^* \circ h(.,w))^*\Big) + \mathrm{epi}\delta_C^*\Bigg].
\end{aligned} \tag{3.2}$$

In addition, by (3.1) and (3.2), we obtain

$$\begin{aligned}
(0,0) \in \mathrm{cl}\Bigg[&\mathrm{epi}(\max_{u \in U} f(.,u))^* + \mathrm{epi}\big(\bar{r}(-\min_{v \in V} g(.,v))\big)^* \\
&+ \mathrm{co}\Big(\bigcup_{y^* \in Y_+^*, w \in W} \mathrm{epi}(y^* \circ h(.,w))^*\Big) + \mathrm{epi}\delta_C^*\Bigg].
\end{aligned} \tag{3.3}$$

Thus there exist a sequence $\{(x_n^*, \xi_n)\} \subseteq X^* \times \mathbb{R}$ with $x_n^* \to_* 0$ and $\xi_n \to 0$, such that

$$\begin{aligned}
(x_n^*, \xi_n) \in\ &\mathrm{epi}(\max_{u \in U} f(.,u))^* + \mathrm{epi}\big(\bar{r}(-\min_{v \in V} g(.,v))\big)^* \\
&+ \mathrm{co}\Big(\bigcup_{y^* \in Y_+^*, w \in W} \mathrm{epi}(y^* \circ h(.,w))^*\Big) + \mathrm{epi}\delta_C^*.
\end{aligned}$$

Thus there exist a sequence $\{(u_n^*,r_n)\}$, $\{(v_n^*,s_n)\}$, $\{(w_n^*,t_n)\}$, $\{(q_n^*,\gamma_n)\}\subseteq X^*\times\mathbb{R}$ such that

$$\begin{cases} (u_n^*,r_n)\in \operatorname{epi}(\max_{u\in U} f(.,u))^*,\ (v_n^*,s_n)\in \operatorname{epi}(\bar{r}(-\min_{v\in V} g(.,v)))^*,\ (q_n^*,\gamma_n)\in \operatorname{epi}\delta_C^*,\\ (w_n^*,t_n)\in \operatorname{co}\Big(\bigcup_{y^*\in Y_+^*,w\in W}\operatorname{epi}(y^*\circ h(.,w))^*\Big),\\ u_n^*+v_n^*+w_n^*+q_n^*\to_* 0,\\ r_n+s_n+t_n+\gamma_n\to 0.\end{cases}$$

Since $(w_n^*,t_n)\in \operatorname{co}\big(\bigcup_{y^*\in Y_+^*,w\in W}\operatorname{epi}(y^*\circ h(.,w))^*\big)$, then there exist $p\geq 1$, $y_n^*\in Y_+^*$, $w_n\in W$, $\alpha_{i,n}\geq 0$ with $\sum_{i=1}^p\alpha_{i,n}=1$ and $(w_{i,n}^*,t_{i,n})\in \operatorname{epi}(y_n^*\circ h(.,w_n))^*$ $(i=1,\ldots,p)$, such that $w_n^*=\sum_{i=1}^p\alpha_{i,n}w_{i,n}^*$ and $t_n=\sum_{i=1}^p\alpha_{i,n}t_{i,n}$. Thus

$$u_n^*+v_n^*+\sum_{i=1}^p\alpha_{i,n}w_{i,n}^*+q_n^*\to_* 0 \quad\text{ and }\quad r_n+s_n+\sum_{i=1}^p\alpha_{i,n}t_{i,n}+\gamma_n\to 0.$$

$(\Leftarrow)$. Assume that there exist $p\geq 1$, $\{(u_n^*,r_n)\}$, $\{(v_n^*,s_n)\}$, $\{(w_{i,n}^*,t_{i,n})\}$, $\{(q_n^*,\gamma_n)\}\subseteq X^*\times\mathbb{R}$ $(i=1,\ldots,p)$ and $y_n^*\in Y_+^*$, $w_n\in W$, $\alpha_{i,n}\geq 0$ $(\sum_{i=1}^p\alpha_{i,n}=1)$, such that

$$\begin{cases} (u_n^*,r_n)\in \operatorname{epi}(\max_{u\in U} f(.,u))^*,\ (v_n^*,s_n)\in \operatorname{epi}(\bar{r}(-\min_{v\in V} g(.,v)))^*,\ (q_n^*,\gamma_n)\in \operatorname{epi}\delta_C^*,\\ (w_{i,n}^*,t_{i,n})\in \operatorname{epi}(y_n^*\circ h(.,w_n))^*,\ (i=1,\ldots,p),\\ u_n^*+v_n^*+\sum_{i=1}^p\alpha_{i,n}w_{i,n}^*+q_n^*\to_* 0,\\ r_n+s_n+\sum_{i=1}^p\alpha_{i,n}t_{i,n}+\gamma_n\to 0.\end{cases}$$

Let $x\in\mathbf{A}$ be arbitrary. By the definition of the epigraph of a conjugate function, we have, for any $n\geq 0$,

$$\begin{cases}\max_{u\in U} f(x,u)\geq\langle u_n^*,x\rangle-r_n,\\ -\bar{r}\min_{v\in V} g(x,v)\geq\langle v_n^*,x\rangle-s_n,\\ 0=\delta_C(x)\geq\langle q_n^*,x\rangle-\gamma_n,\end{cases}\tag{3.4}$$

and $0\geq(y_n^*\circ h(.,w_n))(x)\geq\langle w_{i,n}^*,x\rangle-t_{i,n}$, $i=1,\ldots,p$. Since $\alpha_{i,n}\geq 0$ $(i=1,\ldots,p)$, then $0\geq\langle\alpha_{i,n}w_{i,n}^*,x\rangle-\alpha_{i,n}t_{i,n}$. Thus,

$$0\geq\langle\sum_{i=1}^p\alpha_{i,n}w_{i,n}^*,x\rangle-\sum_{i=1}^p\alpha_{i,n}t_{i,n}.\tag{3.5}$$

By adding (3.4) and (3.5), we obtain

$$\max_{u\in U} f(x,u)-\bar{r}\min_{v\in V} g(x,v)\geq\langle u_n^*+v_n^*+\sum_{i=1}^p\alpha_{i,n}w_{i,n}^*+q_n^*,x\rangle-(r_n+s_n+\sum_{i=1}^p\alpha_{i,n}t_{i,n}+\gamma_n).$$

