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STRONG CONVERGENCE OF AN INERTIAL TSENG'S EXTRAGRADIENT ALGORITHM FOR PSEUDOMONOTONE VARIATIONAL INEQUALITIES WITH APPLICATIONS TO OPTIMAL CONTROL PROBLEMS

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Abstract. We investigate an inertial viscosity-type Tseng's extragradient algorithm with a new step size for solving pseudomonotone variational inequality problems in real Hilbert spaces. A strong convergence theorem of the proposed algorithm is obtained without the prior information of the Lipschitz constant of the operator and also without any requirement of additional projections. Finally, several computational tests are carried out to demonstrate the reliability and benefits of the proposed algorithm and compare it with some existing ones. Moreover, our algorithm is also applied to address the variational inequality problem that appears in optimal control problems.
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1. Introduction

The goal of this paper is to investigate a fast iterative method for discovering a solution to the variational inequality problem (in short, VIP). In this paper, one always assumes that H is a real Hilbert space with inner product $\langle \cdot , \cdot \rangle$ and induced norm $\| \cdot \|$, and C is a closed, convex, and nonempty subset in H. Let us first elaborate on the issue involved in this research as follows:

$$\text{find } y^* \in C \text{ such that } \langle \mathscr{A} y^*, z - y^* \rangle \geq 0, \quad \forall z \in C, \tag{VIP}$$

where $\mathscr{A} : H \to H$ is a nonlinear mapping. We denote the solution set of (VIP) as $\mathrm{VI}(C, \mathscr{A})$.

Variational inequalities are powerful tools and models in applied mathematics and act essential roles in society, optimization, economics, transportation, mathematical programming, engineering mechanics, and other fields (see, for instance, [1, 2, 3]). In the last decades, various effective solution methods have been investigated and developed to solve variational inequalities; see, e.g., [4, 5] and the references therein. It should be pointed out that these approaches usually require that mapping $\mathscr{A}$ has certain monotonicity. In this paper, we consider that the mapping $\mathscr{A}$

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associated with (VIP) is pseudomonotone (see the definition below), which is a broader category than monotone mappings.

The oldest and simplest projection approach to address variational inequality problems is the projected-gradient method, which reads as follows:

$$x_{n+1} = Proj_C(x_n - \gamma \mathscr{A} x_n), \quad \forall n \geq 1, \tag{PGM}$$

where $Proj_C$ represents the metric projection from H onto C, mapping $\mathscr{A}$ is L-Lipschitz continuous and η-strongly monotone and the step size $\gamma \in (0, 2\eta/L^2)$. The iterative sequence $\{x_n\}$ defined by (PGM) converges to the solution of (VIP) provided that $\mathrm{VI}(C, \mathscr{A})$ is nonempty. It should be noted that the iterative sequence $\{x_n\}$ formulated by (PGM) does not necessarily converge when mapping $\mathscr{A}$ is "only" monotone. Recently, Malitsky [6] introduced a projected reflected gradient method, which can be viewed as an improvement of (PGM). Indeed, the sequence generated by this method is as follows:

$$x_{n+1} = Proj_C(x_n - \gamma \mathscr{A}(2x_n - x_{n-1})), \quad \forall n \geq 1. \tag{PRGM}$$

He proved that the sequence $\{x_n\}$ created by iterative scheme (PRGM) converges to $u \in \mathrm{VI}(C, \mathscr{A})$ when the mapping $\mathscr{A}$ is monotone.

In many kinds of research on solving variational inequalities controlled by pseudomonotone and Lipschitz continuous operators, the most commonly used algorithm is the extragradient method (see [7]) and its variants. Korpelevich [7] proposed the extragradient method (EGM) to find the solution of the saddle point problem in finite-dimensional spaces. The details of EGM are described as follows:

$$\begin{cases} y_n = Proj_C(x_n - \gamma \mathscr{A} x_n), \\ x_{n+1} = Proj_C(x_n - \gamma \mathscr{A} y_n), \quad \forall n \geq 1, \end{cases} \tag{EGM}$$

where mapping $\mathscr{A}$ is monotone and L-Lipschitz continuous, and fixed step size $\gamma \in (0, 1/L)$. Under the condition of $\mathrm{VI}(C, \mathscr{A}) \neq \emptyset$, the iterative sequence $\{x_n\}$ defined by (EGM) converges to an element of $\mathrm{VI}(C, \mathscr{A})$. In the past few decades, EGM has been considered and extended by many authors for solving (VIP) in infinite-dimensional spaces; see, e.g., [4, 5, 8, 9] and the references therein. Recently, Vuong [10] extended EGM to solve pseudomonotone variational inequalities in Hilbert spaces, and proved that the iterative sequence constructed by the algorithm converges weakly to a solution of (VIP). On the other hand, it is not easy to calculate the projection on the general closed convex set C, especially when C has a complex structure. Note that in the extragradient method, two projections need to be calculated on the closed convex set C for each iteration, which may severely affect the computational performance of the algorithm used.

Next, we introduce two types of methods to enhance the numerical efficiency of EGM. The first approach is the Tseng's extragradient method (referred to as TEGM, also known as the forward-backward-forward method) proposed by Tseng [11]. The advantage of this method is that the projection on the feasible set only needs to be calculated once in each iteration. More precisely, TEGM is expressed as follows:

$$\begin{cases} y_n = Proj_C(x_n - \gamma \mathscr{A} x_n), \\ x_{n+1} = y_n - \gamma(\mathscr{A} y_n - \mathscr{A} x_n), \quad \forall n \geq 1, \end{cases} \tag{TEGM}$$

where mapping $\mathscr{A}$ is L-Lipschitz continuous and monotone, and fixed step size $\gamma \in (0, 1/L)$. The iterative sequence $\{x_n\}$ formulated by (TEGM) converges to a solution of (VIP) provided that $\mathrm{VI}(C, \mathscr{A})$ is nonempty. Recently, Boţ, Csetnek, and Vuong [12] proposed a modified Tseng's forward-backward-forward algorithm for solving pseudomonotone variational inequalities in Hilbert spaces and performed an asymptotic analysis of the formed trajectories. The second method is the subgradient extragradient method (SEGM) proposed by Censor, Gibali, and Reich [13, 14, 15]. This can be regarded as a modification of EGM. Indeed, they replaced the second projection in (EGM) by a projection onto a half-space. The SEGM is generated as follows:

$$\begin{cases} y_n = Proj_C(x_n - \gamma \mathscr{A} x_n), \\ T_n = \{x \in H \mid \langle x_n - \gamma \mathscr{A} x_n - y_n, x - y_n \rangle \leq 0\}, \\ x_{n+1} = Proj_{T_n}(x_n - \gamma \mathscr{A} y_n), \quad \forall n \geq 1, \end{cases} \tag{SEGM}$$

where mapping $\mathscr{A}$ is L-Lipschitz continuous and monotone, and fixed step size $\gamma \in (0, 1/L)$. The algorithm (SEGM) not only converges to the solution of monotone variational inequalities (see [14]), but also to pseudomonotone variational inequalities (see [15]).

It is worth mentioning that (EGM), (TEGM), and (SEGM) are weakly convergent in infinite-dimensional Hilbert spaces. Some practical problems that occur in the fields of image processing, quantum mechanics, medical imaging, and machine learning need to be modeled and analyzed in infinite-dimensional space. Therefore, strong convergence results are preferable to weak convergence results in infinite-dimensional space. Recently, Thong and Vuong [16] introduced a modified Mann-type Tseng's extragradient method to solve the (VIP) involving a pseudomonotone mapping in Hilbert spaces. Their method uses an Armijo-like line search to eliminate the reliance on the Lipschitz continuous constant of the mapping $\mathscr{A}$. The proposed algorithm is stated as follows:

$$\begin{cases} y_n = Proj_C(x_n - \gamma_n \mathscr{A} x_n), \\ z_n = y_n - \gamma_n(\mathscr{A} y_n - \mathscr{A} x_n), \\ x_{n+1} = (1 - \varphi_n - \tau_n) x_n + \tau_n z_n, \quad \forall n \geq 1, \end{cases} \tag{MaTEGM}$$

where the mapping $\mathscr{A}$ is pseudomonotone, sequentially weakly continuous on C, and uniformly continuous on bounded subsets of H, $\{\varphi_n\}$ and $\{\tau_n\}$ are two real positive sequences in $(0,1)$ such that $\{\tau_n\} \subset (a, 1 - \varphi_n)$ for some $a > 0$ and $\lim_{n\to\infty} \varphi_n = 0$, $\sum_{n=1}^{\infty} \varphi_n = \infty$, $\gamma_n := \alpha \ell^{q_n}$ and q_n is the smallest non-negative integer q satisfying $\alpha \ell^q \|\mathscr{A} x_n - \mathscr{A} y_n\| \leq \phi \|x_n - y_n\|$ ($\alpha > 0$, $\ell \in (0,1)$, $\phi \in (0,1)$). They showed that the iterative scheme formed by (MaTEGM) converges strongly to an element u under $\mathrm{VI}(C, \mathscr{A}) \neq \emptyset$, where $u = \arg\min\{\|z\| : z \in \mathrm{VI}(C, \mathscr{A})\}$.

To accelerate the convergence rate of the algorithms, Polyak [17] considered the following second-order dynamical system

$$\ddot{x}(t) + \gamma \dot{x}(t) + \nabla f(x(t)) = 0,$$

where $\gamma > 0$, ∇f represents the gradient of f, $\dot{x}(t)$ and $\ddot{x}(t)$ denote the first and second derivatives of x at t, respectively. This dynamic system is called the heavy ball with friction. Next, we consider the discretization of this dynamic system, that is,

$$\frac{x_{n+1} - 2x_n + x_{n-1}}{h^2} + \gamma \frac{x_n - x_{n-1}}{h} + \nabla f(x_n) = 0, \quad \forall n \geq 1.$$

Through a direct calculation, we can obtain the following form:

$$x_{n+1} = x_n + \tau(x_n - x_{n-1}) - \varphi \nabla f(x_n), \quad \forall n \geq 1,$$

where $\tau = 1 - \gamma h$ and $\varphi = h^2$. This can be considered as the following two-step iterative scheme:

$$\begin{cases} y_n = x_n + \tau(x_n - x_{n-1}), \\ x_{n+1} = y_n - \varphi \nabla f(x_n), \quad \forall n \geq 0. \end{cases}$$

This iteration is now called the inertial extrapolation algorithm, the term $\tau(x_n - x_{n-1})$ is referred to as the extrapolation point. In recent years, inertial technology as an acceleration method has attracted extensive research in the optimization community. Many scholars have built various fast numerical algorithms by employing the inertial technology. These algorithms have shown advantages in theoretical and computational experiments and have been successfully applied to many problems; see, e.g., [5, 18, 19] and the references therein.

