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SOLVING AN UNCERTAIN QUADRATIC MULTIOBJECTIVE OPTIMIZATION PROBLEM USING NEWTON'S DESCENT METHOD VIA A ROBUST OPTIMIZATION APPROACH

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Abstract. In this paper, we develop a Newton's descent method (NDM) for an uncertain quadratic multiobjective optimization problem (UQMOP). To accomplish this, we utilize a minimum of the objective wise worst case (OWWC) type robust counterpart (RC) of the UQMOP. The resulting RC is a nonsmooth multiobjective optimization problem (MOP). Our approach involves constructing a sub-problem to determine Newton's descent direction (NDD) for the RC. An Armijo-type inexact line search (AILS) technique is employed to identify an appropriate step length. Using NDD and step length, we formulate a Newton's descent algorithm (NDA) for the RC. Under some assumptions, we establish the convergence of NDA for the RC. Under specific assumptions, we demonstrate that the sequence defined by the NDA converges rapidly to the solution, exhibiting both superlinear and quadratic rate of convergence. Finally, we assess the efficacy of NDA by conducting a comparative analysis with the weighted sum method via various numerical problems. We obtain the non-dominated Pareto front for both methods, which support our method.
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1. Introduction

The use of optimization methods in real-world situations is constrained by (at least) two problems. First, dealing with input data that are not exactly known is a problem in almost all real-world applications of mathematical optimization problems. This might be caused by inaccurate data, future developments, fluctuations, or disturbances, or it might be the result of measurement errors. Since there are numerous decision-makers and a variety of goals, most real-world optimization problems lack one objective function that is clearly defined and instead depend on multiple goals. The fields of stochastic (see [8]) and robust optimization (RO) deal with uncertain data, while the field of multiobjective decision-making and optimization analyzes

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the various optimization criteria. Both of these topics have been thoroughly researched recently; see, e.g., [14, 20, 23] and the references therein.

To solve MOPs, there are various solution strategies available in the literature; see, e.g., [13, 14, 21, 27]. One of the basic approaches is the weighting method, where several objective functions are weighted to form a single objective optimization problem [19]. Another similar approach is the ε-constraint method, introduced in [22], where only the chosen objective functions are minimized while the others are bounded by a constraint. Some algorithms require the preordering of the functions based on their importance [15], that is, the functions are optimized according to the importance. However, a major drawback of these approaches is that the choice of weights, constraints, or the importance of the functions needs to be pre-specified, which is not always known in advance.

In recent years, several techniques were developed to solve MOPs without requiring any a priori information; see, e.g., [1, 2, 3, 4, 12, 16, 17, 18, 21, 28]. These include the steepest descent algorithm, which has at most a linear convergence rate [12, 16], the Newton's algorithm, which has a superlinear and quadratic convergence rate [17], and the quasi-Newton's method, which has a superlinear rate [31]. These methods are specifically designed for smooth MOPs. In [6, 14], the idea of RO was introduced to help with uncertain optimization problems. The minimax robustness concept, which was initially created by Soyster [32] and later expanded by Bental et al. [6, 7] for single objective optimization problems with uncertainty, is one of the most well-known notions. The minimax robustness concept (RC) of the uncertain optimization problem determines to minimize the worst-case scenario among every possible scenario. The feasible solution of the RC, which is also the solution to the uncertain single objective optimization problem, is feasible under each scenario. Its optimal solution is referred to as the robust optimal solution. In [14] and [24], the minimax robustness concept was extended to the context of multiobjective optimization. Kuroiwa et al. [24] demonstrated how to establish the optimality conditions of RC, while Ehrgott et al. [14] showed how to compute robust solutions for UMOPs. The robustness concept of uncertain single objective optimization cannot be applied to UMOPs, because there is no standard methodology for this. The minimax robustness concept for UMOPs entails first solving a maximizing type MOP over an uncertainty set for any fixed feasible point and then minimizing all optimal solutions of that problem over the feasible set for a different point of feasibility. Another method to solve a UMOP is the OWWC type RC [14], in which an objective-wise maximum of each objective function was taken for any fixed feasible point, and then was minimized over the feasible set. The OWWC type RC provides the same solution as the minimax RC, but it is easier to find the solution for UMOP by using this method. The OWWC type RC provides the same solution as the minimax RC, but it is easier to find the solution for UMOP by using this method. In [14], the UMOP was solved by using various approaches such as the ε-constrained method, the weighted sum method with the help of minimax RC, and objective wise worst case cost type RC. However, these scalarization approaches suffer from the disadvantage of not knowing the choice of parameter in advance. In [29, 30], a land combat vehicle (LCV) system in defense was solved by using a hybrid decision-making model and a multimethod approach. Choung et al. [9] solved a constrained convex quadratic MOP with parameter and decision uncertainty, inspired by [29, 30]. In [9], Choung et al. developed a necessary and sufficient condition for Pareto optimality in terms of linear matrix inequalities and numerically solved the constrained convex quadratic multiobjective optimization problems with parameter and decision uncertainty.

In this paper, we address an unconstrained quadratic MOP that involves parameter uncertainty. Specifically, only the objective functions are subject to uncertainty, and we utilize an OWWC type RC to solve the problem. In general, the OWWC type RC of a UQMOP is a deterministic nonsmooth quadratic MOP. As discussed in [10], the subgradient method, which is typically used for nonsmooth real-valued minimization, was extended to address nonsmooth MOPs. However, there is no standard method for selecting the descent direction at each iteration in the subgradient algorithm, and the sequence generated by the subgradient algorithm with exact or inexact line search may converge to a noncritical point. Additionally, there is a lack of practical evidence demonstrating the convergence of the subgradient method for MOPs. Apart from this, Kumar et al. [25] recently proposed a Newton-type method for solving general uncertain multiobjective optimization problems under finite uncertainty sets. Their framework provides a robust descent-based approach, which is applicable to a broad class of nonlinear uncertain problems. However, when the objective functions possess a specific structure, such as being quadratic, it becomes both meaningful and beneficial to design specialized algorithms that exploit this structure. Quadratic objective functions offer well-defined mathematical properties, such as constant Hessians and linear gradients, which can significantly enhance algorithmic efficiency when properly utilized. Although general descent methods can be applied to quadratic problems, they typically do not take advantage of these characteristics, leading to slower convergence and increased computational effort. In contrast, the present work focuses on uncertain quadratic multiobjective optimization problems and develops a Newton's descent method that fully utilizes the algebraic structure of quadratic functions to obtain faster and more reliable convergence. Moreover, our approach incorporates matrix uncertainty into the set-based uncertainty model - an extension beyond the scalar or vector uncertainty considered in [25]. This inclusion enables a more realistic and general representation of uncertainty in practical multiobjective problems. Thus, the proposed method not only builds upon but also substantially extends the existing literature in both methodological rigor and modeling capability. To the best of our knowledge, no Newton's descent method has been developed for uncertain quadratic multiobjective optimization problems (UQMOPs). By solving the resulting nonsmooth quadratic MOP, one can obtain robust efficient (Pareto optimal) or robust weakly efficient (Pareto optimal) solutions for unconstrained uncertain quadratic multiobjective optimization problems (UQMOPs) with parameter uncertainty.

The structure of the paper is outlined as follows. In Section 2, we present some results, basic definitions, and theorems related to the robust deterministic problem (RDP) (defined in (2.2)).