Passing to the limit when $n\to\infty$, we obtain that $\max_{u\in U} f(x,u)-\bar{r}\min_{v\in V} g(x,v)\geq 0$. Since $\bar{r}=\max_{(u,v)\in U\times V}\frac{f(\bar{x},u)}{g(\bar{x},v)}-\varepsilon$, then $\max_{u\in U} f(\bar{x},u)-\bar{r}\min_{v\in V} g(\bar{x},v)-\varepsilon\min_{v\in V} g(\bar{x},v)=0$. For any $x\in\mathbf{A}$, we have

$$\max_{u\in U} f(x,u)-\bar{r}\min_{v\in V} g(x,v)\geq\max_{u\in U} f(\bar{x},u)-\bar{r}\min_{v\in V} g(\bar{x},v)-\varepsilon\min_{v\in V} g(\bar{x},v),$$

which means that $\bar{x}$ is an $\bar{\varepsilon}$-solution to $(\mathrm{P}_{\bar{r}})$, where $\bar{\varepsilon}=\varepsilon\min_{v\in V} g(\bar{x},v)$. Then, by Lemma 3.1, $\bar{x}$ is an ε-solution to (P), and the proof is complete. $\square$

Remark 3.2. For problem (P), if, for all $x \in X$, $f(x,.)$ is a concave function and $g(x,.)$ is a convex function, one see by using Proposition 2.2 that

$$\operatorname{epi}(\max_{u\in U} f(.,u))^* = \bigcup_{u\in U} \operatorname{epi}(f(.,u))^*,\ \operatorname{epi}(\bar{r}(-\min_{v\in V} g(.,v)))^* = \bigcup_{v\in V} \operatorname{epi}(-\bar{r}g(.,v))^*.$$

By Theorem 3.1 and Remark 3.2, we can obtain the following corollary.

Corollary 3.1. *For* (P)*, let* $\bar{x} \in \mathbf{A}$*,* $\varepsilon \geq 0$ *and* $\bar{r} = \max_{(u,v)\in U\times V} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \varepsilon \geq 0$*. If, for all* $x \in X$*,* $f(x,.)$ *is a concave function and* $g(x,.)$ *is a convex function, then* $\bar{x}$ *is an* ε*-solution to problem* (P) *if and only if there exist* $p \geq 1$*,* $\{(u_n^*, r_n)\}$*,* $\{(v_n^*, s_n)\}$*,* $\{(w_{i,n}^*, t_{i,n})\}$*,* $\{(q_n^*, \gamma_n)\} \subseteq X^* \times \mathbb{R}$ $(i = 1,...,p)$*,* $\bar{u}_n \in U$*,* $\bar{v}_n \in V$*,* $y_n^* \in Y_+^*$*,* $w_n \in W$ *and* $\alpha_{i,n} \geq 0$ $(\sum_{i=1}^p \alpha_{i,n} = 1)$*, such that*

$$\begin{cases} (u_n^*, r_n) \in \operatorname{epi}(f(.,\bar{u}_n))^*,\ (v_n^*, s_n) \in \operatorname{epi}(-\bar{r}g(.,\bar{v}_n))^*,\ (q_n^*, \gamma_n) \in \operatorname{epi}\delta_C^*, \\ (w_{i,n}^*, t_{i,n}) \in \operatorname{epi}(y_n^* \circ h(.,w_n))^*,\ (i = 1,\dots,p), \\ u_n^* + v_n^* + \displaystyle\sum_{i=1}^p \alpha_{i,n} w_{i,n}^* + q_n^* \to_* 0, \\ r_n + s_n + \displaystyle\sum_{i=1}^p \alpha_{i,n} t_{i,n} + \gamma_n \to 0. \end{cases}$$

Now, we establish some necessary and sufficient optimality conditions charaterizing an approximate solution of problem (P) in the absence of constraint qualification, in terms of approximate subdifferentials.

Theorem 3.2. *For* (P)*, let* $\bar{x} \in \mathbf{A}$*,* $\varepsilon \geq 0$ *and* $\bar{r} = \max_{(u,v)\in U\times V} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \varepsilon \geq 0$*. Then,* $\bar{x}$ *is an* ε*-solution of (P) if and only if there exist* $p \geq 1$*,* $\{\beta_n\}$*,* $\{\eta_n\}$*,* $\{\theta_{i,n}\}$*,* $\{\nu_n\} \subseteq \mathbb{R}_+$ $(i = 1,...,p)$*,* $y_n^* \in Y_+^*$*,* $w_n \in W$*,* $\alpha_{i,n} \geq 0$ $(\sum_{i=1}^p \alpha_{i,n} = 1)$ *and* $\{u_n^*\}$*,* $\{v_n^*\}$*,* $\{w_{i,n}^*\}$*,* $\{q_n^*\} \subseteq X^*$ $(i = 1,...,p)$*, such that*

$$\begin{cases} u_n^* \in \partial_{\beta_n}(\max_{u\in U} f(.,u))(\bar{x}),\ v_n^* \in \partial_{\eta_n}(\bar{r}(-\min_{v\in V} g(.,v)))(\bar{x}),\ q_n^* \in N_{\nu_n}(C,\bar{x}), \\ w_{i,n}^* \in \partial_{\theta_{i,n}}(y_n^* \circ h(.,w_n))(\bar{x}),\ (i = 1,\dots,p), \\ u_n^* + v_n^* + \displaystyle\sum_{i=1}^p \alpha_{i,n} w_{i,n}^* + q_n^* \to_* 0, \\ \beta_n + \eta_n + \displaystyle\sum_{i=1}^p \alpha_{i,n}\theta_{i,n} + \nu_n - (y_n^* \circ h(.,w_n))(\bar{x}) \to \bar{\varepsilon}. \end{cases}$$