Inspired by the inertial method, the SEGM, and the viscosity method, Thong, Hieu, and Rassias [19] presented a viscosity-type inertial subgradient extragradient algorithm to solve pseudomonotone (VIP) in Hilbert spaces. The algorithm is of the form:

$$\begin{cases} s_n = x_n + \delta_n(x_n - x_{n-1}), \\ y_n = Proj_C(s_n - \gamma_n \mathscr{A} s_n), \\ T_n = \{x \in H \mid \langle s_n - \gamma_n \mathscr{A} s_n - y_n, x - y_n \rangle \leq 0\}, \\ z_n = Proj_{T_n}(s_n - \gamma_n \mathscr{A} y_n), \\ x_{n+1} = \varphi_n f(z_n) + (1 - \varphi_n) z_n, \quad \forall n \geq 1. \\ \gamma_{n+1} = \begin{cases} \min\left\{ \frac{\phi \|s_n - y_n\|}{\|\mathscr{A} s_n - \mathscr{A} y_n\|}, \gamma_n \right\}, & \text{if } \mathscr{A} s_n - \mathscr{A} y_n \neq 0; \\ \gamma_n, & \text{otherwise,} \end{cases} \end{cases} \tag{ViSEGM}$$

where mapping $\mathscr{A}$ is pseudomonotone, L-Lipschitz continuous, and sequentially weakly continuous on C, and the inertia parameters δ_n are updated in the following way:

$$\delta_n = \begin{cases} \min\left\{ \dfrac{\varepsilon_n}{\|x_n - x_{n-1}\|}, \delta \right\}, & \text{if } x_n \neq x_{n-1}; \\ \delta, & \text{otherwise.} \end{cases}$$

Note that the algorithm (ViSEGM) uses a simple step size rule, which is generated through some calculations of previously known information in each iteration. Therefore, it can work well without the prior information of the Lipschitz constant of the mapping $\mathscr{A}$. They confirmed the strong convergence of (ViSEGM) under mild assumptions on cost mapping and parameters.

Motivated and stimulated by the above works, we introduce a new inertial Tseng's extragradient algorithm with a new step size for solving the pseudomonotone (VIP) in Hilbert spaces. The advantages of our algorithm are: (1) only one projection on the feasible set needs to be computed at each iteration, (2) no prior information about the Lipschitz constant of the cost mapping needs to be known, and (3) the inclusion of inertial gives a faster convergence speed. Under mild assumptions, we confirm a strong convergence theorem of the suggested algorithm. Lastly, some computational tests appearing in finite- and infinite-dimensions are proposed to verify our theoretical results. Furthermore, our algorithm is also designed to solve optimal control problems. Our algorithm improves some existing results in [16, 19, 20, 21].

The organizational structure of our paper is built up as follows. Some essential definitions and technical lemmas that need to be used are given in the next section. In Sect. 3, we propose a self-adaptive inertial Tseng's extragradient algorithm and analyze its convergence. Some computational tests and applications to verify our theoretical results are presented in Sect. 4. Finally, the paper ends with a brief summary in Sect. 5.

2. PRELIMINARIES

Let C be a closed, convex, and nonempty subset of a real Hilbert space H. The weak convergence and strong convergence of $\{x_n\}_{n=1}^{\infty}$ to x are represented by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. For each $x, y \in H$ and $\delta \in \mathbb{R}$, we have the following facts:

(1) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$;
(2) $\|\delta x + (1-\delta)y\|^2 = \delta\|x\|^2 + (1-\delta)\|y\|^2 - \delta(1-\delta)\|x-y\|^2$.

Let us review some mappings in nonlinear analysis for further use. For any elements $p, q \in H$, one recalls that a mapping $\mathscr{A}: H \to H$ is said to be:

(1) η*-strongly monotone* if there is a positive number η such that

$$\langle \mathscr{A}p - \mathscr{A}q, p-q\rangle \geq \eta\|p-q\|.$$

(2) η*-inverse strongly monotone* if there is a positive number η such that

$$\langle \mathscr{A}p - \mathscr{A}q, p-q\rangle \geq \eta\|\mathscr{A}p - \mathscr{A}q\|^2.$$

(3) *monotone* if $\langle \mathscr{A}p - \mathscr{A}q, p-q\rangle \geq 0$.
(4) η*-strongly pseudomonotone* if there is a positive number η such that

$$\langle \mathscr{A}p, q-p\rangle \geq 0 \Rightarrow \langle \mathscr{A}q, q-p\rangle \geq \eta\|p-q\|^2.$$

(5) *pseudomonotone* if $\langle \mathscr{A}p, q-p\rangle \geq 0 \Rightarrow \langle \mathscr{A}q, q-p\rangle \geq 0$.
(6) *L-Lipschitz continuous* if there is $L > 0$ such that $\|\mathscr{A}p - \mathscr{A}q\| \leq L\|p-q\|$.
(7) *sequentially weakly continuous* if for any sequence $\{p_n\}$ weakly converges to a point $p \in H$, $\{\mathscr{A}p_n\}$ weakly converges to $\mathscr{A}p$.

It can be easily checked that the following relations: (1) $\Rightarrow$ (3) $\Rightarrow$ (5) and (1) $\Rightarrow$ (4) $\Rightarrow$ (5). Note that the opposite statement is generally incorrect. Recall that a mapping $P_C: H \to C$ is called the metric projection from H onto C, if for all $x \in H$, there is a unique nearest point in C, which is represented by $Proj_C(x)$, such that $Proj_C(x) := \operatorname{argmin}\{\|x-y\|, y \in C\}$.

It is known that $Proj_C(x)$ has the following basic property:

$$\langle x - Proj_C(x), y - Proj_C(x)\rangle \leq 0, \quad \forall x \in H, y \in C. \tag{2.1}$$

We give some explicit formulas to calculate projections on special feasible sets.

(1) The projection of x onto a half-space $H_{u,v} = \{x\colon \langle u, x\rangle \leq v\}$ is represented by

$$Proj_{H_{u,v}}(x) = x - \max\left\{\frac{\langle u, x\rangle - v}{\|u\|^2}, 0\right\} u.$$

(2) The projection of x onto a box $\mathrm{Box}[a,b] = \{x\colon a \leq x \leq b\}$ is formulated as

$$Proj_{\mathrm{Box}[a,b]}(x)_i = \min\{b_i, \max\{x_i, a_i\}\}.$$

(3) The projection of x onto a ball $B[p,q]=\{x\colon \|x-p\|\le q\}$ is given by

$$Proj_{B[p,q]}(x)=p+\frac{q}{\max\{\|x-p\|,q\}}(x-p).$$

The following lemmas play important roles in our proof.

Lemma 2.1 ([22]). *Assume that C is a closed and convex subset of a real Hilbert space H. Let operator $\mathscr{A}: C\to H$ be continuous and pseudomonotone. Then, x^* is a solution of (VIP) if and only if $\langle \mathscr{A}x, x-x^*\rangle \ge 0$ for all $x\in C$.*

Lemma 2.2 ([23]). *Let $\{p_n\}$ be a positive sequence, $\{q_n\}$ be a sequence of real numbers, and $\{\sigma_n\}$ be a sequence in $(0,1)$ such that $\sum_{n=1}^{\infty}\sigma_n=\infty$. Assume that $p_{n+1}\le(1-\sigma_n)p_n+\sigma_n q_n$ for all $n\ge 1$. If $\limsup_{k\to\infty} q_{n_k}\le 0$ for every subsequence $\{p_{n_k}\}$ of $\{p_n\}$ satisfying $\liminf_{k\to\infty}\left(p_{n_k+1}-p_{n_k}\right)\ge 0$, then $\lim_{n\to\infty}p_n=0$.*

3. Main results

In this section, we present a self-adaptive inertial viscosity-type Tseng's extragradient algorithm, which is based on the inertial method, the viscosity method, and the Tseng's extragradient method. The major benefit of this algorithm is that the step size is automatically updated at each iteration without performing any line search procedure. Moreover, our iterative scheme only needs to calculate the projection once in each iteration.

Our algorithm is described as follows.

Algorithm 1 Self adaptive inertial viscosity-type Tseng's extragradient algorithm

Initialization: Take $\delta>0$, $\gamma_1>0$, and $\phi\in(0,1)$. Let $x_0,x_1\in H$ be two initial points.
Iterative Steps: Given the current iterate x_n, calculate x_{n+1} as follows:
Step 1. Compute $s_n=x_n+\delta_n(x_n-x_{n-1})$, where

$$\delta_n=\begin{cases}\min\left\{\dfrac{\varepsilon_n}{\|x_n-x_{n-1}\|},\delta\right\}, & \text{if } x_n\ne x_{n-1};\\ \delta, & \text{otherwise.}\end{cases}\tag{3.1}$$

Step 2. Compute $y_n=Proj_C(s_n-\gamma_n\mathscr{A}s_n)$.
Step 3. Compute $z_n=y_n-\gamma_n(\mathscr{A}y_n-\mathscr{A}s_n)$.
Step 4. Compute $x_{n+1}=\varphi_n f(z_n)+(1-\varphi_n)z_n$, and update γ_{n+1} by

$$\gamma_{n+1}=\begin{cases}\min\left\{\dfrac{\phi\|s_n-y_n\|}{\|\mathscr{A}s_n-\mathscr{A}y_n\|},\gamma_n\right\}, & \text{if } \mathscr{A}s_n-\mathscr{A}y_n\ne 0;\\ \gamma_n, & \text{otherwise.}\end{cases}\tag{3.2}$$

Before starting to state our main result, we assume that our algorithm satisfies the following five assumptions.