In Subsection 3.1, we describe the NDD finding subproblem and provide its solution. In Subsection 3.2, we introduce an AILS rule to find a suitable step length that ensures the function value decreases in the NDD. Next, in Subsection 3.3, we present the NDA for RDP to find the critical point. We generate a sequence by using this algorithm, and in Subsection 3.4, we prove that the sequence converges to a critical point, which is a solution to RDP and the robust Pareto optimal solution (POS) to Q(S) (defined in (2.1)). In Subsection 3.5, we additionally establish that the sequence produced by NDA converges to the solution with both superlinear and quadratic rates. Finally, in Subsection 3.6, we verify the effectiveness of the NDA for RDP with some numerical examples. We demonstrate that the algorithm can generate good approximate

Pareto fronts without requiring pre-order choosing weight. In Section 4, we conclude with some comments and remarks.

2. Preliminaries

We start with some important notation. The set of real numbers, non-negative real numbers, and positive real numbers are denoted by $\mathbb{R}$, $\mathbb{R}_{\geq}$, and $\mathbb{R}_{>}$, respectively. Some extended version of the set of real numbers can be defined as follows:

- $\mathbb{R}^n = \{y = (y_1, y_2, \ldots, y_n) : y_i \in \mathbb{R}\}$
- $\mathbb{R}^n_{\geq} = \{y \in \mathbb{R}^n : y_i \in \mathbb{R}_{\geq}\}$
- $\mathbb{R}^n_{>} = \{y \in \mathbb{R}^n : y_i \in \mathbb{R}_{>}\}$
- $-\mathbb{R}^n_{>} = \{-y : y \in \mathbb{R}^n_{>}\}$ and $-\mathbb{R}^n_{\geq} = \{-y : y \in \mathbb{R}^n_{\geq}\}$
- $\mathbb{R}^{n\times n}$ denotes the set of all matrices of order $n \times n$
- $\langle m\rangle = \{1, 2, \ldots, m\}$ and $\langle p\rangle = \{1, 2, \ldots, p\}$ are the indexed sets with m and p elements, respectively.
- A continuously differentiable function is denoted as a CD function.

Now, for any $x, y \in \mathbb{R}^n$, $x \geq y \Longleftrightarrow x - y \in \mathbb{R}^n_{\geq} \Longleftrightarrow x_j - y_j \geq 0,$ for each j and $x > y \Longleftrightarrow x - y \in \mathbb{R}^n_{>} \Longleftrightarrow x_j - y_j > 0,$ for each j. An unconstrained deterministic MOP is defined as:

$$\text{MOP}: \quad \min_{y\in\mathbb{R}^n} f(y) = (f_1(y), f_2(y), \ldots, f_m(y)),$$

where $f : \mathbb{R}^n \to \mathbb{R}^m$ and $f_j : \mathbb{R}^n \to \mathbb{R}$. The solution of the MOP manifests as either a Pareto optimal solution (WPOS) or Pareto optimal solution (POS). A point $y^* \in \mathbb{R}^n$ is a POS for MOP if there exists no $y \in \mathbb{R}^n$, $y \neq y^*$ such that $f(y) \leq f(y^*)$ and $f(y) \neq f(y^*)$. Additionally, a point $y^* \in \mathbb{R}^n$ is a WPOS for MOP if there exists no $y \in \mathbb{R}^n$ such that $f(y) < f(y^*)$. It is evident that every POS is also a WPOS.

Motivated by [29, 30] and Choung et al. [9], in this paper, we consider an UQMOP under parameter uncertainty (i.e., only objective functions faces the uncertainty). We solve this with the help of its OWWC type RC. An UQMOP and its OWWC type RC is defined in the following way

$$Q(S) = \{Q(A_i) : A_i \in S,\ i \in \langle p\rangle\}, \tag{2.1}$$

where $S = \{A_i \in \mathbb{R}^{n\times n} : A_i$ is the symmetric and positive definite matrix for each $i \in \langle p\rangle\}$, and for any fix $A_i \in S$, $Q(A_i) : \min_{y\in\mathbb{R}^n} f(y, A_i)$, where $f : \mathbb{R}^n \times S \to \mathbb{R}^m$ is defined by $f(y, A_i) = (f_1(y, A_i), f_2(y, A_i), \ldots, f_m(y, A_i))$ and $f_j : \mathbb{R}^n \times S \to \mathbb{R}$ such that $f_j(y, A_i) = \frac{1}{2} y^T A_i y + q_j^T y + r_j,\ j \in \langle m\rangle,\ i \in \langle p\rangle$. Q(S) is called UQMOP with parameter uncertainty. OWWC type RC to Q(S) is given by

$$RDP: \quad \min_{y\in\mathbb{R}^n} \xi(y), \tag{2.2}$$

where

$$\xi(y) = (\xi_1(y), \xi_2(y), \ldots, \xi_m(y)) \text{ and } \xi_j(y) = \max_{i\in\langle p\rangle} f_j(x, A_i) = \max_{i\in\langle p\rangle} \{\frac{1}{2} y^T A_i y + q_j^T y + r_j\}.$$

We now offer two remarks about our problem that explain the motivation behind our decision to select it.

Remark 2.1. In [9], Choung et al. developed a necessary and sufficient condition for Pareto optimality in terms of linear matrix inequalities and numerically solved the constrained convex quadratic MOP with parameter and decision uncertainty. Apart from this, in this paper, we consider an unconstrained UQMOP. To the best of our knowledge, there is no Newton's method developed for finding the solution of an unconstrained UQMOP. Therefore, our main contribution is the Newton's descent method for unconstrained UQMOP, namely Q(S) (defined in (2.1)). To achieve this, we use OWWC, namely RDP (defined in (2.2)).

Remark 2.2. In addition to Remark 2.1, an UMOP was solved by Ehrgott et al. [14] with the help of the scalarization approach (weighted sum method and the ε-constrained method). When using the scalarization approach, we convert a deterministic MOP into a scalarized single objective optimization problem. As a result, we encounter some challenges when choosing the parameters, such as the possibility that the scalarized problem's solution is impossible to find while the original deterministic MOP's solution is still feasible. The fact that the parameters are not predetermined, leaving it up to the modeler and decision-maker to make those choices, is another disadvantage of this approach. These drawbacks also can be found at the same time when the UMOP is solved by using a robust optimization approach with the help of a scalarization approach. To remove all this difficulty, we solve Q(S) by using Newton's descent method and RDP.

Our key objective in solving problem Q(S) is to find a robust POS or robust WPOS, which is defined as following.

Definition 2.1. A feasible point $y^* \in \mathbb{R}^n$ is termed robust POS (robust WPOS) if there does not exist any $y \in \mathbb{R}^n - \{y^*\}$ such that $f(y;S) \subset f(y^*;S) - \mathbb{R}^k_{\geq}$ $\left(f(y;S) \subset f(y^*;S) - \mathbb{R}^k_{>}\right)$, where $f(y;S) = f(y,A_i) : A_i \in S$ denotes the set of all possible values of the objective function.

The following theorem establishes a relationship between the solutions of Q(S) and RDP, thereby indicating that solving RDP can effectively replace the need to solve Q(S).

Theorem 2.1. *Let Q(S) be an UQMOP with uncertainty set S and RDP be the OWWC type RC of Q(S). Then*

(a) If $y^* \in \mathbb{R}^n$ is a POS to RDP, then y^* is robust POS for Q(S).

(b) If $\max_{i\in\langle p\rangle} f_j(y,A_i)$ exist for all $j \in \langle m\rangle$, $y \in \mathbb{R}^n$ and y^* is a WPOS to RDP, then y^* is robust WPOS for Q(S).