Proof. $(\Rightarrow)$. Assume that $\bar{x}$ is an ε-solution to (P). Then, by Theorem 3.1, there exist $p \geq 1$, $\{(u_n^*, r_n)\}$, $\{(v_n^*, s_n)\}$, $\{(w_{i,n}^*, t_{i,n})\}$, $\{(q_n^*, \gamma_n)\} \subseteq X^* \times \mathbb{R}$ $(i = 1,...,p)$, $y_n^* \in Y_+^*$, $w_n \in W$, and $\alpha_{i,n} \geq 0$ $(\sum_{i=1}^p \alpha_{i,n} = 1)$ such that

$$\begin{cases} (u_n^*, r_n) \in \operatorname{epi}(\max_{u\in U} f(.,u))^*,\ (v_n^*, s_n) \in \operatorname{epi}(\bar{r}(-\min_{v\in V} g(.,v)))^*,\ (q_n^*, \gamma_n) \in \operatorname{epi}\delta_C^*, \\ (w_{i,n}^*, t_{i,n}) \in \operatorname{epi}(y_n^* \circ h(.,w_n))^*,\ (i = 1,\dots,p), \\ u_n^* + v_n^* + \displaystyle\sum_{i=1}^p \alpha_{i,n} w_{i,n}^* + q_n^* \to_* 0, \\ r_n + s_n + \displaystyle\sum_{i=1}^p \alpha_{i,n} t_{i,n} + \gamma_n \to 0. \end{cases}$$

By applying Proposition 2.1, we have

$$\operatorname{epi}(\max_{u\in U} f(.,u))^* = \bigcup_{\beta\geq 0}\Big\{(s,\langle s,\bar{x}\rangle+\beta-\max_{u\in U} f(\bar{x},u))\ /\ s\in\partial_\beta(\max_{u\in U} f(.,u))(\bar{x})\Big\},$$

$$\operatorname{epi}(\bar{r}(-\min_{v\in V} g(.,v)))^* = \bigcup_{\eta\geq 0}\Big\{(s,\langle s,\bar{x}\rangle+\eta+\bar{r}\min_{v\in V} g(.,v))\ /\ s\in\partial_\eta(\bar{r}(-\min_{v\in V} g(\bar{x},v)))(\bar{x}))\Big\},$$

$$\operatorname{epi}(y_n^*\circ h(.,w_n))^* = \bigcup_{\theta\geq 0}\Big\{(s,\langle s,\bar{x}\rangle+\theta-(y_n^*\circ h(.,w_n))(\bar{x}))\ /\ s\in\partial_\theta(y_n^*\circ h(.,w_n))(\bar{x})\Big\},$$

$$\operatorname{epi}\delta_C^* = \bigcup_{\nu\geq 0}\Big\{(s,\langle s,\bar{x}\rangle+\nu)\ /\ s\in\partial_\nu\delta_C(\bar{x})=N_\nu(C,\bar{x})\Big\}.$$

Thus there exist sequences $\{\beta_n\}$, $\{\eta_n\}$, $\{\theta_{i,n}\}$, and $\{\nu_n\}\subseteq\mathbb{R}_+$ $(i=1,...,p)$ such that

$$\begin{array}{lll} u_n^*\in\partial_{\beta_n}(\max_{u\in U} f(.,u))(\bar{x}) & \text{and} & r_n=\langle u_n^*,\bar{x}\rangle+\beta_n-\max_{u\in U} f(\bar{x},u),\\ v_n^*\in\partial_{\eta_n}(\bar{r}(-\min_{v\in V} g(.,v)))(\bar{x}) & \text{and} & s_n=\langle v_n^*,\bar{x}\rangle+\eta_n+\bar{r}\min_{v\in V} g(\bar{x},v),\\ q_n^*\in N_{\nu_n}(C,\bar{x}) & \text{and} & \gamma_n=\langle q_n^*,\bar{x}\rangle+\nu_n,\\ w_{i,n}^*\in\partial_{\theta_{i,n}}(y_n^*\circ h(.,w_n))(\bar{x}) & \text{and} & t_{i,n}=\langle w_{i,n}^*,\bar{x}\rangle+\theta_{i,n}-(y_n^*\circ h(.,w_n))(\bar{x}),\ (i=1,\dots,p). \end{array}$$

By summing the above equalities, we obtain

$$\begin{aligned} r_n+s_n+\sum_{i=1}^p\alpha_{i,n}t_{i,n}+\gamma_n &= \langle u_n^*+v_n^*+\sum_{i=1}^p\alpha_{i,n}w_{i,n}^*+q_n^*,\bar{x}\rangle-\big(\max_{u\in U} f(\bar{x},u)-\bar{r}\min_{v\in V} g(\bar{x},v)\big)\\ &\quad -(y_n^*\circ h(.,w_n))(\bar{x})+\Big(\beta_n+\eta_n+\sum_{i=1}^p\alpha_{i,n}\theta_{i,n}+\nu_n\Big). \end{aligned}$$

Since $\bar{r}=\max_{(u,v)\in U\times V}\frac{f(\bar{x},u)}{g(\bar{x},v)}-\varepsilon$, then $\max_{u\in U} f(\bar{x},u)-\bar{r}\min_{v\in V} g(\bar{x},v)-\bar{\varepsilon}=0$. Thus

$$\begin{aligned} r_n+s_n+\sum_{i=1}^p\alpha_{i,n}t_{i,n}+\gamma_n &= \langle u_n^*+v_n^*+\sum_{i=1}^p\alpha_{i,n}w_{i,n}^*+q_n^*,\bar{x}\rangle-\bar{\varepsilon}-(y_n^*\circ h(.,w_n))(\bar{x})\\ &\quad +\Big(\beta_n+\eta_n+\sum_{i=1}^p\alpha_{i,n}\theta_{i,n}+\nu_n\Big). \end{aligned}$$

Now, passing to the limit as $n\to\infty$, we obtain $\beta_n+\eta_n+\sum_{i=1}^p\alpha_{i,n}\theta_{i,n}+\nu_n-(y_n^*\circ h(.,w_n))\to\bar{\varepsilon}$.