(C1) The feasible set C is closed, convex, and nonempty.
(C2) The solution set of the (VIP) is nonempty, that is, $\mathrm{VI}(C,\mathscr{A})\ne\emptyset$.
(C3) The mapping $\mathscr{A}: H\to H$ is pseudomonotone and L-Lipschitz continuous on H, and sequentially weakly continuous on C.
(C4) The mapping $f\colon H\to H$ is ρ-contractive with $\rho\in(0,1)$.

(C5) The positive sequence $\{\varepsilon_n\}$ satisfies $\lim_{n\to\infty}\frac{\varepsilon_n}{\varphi_n}=0$, where $\{\varphi_n\}\subset(0,1)$ such that $\lim_{n\to\infty}\varphi_n=0$ and $\sum_{n=1}^{\infty}\varphi_n=\infty$.

Remark 3.1. It follows from (3.1) and Assumption (C5) that

$$\lim_{n\to\infty}\frac{\delta_n}{\varphi_n}\|x_n-x_{n-1}\|=0.$$

Indeed, we obtain $\delta_n\|x_n-x_{n-1}\|\le\varepsilon_n,\forall n\ge 1$, which together with $\lim_{n\to\infty}\frac{\varepsilon_n}{\varphi_n}=0$ yields

$$\lim_{n\to\infty}\frac{\delta_n}{\varphi_n}\|x_n-x_{n-1}\|\le\lim_{n\to\infty}\frac{\varepsilon_n}{\varphi_n}=0.$$

The following lemmas play significant roles in the convergence proof of our algorithm.

Lemma 3.1. *The sequence $\{\gamma_n\}$ formulated by (3.2) is nonincreasing and*

$$\lim_{n\to\infty}\gamma_n=\gamma\ge\min\left\{\gamma_1,\frac{\phi}{L}\right\}.$$

Proof. On account of (3.2), we have $\gamma_{n+1}\le\gamma_n,\forall n\in\mathbb{N}$. Hence, $\{\gamma_n\}$ is nonincreasing. Moreover, we obtain that $\|\mathscr{A}s_n-\mathscr{A}y_n\|\le L\|s_n-y_n\|$ by means of $\mathscr{A}$ is L-Lipschitz continuous. Thus,

$$\phi\frac{\|s_n-y_n\|}{\|\mathscr{A}s_n-\mathscr{A}y_n\|}\ge\frac{\phi}{L},\ \text{ if }\ \mathscr{A}s_n\ne\mathscr{A}y_n,$$

which together with (3.2) implies that $\gamma_n\ge\min\{\gamma_1,\frac{\phi}{L}\}$. Therefore, $\lim_{n\to\infty}\gamma_n=\gamma\ge\min\{\gamma_1,\frac{\phi}{L}\}$ since sequence $\{\gamma_n\}$ is lower bounded and nonincreasing. □

Lemma 3.2. *Suppose that Assumptions (C1)–(C3) hold. Let $\{s_n\}$ and $\{y_n\}$ be two sequences formulated by Algorithm 1. If there exists a subsequence $\{s_{n_k}\}$ converges weakly to $z\in H$ and $\lim_{k\to\infty}\|s_{n_k}-y_{n_k}\|=0$, then $z\in\mathrm{VI}(C,\mathscr{A})$.*

Proof. From the property of projection (2.1) and $y_n=Proj_C(s_n-\gamma_n\mathscr{A}s_n)$, we have

$$\langle s_{n_k}-\gamma_{n_k}\mathscr{A}s_{n_k}-y_{n_k},x-y_{n_k}\rangle\le 0,\quad\forall x\in C,$$

which can be written as follows

$$\frac{1}{\gamma_{n_k}}\langle s_{n_k}-y_{n_k},x-y_{n_k}\rangle\le\langle\mathscr{A}s_{n_k},x-y_{n_k}\rangle,\quad\forall x\in C.$$

Through a direct calculation, we obtain

$$\frac{1}{\gamma_{n_k}}\langle s_{n_k}-y_{n_k},x-y_{n_k}\rangle+\langle\mathscr{A}s_{n_k},y_{n_k}-s_{n_k}\rangle\le\langle\mathscr{A}s_{n_k},x-s_{n_k}\rangle,\quad\forall x\in C. \tag{3.3}$$

We have that $\{s_{n_k}\}$ is bounded since $\{s_{n_k}\}$ is converges weakly to $z\in H$. Then, from the Lipschitz continuity of $\mathscr{A}$ and $\|s_{n_k}-y_{n_k}\|\to 0$, we obtain that $\{\mathscr{A}s_{n_k}\}$ and $\{y_{n_k}\}$ are also bounded. Since $\gamma_{n_k}\ge\min\{\gamma_1,\frac{\phi}{L}\}$, one concludes from (3.3) that

$$\liminf_{k\to\infty}\langle\mathscr{A}s_{n_k},x-s_{n_k}\rangle\ge 0,\quad\forall x\in C. \tag{3.4}$$

Moreover, one has

$$\begin{aligned}\langle\mathscr{A}y_{n_k},x-y_{n_k}\rangle=&\langle\mathscr{A}y_{n_k}-\mathscr{A}s_{n_k},x-s_{n_k}\rangle+\langle\mathscr{A}s_{n_k},x-s_{n_k}\rangle\\&+\langle\mathscr{A}y_{n_k},s_{n_k}-y_{n_k}\rangle.\end{aligned} \tag{3.5}$$

Since $\lim_{k\to\infty}\|s_{n_k}-y_{n_k}\|=0$ and $\mathscr{A}$ is Lipschitz continuous, we obtain $\lim_{k\to\infty}\|\mathscr{A}s_{n_k}-\mathscr{A}y_{n_k}\|=0$. This together with (3.4) and (3.5) yields that $\liminf_{k\to\infty}\langle\mathscr{A}y_{n_k},x-y_{n_k}\rangle\geq 0$.

Next, we select a positive number decreasing sequence $\{\zeta_k\}$ such that $\zeta_k\to 0$ as $k\to\infty$. For any k, we represent the smallest positive integer with N_k such that

$$\langle\mathscr{A}y_{n_j},x-y_{n_j}\rangle+\zeta_k\geq 0,\quad \forall j\geq N_k. \tag{3.6}$$

It can be easily seen that the sequence $\{N_k\}$ is increasing because $\{\zeta_k\}$ is decreasing. Moreover, for any k, from $\{y_{N_k}\}\subset C$, we can assume $\mathscr{A}y_{N_k}\neq 0$ (otherwise, y_{N_k} is a solution) and set $u_{N_k}=\mathscr{A}y_{N_k}/\|\mathscr{A}y_{N_k}\|^2$. Then, we obtain $\langle\mathscr{A}y_{N_k},u_{N_k}\rangle=1$ for all k. Now, we can deduce from (3.6) that $\langle\mathscr{A}y_{N_k},x+\zeta_k u_{N_k}-y_{N_k}\rangle\geq 0$ for all k. According to the fact that $\mathscr{A}$ is pseudomonotone on H, we can show that

$$\langle\mathscr{A}\left(x+\zeta_k u_{N_k}\right),x+\zeta_k u_{N_k}-y_{N_k}\rangle\geq 0,$$

which further yields that

$$\langle\mathscr{A}x,x-y_{N_k}\rangle\geq\langle\mathscr{A}x-\mathscr{A}\left(x+\zeta_k u_{N_k}\right),x+\zeta_k u_{N_k}-y_{N_k}\rangle-\zeta_k\langle\mathscr{A}x,u_{N_k}\rangle. \tag{3.7}$$

Now, we prove that $\lim_{k\to\infty}\zeta_k u_{N_k}=0$. We obtain that $y_{N_k}\rightharpoonup z$ since $s_{n_k}\rightharpoonup z$ and $\lim_{k\to\infty}\|s_{n_k}-y_{n_k}\|=0$. From $\{y_n\}\subset C$, we have $z\in C$. In view of $\mathscr{A}$ is sequentially weakly continuous on C, one has that $\{\mathscr{A}y_{n_k}\}$ converges weakly to $\mathscr{A}z$. One assumes that $\mathscr{A}z\neq 0$ (otherwise, z is a solution). According to the fact that norm mapping is sequentially weakly lower semicontinuous, we obtain $0<\|\mathscr{A}z\|\leq\liminf_{k\to\infty}\|\mathscr{A}y_{n_k}\|$. Using $\{y_{N_k}\}\subset\{y_{n_k}\}$ and $\zeta_k\to 0$ as $k\to\infty$, we have

$$0\leq\limsup_{k\to\infty}\|\zeta_k u_{N_k}\|=\limsup_{k\to\infty}\left(\frac{\zeta_k}{\|\mathscr{A}y_{n_k}\|}\right)\leq\frac{\limsup_{k\to\infty}\zeta_k}{\liminf_{k\to\infty}\|\mathscr{A}y_{n_k}\|}=0.$$

That is, $\lim_{k\to\infty}\zeta_k u_{N_k}=0$. Since $\mathscr{A}$ is Lipschitz continuous, sequences $\{y_{N_k}\}$ and $\{u_{N_k}\}$ are bounded, and $\lim_{k\to\infty}\zeta_k u_{N_k}=0$, we can conclude from (3.7) that $\liminf_{k\to\infty}\langle\mathscr{A}x,x-y_{N_k}\rangle\geq 0$. Therefore,

$$\langle\mathscr{A}x,x-z\rangle=\lim_{k\to\infty}\langle\mathscr{A}x,x-y_{N_k}\rangle=\liminf_{k\to\infty}\langle\mathscr{A}x,x-y_{N_k}\rangle\geq 0,\ \forall x\in C.$$

Consequently, we observe that $z\in\mathrm{VI}(C,\mathscr{A})$ by Lemma 2.1. This completes the proof. □

Remark 3.2. If $\mathscr{A}$ is monotone, then $\mathscr{A}$ does not need to satisfy sequential weak continuity (see [24]).