Proof. (a) On the contrary, we suppose that y^* is not a robust POS for Q(S). By the definition of robust POS, there exists $\bar{y} \in \mathbb{R}^n$ such that

$$
\begin{aligned}
& f(\bar{y};S) \subseteq f(y^*;S) - \mathbb{R}^k_{\geq}, \\
& \Longrightarrow f(\bar{y};S) - \mathbb{R}^k_{\geq} \subseteq f(y^*;S) - \mathbb{R}^k_{\geq} \\
& \Longrightarrow \quad \text{for each } \ i \in \langle p\rangle \text{ there exist a } \ l \in \langle p\rangle \text{ such that} \\
& \Longrightarrow f(\bar{y},A_i) \leq f(y^*,A_l) \\
& \Longrightarrow \max_{i\in\langle p\rangle} f_j(\bar{y},A_i) \leq \max_{l\in\langle p\rangle} f_j(y^*,A_l) \text{ for all } j \in \langle m\rangle.
\end{aligned}
$$

From the last expression, it is evident that $\xi(\bar{y})$ is either equal to or dominates $\xi(y^*)$ in RDP. Consequently, y^* cannot be a POS for RDP, contradicting the assertion that y^* is a POS for RDP. Thus, y^* emerges as a robust POS for $Q(S)$.

(b) Similarly to item (a), the proof for item (b) follows by contradiction. Assuming contrary to the claim, let us suppose that y^* is not a robust WPOS for Q(S). According to the definition of a robust POS, there exists $\bar{y} \in \mathbb{R}^n$ such that

$$
\begin{aligned}
& f(\bar{y};S) \subseteq f(y^*;S) - \mathbb{R}^k_>, \\
\implies & f(\bar{y};S) - \mathbb{R}^k_\geq \subseteq f(y^*;S) - \mathbb{R}^k_> \\
\implies & \text{ for each } \ i \in \langle p \rangle \text{ there exist a } \ l \in \langle p \rangle \text{ such that} \\
\implies & f(\bar{y}, A_i) < f(y^*, A_l) \\
\implies & \max_{i \in \langle p \rangle} f_j(\bar{y}, A_i) < \max_{l \in \langle p \rangle} f_j(y^*, A_l) \text{ for all } j \in \langle m \rangle.
\end{aligned}
$$

The last expression clearly indicates that $\xi(\bar{y})$ strictly dominates $\xi(y^*)$ in the context of RDP. Consequently, this implies that y^* cannot be a WPOS for RDP. However, this contradicts the assertion that y^* is indeed a WPOS for RDP. Therefore, we conclude that y^* serves as a robust WPOS for Q(S). $\square$

As a result of the theorem connecting the solutions to Q(S) and RDP, it is clear that solving RDP is preferable to solving Q(S). This is a clear advantage because computing $\xi(y)$ in RDP for a given y is much easier than solving a class of MOPs in Q(S). The problem RDP, only requires solving m deterministic single objective optimization problems and hence it is a deterministic multiobjective optimization problem.

Definition 2.2. In RDP, a vector $d \in \mathbb{R}^n$ is said to be robust descent direction for ξ at y if $(A_i y + q_j)^T d < 0$ for all $j \in \langle m \rangle$ and $i \in S_j(y)$. Also, if $d \in \mathbb{R}^n$ is a robust descent direction for ξ at y, then there exists $\bar{\alpha} > 0$ such that $\xi_j(y + \alpha v) < \xi_j(y)$ for all $j \in \langle m \rangle$, and for all $\alpha \in (0, \bar{\alpha}]$.

Definition 2.3. $\xi : \mathbb{R}^n \to \mathbb{R}^m$ is strongly $\mathbb{R}^m-$strongly convex if and only if there exits $\tau > 0$ such that $\Psi_{\min}(\nabla^2 f_j(y, A_i)) \geq \tau$ for all $y \in \mathbb{R}^n$ and $j \in \langle m \rangle$, where $\Psi_{\min} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denotes the minimum eigenvalue function.

Theorem 2.2. [11] *Let $\xi_j : \mathbb{R}^n \to \mathbb{R}^n$ be a function such that $\xi_j(y) = \max_{i \in \langle p \rangle} f_j(y, A_i)$.*

(i) In the direction v, the directional derivative of ξ_j at y is given by $\xi_j'(y, v) = \max_{i \in S_j(y)} (A_i y + q_j)^T v$, where $S_j(y) = \{i \in \langle p \rangle : f_j(y, A_i) = \xi_j(y)\}$.

(ii) The subdifferential of ξ_j is $\partial \xi_j(y) = conv\{A_i y + q_j : i \in S_j(y)\}$. Also, $y^* = \underset{y \in \mathbb{R}^n}{\text{argmin}}\ \xi_j(y) \iff 0 \in \partial \xi_j(y^*))$.

Definition 2.4. A point $y^* \in \mathbb{R}^n$ is robust critical point (RCP) for ξ if $R(\Phi(y^*)) \cap (-\mathbb{R}^m_>) = \emptyset$, where $R(\Phi(y^*))$ denotes the range of $\Phi(y^*) = \cup_{j \in \langle m \rangle} \partial \xi_j(y^*)$. It is noticed that y^* is a RCP for RDP if and only there exists no $v \in \mathbb{R}^n$ such that $(A_i y^* + q_j)^T v < 0$, for each $i \in S_j(y^*)$, $j \in \langle m \rangle$, where $S_j(y^*) = \{i \in \langle p \rangle : f_j(y^*, A_i) = \xi_j(y^*)\}$.

Now, we extend the idea of the Newton's method for smooth multiobjective optimization [17] to RDP (nonsmooth MOP). With the help of the Newton's descent method, we can find the RCP for RDP which is the POS to RDP as well as the robust POS to Q(S).

3. Newton's Descent Method (NDM) for RDP

In this section, we construct NDM for RDP. In RDP, the function $\xi : \mathbb{R}^n \to \mathbb{R}^m$ is consider as $\xi(y) = (\xi_1(y), \xi_2(y), \ldots, \xi_m(y))$, where

$$\xi_j(y) = \max_{i\in\langle p\rangle} f_j(y,A_i) = \max_{i\in\langle p\rangle}\{\frac{1}{2}y^T A_i y + q_j^T y + r_j\},\ j \in \langle m\rangle.$$

We assume that the function $f_j(y,A_i)$ is a twice CD for each y and $i \in \langle p\rangle$. It can be observe that uncertainty S contains symmetric and positive definite matrices, i.e., for each $i \in \langle p\rangle$, $A_i \in S$ is symmetric and positive definite. Therefore, $f_j(y,A_i)$ is a convex function for each $i \in \langle p\rangle$ and $j \in \langle m\rangle$. Consequently, being a maximum of convex functions, $\xi_j(y)$ is convex for each $j \in \langle m\rangle$. The subsequent lemma and theorem establish a necessary and sufficient condition for Pareto optimality or efficiency for RDP.

Lemma 3.1. *The point y^* is a RCP for $\xi \iff 0 \in Conv\{\cup_{j\in\langle m\rangle}\partial\xi_j(y^*)\}$.*

Proof. Letting y^* be the RCP for ξ, we see that there must exist $d \in \bigcup_{j\in\langle m\rangle}\partial\xi_j(y^*)$ satisfying

$$u^T d \geq 0,\ \forall\, u \in \mathbb{R}^n. \tag{3.1}$$

Now, we suppose the contrary, that is, $0 \notin Conv\{\cup_{j\in\langle m\rangle}\partial\xi_j(y^*)\}$. Since $Conv\{\cup_{j\in\langle m\rangle}\partial\xi_j(y^*)\}$ and $\{0\}$ are both closed and convex sets, we can utilize the theorem of separation. This theorem guarantees the existence of $u \in \mathbb{R}^n$ and $w \in \mathbb{R}$ such that $u^T 0 \geq w$ and $u^T d < w\ \forall d \in Conv\{\cup_{j\in\langle m\rangle}\partial\xi_j(y^*)\}$ They contradict (3.1). Hence, $0 \in Conv\{\cup_{j\in\langle m\rangle}\partial\xi_j(y^*)\}$.