($\Leftarrow$). Assume that there exist $p\geq 1$, $\{\beta_n\}$, $\{\eta_n\}$, $\{\theta_{i,n}\}$, $\{\nu_n\}\subseteq\mathbb{R}_+$ $(i=1,...,p)$, $y_n^*\in Y_+^*$, $w_n\in W$, $\alpha_{i,n}\geq 0$ $(\sum_{i=1}^p\alpha_{i,n}=1)$ and $\{u_n^*\}$, $\{v_n^*\}$, $\{w_{i,n}^*\}$, $\{q_n^*\}\subseteq X^*$ $(i=1,...,p)$, such that

$$\begin{cases} u_n^*\in\partial_{\beta_n}(\max_{u\in U} f(.,u))(\bar{x}),\ v_n^*\in\partial_{\eta_n}(\bar{r}(-\min_{v\in V} g(.,v)))(\bar{x}),\ q_n^*\in N_{\nu_n}(C,\bar{x}),\\ w_{i,n}^*\in\partial_{\theta_{i,n}}(y_n^*\circ h(.,w_n))(\bar{x}),\ (i=1,\dots,p),\\ u_n^*+v_n^*+\displaystyle\sum_{i=1}^p\alpha_{i,n}w_{i,n}^*+q_n^*\to_* 0,\\ \beta_n+\eta_n+\displaystyle\sum_{i=1}^p\alpha_{i,n}\theta_{i,n}+\nu_n-(y_n^*\circ h(.,w_n))\to\bar{\varepsilon}. \end{cases}$$

Let $x \in \mathbf{A}$ be arbitrary. By the definition of approximate subdifferentials and $N_{\nu_n}(C,\bar{x})$, we have, for any $n \geq 0$,

$$\begin{cases} \max_{u\in U} f(x,u) - \max_{u\in U} f(\bar{x},u) \geq \langle u_n^*, x-\bar{x}\rangle - \beta_n, \\ -\bar{r}\min_{v\in V} g(x,v) + \bar{r}\min_{v\in V} g(\bar{x},v) \geq \langle v_n^*, x-\bar{x}\rangle - \eta_n, \\ 0 \geq \langle q_n^*, x-\bar{x}\rangle - \nu_n, \end{cases} \tag{3.6}$$

and $(y_n^*\circ h(.,w_n))(x) - (y_n^*\circ h(.,w_n))(\bar{x}) \geq \langle w_{i,n}^*, x-\bar{x}\rangle - \theta_{i,n}$, $i = 1,\ldots,p$. Since $\alpha_{i,n} \geq 0$ ($i = 1,\ldots,p$), then $\alpha_{i,n}(y_n^*\circ h(.,w_n))(x) - \alpha_{i,n}(y_n^*\circ h(.,w_n))(\bar{x}) \geq \langle \alpha_{i,n}w_{i,n}^*, x-\bar{x}\rangle - \alpha_{i,n}\theta_{i,n}$. Therefore,

$$(y_n^*\circ h(.,w_n))(x) - (y_n^*\circ h(.,w_n))(\bar{x}) \geq \langle \sum_{i=1}^{p} \alpha_{i,n}w_{i,n}^*, x-\bar{x}\rangle - \sum_{i=1}^{p} \alpha_{i,n}\theta_{i,n}. \tag{3.7}$$

Adding (3.6) and (3.7), we obtain

$$\begin{aligned} &\max_{u\in U} f(x,u) - \max_{u\in U} f(\bar{x},u) + \bar{r}\,(-\min_{v\in V} g(x,v)) - \bar{r}\,(-\min_{v\in V} g(\bar{x},v)) + (y_n^*\circ h(.,w_n))(x) \\ &-(y_n^*\circ h(.,w_n))(\bar{x}) \geq \langle u_n^* + v_n^* + \sum_{i=1}^{p} \alpha_{i,n}w_{i,n}^* + q_n^*, x-\bar{x}\rangle - (\beta_n + \eta_n + \sum_{i=1}^{p} \alpha_{i,n}\theta_{i,n} + \nu_n). \end{aligned}$$

In addition, by using the fact that $(y_n^*\circ h(.,w_n))(x) \leq_{Y_+} 0$, we have

$$\begin{aligned} &\max_{u\in U} f(x,u) - \max_{u\in U} f(\bar{x},u) + \bar{r}\,(-\min_{v\in V} g(x,v)) - \bar{r}\,(-\min_{v\in V} g(\bar{x},v)) \\ &\geq \langle u_n^* + v_n^* + \sum_{i=1}^{p} \alpha_{i,n}w_{i,n}^* + q_n^*, x-\bar{x}\rangle - (\beta_n + \eta_n + \sum_{i=1}^{p} \alpha_{i,n}\theta_{i,n} + \nu_n - (y_n^*\circ h(.,w_n))(\bar{x})). \end{aligned}$$

Passing to the limit as $n \to \infty$, we have $\max_{u\in U} f(x,u) - \max_{u\in U} f(\bar{x},u) - \bar{r}\min_{v\in V} g(x,v) + \bar{r}\ \min_{v\in V} g(\bar{x},v) \geq -\bar{\varepsilon}$. Hence, for any $x \in \mathbf{A}$,

$$\max_{u\in U} f(x,u) - \bar{r}\min_{v\in V} g(x,v) \geq \max_{u\in U} f(\bar{x},u) - \bar{r}\ \min_{v\in V} g(\bar{x},v) - \bar{\varepsilon},$$

which means that $\bar{x}$ is an $\bar{\varepsilon}$-solution to ($\mathrm{P}_{\bar{r}}$), where $\bar{\varepsilon} = \varepsilon \min_{v\in V} g(\bar{x},v)$. By Lemma 3.1, $\bar{x}$ is an ε-solution to (P). The proof is complete. □

By taking $g \equiv 1$ in the above theorem, we obtain the following result immediately.