Lemma 3.3. *Suppose that Assumptions (C1)–(C3) hold. Let sequences $\{z_n\}$ and $\{y_n\}$ be formulated by Algorithm 1. Then*

$$\|z_n-u\|^2\leq\|s_n-u\|^2-\left(1-\phi^2\frac{\gamma_n^2}{\gamma_{n+1}^2}\right)\|s_n-y_n\|^2,\quad \forall u\in\mathrm{VI}(C,\mathscr{A}),$$

and

$$\|z_n-y_n\|\leq\phi\frac{\gamma_n}{\gamma_{n+1}}\|s_n-y_n\|.$$

Proof. Using the definition of $\{\gamma_n\}$, one obtains

$$\|\mathscr{A}s_n-\mathscr{A}y_n\|\leq\frac{\phi}{\gamma_{n+1}}\|s_n-y_n\|,\quad \forall n\geq 1. \tag{3.8}$$

Indeed, if $\mathscr{A}s_n = \mathscr{A}y_n$ then (3.8) clearly holds. Otherwise, it follows from (3.2) that

$$\gamma_{n+1} = \min\left\{\frac{\phi\,\|s_n - y_n\|}{\|\mathscr{A}s_n - \mathscr{A}y_n\|}, \gamma_n\right\} \le \frac{\phi\,\|s_n - y_n\|}{\|\mathscr{A}s_n - \mathscr{A}y_n\|}.$$

Consequently, we have

$$\|\mathscr{A}s_n - \mathscr{A}y_n\| \le \frac{\phi}{\gamma_{n+1}}\|s_n - y_n\|.$$

Therefore, inequality (3.8) holds when $\mathscr{A}s_n = \mathscr{A}y_n$ and $\mathscr{A}s_n \ne \mathscr{A}y_n$. From the definition of z_n, one sees that

$$\begin{aligned}
\|z_n - u\|^2 &= \|y_n - u\|^2 + \gamma_n^2\|\mathscr{A}y_n - \mathscr{A}s_n\|^2 - 2\gamma_n\langle y_n - u, \mathscr{A}y_n - \mathscr{A}s_n\rangle \\
&= \|s_n - u\|^2 + \|y_n - s_n\|^2 + 2\langle y_n - s_n, s_n - u\rangle \\
&\quad + \gamma_n^2\|\mathscr{A}y_n - \mathscr{A}s_n\|^2 - 2\gamma_n\langle y_n - u, \mathscr{A}y_n - \mathscr{A}s_n\rangle \\
&= \|s_n - u\|^2 + \|y_n - s_n\|^2 - 2\langle y_n - s_n, y_n - s_n\rangle + 2\langle y_n - s_n, y_n - u\rangle \\
&\quad + \gamma_n^2\|\mathscr{A}y_n - \mathscr{A}s_n\|^2 - 2\gamma_n\langle y_n - u, \mathscr{A}y_n - \mathscr{A}s_n\rangle \\
&= \|s_n - u\|^2 - \|y_n - s_n\|^2 + 2\langle y_n - s_n, y_n - u\rangle \\
&\quad + \gamma_n^2\|\mathscr{A}y_n - \mathscr{A}s_n\|^2 - 2\gamma_n\langle y_n - u, \mathscr{A}y_n - \mathscr{A}s_n\rangle.
\end{aligned} \tag{3.9}$$

Combining $y_n = Proj_C(s_n - \gamma_n\mathscr{A}s_n)$ and the property of projection (2.1), we obtain

$$\langle y_n - s_n + \gamma_n\mathscr{A}s_n, y_n - u\rangle \le 0,$$

or equivalently

$$\langle y_n - s_n, y_n - u\rangle \le -\gamma_n\langle\mathscr{A}s_n, y_n - u\rangle. \tag{3.10}$$

From (3.8), (3.9), and (3.10), we have

$$\begin{aligned}
\|z_n - u\|^2 &\le \|s_n - u\|^2 - \|y_n - s_n\|^2 - 2\gamma_n\langle\mathscr{A}s_n, y_n - u\rangle \\
&\quad + \phi^2\frac{\gamma_n^2}{\gamma_{n+1}^2}\|s_n - y_n\|^2 - 2\gamma_n\langle y_n - u, \mathscr{A}y_n - \mathscr{A}s_n\rangle \\
&\le \|s_n - u\|^2 - \left(1 - \phi^2\frac{\gamma_n^2}{\gamma_{n+1}^2}\right)\|s_n - y_n\|^2 - 2\gamma_n\langle y_n - u, \mathscr{A}y_n\rangle.
\end{aligned} \tag{3.11}$$

By means of $u \in \mathrm{VI}(C, \mathscr{A})$, one has $\langle\mathscr{A}u, y_n - u\rangle \ge 0$. Using the pseudomonotonicity of $\mathscr{A}$, we obtain

$$\langle\mathscr{A}y_n, y_n - u\rangle \ge 0. \tag{3.12}$$

Combining (3.11) and (3.12), we can show that

$$\|z_n - u\|^2 \le \|s_n - u\|^2 - \left(1 - \phi^2\frac{\gamma_n^2}{\gamma_{n+1}^2}\right)\|s_n - y_n\|^2.$$

According to the definition of z_n and (3.8), we obtain

$$\|z_n - y_n\| \le \phi\frac{\gamma_n}{\gamma_{n+1}}\|s_n - y_n\|.$$

This completes the proof of the Lemma 3.3. □

Theorem 3.1. *Suppose that Assumptions (C1)–(C5) hold. Then the iterative sequence $\{x_n\}$ formulated by Algorithm 1 converges strongly to $u \in \mathrm{VI}(C,\mathscr{A})$, where $u = Proj_{\mathrm{VI}(C,\mathscr{A})} \circ f(u)$.*

Proof. We divide the proof into four claims.
Claim 1. The sequence $\{x_n\}$ is bounded. According to Lemma 3.3, we obtain

$$\lim_{n\to\infty} \left(1-\phi^2 \frac{\gamma_n^2}{\gamma_{n+1}^2}\right) = 1-\phi^2 > 0.$$

Therefore, there is a constant $n_0 \in \mathbb{N}$ that satisfies $1-\phi^2 \frac{\gamma_n^2}{\gamma_{n+1}^2} > 0$ for all $n \geq n_0$. From Lemma 3.3, one has

$$\|z_n - u\| \leq \|s_n - u\|, \quad \forall n \geq n_0. \tag{3.13}$$

By the definition of s_n, one sees that

$$\|s_n - u\| \leq \|x_n - u\| + \varphi_n \cdot \frac{\delta_n}{\varphi_n} \|x_n - x_{n-1}\|. \tag{3.14}$$

From Remark 3.1, one obtains $\frac{\delta_n}{\varphi_n}\|x_n - x_{n-1}\| \to 0$ as $n \to \infty$. Thus, there is a constant $Q_1 > 0$ that satisfies

$$\frac{\delta_n}{\varphi_n}\|x_n - x_{n-1}\| \leq Q_1, \quad \forall n \geq 1. \tag{3.15}$$

Using (3.13), (3.14), and (3.15), we obtain

$$\|z_n - u\| \leq \|s_n - u\| \leq \|x_n - u\| + \varphi_n Q_1, \quad \forall n \geq n_0. \tag{3.16}$$

According to the definition of x_{n+1} and (3.16), we have

$$\begin{aligned}
\|x_{n+1} - u\| &\leq \varphi_n \|f(z_n) - f(u)\| + \varphi_n \|f(u) - u\| + (1-\varphi_n)\|z_n - u\| \\
&\leq \varphi_n \rho \|z_n - u\| + \varphi_n \|f(u) - u\| + (1-\varphi_n)\|z_n - u\| \\
&\leq (1-(1-\rho)\varphi_n)\|x_n - u\| + \varphi_n Q_1 + \varphi_n \|f(u) - u\| \\
&= (1-(1-\rho)\varphi_n)\|x_n - u\| + (1-\rho)\varphi_n \frac{Q_1 + \|f(u) - u\|}{1-\rho} \\
&\leq \max\left\{\|x_n - u\|, \frac{Q_1 + \|f(u) - u\|}{1-\rho}\right\}, \ \forall n \geq n_0 \\
&\leq \cdots \leq \max\left\{\|x_{n_0} - u\|, \frac{Q_1 + \|f(u) - u\|}{1-\rho}\right\}.
\end{aligned}$$

That is, $\{x_n\}$ is bounded. We have that $\{s_n\}$, $\{z_n\}$, and $\{f(z_n)\}$ are also bounded.
Claim 2.

$$\left(1-\phi^2 \frac{\gamma_n^2}{\gamma_{n+1}^2}\right)\|s_n - y_n\|^2 \leq \|x_n - u\|^2 - \|x_{n+1} - u\|^2 + \varphi_n Q_4$$

for some $Q_4 > 0$. It follows from (3.16) that

$$\begin{aligned}
\|s_n - u\|^2 &\leq (\|x_n - u\| + \varphi_n Q_1)^2 \\
&= \|x_n - u\|^2 + \varphi_n (2Q_1 \|x_n - u\| + \varphi_n Q_1^2) \\
&\leq \|x_n - u\|^2 + \varphi_n Q_2
\end{aligned} \tag{3.17}$$

for some $Q_2 > 0$. Combining Lemma 3.3 and (3.17), we see that

$$\begin{aligned}
\|x_{n+1}-u\|^2 &\le \varphi_n\|f(z_n)-u\|^2+(1-\varphi_n)\|z_n-u\|^2 \\
&\le \varphi_n(\|f(z_n)-f(u)\|+\|f(u)-u\|)^2+(1-\varphi_n)\|z_n-u\|^2 \\
&\le \varphi_n(\|z_n-u\|+\|f(u)-u\|)^2+(1-\varphi_n)\|z_n-u\|^2 \\
&= \varphi_n\|z_n-u\|^2+(1-\varphi_n)\|z_n-u\|^2 \\
&\quad +\varphi_n(\|f(u)-u\|^2+2\|z_n-u\|\cdot\|f(u)-u\|) \\
&\le \|z_n-u\|^2+\varphi_n Q_3 \\
&\le \|s_n-u\|^2-\left(1-\phi^2\frac{\gamma_n^2}{\gamma_{n+1}^2}\right)\|s_n-y_n\|^2+\varphi_n Q_3 \\
&\le \|x_n-u\|^2-\left(1-\phi^2\frac{\gamma_n^2}{\gamma_{n+1}^2}\right)\|s_n-y_n\|^2+\varphi_n Q_4,
\end{aligned}$$

where $Q_4 := Q_2+Q_3$. Therefore

$$\left(1-\phi^2\frac{\gamma_n^2}{\gamma_{n+1}^2}\right)\|s_n-y_n\|^2 \le \|x_n-u\|^2-\|x_{n+1}-u\|^2+\varphi_n Q_4.$$