On the other hand, we need to demonstrate that y^* is RCP for ξ if $0 \in Conv\{\cup_{j\in\langle m\rangle}\partial\xi_j(y^*)\}$. For that, we define $\breve{\xi}(y) = \max_{j\in\langle m\rangle}\xi_j(y) - \xi_j(y^*)$. Utilizing Theorem 2.2(ii), we obtain $\partial\breve{\xi}(y) = Conv\{\cup_{j\in\langle m\rangle}\partial\xi_j(y)\}$. Therefore, according to our assumption, $0 \in Conv\{\cup_{j\in\langle m\rangle}\partial\xi_j(y^*)\}$, it follows that $y^* = \operatorname*{argmin}_{y\in\mathbb{R}^n}\breve{\xi}(y)$. On the contrary, we suppose that y^* is not a (RCP). According to Definition 2.4, there exists $q \in \mathbb{R}^n$ such that $(A_i y^* + q_j)^T q < 0$ for all $i \in S_j(y^*)$, $j \in \langle m\rangle$, implying $\xi_j'(y^*,q) < 0$ for all j. Consequently, there exists some $\eta > 0$ sufficiently small such that $\xi_j(y^* + \eta q) < \xi_j(y^*)$ for all $j \in \langle m\rangle$, which further implies $\breve{\xi}(y^* + \eta q) < 0 = \breve{\xi}(y^*)$ holds for some $(y^* + \eta q) \in \mathbb{R}^n$. However, this contradicts the fact that $y^* = \operatorname*{argmin}_{y\in\mathbb{R}^n}\breve{\xi}(y)$. Thus, our assumption that y^* is not a RCP is incorrect, and y^* is indeed an RCP for ξ. □

Theorem 3.1. *Let $f_j(y,A_i) = \frac{1}{2}y^T A_i y + q_j^T y + r_j$ be a CD function such that, for each $i \in \langle p\rangle$, A_i is symmetric and positive definite matrix then y^* is WPOS for RDP if and only if $0 \in Conv\{\cup_{j\in\langle m\rangle}\partial\xi_j(y^*)\}$.*

Proof. Let y^* be a WPOS for ξ. We need to demonstrate that $0 \in Conv\{\cup_{j\in\langle m\rangle}\partial\xi_j(y^*)\}$. Since $\frac{1}{2}y^T A_i y + q_j^T y + r_j$ is CD and convex for each $j \in \langle m\rangle$ and $i \in \langle p\rangle$, then $\frac{1}{2}y^T A_i y + q_j^T y + r_j$ is locally Lipschitz continuous for each $i \in \langle p\rangle$. Consequently, by [26, Theorem 4.3], $0 \in Conv\{\cup_{j\in\langle m\rangle}\partial\xi_j(y^*)\}$.

On the other hand, because A_i is symmetric and positive definite for each $i \in \langle p\rangle$, we see that $\frac{1}{2}y^T A_i y + q_j^T y + r_j$ is convex. Therefore, $\xi_j = \max_{i\in\langle p\rangle}\frac{1}{2}y^T A_i y + q_j^T y + r_j$ is also convex. Assume that $0 \in Conv\{\cup_{j\in\langle m\rangle}\partial\xi_j(y^*)\}$, which implies y^* is a RCP for ξ. For at least one j_0 we have $\xi'_{j_0}(y^*,d) \geq 0$ for all $d \in R^n$. Thus

$$(A_i y^* + q_{j_0})^T d \geq 0,\ i \in S_{j_0}(y^*),\ \forall\, d \in \mathbb{R}^n. \tag{3.2}$$

By convexity of ξ_j and $\frac{1}{2}y^T A_i y + q_j^T y + r_j$, we have

$$\frac{1}{2}y^T A_i y + q_{j_0}^T y + r_{j_0} \geq \frac{1}{2}y^{*T} A_i y^* + q_{j_0}^T y^* + r_{j_0} + (A_i y + q_{j_0})^T (y - y^*),\ \forall i \in S_{j_0}(y^*)$$

It follows from (3.2) that

$$\begin{aligned}\frac{1}{2}y^T A_i y + q_{j_0}^T y + r_{j_0} &\geq \frac{1}{2}y^{*T} A_i y^* + q_{j_0}^T y^* + r_{j_0},\ \forall i \in S_{j_0}(y)\\ &\implies \xi_{j_0}(y) \geq \xi_{j_0}(y^*),\end{aligned}$$

that is, y^* is WPOS for RDP. □

In order to apply NDM to solve RDP, we need to determine both the NDD and the appropriate step length in that direction. In subsection 3.1, we solve a subproblem that involves minimizing a real-valued function. The solution to this subproblem provides us with the NDD.

3.1. **The subproblem of finding the NDD.** To find the NDD for RDP, we solve a real-valued minimization subproblem. This subproblem can be stated as follows

$$\min_{d\in\mathbb{R}^n} \max_{j\in\langle m\rangle} \max_{i\in\langle p\rangle} \{\frac{1}{2}y^T A_i y + q_j^T y + r_j + (A_i y + q_j)^T d + \frac{1}{2}d^T A_i d - \xi_j(y)\}. \tag{3.3}$$

Let $d(y)$ and $\Theta(y)$ represent the optimal solution and optimal value, respectively, of subproblem (3.3). Therefore,

$$d(y) = \underset{d\in\mathbb{R}^n,\ t\in\mathbb{R}}{\operatorname{argmin}}\ t \tag{3.4}$$

$$\Theta(y) = \max_{j\in\langle m\rangle} \max_{i\in\langle p\rangle} \{\frac{1}{2}y^T A_i y + q_j^T y + r_j + (A_i y + q_j)^T d(y) + \frac{1}{2}d(y)^T A_i d(y) - \xi_j(y)\}, \tag{3.5}$$

where $t = \max_{j\in\langle m\rangle} \max_{i\in\langle p\rangle} \{\frac{1}{2}y^T A_i y + q_j^T y + r_j + (A_i y + q_j)^T d + \frac{1}{2}d^T A_i d - \xi_j(y)\}$. Suproblem (3.3) can be rewritten as

$$\begin{aligned} P(y): \quad & \min_{d\in\mathbb{R}^n,\ t\in\mathbb{R}} \quad t \\ & \text{subject to:}\ \ \frac{1}{2}y^T A_i y + q_j^T y + r_j + (A_i y + q_j)^T d + \frac{1}{2}d^T A_i d - \xi_j(y) \leq t,\ i \in \langle p\rangle,\ j \in \langle m\rangle. \end{aligned}$$

It can be observe that the constraints of P(y) satisfy the Slater's constraint qualification condition at $t = 1$, and $d = (0,0,\ldots,0) \in \mathbb{R}^n$. Therefore, the solution to the given convex programming problem P(y) is determined by the Karush-Kuhn-Tucker (KKT) optimality condition. To apply this condition, we need to define the Lagrangian function by

$$L(t,d,\lambda) = t + \sum_{j\in\langle m\rangle} \sum_{i\in\langle p\rangle} \lambda_{ij} \left(\frac{1}{2}y^T A_i y + q_j^T y + r_j + (A_i y + q_j)^T d + \frac{1}{2}d^T A_i d - \xi_j(y) - t\right).$$