Corollary 3.2. *For problem* (P') : $\min\{\max_{u\in U} f(x,u) : x \in C,\ h(x,w) \in Y_+,\ \forall w \in W\}$, *let* $\bar{x} \in \mathbf{A}$, $\varepsilon \geq 0$. *Then,* $\bar{x}$ *is an* ε*-solution to problem* (P') *if and only if there exist* $p \geq 1$, $\{\beta_n\}$, $\{\theta_{i,n}\}$, $\{\nu_n\} \subseteq \mathbb{R}_+$ $(i = 1,...,p)$, $y_n^* \in Y_+^*$, $w_n \in W$, $\alpha_{i,n} \geq 0$ $(\sum_{i=1}^{p} \alpha_{i,n} = 1)$ *and* $\{u_n^*\}$, $\{w_{i,n}^*\}$, $\{q_n^*\} \subseteq X^*$ $(i = 1,...,p)$, *such that*

$$\begin{cases} u_n^* \in \partial_{\beta_n}(\max_{u\in U} f(.,u))(\bar{x}),\ w_{i,n}^* \in \partial_{\theta_{i,n}}(y_n^*\circ h(.,w_n))(\bar{x}),\ q_n^* \in N_{\nu_n}(C,\bar{x}),\ (i = 1,\ldots,p), \\ u_n^* + \sum_{i=1}^{p} \alpha_{i,n}w_{i,n}^* + q_n^* \to_* 0, \\ \beta_n + \sum_{i=1}^{p} \alpha_{i,n}\theta_{i,n} + \nu_n - (y_n^*\circ h(.,w_n))(\bar{x}) \to \bar{\varepsilon}. \end{cases}$$

4. Approximate Optimality Conditions of Problem (P) under a Constraint Qualification

In this section, we present an exact necessary and sufficient optimality conditions charaterizing an approximate solution of (P) under a constraint qualification. For this, we consider the following constraint qualification $(CQ)_1$

$$\operatorname{epi}(\max_{u\in U} f(.,u))^* + \operatorname{epi}(\bar{r}(-\min_{v\in V} g(.,v)))^* + \bigcup_{y^*\in Y_+^*, w\in W} \operatorname{epi}(y^*\circ h(.,w))^* + \operatorname{epi}\delta_C^*$$

is a weak* closed and convex subset.

The following theorem establishes the necessary and sufficient optimality conditions characterizing an approximate solution under a constraint qualification of problem (P).

Theorem 4.1. *For* (P)*, let* $\varepsilon \geq 0$*,* $\bar{x} \in \mathbf{A}$ *and* $\bar{r} = \max_{(u,v)\in U\times V} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \varepsilon \geq 0$*. Suppose that* $(CQ)_1$ *holds. Then,* $\bar{x}$ *is an* ε*-solution to* (P) *if and only if there exist* $\bar{y}^* \in Y_+^*$*,* $\bar{w} \in W$*, and* $\beta, \eta, \theta, \nu \in \mathbb{R}_+$*, such that*

$$0 \in \partial_\beta(\max_{u\in U} f(.,u))(\bar{x}) + \partial_\eta(\bar{r}(-\min_{v\in V} g(.,v)))(\bar{x}) + \partial_\theta(\bar{y}^*\circ h(.,\bar{w}))(\bar{x}) + N_\nu(C,\bar{x}), \tag{4.1}$$

and

$$\beta + \eta + \theta + \nu - \bar{\varepsilon} = (\bar{y}^*\circ h(.,\bar{w}))(\bar{x}). \tag{4.2}$$

Proof. $(\Rightarrow)$. Following the proof of Theorem 3.1, if $\bar{x}$ is an ε-solution to problem (P), then (3.3) hold, which means that

$$(0,0) \in \operatorname{cl}\operatorname{co}\left[\operatorname{epi}(\max_{u\in U} f(.,u))^* + \operatorname{epi}\big(\bar{r}(-\min_{v\in V} g(.,v))\big)^* + \bigcup_{y^*\in Y_+^*, w\in W} \operatorname{epi}(y^*\circ h(.,w))^* + \operatorname{epi}\delta_C^*\right].$$

Since $(CQ)_1$ holds, we obtain

$$(0,0) \in \operatorname{epi}(\max_{u\in U} f(.,u))^* + \operatorname{epi}\big(\bar{r}(-\min_{v\in V} g(.,v))\big)^* + \bigcup_{y^*\in Y_+^*, w\in W} \operatorname{epi}(y^*\circ h(.,w))^* + \operatorname{epi}\delta_C^*.$$

So, there exist $\bar{y}^* \in Y_+^*$ and $\bar{w} \in W$, such that

$$(0,0) \in \operatorname{epi}(\max_{u\in U} f(.,u))^* + \operatorname{epi}\big(\bar{r}(-\min_{v\in V} g(.,v))\big)^* + \operatorname{epi}(\bar{y}^*\circ h(.,\bar{w}))^* + \operatorname{epi}\delta_C^*.$$

This yields that there exist $(u^*,r) \in \operatorname{epi}(\max_{u\in U} f(.,u))^*$, $(v^*,s) \in \operatorname{epi}(\bar{r}(-\min_{v\in V} g(.,v)))^*$, $(w^*,t) \in \operatorname{epi}(\bar{y}^*\circ h(.,\bar{w}))^*$ and $(q^*,\gamma) \in \operatorname{epi}\delta_C^*$, such that

$$u^* + v^* + w^* + q^* = 0 \tag{4.3}$$

and

$$r + s + t + \gamma = 0. \tag{4.4}$$

Moreover, by applying Proposition 2.1, there exist $\beta, \eta, \theta, \nu \in \mathbb{R}_+$, such that

$$\begin{array}{lll} u^* \in \partial_\beta(\max_{u\in U} f(.,u))(\bar{x}) & \text{and} & r = \langle u^*,\bar{x}\rangle + \beta - \max_{u\in U} f(\bar{x},u), \\ v^* \in \partial_\eta(\bar{r}(-\min_{v\in V} g(.,v)))(\bar{x}) & \text{and} & s = \langle v^*,\bar{x}\rangle + \eta + \bar{r}\min_{v\in V} g(\bar{x},v), \\ w^* \in \partial_\theta(\bar{y}^*\circ h(.,\bar{w}))(\bar{x}) & \text{and} & t = \langle w^*,\bar{x}\rangle + \theta - (\bar{y}^*\circ h(.,\bar{w}))(\bar{x}), \\ q^* \in N_\nu(C,\bar{x}) & \text{and} & \gamma = \langle q^*,\bar{x}\rangle + \nu. \end{array}$$

By adding the above equalities and using (4.3), we obtain

$$0 \in \partial_\beta(\max_{u\in U} f(.,u))(\overline{x}) + \partial_\eta(\overline{r}(-\min_{v\in V} g(.,v)))(\overline{x}) + \partial_\theta(\overline{y}^* \circ h(.,\overline{w}))(\overline{x}) + N_\nu(C,\overline{x}),$$

and

$$\begin{aligned} r+s+t+\gamma &= \langle u^*+v^*+w^*+q^*, \overline{x}\rangle - \Big(\max_{u\in U} f(\overline{x},u) - \overline{r}\min_{v\in V} g(\overline{x},v)\Big) \\ &\quad -(\overline{y}^*\circ h(.,\overline{w}))(\overline{x}) + \Big(\beta+\eta+\theta+\nu\Big). \end{aligned} \tag{4.5}$$