Claim 3.

$$\begin{aligned}
\|x_{n+1}-u\|^2 \le (1-(1-\rho)\varphi_n)\|x_n-u\|^2+(1-\rho)\varphi_n\cdot\Bigg[\frac{3Q}{1-\rho}\cdot\frac{\delta_n}{\varphi_n}\|x_n-x_{n-1}\| \\
+\frac{2}{1-\rho}\langle f(u)-u,x_{n+1}-u\rangle\Bigg], \quad \forall n\ge n_0.
\end{aligned}$$

for some $Q > 0$. Using the definition of s_n, we can show that

$$\begin{aligned}
\|s_n-u\|^2 &= \|x_n+\delta_n(x_n-x_{n-1})-u\|^2 \\
&\le \|x_n-u\|^2+2\delta_n\|x_n-u\|\|x_n-x_{n-1}\|+\delta_n^2\|x_n-x_{n-1}\|^2 \qquad (3.18) \\
&\le \|x_n-u\|^2+3Q\delta_n\|x_n-x_{n-1}\|,
\end{aligned}$$

where $Q := \sup_{n\in\mathbb{N}}\{\|x_n-u\|,\delta\|x_n-x_{n-1}\|\} > 0$. Using (3.13) and (3.18), we obtain

$$\begin{aligned}
&\|x_{n+1}-u\|^2 = \|\varphi_n f(z_n)+(1-\varphi_n)z_n-u\|^2 \\
&= \|\varphi_n(f(z_n)-f(u))+(1-\varphi_n)(z_n-u)+\varphi_n(f(u)-u)\|^2 \\
&\le \|\varphi_n(f(z_n)-f(u))+(1-\varphi_n)(z_n-u)\|^2+2\varphi_n\langle f(u)-u,x_{n+1}-u\rangle \\
&\le \varphi_n\|f(z_n)-f(u)\|^2+(1-\varphi_n)\|z_n-u\|^2+2\varphi_n\langle f(u)-u,x_{n+1}-u\rangle \\
&\le \varphi_n\rho^2\|z_n-u\|^2+(1-\varphi_n)\|z_n-u\|^2+2\varphi_n\langle f(u)-u,x_{n+1}-u\rangle \qquad (3.19) \\
&\le (1-(1-\rho)\varphi_n)\|z_n-u\|^2+2\varphi_n\langle f(u)-u,x_{n+1}-u\rangle \\
&\le (1-(1-\rho)\varphi_n)\|x_n-u\|^2+(1-\rho)\varphi_n\cdot\Bigg[\frac{3Q}{1-\rho}\cdot\frac{\delta_n}{\varphi_n}\|x_n-x_{n-1}\| \\
&\quad +\frac{2}{1-\rho}\langle f(u)-u,x_{n+1}-u\rangle\Bigg], \quad \forall n\ge n_0.
\end{aligned}$$

Claim 4. $\{\|x_n - u\|\}$ converges to zero. From Lemma 2.2 and Remark 3.1, it remains to show that $\limsup_{k\to\infty}\langle f(u)-u, x_{n_k+1}-u\rangle \le 0$ for any subsequence $\{\|x_{n_k}-u\|\}$ of $\{\|x_n-u\|\}$ satisfies $\liminf_{k\to\infty}\left(\|x_{n_k+1}-u\|-\|x_{n_k}-u\|\right)\ge 0$.

For this purpose, we assume that $\{\|x_{n_k}-u\|\}$ is a subsequence of $\{\|x_n-u\|\}$ such that

$$\liminf_{k\to\infty}\left(\|x_{n_k+1}-u\|-\|x_{n_k}-u\|\right)\ge 0.$$

Then,

$$\begin{aligned}&\liminf_{k\to\infty}\left(\|x_{n_k+1}-u\|^2-\|x_{n_k}-u\|^2\right)\\ =&\liminf_{k\to\infty}\left[(\|x_{n_k+1}-u\|-\|x_{n_k}-u\|)(\|x_{n_k+1}-u\|+\|x_{n_k}-u\|)\right]\ge 0.\end{aligned}$$

It follows from Claim 2 and Assumption (C5) that

$$\begin{aligned}&\limsup_{k\to\infty}\left(1-\phi^2\frac{\gamma_{n_k}^2}{\gamma_{n_k+1}^2}\right)\|s_{n_k}-y_{n_k}\|^2\\ \le&\limsup_{k\to\infty}\left[\|x_{n_k}-u\|^2-\|x_{n_k+1}-u\|^2\right]+\limsup_{k\to\infty}\varphi_{n_k}Q_4\\ =&-\liminf_{k\to\infty}\left[\|x_{n_k+1}-u\|^2-\|x_{n_k}-u\|^2\right]\\ \le&\, 0,\end{aligned}$$

which yields that $\lim_{k\to\infty}\|s_{n_k}-y_{n_k}\|=0$. From Lemma 3.3, we obtain $\lim_{k\to\infty}\|z_{n_k}-y_{n_k}\|=0$. Hence $\lim_{k\to\infty}\|z_{n_k}-s_{n_k}\|=0$.

Moreover, using Remark 3.1 and Assumption (C5), we have

$$\|x_{n_k}-s_{n_k}\|=\delta_{n_k}\|x_{n_k}-x_{n_k-1}\|=\varphi_{n_k}\cdot\frac{\delta_{n_k}}{\varphi_{n_k}}\|x_{n_k}-x_{n_k-1}\|\to 0,$$

and

$$\|x_{n_k+1}-z_{n_k}\|=\varphi_{n_k}\|z_{n_k}-f(z_{n_k})\|\to 0.$$

Therefore, we conclude that

$$\|x_{n_k+1}-x_{n_k}\|\le\|x_{n_k+1}-z_{n_k}\|+\|z_{n_k}-s_{n_k}\|+\|s_{n_k}-x_{n_k}\|\to 0. \tag{3.20}$$

Since $\{x_{n_k}\}$ is bounded, one asserts that there is a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ that satisfies $x_{n_{k_j}}\rightharpoonup q$. Furthermore,

$$\limsup_{k\to\infty}\langle f(u)-u, x_{n_k}-u\rangle=\lim_{j\to\infty}\langle f(u)-u, x_{n_{k_j}}-u\rangle=\langle f(u)-u, q-u\rangle. \tag{3.21}$$

We obtain $s_{n_k}\rightharpoonup q$ since $\|x_{n_k}-s_{n_k}\|\to 0$. This together with $\lim_{k\to\infty}\|s_{n_k}-y_{n_k}\|=0$ and Lemma 3.2 obtains $q\in \mathrm{VI}(C,\mathscr{A})$. By the definition of $u=Proj_{\mathrm{VI}(C,\mathscr{A})}\circ f(u)$ and (3.21), we infer that

$$\limsup_{k\to\infty}\langle f(u)-u, x_{n_k}-u\rangle=\langle f(u)-u, q-u\rangle\le 0. \tag{3.22}$$

Combining (3.20) and (3.22), we see that

$$\limsup_{k\to\infty}\langle f(u)-u, x_{n_k+1}-u\rangle\le\limsup_{k\to\infty}\langle f(u)-u, x_{n_k}-u\rangle\le 0. \tag{3.23}$$

Thus, from Remark 3.1, (3.23), Claim 3, and Lemma 2.2, we conclude that $x_n\to u$ as $n\to\infty$. The proof of Theorem 3.1 is now complete. □

If inertial parameter $\delta_n=0$ in Algorithm 1, we have the following result.

Corollary 3.1. *Assume that mapping $\mathscr{A}: H \to H$ is L-Lipschitz continuous, pseudomonotone on H, and sequentially weakly continuous on C. Let mapping $f: H \to H$ be ρ-contractive with $\rho \in (0,1)$. Take $\gamma_0 > 0$, $\{\varphi_n\} \subset (0,1)$ satisfies $\lim_{n\to\infty} \varphi_n = 0$ and $\sum_{n=1}^{\infty} \varphi_n = \infty$. Let x_0 be the initial point and $\{x_n\}$ be the sequence generated by*

$$\begin{cases} y_n = Proj_C\left(x_n - \gamma_n \mathscr{A} x_n\right), \\ z_n = y_n - \gamma_n\left(\mathscr{A} y_n - \mathscr{A} x_n\right), \\ x_{n+1} = \varphi_n f\left(z_n\right) + \left(1-\varphi_n\right) z_n, \end{cases} \tag{3.24}$$

where step size $\{\gamma_n\}$ is updated through (3.2). Then the iterative sequence $\{x_n\}$ formulated by Algorithm (3.24) converges strongly to $u \in \mathrm{VI}(C,\mathscr{A})$, where $u = Proj_{\mathrm{VI}(C,\mathscr{A})} \circ f(u)$.

Remark 3.3. It should be pointed out that our algorithm (3.24) improves and summarizes the results in [20, Algorithm 3] and [21, Algorithm 1]. Moreover, our algorithm is designed to address pseudomonotone variational inequalities, while the algorithms presented in [20, 21] can only be used to solve monotone variational inequalities. It known that the classes of pseudomonotone mappings cover the classes of monotone mappings. Therefore, our algorithm is more applicable.