Then, the KKT optimality conditions for problem P(y) are

$$\sum_{j\in\langle m\rangle} \sum_{i\in\langle p\rangle} \lambda_{ij} = 1, \tag{3.6}$$

$$\sum_{j\in\langle m\rangle}\sum_{i\in\langle p\rangle}\lambda_{ij}A_i d+\sum_{j\in\langle m\rangle}\sum_{i\in\langle p\rangle}\lambda_{ij}(A_i y+q_j)=0, \tag{3.7}$$

$$\frac{1}{2}y^T A_i y+q_j^T y+r_j+(A_i y+q_j)^T d+\frac{1}{2}d^T A_i d-\xi_j(y)\leq t,\ i\in\langle p\rangle,\ j\in\langle m\rangle, \tag{3.8}$$

$$\lambda_{ij}\left(\frac{1}{2}y^T A_i y+q_j^T y+r_j+(A_i y+q_j)^T d+\frac{1}{2}d^T A_i d-\xi_j(y)-t\right)=0,\ \ \lambda_{ij}\geq 0,\ i\in\langle p\rangle,\ j\in\langle m\rangle. \tag{3.9}$$

Problem P(y) has a unique solution, $(d(y),\Theta(y))$. As this is a convex problem and has a Slater's point, there exists a KKT multiplier $\lambda=\lambda(y)=\lambda_{ij}(y)$, which together with $d=d(y)$ and $t=\Theta(y)$ satisfies n (3.6), (3.7), (3.8), and (3.9). Particularly, from (3.7), we obtain

$$d(y)=-\Big(\sum_{j\in\langle m\rangle}\sum_{i\in\langle p\rangle}\lambda_{ij}A_i\Big)^{-1}\sum_{j\in\langle m\rangle}\sum_{i\in\langle p\rangle}\lambda_{ij}(A_i y+q_j). \tag{3.10}$$

Because A_i is positive definite for every j and $y\in\mathbb{R}^n$, where $i\in\langle p\rangle$, it ensures the existence of the inverse of A_i. Since y is not a RCP, there must exist $\lambda_{ij}>0$ for at least one i and j. Consequently, the inverse of $\sum_{j\in\langle m\rangle}\sum_{i\in\langle p\rangle}\lambda_{ij}A_i$ exists, ensuring the well-definedness of $d(y)$. Therefore, the solution of P(y) is exists, and $d(y)$ is the NDD for RDP.

Theorem 3.2 provides a connection between RCP and d(y) as defined in (3.10).

Theorem 3.2. *Let the optimal value and the solution of subproblem (3.3) bee given in (3.4) and (3.5). Then the following statements hold:*

(1) If $C\subset\mathbb{R}^n$ is compact, then $d(y)$ is bounded on C and $\Theta(y)\leq 0$.
(2) The subsequent conditions are equivalent:
  (i) $y\in\mathbb{R}^n$, is not a RCP;
  (ii) $\Theta(y)<0$;
  (iii) $d(y)\neq 0$;
  (iv) $d(y)$ serves as a descent direction for ξ at y for RDP.

Moreover, y is a RCP if and only if $\Theta(y)=0$.

Proof. (1) Let $C\subset\mathbb{R}^n$ be a compact set. Considering $j\in\langle m\rangle$, we see that $f_j(y,A_i)=\frac{1}{2}y^T A_i y+q_j^T y+r_j$ is twice CD for all $y\in\mathbb{R}^n$ and $i\in\langle p\rangle$. Consequently, its first and second-order derivatives are bounded on C. Therefore, for every $y\in\mathbb{R}^n$, $j\in\langle m\rangle$, $i\in\langle p\rangle$, and according to (3.10), $d(y)$ is bounded on the compact set C. Since $d=\bar{0}=(0,0,\ldots,0)\in\mathbb{R}^n$ and $t=1$ satisfy the constraints of subproblem P(y), we have that (t,d) lies in the feasible region. Consequently, $\Theta(y)\leq\max_{j\in\langle m\rangle}\{\max_{i\in\langle p\rangle}\{\frac{1}{2}y^T A_i y+q_j^T y+r_j\}+\max_{i\in\langle p\rangle}\{(A_i y+q_j)^T\bar{0}\}+\max_{i\in\langle p\rangle}\frac{1}{2}\bar{0}^T A_i\bar{0}-\max_{i\in\langle p\rangle}\xi_j(y)\}=0$. Hence $\Theta(y)\leq 0$.

(2) $(i)\implies(ii)$ Assume that y is not a RCP. In such a case, there exists $\bar{d}$ such that $(A_i y+q_j)^T\bar{d}<0$ for all $j\in\langle m\rangle$ and $i\in S_j(y)$. From the definition of $\Theta(y)$, it follows that, for every $\delta>0$, $\Theta(y)\leq\max_{j\in\langle m\rangle}\{\max_{i\in\langle p\rangle}\{\frac{1}{2}y^T A_i y+q_j^T y+r_j\}+\max_{i\in\langle p\rangle}\{(A_i y+q_j)^T(\delta\bar{d})+\frac{1}{2}(\delta\bar{d})^T A_i(\delta\bar{d})\}-\max_{i\in\langle p\rangle}\xi_j(y)\}$, which implies $\Theta(y)\leq\delta\big(\max_{j\in\langle m\rangle}\max_{i\in\langle p\rangle}\{(A_i y+q_j)^T\bar{d}+\delta\frac{1}{2}\bar{d}^T A_i\bar{d}\}\big)$. For sufficiently small $\delta>0$, the right-hand side of the aforementioned inequality is negative due to $(A_i y+q_j)^T\bar{d}<0$, $\frac{1}{2}\bar{d}^T A_i\bar{d}>0$, and $\Theta(y)\leq 0$. Therefore, $\Theta(y)<0$.

$(ii) \implies (iii)$ Observe that $\Theta(y)$ represents the optimal value of subproblem (3.3). We conclude that $d(y) \neq 0$. If $d(y) = 0$, then $\Theta(y)$ would be zero, which contradicts the assertion from (ii). Therefore, if $\Theta(y) < 0$, then $d(y) \neq 0$.

$(iii) \implies (iv)$ Let $d(y) \neq 0$. Then, $\Theta(y) \neq 0$. As $\Theta(y) \leq 0$, one has $\Theta(y) < 0$ and

$$\Theta(y) = \max_{j \in \langle m \rangle} \{ \{ \max_{i \in \langle p \rangle} \{ \frac{1}{2} y^T A_i y + q_j^T y + r_j \} + \max_{i \in \langle p \rangle} \{ (A_i y + q_j)^T d(y) + \frac{1}{2} d(y)^T A_i d(y) \} - \max_{i \in \langle p \rangle} \xi_j(y) \}$$
$$< 0$$
$$\implies \max_{j \in \langle m \rangle} \max_{i \in \langle p \rangle} \{ (A_i y + q_j)^T d(y) + \frac{1}{2} d(y)^T A_i d(y) \} < 0$$
$$\implies (A_i y + q_j)^T d(y) < 0, \ \forall \ j \in \langle m \rangle \text{ and } i \in \langle p \rangle$$
$$\implies (A_i y + q_j)^T d(y) < 0, \ \forall \ j \in \langle m \rangle \text{ and } i \in S_j(y)$$
$$\implies d(y) \text{ serves as a descent direction for } \xi \text{ at } y \text{ for RDP.}$$

$(iv) \implies (i)$ As $d(y)$ serves as a descent direction for ξ at y, we have $(A_i y + q_j)^T d(y) < 0$ for all $j \in \langle m \rangle$ and $i \in S_j(y)$. Hence, y is not a RCP. Furthermore, if $\Theta(y) < 0$, then $d(y) \neq 0$. Conversely, if $d(y) \neq 0$, then $\Theta(y) < 0$. Therefore, we conclude that y is a RCP if and only if $\Theta(y) = 0$. □

Theorem 3.2 provides a way to determine whether a RCP has been reached or a descent direction can be found, which can be employed as a stopping condition in NDA. The following theorem aims to demonstrate that $\Theta(y)$ is a continuous function for all y belonging to $\mathbb{R}^n$.