Since $\overline{r} = \max_{(u,v)\in U\times V} \frac{f(\overline{x},u)}{g(\overline{x},v)} - \varepsilon$, and $\overline{\varepsilon} = \varepsilon \min_{v\in V} g(\overline{x},v)$, then $\max_{u\in U} f(\overline{x},u) - \overline{r}\min_{v\in V} g(\overline{x},v) - \overline{\varepsilon} = 0$. Using (4.3), (4.4), and (4.5), we obtain

$$\beta+\eta+\theta+\nu-\overline{\varepsilon} = (\overline{y}^*\circ h(.,\overline{w}))(\overline{x}).$$

Hence, (4.1) and (4.2) hold.

($\Leftarrow$). Suppose that there exist $\overline{y}^* \in Y_+^*$, $\overline{w} \in W$, and $\beta,\eta,\theta,\nu \in \mathbb{R}_+$, such that (4.1) and (4.2) hold. By (4.1), there exist $u^* \in \partial_\beta(\max_{u\in U} f(.,u))(\overline{x})$, $v^* \in \partial_\eta(\overline{r}(-\min_{v\in V} g(.,v)))(\overline{x})$, $w^* \in \partial_\theta(\overline{y}^*\circ h(.,\overline{w}))(\overline{x})$ and $q^* \in N_\nu(C,\overline{x})$, such that

$$u^*+v^*+w^*+q^* = 0. \tag{4.6}$$

By the definition of approximate subdifferentials and $N_\nu(C,\overline{x})$, we obtain, for any $x \in \mathbf{A}$,

$$\begin{cases} \max_{u\in U} f(x,u) - \max_{u\in U} f(\overline{x},u) \geq \langle u^*, x-\overline{x}\rangle - \beta, \\ -\overline{r}\min_{v\in V} g(x,v) + \overline{r}\min_{v\in V} g(\overline{x},v) \geq \langle v^*, x-\overline{x}\rangle - \eta, \\ (\overline{y}^*\circ h(.,\overline{w}))(x) - (\overline{y}^*\circ h(.,\overline{w}))(\overline{x}) \geq \langle w^*, x-\overline{x}\rangle - \theta, \\ 0 \geq \langle q^*, x-\overline{x}\rangle - \nu. \end{cases}$$

Adding the above inequalities, we have, for any $x \in \mathbf{A}$,

$$\begin{aligned} &\max_{u\in U} f(x,u) - \max_{u\in U} f(\overline{x},u) - \overline{r}\min_{v\in V} g(x,v) + \overline{r}\min_{v\in V} g(\overline{x},v) + (\overline{y}^*\circ h(.,\overline{w}))(x) \\ &-(\overline{y}^*\circ h(.,\overline{w}))(\overline{x}) \geq \langle u^*+v^*+w^*+q^*, x-\overline{x}\rangle - (\beta+\eta+\theta+\nu). \end{aligned}$$

Observe $(\overline{y}^*\circ h(.,\overline{w}))(x) \leq_{Y_+} 0$. Using equality (4.6), we obtain, for any $x \in \mathbf{A}$,

$$\begin{aligned} &\max_{u\in U} f(x,u) - \max_{u\in U} f(\overline{x},u) - \overline{r}\min_{v\in V} g(x,v) + \overline{r}\min_{v\in V} g(\overline{x},v) \\ &\geq -(\beta+\eta+\theta+\nu-(\overline{y}^*\circ h(.,\overline{w}))(\overline{x})). \end{aligned}$$

It follows from (4.2) that $\max_{u\in U} f(x,u) - \max_{u\in U} f(\overline{x},u) - \overline{r}\min_{v\in V} g(x,v) + \overline{r}\ \min_{v\in V} g(\overline{x},v) \geq -\overline{\varepsilon}$. Hence, for any $x \in \mathbf{A}$,

$$\max_{u\in U} f(x,u) - \overline{r}\min_{v\in V} g(x,v) \geq \max_{u\in U} f(\overline{x},u) - \overline{r}\ \min_{v\in V} g(\overline{x},v) - \overline{\varepsilon},$$

which means that $\overline{x}$ is an $\overline{\varepsilon}$-solution of $(\mathrm{P}_{\overline{r}})$, where $\overline{\varepsilon} = \varepsilon\min_{v\in V} g(\overline{x},v)$. According to Lemma 3.1, we deduce that $\overline{x}$ is an ε-solution of problem (P). The proof is complete. □

Remark 4.1. If we suppose the following constraint qualification $(CQ)_2$

$$\begin{cases} \operatorname{epi}(\max_{u\in U} f(.,u))^* + \operatorname{epi}(\overline{r}(-\min_{v\in V} g(.,v)))^* + \bigcup_{y^*\in Y_+^*, w\in W} \operatorname{epi}(y^*\circ h(.,w))^* + \operatorname{epi}\delta_C^* \\ \text{is a weak}^* \text{ closed subset and } \bigcup_{y^*\in Y_+^*, w\in W} \operatorname{epi}(y^*\circ h(.,w))^* \text{ is a convex subset,} \end{cases}$$

then constraint qualification $(CQ)_1$ holds, i.e.,

$$\operatorname{epi}(\max_{u\in U} f(.,u))^* + \operatorname{epi}(\bar{r}(-\min_{v\in V} g(.,v)))^* + \bigcup_{y^*\in Y_+^*, w\in W} \operatorname{epi}(y^*\circ h(.,w))^* + \operatorname{epi}\delta_C^*$$

is a weak* closed and convex subset, which ensures that the result of Theorem 4.1 holds.

Note that the convexity plays an important role in Theorem 4.1. Now, we give an example to demonstrate that the obtained result fail if the convexity assumption in constraint qualification $(CQ)_2$ is not fulfilled.