4. Numerical examples

In this section, we give some computational tests and applications to show the numerical behavior of our Algorithm 1 (shortly, ViTEGM), and also to compare it with some strong convergent algorithms (Algorithms (MaTEGM) and (ViSEGM)). It should be emphasized that all algorithms can work without the prior information of the Lipschitz constant of the mapping. We use the FOM solver [25] to effectively calculate the projections onto C and T_n. All the programs are implemented in MATLAB 2018a on a personal computer.

In our numerical examples, if the solution x^* of our problem is known, we take $E_n = \|x_n - x^*\|$ to represent the computational error of all algorithms at iteration step n; otherwise, according to the feature of solutions to (VIP), we use $E_n = \|s_n - Proj_C(s_n - \gamma_n \mathscr{A} s_n)\|$ to study the performance of all algorithms at iteration step n. Note that, if $\|E_n\| \to 0$, then x_n can be regards as an approximate solution of (VIP). The parameters of all algorithms are chosen as follows:

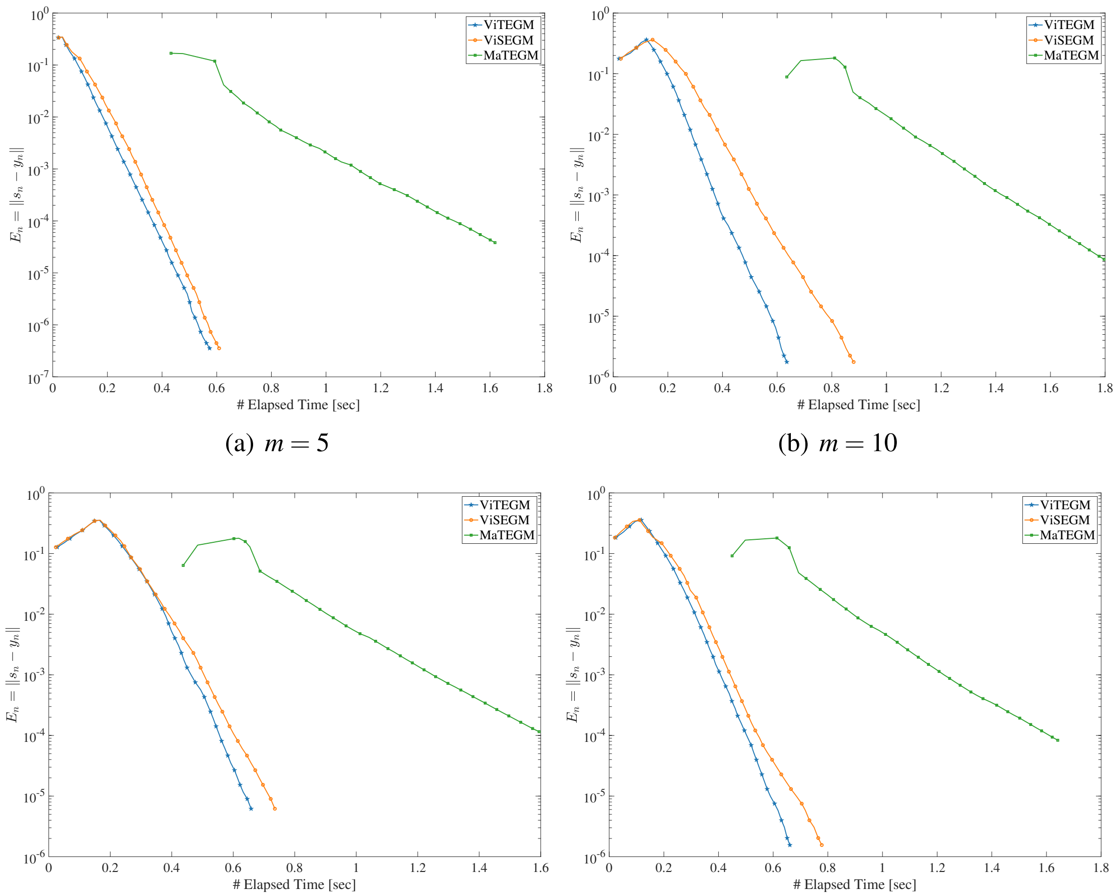
- Take $\phi = 0.8$, $\gamma_1 = 1$, $\delta = 0.3$, $\varepsilon_n = 1/(n+1)^2$, $\varphi_n = 1/(n+1)$, and $f(x) = 0.9x$ for the proposed Algorithm 1 and Algorithm (ViSEGM).
- Pick $\alpha = \ell = 0.5$, $\phi = 0.4$, $\varphi_n = 1/(n+1)$, and $\tau_n = 0.5(1-\varphi_n)$ for Algorithm (MaTEGM).

4.1. **Theoretical examples.**

Example 4.1. Let $\mathscr{A}: R^m \to R^m$ $(m = 5, 10, 15, 20)$ be an operator given by

$$\mathscr{A}(x) = \frac{1}{\|x\|^2+1} \operatorname{argmin}_{y \in R^m} \left\{ \frac{\|y\|^4}{4} + \frac{1}{2}\|x-y\|^2 \right\}.$$

We emphasize that the operator $\mathscr{A}$ is not monotone. However, the operator $\mathscr{A}$ is Lipschitz continuous and pseudomonotone (see [26]). In this example, we choose the feasible set is a box constraint $C = [-5,5]^m$. Take initial values $x_0 = x_1$ are randomly generated by *rand(m,1)* in MATLAB. The maximum number of iterations 50 as a common stopping criterion. For the four different dimensions of the operator $\mathscr{A}$, the numerical results are presented in Fig. 1.



(a) $m = 5$ (b) $m = 10$

(c) $m = 15$ (d) $m = 20$

FIGURE 1. Numerical results for Example 4.1

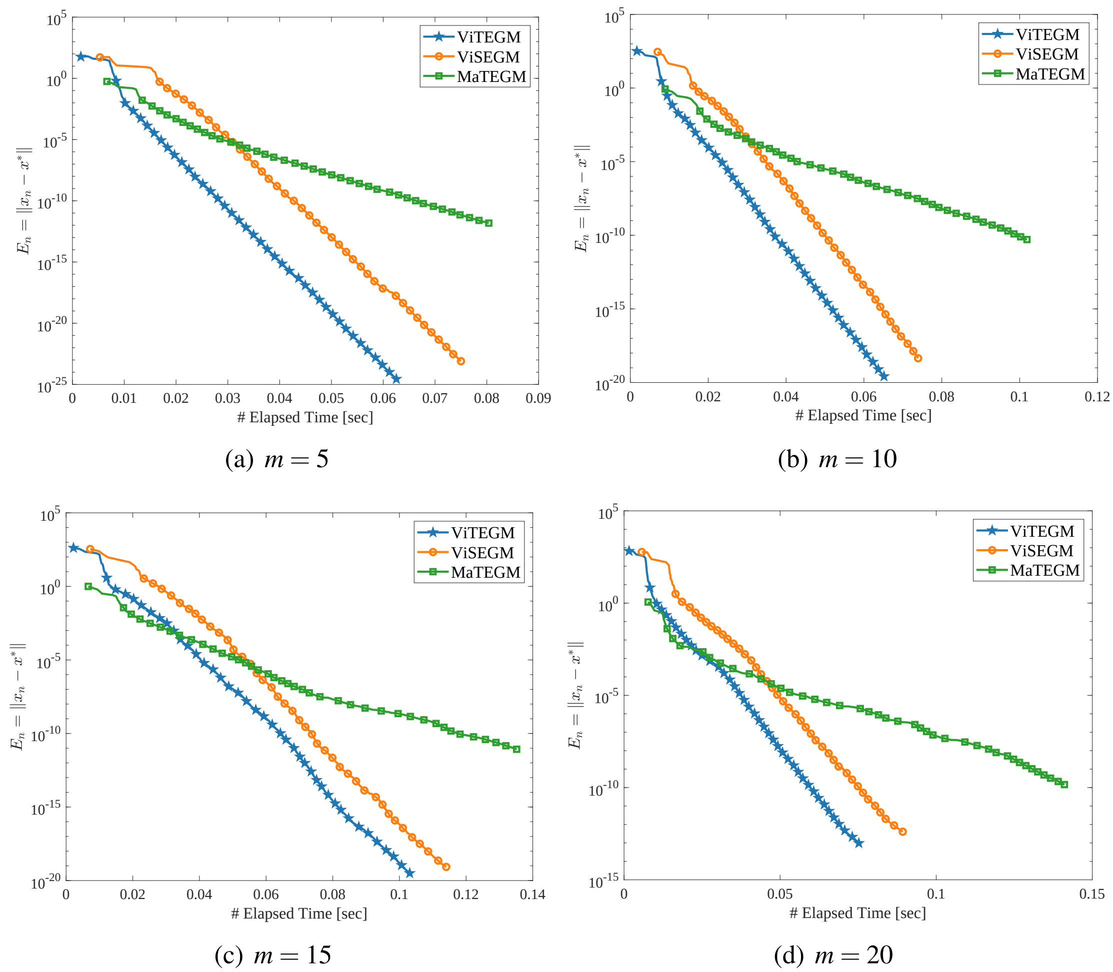
Example 4.2. In the second example, we consider the form of linear operator $\mathscr{A}: R^m \to R^m$ $(m = 5, 10, 15, 20)$ as follows: $\mathscr{A}(x) = Gx + g$, where $g \in R^m$ and $G = BB^\top + M + E$, matrix $B \in R^{m\times m}$, matrix $M \in R^{m\times m}$ is skew-symmetric, and matrix $E \in R^{m\times m}$ is diagonal matrix whose diagonal terms are non-negative (hence G is positive symmetric definite). We choose the feasible set as $C = \{x \in R^m: -2 \le x_i \le 5, i = 1, \ldots, m\}$. We obtain that mapping $\mathscr{A}$ is monotone and Lipschitz continuous. In this numerical example, both B and M entries are randomly created in $[-2, 2]$, E is generated randomly in $[0, 2]$, and $g = \mathbf{0}$. It can be easily seen that the solution to the problem is $x^* = \{\mathbf{0}\}$. The maximum number of iterations 1000 as a common stopping criterion and the initial values $x_0 = x_1$ are randomly generated by *rand(m,1)* in MATLAB. The numerical results of all algorithms with elapsed time are described in Fig. 2.