Theorem 3.3. *Let $\Theta(y) : \mathbb{R}^n \to \mathbb{R}$ be a function defined by (3.4). Then, $\Theta(y)$ is continuous.*

Proof. To establish the continuity of $\Theta(y)$, it suffices to demonstrate its continuity in any arbitrary compact subset C of $\mathbb{R}^n$. According to Theorem 3.2, $\Theta(y) \leq 0$, and then, for each $j \in \langle m \rangle$ and $i \in \langle p \rangle$,

$$\frac{1}{2} y^T A_i y + q_j^T y + r_j + (A_i y + q_j)^T d(y) + \frac{1}{2} d(y)^T A_i d(y) - \xi_j(y) \leq 0.$$

At the active indices, where $\frac{1}{2} y^T A_i y + q_j^T y + r_j$ attains its maximum value, i.e., $\frac{1}{2} y^T A_i y + q_j^T y + r_j = \xi_j(y)$ for each $i \in S_j(y)$. It follows that $(A_i y + q_j)^T d(y) + \frac{1}{2} d(y)^T A_i d(y) \leq 0$, which implies

$$\frac{1}{2} d(y)^T A_i d(y) \leq -(A_i y + q_j)^T d(y), \text{ for each } j \in \langle m \rangle, \ i \in \langle p \rangle. \tag{3.11}$$

For every $j \in \langle m \rangle$ and $i \in \langle p \rangle$, the function $f_j(y, A_i) = \frac{1}{2} y^T A_i y + q_j^T y + r_j$ is twice CD, and A_i is positive definite for every $i \in \langle p \rangle$. Additionally, $C \subset \mathbb{R}^n$ is a compact, and the eigenvalues of the A_i are bounded on C. Therefore, there exist $\mu > 0$ and $\bar{\mu}$ such that

$$\bar{\mu} = \max_{y \in C, \ j \in \langle m \rangle} \max_{i \in \langle p \rangle} \|A_i y + q_j\| \tag{3.12}$$

and

$$\mu \|w\|^2 \leq w^T A_i w, \text{ for each } j \in \langle m \rangle, \ i \in \langle p \rangle, \ \forall y \in C. \tag{3.13}$$

Using (3.11), (3.12), (3.13), and the Cauchy-Schwarz inequality, we obtain

$$\mu \|d(y)\|^2 \leq \|A_i y + q_j\| \|d(y)\| \leq \bar{\mu} \|d(y)\| \ \forall y \in C, \ j \in \langle m \rangle, \ i \in \langle p \rangle.$$

This implies that $\|d(y)\| \leq \frac{\bar{\mu}}{\mu}$ for all $y \in C$, i.e., d(y), the NDDs remain bounded on C. For $y \in C$, $j \in \langle m \rangle$ and $i \in \langle p \rangle$, we set

$$F_{y,j,A_i} : C \to \mathbb{R}$$

$$s.t. \qquad z \to \frac{1}{2}z^T A_i z + q_j^T z + r_j + (A_i z + q_j)^T d(y) + \frac{1}{2}d(y)^T A_i d(y) - \xi_j(z).$$

Thus $\{F_{y,j,A_i}\}_{y\in C,\ j\in\langle m\rangle, i\in\langle p\rangle}$ is equicontinuous. Consequently, $\{\Gamma_y = \max_{j\in\langle m\rangle} \max_{i\in\langle \bar{m}\rangle} F_{y,j,A_i}\}_{y\in C}$ is also equicontinuous. Consider $\varepsilon > 0$, $\exists\ \delta > 0$ such that, for all $u, w \in C$, $\|u - w\| < \delta \implies |\Gamma_y(u) - \Gamma_y(w)| < \varepsilon$ for all $y \in C$. Hence, for $\|u - w\| < \delta$,

$$\begin{aligned}\Theta(w) &\leq \frac{1}{2}w^T A_i w + q_j^T w + r_j + (A_i w + q_j)^T d(u) + \frac{1}{2}d(u)^T A_i d(u) - \xi_j(w)\\ &= \Gamma_u(w)\\ &\leq |\Gamma_u(w) - \Gamma_u(u)| + \Gamma_u(u)\\ &< \varepsilon + \Theta(u),\end{aligned}$$

Thus $\Theta(u) - \Theta(w) < \varepsilon$. If w and u are interchanged, then $\Theta(u) - \Theta(w) > -\varepsilon$. Consequently, $|\Theta(u) - \Theta(w)| < \varepsilon$ whenever $\|u - w\| < \delta$. Hence, $\Theta(y)$ is continuous in C. As C represents any arbitrary compact subset of $\mathbb{R}^n$, it follows that $\Theta(y)$ is continuous. □

In Section 3.1, we determined the NDD for RDP. In order to formulate the NDA for RDP, we must determine the appropriate step length. To accomplish this, we introduce an AILS technique in Subsection 3.2.

3.2. **Derivation of a new Armijo-type inexact line search (AILS) method for determining the step length size.** As the optimal value of subproblem (3.3) is given by

$$\Theta(y) = \max_{j\in\langle m\rangle} \max_{i\in\langle p\rangle} \{\frac{1}{2}y^T A_i y + q_j^T y + r_j + (A_i y + q_j)^T d(y) + \frac{1}{2}d(y)^T A_i d(y) - \xi_j(y)\}$$

and $\Theta(y) \leq 0$. We introduce an auxiliary function $\xi_j''^*(y, \alpha d)$ to determine the step length size, which is given by

$$\xi_j''^*(y, \alpha d) = \max_{i\in\langle p\rangle} \{\frac{1}{2}y^T A_i y + q_j^T y + r_j + \alpha(A_i y + q_j)^T d + \alpha^2 \frac{1}{2}d^T A_i d\} - \xi_j(y) - \xi_j'(y, \alpha d), \tag{3.14}$$

where $\alpha \in [0, \varepsilon]$ and $\varepsilon < 1$. Then, for every $j \in \langle m \rangle$, (3.14) can be written as

$$\begin{aligned}\xi_j''^*(y, \alpha d) + \xi_j'(y, \alpha d) &\leq \max_{i\in\langle p\rangle} \{\frac{1}{2}y^T A_i y + q_j^T y + r_j\} + \max_{i\in\langle p\rangle} \{\alpha(A_i y + q_j)^T d + \alpha^2 \frac{1}{2}d^T A_i d\} - \xi_j(y)\\ &\leq \max_{i\in\langle p\rangle} \{\alpha(A_i y + q_j)^T d + \alpha^2 \frac{1}{2}d^T A_i d\}.\end{aligned}$$

Thus,

$$\xi_j''^*(y, \alpha d) + \xi_j'(y, \alpha d) \leq \max_{i\in\langle p\rangle} \{\alpha(A_i y + q_j)^T d + \alpha^2 \frac{1}{2}d^T A_i d\}. \tag{3.15}$$

As $f_j(x, A_i) = \frac{1}{2}y^T A_i y + q_j^T y + r_j$ is twice CD and convex, it is upper uniformly twice differentiable. Therefore, there exists $k_i^j > 0$ such that $\frac{1}{2}(y + d)^T A_i(y + d) + q_j^T(y + d) + r_j \leq$