Example 4.1. Consider the following fractional programming problem with uncertainty

$$(\mathrm{P}_3) \qquad \begin{cases} \min \max\limits_{(u,v)\in U\times V} \dfrac{-ux+2}{-y+v} \\ \text{s.t. } w^2|x| + \max\{y,0\} - 2w \le 0,\ w\in[0,1], \\ \qquad (x,y)\in]-\infty,0]\times[-1,0], \end{cases}$$

where $U = V = [1,2]$ and $W = [0,1]$. Let $f((x,y),u) := -ux+2$, $g((x,y),v) := -y+v$, $h((x,y),w) := w^2|x| + \max\{y,0\} - 2w$, $C :=]-\infty,0]\times[-1,0]$, $\varepsilon := 1$, and $(\bar{x},\bar{y}) := (0,0)$. The feasible set $\mathbf{A} =]-2,0]\times[-1,0]$. Clearly, $\bar{r} = \max_{(u,v)\in U\times V} \frac{f((\bar{x},\bar{y}),u)}{g((\bar{x},\bar{y}),v)} - \varepsilon = 1$, and then $(\bar{x},\bar{y})$ is an approximate solution to problem (P_3). In [23, Example 2.1], it was proved that $\bigcup_{\mu\ge 0, w\in[0,1]} \operatorname{epi}(\mu h(.,w))^*$ is not convex. Moreover, for any $\beta,\eta,\theta,\nu\in\mathbb{R}_+$, we have

$$\partial_\beta(\max_{u\in U} f(.,u))(\bar{x},\bar{y}) = \{(-2,0)\},\ \partial_\eta(\bar{r}(-\min_{v\in V} g(.,v)))(\bar{x},\bar{y}) = \{(0,1)\}$$

and

$$\partial_\theta(\mu h(.,w))(\bar{x},\bar{y}) = [-\mu w^2, \mu w^2]\times[0,\mu],\ N_\nu(C,(\bar{x},\bar{y})) = \mathbb{R}_+^2.$$

Thus, for any (a,b) such that

$$(a,b)\in \partial_\beta(\max_{u\in U} f(.,u))(\bar{x},\bar{y}) + \partial_\eta(\bar{r}(-\min_{v\in V} g(.,v)))(\bar{x},\bar{y}) + \partial_\theta(\mu h(.,\overline{w}))(\bar{x},\bar{y}) + N_\nu(C,\bar{x},\bar{y}),$$

we have $b>0$. It follows that

$$(0,0)\notin \partial_\beta(\max_{u\in U} f(.,u))(\bar{x},\bar{y}) + \partial_\eta(\bar{r}(-\min_{v\in V} g(.,v)))(\bar{x},\bar{y}) + \partial_\theta(\mu h(.,\overline{w}))(\bar{x},\bar{y}) + N_\nu(C,\bar{x},\bar{y}).$$

Thus, Theorem 4.1 is not applicable with respect to the constraint qualification, $(CQ)_2$.

When $\varepsilon = 0$, we obtain the following result immediately.

Corollary 4.1. *For* (P), *let* $\bar{x}\in\mathbf{A}$ *and* $\bar{r} = \max_{(u,v)\in U\times V}\frac{f(\bar{x},u)}{g(\bar{x},v)}$. *Suppose that* $(CQ)_1$ *holds. Then,* $\bar{x}$ *is an exact solution of* (P) *if and only if there exist* $\bar{y}^*\in Y_+^*$ *and* $\overline{w}\in W$, *such that*

$$0\in\partial(\max_{u\in U} f(.,u))(\bar{x}) + \partial(\bar{r}(-\min_{v\in V} g(.,v)))(\bar{x}) + \partial(\bar{y}^*\circ h(.,\overline{w}))(\bar{x}) + N(C,\bar{x}),$$

and $(\bar{y}^*\circ h(.,\overline{w}))(\bar{x}) = 0$.

Proof. By taking $\varepsilon = 0$ in Theorem 4.1, we deduce that $\bar{x}$ is an exact solution to problem (P) if and only if there exist $\bar{y}^*\in Y_+^*$, $\overline{w}\in W$ and $\beta,\eta,\theta,\nu\in\mathbb{R}_+$ such that

$$0\in\partial_\beta(\max_{u\in U} f(.,u))(\bar{x}) + \partial_\eta(\bar{r}(-\min_{v\in V} g(.,v)))(\bar{x}) + \partial_\theta(\bar{y}^*\circ h(.,\overline{w}))(\bar{x}) + N_\nu(C,\bar{x}),$$

and $\beta+\eta+\theta+\nu-(\bar{y}^*\circ h(.,\overline{w}))(\bar{x}) = 0$. Due to the fact that $\beta+\eta+\theta+\nu-(\bar{y}^*\circ h(.,\overline{w}))(\bar{x}) = 0$, $\beta,\eta,\theta,\nu\in\mathbb{R}_+$ and $-(\bar{y}^*\circ h(.,\overline{w}))(\bar{x})\ge 0$, (since $h(\bar{x},w)\in -Y^+$ and $\bar{y}^*\in Y_+^*$), we have

$\beta = \eta = \theta = \nu = -(\overline{y}^* \circ h(.,\overline{w}))(\overline{x}) = 0$. Therefore, $\overline{x}$ is an exact solution to (P) if and only if there exist $\overline{y}^* \in Y_+^*$ and $\overline{w} \in W$ such that

$$0 \in \partial(\max_{u\in U} f(.,u))(\overline{x}) + \partial(\overline{r}(-\min_{v\in V} g(.,v)))(\overline{x}) + \partial(\overline{y}^* \circ h(.,\overline{w}))(\overline{x}) + N(C,\overline{x}),$$

and $(\overline{y}^* \circ h(.,\overline{w}))(\overline{x}) = 0$, which completes the proof. □

Now, we present an example illustrating our main result.