Example 4.3. Finally, we focus on a case in Hilbert space $H = L^2[0, 1]$ with inner product

$$\langle x, y\rangle = \int_0^1 x(t)y(t)\mathrm{d}t, \quad \forall x, y \in H$$

and norm

$$\|x\| = \left(\int_0^1 x(t)^2 \mathrm{d}t\right)^{1/2}, \quad \forall x \in H.$$



(a) $m = 5$ (b) $m = 10$

(c) $m = 15$ (d) $m = 20$

FIGURE 2. Numerical results for Example 4.2

Let b and a be two positive numbers such that $a/(m+1) < b/m < b < a$ for some $m > 1$. We select the feasible set as $C = \{x \in H \colon \|x\| \leq b\}$. The operator $\mathscr{A} \colon H \to H$ is of the form

$$\mathscr{A}(x) = (a - \|x\|)x, \quad \forall x \in H.$$

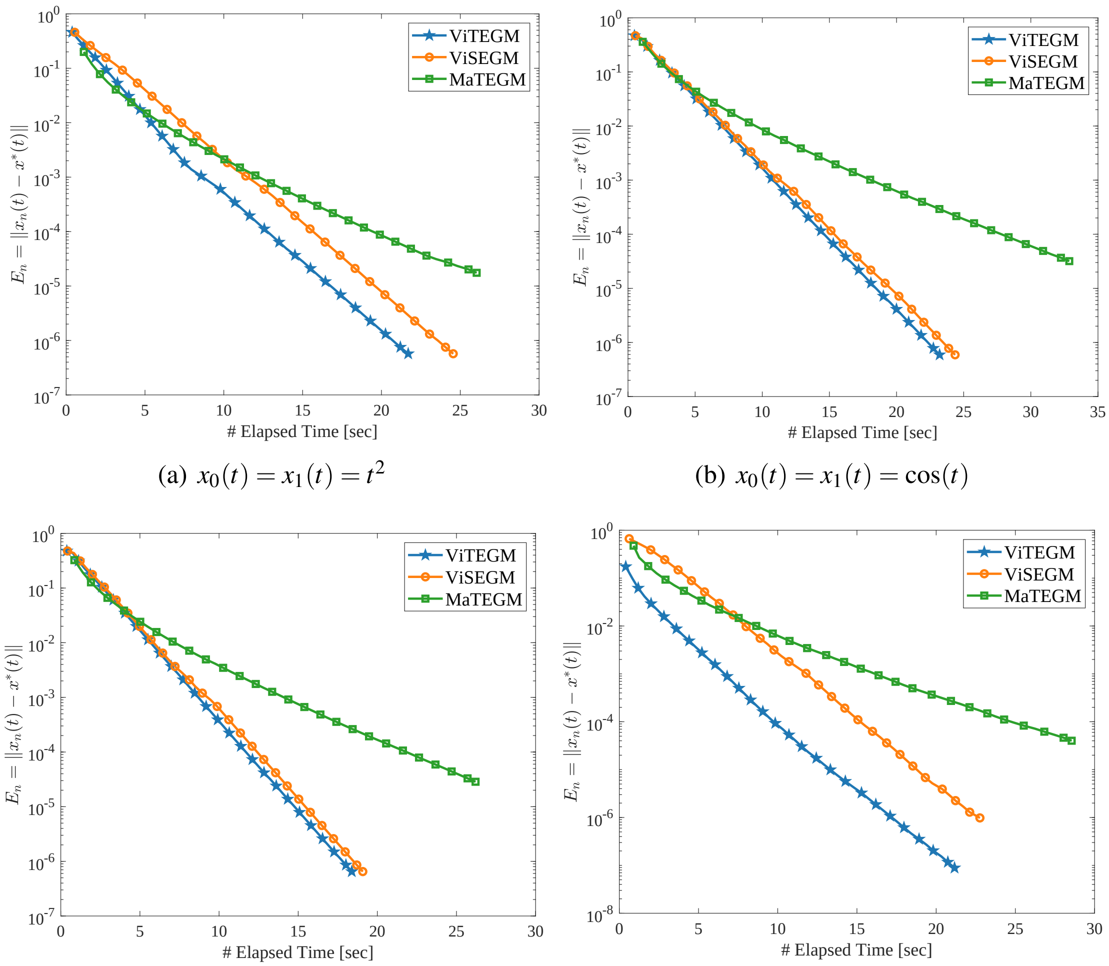
It should be pointed out that operator $\mathscr{A}$ is not monotone. Indeed, take a particular pair $(x^{\ddagger}, mx^{\ddagger})$, we pick $x^{\ddagger} \in C$ to satisfy $a/(m+1) < \|x^{\ddagger}\| < b/m$, one can sees that $m\|x^{\ddagger}\| \in C$. By a simple operation, we obtain

$$\langle \mathscr{A}(x^{\ddagger}) - \mathscr{A}(y^{\ddagger}), x^{\ddagger} - y^{\ddagger} \rangle = (1-m)^2 \|x^{\ddagger}\|^2 (a - (1+m)\|x^{\ddagger}\|) < 0.$$

Hence, the operator $\mathscr{A}$ is not monotone on C. Next, we show that $\mathscr{A}$ is pseudomonotone. Indeed, one assumes that $\langle \mathscr{A}(x), y - x \rangle \geq 0$ for all $x, y \in C$, that is, $\langle (a - \|x\|)x, y - x \rangle \geq 0$. From $\|x\| < a$, we obtain that $\langle x, y - x \rangle \geq 0$. Therefore

$$\begin{aligned}
\langle \mathscr{A}(y), y - x \rangle &= \langle (a - \|y\|)y, y - x \rangle \\
&\geq (a - \|y\|)(\langle y, y - x \rangle - \langle x, y - x \rangle) \\
&= (a - \|y\|)\|y - x\|^2 \geq 0.
\end{aligned}$$

For the experiment, we take $a = 1.5$, $b = 1$, and $m = 1.1$. We know that the solution to the problem is $x^*(t) = 0$. The maximum number of iterations 50 as the stopping criterion. Figure 3 shows the behaviors of function $E_n = \|x_n(t) - x^*(t)\|$ formulated by all algorithms with four initial points $x_0(t) = x_1(t)$.



(a) $x_0(t) = x_1(t) = t^2$ (b) $x_0(t) = x_1(t) = \cos(t)$

(c) $x_0(t) = x_1(t) = \sin(2t)$ (d) $x_0(t) = x_1(t) = 2^t$

FIGURE 3. Numerical results for Example 4.3

Remark 4.1. We have the following observations for Examples 4.1–4.3.

(1) From Figs. 1–3, we can see that our proposed algorithm converges quickly and has a better computational performance than the existing ones in [16, 19]. In addition, these results are independent of the selection of initial values and the size of dimensions. Therefore, our algorithm is robust and useful.

(2) It should be emphasized that Algorithm (MaTEGM) needs to spend more running time to achieve the same error accuracy because it uses an Armoji-type rule to automatically update the step size, and this update criterion requires to calculate the value of operator $\mathscr{A}$ many times in each iteration. However, the proposed Algorithm 1 uses previously known information to update the step size by a simple calculation in each iteration, which makes it converge faster.

(3) It is noted that operator $\mathscr{A}$ is pseudomonotone in Examples 4.1–4.3. At this point, the algorithms introduced in [20, 21] for solving monotone (VIP) will not be available. Therefore, our proposed algorithm is more applicable for practical applications.

4.2. **Application to optimal control problems.** Next, we use our proposed Algorithm 1 to solve the (VIP) that appears in optimal control problems. Recently, many scholars have proposed different methods to solve it. We recommend readers to refer to [2, 27] for the algorithms and detailed description of the problem.

Example 4.4 (Control of a harmonic oscillator, see [28])**.**

$$\begin{aligned} \text{minimize} \quad & x_2(3\pi) \\ \text{subject to} \quad & \dot{x}_1(t) = x_2(t), \\ & \dot{x}_2(t) = -x_1(t) + u(t), \ \forall t \in [0, 3\pi], \\ & x(0) = 0, \\ & u(t) \in [-1, 1]. \end{aligned}$$

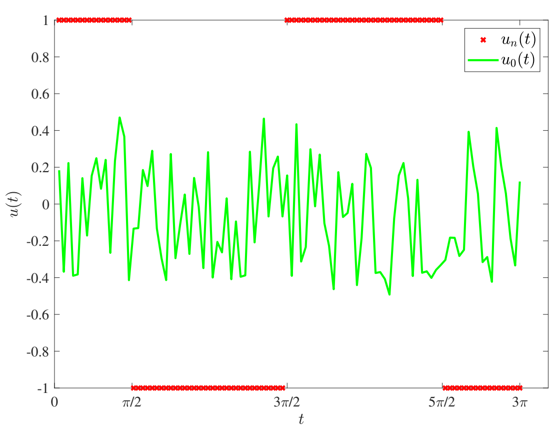
The exact optimal control of Example 4.4 is known:

$$u^*(t) = \begin{cases} 1, & \text{if } t \in [0, \pi/2) \cup (3\pi/2, 5\pi/2); \\ -1, & \text{if } t \in (\pi/2, 3\pi/2) \cup (5\pi/2, 3\pi]. \end{cases}$$

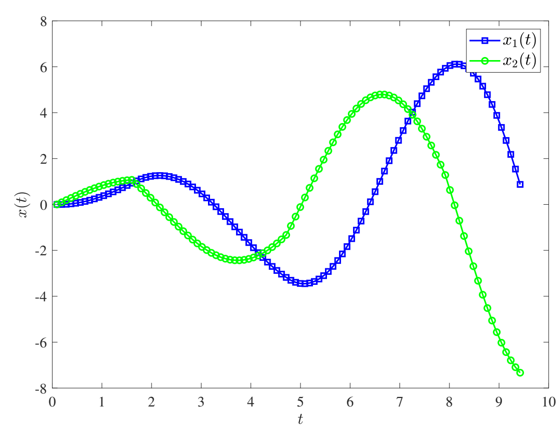
Our parameters are set as follows:

$$N = 100, \phi = 0.1, \gamma_1 = 0.4, \delta = 0.3, \varepsilon_n = \frac{10^{-4}}{(n+1)^2}, \varphi_n = \frac{10^{-4}}{n+1}, f(x) = 0.1x.$$

The initial controls $u_0(t) = u_1(t)$ are randomly generated in $[-1, 1]$, and the stopping criterion is $\|u_{n+1} - u_n\| \leq 10^{-4}$ or reach the maximum number of iterations 1000. After 122 iterations, Algorithm 1 took 0.059839 seconds to reach the required error accuracy. Figure 4 shows the approximate optimal control and the corresponding trajectories of Algorithm 1.