$\frac{1}{2}y^T A_i y + q_j^T y + r_j + (A_i y + q_j)^T d + \frac{1}{2} d^T A_i d + \frac{1}{3!} k_i^j \|d\|^3$. Since the above inequality holds for every $d \in \mathbb{R}^n$, it also holds for αd. Therefore,

$$\begin{aligned}
&\frac{1}{2}(y+\alpha d)^T A_i (y+\alpha d) + q_j^T (y+\alpha d) + r_j \\
&\leq \frac{1}{2} y^T A_i y + q_j^T y + r_j + \alpha (A_i y + q_j)^T d + \frac{1}{2}\alpha^2 d^T A_i d + \frac{1}{3!} k_i^j \|\alpha d\|^3 \\
&\leq \max_{i\in\langle p\rangle} \{\frac{1}{2} y^T A_i y + q_j^T y + r_j + \alpha (A_i y + q_j)^T d + \alpha^2 \frac{1}{2} d^T A_i d\} + \frac{1}{3!} K^j \|\alpha d\|^3,
\end{aligned}$$

where $K^j = \max_{i\in\langle p\rangle} k_i^j$. Now, from (3.14), we have

$$\frac{1}{2}(y+\alpha d)^T A_i (y+\alpha d) + q_j^T (y+\alpha d) + r_j \leq \xi_j(y) + \xi_j'(y,\alpha d) + \xi_j''^*(y,\alpha d) + \frac{1}{3!}\alpha^3 K^j \|d\|^3.$$

It holds for each $i \in \langle p\rangle$. Therefore,

$$\max_{i\in\langle p\rangle} \{\frac{1}{2}(y+\alpha d)^T A_i (y+\alpha d) + q_j^T (y+\alpha d) + r_j\} \leq \xi_j'(y,\alpha d) + \xi_j(y) + \xi_j''^*(y,\alpha d) + \frac{1}{3!}\alpha^3 K^j \|\alpha d\|^3$$

$$\xi_j(y+\alpha d) \leq \xi_j'(y,\alpha d) + \xi_j(y) + \xi_j''^*(y,\alpha d) + \frac{1}{3!} K^j \|\alpha d\|^3. \tag{3.16}$$

Now, from (3.15), one has

$$\begin{aligned}
&\xi_j''^*(y,\alpha d) + \xi_j'(y,\alpha d) \\
&\leq \max_{i\in\check{\langle m\rangle}} \{\alpha (A_i y + q_j)^T d + \frac{1}{2}\alpha^2 d^T A_i d\} \\
&\leq \max_{j\in\langle m\rangle} \max_{i\in\check{\langle m\rangle}} \{\alpha(\frac{1}{2} y^T A_i y + q_j^T y + r_j) + \alpha (A_i y + q_j)^T d + \frac{1}{2}\alpha^2 d^T A_i d - \alpha \xi_j(y)\} \\
&\leq \alpha \max_{j\in\langle m\rangle} \max_{i\in\langle p\rangle} \{\frac{1}{2} y^T A_i y + q_j^T y + r_j + (A_i y + q_j)^T d + \frac{1}{2} d^T A_i d - \xi_j(y)\} \\
&+ \frac{\alpha(\alpha-1)}{2} \max_{j\in\langle m\rangle} \max_{i\in\langle p\rangle} \{d^T A_i d\} \\
&\leq \alpha \Theta(y) + \frac{1}{2}\alpha(\alpha-1) \max_{j\in\langle m\rangle} \max_{i\in\langle p\rangle} \{d^T A_i d\}.
\end{aligned} \tag{3.17}$$

As y is not a RCP, we find by Theorem 3.2 and (3.15), (3.16), and (3.17) that

$$\xi_j(y+\alpha d) \leq \xi_j(y) + \alpha\Theta(y) + \frac{1}{2}\alpha(\alpha-1) \max_{j\in\langle m\rangle} \max_{i\in\langle p\rangle} \{d^T A_i d\} + \frac{1}{3!}\alpha^3 K^j \|d\|^3. \tag{3.18}$$

Since A is positive definite for all $y \in \mathbb{R}^n$ and $i \in \langle p\rangle$, we have $d^T A_i d > 0$, where $d(\neq 0) \in \mathbb{R}^n$. Now, In view of the third term on the right-hand side of (3.18), it is negative due to $\alpha < 1$. Thus (3.18) can be expressed as $\xi_j(y+\alpha d) \leq \xi_j(y) + \alpha\Theta(y) + \alpha^3 \frac{1}{3!} K^j \|d\|^3$. Since y is not a RCP, $\Theta(y) < 0$; and for $\alpha > 0$ sufficiently small, the last term in the right hand side of the above inequality tends to zero. Therefore, for any $0 < \beta < 1$,

$$\xi_j(y+\alpha v) \leq \xi_j(y) + \alpha\beta\Theta(y), \tag{3.19}$$

which represents the AILS technique for finding the step length size for NDA for the problem RDP.

After obtaining the NDD and the AILS technique to determine the step length size for the problem RDP, we can now proceed to write the NDA for RDP.

3.3. **Newton's descent algorithm (NDA) for RDP.**

Algorithm 1 (NDA for RDP)
Step 1. Choose $\varepsilon > 0$, $\beta \in (0,1)$ and $y^0 \in \mathbb{R}^n$. Set $k := 0$.
Step 2. Solve $P(y^k)$ and find d^k, $t(d(y^k))$ and $\Theta(y^k)$.
Step 3. If $|\Theta(y^k)| < \varepsilon$, then stop. Otherwise proceed to Step 4.
Step 4. Choose α_k as the greatest $\alpha \in \{\frac{1}{2^r} : r = 1,2,3,\dots\}$ such that

$$\xi_j(y^k + \alpha d^k) \leq \xi_j(y^k) + \alpha\beta\Theta(y^k), \ \ \forall\, j \in \Lambda. \tag{3.20}$$

Step 5. Define $y^{k+1} := y^k + \alpha_k d^k$, update $k := k+1$ and go to Step 2.
We now present a remark on the understanding of Algorithm 1.

Remark 3.1. The well-definedness of Algorithm 1 hinges on the successful execution of Step 2, Step 3, Step 4, and Step 5. In Step 2, the task is to compute a minimizer of a function $d \mapsto \max\limits_{j\in\langle m\rangle} \max\limits_{i\in\langle p\rangle} \frac{1}{2} y^T A_i y + q_j^T y + r_j + (A_i y + q_j)^T d + \frac{1}{2} d^T A_i d - \xi_j(y)$ at the point y^k. Since this function is strongly convex, it is guaranteed to possess a unique minimizer. Therefore, d^k is always well-defined, ensuring the well-definedness of Step 2. With d^k determined, we can find $\Theta(y^k)$, making Step 3 well-defined. Notably, according to Theorem 3.2, the stopping criterion $|\Theta(y^k)| < \varepsilon$ can be equivalently replaced by $|d^k| < \varepsilon$. If at iteration k, Algorithm 1 halts at Step 3 without reaching Step 4, then, by Theorem 3.2, y^k serves as an approximately RCP for ξ. Upon reaching Step 4 at iteration k, we select α_k as the largest $\alpha \in \frac{1}{2^r} : r = 1,2,3,\dots$ satisfying (3.19). Additionally, it Is worth noting that due to (3.19), the objective function values consistently decrease in the component-wise partial order from Step 4. Since y^k is not a RCP of ξ (as per Theorem 3.2), $\Theta(y^k) < 0$. Hence, from Step 4, we observe that $\xi(y_{k+1}) < \xi(y_k)$, ensuring the well-definedness of Step 4. Finally, leveraging d^k, α_k, and the current iteration point y^k, we compute y^{k+1} in Step 5, thereafter iterating back to Step 2 until the stopping criteria at Step 3 are met.

The forthcoming subsection delves into the convergence analysis of NDA applied to the problem denoted as RDP.