Example 4.2. Consider the following robust fractional programming problem

$$(\mathrm{P}_4) \qquad \begin{cases} \min \max\limits_{(u,v)\in U\times V} \dfrac{u(-\frac{x}{8}+\frac{y}{4}+\frac{3}{4})}{v(\frac{y}{8}+\frac{1}{4})+\frac{1}{4}} \\ \text{s.t. } \max\{0,wx\}+wy^2 \le 0,\ w\in[1,2], \\ \qquad (x,y)\in[-1,1]\times[-1,1], \end{cases}$$

where $U = [1,2]$ and $V = [1,2]$. By taking $f((x,y),u) := u(-\frac{x}{8}+\frac{y}{4}+\frac{3}{4})$, $g((x,y),v) := v(\frac{y}{8}+\frac{1}{4})+\frac{1}{4}$, $h((x,y),w) := \max\{0,wx\}+wy^2$, $C := [-1,1]\times[-1,1]$, $\varepsilon := 1$ and $(\overline{x},\overline{y}) := (0,0) \in \mathbf{A} = \{(x,y)\in C : h((x,y),w) \le 0,\ \forall w\in[1,2]\} = [-1,0]\times\{0\}$ a feasible point, one has $\overline{r} = \max_{(u,v)\in U\times V} \frac{f((\overline{x},\overline{y}),u)}{g((\overline{x},\overline{y}),v)} - \varepsilon = 2$ and $\overline{\varepsilon} = \varepsilon \min_{v\in V} g((\overline{x},\overline{y}),v) = \frac{1}{2}$. Moreover, by a simple calculation, we have

$$\begin{cases} \operatorname{epi}(\max\limits_{u\in U} f(.,u))^* = \left\{\left(-\dfrac{1}{4},\dfrac{1}{2}\right)\right\} \times \left[-\dfrac{3}{2},+\infty\right[, \\ \operatorname{epi}(\overline{r}(-\min\limits_{v\in V} g(.,v)))^* = \left\{\left(0,-\dfrac{1}{4}\right)\right\} \times [1,+\infty[, \\ \operatorname{epi}(\mu h(.,w))^* = \left\{(x^*,y^*,r) : |\, x^*\in[0,\mu w], y^*\in\mathbb{R}, \dfrac{y^{*2}}{4\mu w} \leqslant r\right\} \text{for any } \mu\neq 0, \\ \operatorname{epi}\delta_C^* = \{(x^*,y^*,r):\ |x^*|+|y^*\,| \leqslant r\}. \end{cases}$$

Thus

$$\begin{cases} \operatorname{epi}(\max\limits_{u\in U} f(.,u))^* + \operatorname{epi}(\overline{r}(-\min\limits_{v\in V} g(.,v)))^* = \left\{\left(-\dfrac{1}{4},\dfrac{1}{4}\right)\right\} \times \left[-\dfrac{1}{2},+\infty\right[, \\ \bigcup\limits_{\mu\ge 0, w\in[1,2]} \operatorname{epi}(\mu h(.,w))^* = (\{0\}\times\{0\}\times\mathbb{R}_+) \cup (\mathbb{R}_+^*\times\{0\}\times\mathbb{R}_+^*) \cup (\mathbb{R}_+^*\times\mathbb{R}^*\times\mathbb{R}_+^*), \\ \operatorname{epi}\delta_C^* = \{(x^*,y^*,r)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}_+ :\ |x^*|+|y^*\,| \leqslant r\}, \end{cases}$$

which means that

$$\operatorname{epi}(\max_{u\in U} f(.,u))^* + \operatorname{epi}(\overline{r}(-\min_{v\in V} g(.,v)))^* + \bigcup_{\mu\ge 0, w\in[1,2]} \operatorname{epi}(\mu h(.,w))^* + \operatorname{epi}\delta_C^*$$

is not closed. Hence, the constraint qualification $(CQ)_1$ does not holds.

For any $n\in\mathbb{N}$, let $p=2$, $\beta_n = \frac{1}{n}$, $\eta_n = \frac{1}{n}$, $\theta_{i,n} = 0$ $(i=1,2)$, $\nu_n = \frac{1}{2}+\frac{1}{n}$. Let $u_n^* = (-\frac{1}{4},\frac{1}{2})$, $v_n^* = (0,-\frac{1}{4})$, $w_{i,n}^* = (0,0)$, $\alpha_{i,n} = \frac{1}{2}$ $(i=1,2)$, $q_n^* = (\frac{1}{4},-\frac{1}{4})$, $\mu_n = 0 \in\mathbb{R}_+$, and $w_n = 1\in[1,2]$. It is easy to see that $\partial_{\beta_n}(\max_{u\in U} f(.,u))(\overline{x},\overline{y}) = \{(-\frac{1}{4},\frac{1}{2})\}$, $\partial_{\eta_n}(\overline{r}(-\min_{v\in V} g(.,v)))(\overline{x},\overline{y}) = \{(0,-\frac{1}{4})\}$, $\partial_{\theta_{i,n}}(\mu_n h(.,w_n))(\overline{x},\overline{y}) = \{(0,0)\}$, $(i=1,2)$ and $N_{\nu_n}(C,(\overline{x},\overline{y})) = \{(u,v):\ |u|+|v\,| \leqslant \nu_n\}$, which yields

$$\begin{cases} u_n^* \in \partial_{\beta_n}(\max_{u\in U} f(.,u))(\bar{x},\bar{y}),\ v_n^* \in \partial_{\eta_n}(\bar{r}(-\min_{v\in V} g(.,v)))(\bar{x},\bar{y}),\ q_n^* \in N_{\nu_n}(C,(\bar{x},\bar{y})), \\ w_{i,n}^* \in \partial_{\theta_{i,n}}(\mu_n h(.,w_n))(\bar{x},\bar{y}),\ (i=1,2), \\ u_n^* + v_n^* + \sum_{i=1}^{p} \alpha_{i,n} w_{i,n}^* + q_n^* = (0,0) \to_* (0,0), \\ \beta_n + \eta_n + \sum_{i=1}^{p} \alpha_{i,n}\theta_{i,n} + \nu_n - (\mu_n h(.,w_n))(\bar{x},\bar{y}) \to \frac{1}{2} = \bar{\varepsilon}. \end{cases}$$

Then, by Theorem 3.2, $(\bar{x},\bar{y})$ is an approximate solution to problem (P_4).

5. Conclusion

In this paper, following the framework of robust optimization approach, we obtained sequential approximate optimality conditions without any constraint qualifications for a scalar constrained fractional optimization problem (P) in the face of data uncertainty both in the objective and constraints in terms of the approximate subdifferentials of the associated functions. We also stated some approximate optimality conditions for problem (P) under a constraint qualifications.

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