(a) Initial and optimal controls



(b) Optimal trajectories

FIGURE 4. Numerical results of Algorithm 1 for Example 4.4

We now consider an example in which the terminal function is not linear.

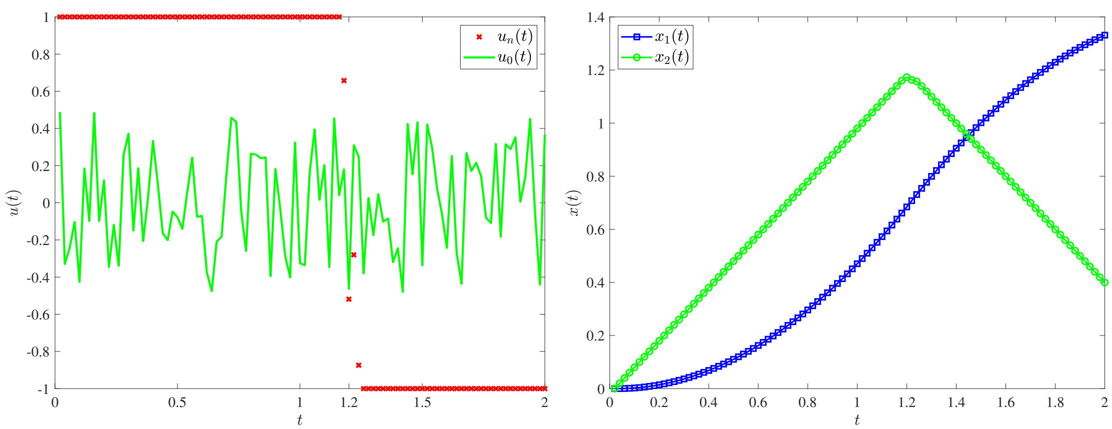
Example 4.5 (See [29]).

$$
\begin{aligned}
\text{minimize} \quad & -x_1(2) + (x_2(2))^2, \\
\text{subject to} \quad & \dot{x}_1(t) = x_2(t), \\
& \dot{x}_2(t) = u(t), \ \ \forall t \in [0,2], \\
& x_1(0) = 0, \ \ x_2(0) = 0, \\
& u(t) \in [-1,1].
\end{aligned}
$$

The exact optimal control of Example 4.5 is

$$
u^*(t) = \begin{cases} 1 & \text{if } \ t \in [0,1.2); \\ -1 & \text{if } \ t \in (1.2,2]. \end{cases}
$$

In this example, the parameters of our algorithm are set the same as in Example 4.4. After the maximum allowed 1000 iterations, the proposed Algorithm 1 took 0.39932 seconds, but the required error accuracy was not achieved. Reaching the allowable error range may require more iterations. The approximate optimal control and the corresponding trajectories of Algorithm 1 are plotted in Fig. 5.



(a) Initial and optimal controls

(b) Optimal trajectories

FIGURE 5. Numerical results of Algorithm 1 for Example 4.5

Remark 4.2. As can be seen from Examples 4.4 and 4.5, the algorithm proposed in this paper can work well on optimal control problems. It should be pointed out that the proposed algorithm can perform better when the terminal function is linear rather than nonlinear (cf. Figs. 4 and 5).

5. CONCLUSIONS

In this paper, based on the inertial method, the Tseng's extragradient method, and the viscosity method, we introduced an accelerated extragradient-type algorithm to address the pseudomonotone variational inequality problem in a real Hilbert space. The main advantage of the suggested method is that only one projection needs to be calculated in each iteration. The convergence of the algorithm was proved without the prior information of the Lipschitz constant of the mapping. Moreover, our algorithm adds an inertial term, which greatly improves the convergence speed of the algorithm. Our numerical experiments showed that the proposed algorithm improves some results of the existing ones in the literature. As an application, the variational inequality problem in the optimal control problem was also studied.

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REFERENCES

[1] A. Nagurney, I. Pour, S. Samadi, A variational inequality trade network model in prices and quantities under commodity losses, J. Nonlinear Var. Anal. 8 (2024), 935-952.

[2] P.T. Vuong, Y. Shehu, Convergence of an extragradient-type method for variational inequality with applications to optimal control problems, Numer. Algorithms 81 (2019), 269-291.

[3] M. Sofonea, D. A. Tarzia, Well-posedness and convergence results for elliptic hemivariational inequalities, Appl. Set-Valued Anal. Optim. 7 (2025), 1-21.

[4] Y. Shehu, O.S. Iyiola, Strong convergence result for monotone variational inequalities, Numer. Algorithms 76 (2017), 259-282.

[5] Q.L. Dong, Y.J. Cho, L.L. Zhong, T.M. Rassias, Inertial projection and contraction algorithms for variational inequalities, J. Global Optim. 70 (2018), 687–704.

[6] Y. Malitsky, Projected reflected gradient methods for monotone variational inequalities, SIAM J. Optim. 25 (2015), 502-520.

[7] G.M. Korpelevich, The extragradient method for finding saddle points and other problems, Ekonom. i Mat. Metody 12 (1976), 747-756.

[8] Y. Shehu, O.S. Iyiola, X.H. Li, Q.L. Dong, Convergence analysis of projection method for variational inequalities, Comput. Appl. Math. 38 (2019), Article ID 161.

[9] G. Cai, Q.L. Dong, Y. Peng, Strong convergence theorems for solving variational inequality problems with pseudo-monotone and non-Lipschitz Operators, J. Optim. Theory Appl. 188 (2021), 447-472.

[10] P.T. Vuong, On the weak convergence of the extragradient method for solving pseudo-monotone variational inequalities, J. Optim. Theory Appl. 176 (2018), 399-409.

[11] P. Tseng, A modified forward-backward splitting method for maximal monotone mappings, SIAM J. Control Optim. 38 (2000), 431–446.

[12] R.I. Boţ, E.R. Csetnek, P.T. Vuong, The forward-backward-forward method from continuous and discrete perspective for pseudo-monotone variational inequalities in Hilbert spaces, European J. Oper. Res. 287 (2020), 49-60.

[13] Y. Censor, A. Gibali, S. Reich, Strong convergence of subgradient extragradient methods for the variational inequality problem in Hilbert space, Optim. Methods Softw. 26 (2011), 827-845.

[14] Y. Censor, A. Gibali, S. Reich, The subgradient extragradient method for solving variational inequalities in Hilbert space, J. Optim. Theory Appl. 148 (2011), 318-335.

[15] Y. Censor, A. Gibali, S. Reich, Extensions of Korpelevich's extragradient method for the variational inequality problem in Euclidean space, Optimization 61 (2012), 1119-1132.

[16] D.V. Thong, P.T. Vuong, Modified Tseng's extragradient methods for solving pseudo-monotone variational inequalities, Optimization 68 (2019), 2207-2226.

[17] B.T. Polyak, Some methods of speeding up the convergence of iteration methods, USSR Comput. Math. Math. Phys. 4 (1964), 1-17.

[18] G. Cai, Q.L. Dong, Y. Peng, Strong convergence theorems for inertial Tseng's extragradient method for solving variational inequality problems and fixed point problems, Optim. Lett. 15 (2021), 1457-1474.

[19] D.V. Thong, D.V. Hieu, T.M. Rassias, Self-adaptive inertial subgradient extragradient algorithms for solving pseudomonotone variational inequality problems, Optim. Lett. 14 (2020), 115-144.

[20] D.V. Thong, D.V. Hieu, Weak and strong convergence theorems for variational inequality problems, Numer. Algorithms 78 (2018), 1045-1060.

[21] J. Yang, H. Liu, Strong convergence result for solving monotone variational inequalities in Hilbert space, Numer. Algorithms 80 (2019), 741-752.

[22] R.W. Cottle, J.C. Yao, Pseudo-monotone complementarity problems in Hilbert space, J. Optim. Theory Appl. 75 (1992), 281-295.

[23] S. Saejung, P. Yotkaew, Approximation of zeros of inverse strongly monotone operators in Banach spaces, Nonlinear Anal. 75 (2012), 742-750.

[24] S.V. Denisov, V.V. Semenov, L.M. Chabak, Convergence of the modified extragradient method for variational inequalities with non-Lipschitz operators, Cybernet. Systems Anal. 51 (2015), 757-765.

[25] A. Beck, N. Guttmann-Beck, FOM–a MATLAB toolbox of first-order methods for solving convex optimization problems, Optim. Methods Softw. 34 (2019), 172-193.

[26] D.V. Hieu, Y.J. Cho, Y.B. Xiao, P. Kumam, Relaxed extragradient algorithm for solving pseudomonotone variational inequalities in Hilbert spaces, Optimization 69 (2020), 2279-2304.

[27] J. Preininger, P.T. Vuong, On the convergence of the gradient projection method for convex optimal control problems with bang-bang solutions, Comput. Optim. Appl. 70 (2018), 221-238.

[28] A. Pietrus, T. Scarinci, V.M. Veliov, High order discrete approximations to Mayer's problems for linear systems, SIAM J. Control Optim. 56 (2018), 102-119.

[29] B. Bressan, B. Piccoli, Introduction to the Mathematical Theory of Control, AIMS Series on Applied Mathematics, 2007.