3.4. **Convergence analysis of NDA (Algorithm 1) for RDP.**

Lemma 3.2. *For every $k \geq 0$ and $j \in \langle m\rangle$,$\sum_{r=0}^{k} \alpha_r |\Theta(y^r)| \leq \beta^{-1}(\xi_j(y^0) - \xi_j(y^{k+1}))$, where $\xi_j(y) = \max_{i\in\langle p\rangle}\{\frac{1}{2} y^T A_i y + q_j^T y + r_j\}$.*

Proof. According to Step 4 of Algorithm 1, it follows that

$$\xi_j(y^{k+1}) \leq \xi_j(y^k) + \alpha_k\beta\Theta(y^k), \ -\alpha_k\Theta(y^k) \leq \beta^{-1}(\xi_j(y^k) - \xi_j(y^{k+1})).$$

Thus $-\sum_{r=0}^{k} \alpha_r\Theta(y^r) \leq \beta^{-1}(\xi_j(y^0) - \xi_j(y^{k+1}))$. Also,

$$\sum_{r=0}^{k} \alpha_r\Theta(y^r) \geq -\beta^{-1}(\xi_j(y^0) - \xi_j(y^{k+1})). \tag{3.21}$$

Since $\Theta(y^r) < 0$, we have from (3.21) that $\beta^{-1}(\xi_j(y^0) - \xi_j(y^{k+1})) \geq \sum_{r=0}^{k} \alpha_r \Theta(y^r)$, which together with (3.21) obtains $\sum_{r=0}^{k} \alpha_r |\Theta(y^r)| \leq \beta^{-1}(\xi_j(y^0) - \xi_j(y^{k+1}))$. □

Theorem 3.4. *If $\{y^k\}$ represents a sequence generated by Algorithm 1, then the accumulation point of $\{y^k\}$ is a POS for ξ for problem RDP.*

Proof. Let y^* denote an accumulation point of the sequence $\{y^k\}$, where $\{y^k\}$ is generated by Algorithm 1. Then, there exists a subsequence $\{y^{k_l}\}$ such that $\lim_{l\to\infty} y^{k_l} = y^*$. Observe that ξ_j is continuous. Utilizing Lemma 3.2, with $l = k_l$, we obtain $\sum_{l=0}^{\infty} \alpha^l |\Theta(y^l)| < \infty$. Therefore, $\lim_{k\to\infty} \alpha^k \Theta(y^k) = 0$. In particular, we have

$$\lim_{l\to\infty} \alpha^{k_l} \Theta(y^{k_l}) = 0. \tag{3.22}$$

If y^* is not a RCP, then

$$\Theta(y^*) < 0 \ \ \textit{and} \ \ d^* \neq 0. \tag{3.23}$$

Now, we define $g : \mathbb{R}^m \to \mathbb{R}$ by $g(v) = \max_{j\in\langle m\rangle} v_j$. Applying the definition of $\Theta(\cdot)$ and the fact that $d(y^*) = d^*$, we infer the existence of a nonnegative integer q such that $g(\xi(y^* + 2^{-q}d(y^*)) - \xi(y^*)) < \beta 2^{-q}\Theta(y^*)$. Since g(.) is continuous, then $\lim_{l\to\infty} d^{k_l} = d(y^*)$ and $\lim_{l\to\infty} \Theta(y^{k_l}) = \Theta(y^*)$. For l large enough, one has $g(\xi(y^{k_l} + 2^{-q}d^{k_l}) - \xi(y^{k_l})) < \sigma 2^{-q}\Theta(y^{k_l})$, which implies

$$\xi_j(y^{k_l} + 2^{-q}d^{k_l}) - \xi_j(y^{k_l}) < \sigma 2^{-q}\Theta(y^{k_l}). \tag{3.24}$$

From Step 4 of Algorithm 1, we have $\xi_j(y^{k_l} + \alpha_{k_l} v^{k_l}) \leq \xi_j(y^{k_l}) + \sigma\alpha_{k_l}\Theta(y^{k_l})$. Since t_{k_l} is the largest of $\{\frac{1}{2^n} : n = 1,2,3,....\}$, and $\sigma \in (0,1)$, we find by (3.24) that $\sigma\alpha_{k_l}\Theta(y^{k_l}) \leq \sigma 2^{-q}\Theta(y^{k_l})$, which implies $\alpha_{k_l} \geq 2^{-q} > 0$ for l large enough. Consequently, $\lim_{l\to\infty} \alpha^{k_l}\Theta(y^{k_l}) > 0$, which contradicts (3.22). Hence, the condition stated in (3.23) is false, indicating that y^* is a stationary point. By the strict convexity of ξ, it follows that y^* is a POS. □

Theorem 3.5. *Let $\{y^k\}$ be a sequence generated by Algorithm 1 and y^0 be in a compact level set of ξ. Then $\{y^k\}$ converges to a POS $y^* \in \mathbb{R}^n$.*

Proof. Let γ_0 be the $\xi(y^0)$-level set of ξ, i.e., $\gamma_0 = \{y \in \mathbb{R}^n : \xi(y) \leq \xi(y^0)\}$. Given that the sequence $\{\xi(y^k)\}$ is decreasing in $\mathbb{R}^m$, it follows that $y^k \in \gamma_0$ for all k. Consequently, $\{y^k\}$ is bounded, and all its accumulation points reside in γ_0. Utilizing the theorem above, we conclude that all the accumulation points are Pareto optima for ξ. Considering $y^k \in \gamma_0$, where $\xi(y^{k+1}) \leq \xi(y^k)$ for all k, [18, Lemma 3.8] asserts that, for any accumulation point y^* of $\{y^k\}$, $\xi(y^) \leq \xi(y^k)$ and $\lim_{k\to\infty} \xi(y^k) = \xi(y^*)$, with $\xi(y)$ being the constant in the set of accumulation points of $\{y^k\}$. As ξ is strongly convex, we see that there exists a unique accumulation point of $\{y^k\}$, denoted as y^*. Hence, the proof is complete. □

3.5. **Convergence rate of NDA (Algorithm 1.** In this section, we investigate the convergence rate of any infinite sequence generated by Algorithm 1. The analysis assumes a bound for $d(y)$, the descent direction for ξ at y, and provides a bound for $\Theta(y^{k+1})$ by using information from the previous iteration point y^k. The section demonstrates that, for sufficiently large k, full Newton's steps are taken, i.e., $\alpha_k = 1$. Using this result, it is proved that y^k converges to y^* with a superlinear rate. Finally, under additional assumptions, it is demonstrated that y^k converges to y^* with a quadratic rate.

Lemma 3.3. *Assume that, for any $k \geq 0$, there exists λ_{ij} such that $\sum\limits_{j\in\langle m\rangle}\sum\limits_{i\in\langle p\rangle}\lambda_{ij}^k = 1$. Then*

$$\Theta(y^{k+1}) \geq -\frac{1}{2\tau}\Big\|\sum_{j\in\langle m\rangle}\sum_{i\in\langle p\rangle}\lambda_{ij}^k(A_iy^k+q_j)\Big\|^2$$

where $\tau > 0$ is defined in Definition 2.3.

Proof. Observe that

$$\begin{aligned}\Theta(y^{k+1}) = \max_{j\in\langle m\rangle}\max_{i\in\langle p\rangle}\{&\frac{1}{2}(y^{k+1})^TA_iy^{k+1}+q_j^Ty^{k+1}+r_j+(A_iy^{k+1}+q_j)^Td(y^{k+1})\\&+\frac{1}{2}d(y^{k+1})^TA_id(y^{k+1})-\xi_j(y^{k+1})\}.\end{aligned}$$

Now, from (3.6), we have

$$\begin{aligned}\Theta(y^{k+1}) &= \sum_{j\in\langle m\rangle}\sum_{i\in\langle p\rangle}\lambda_{ij}^k\max_{j\in\langle m\rangle}\max_{i\in\langle p\rangle}\{\frac{1}{2}(y^{k+1})^TA_iy^{k+1}+q_j^Ty^{k+1}+r_j+(A_iy^{k+1}+q_j)^Td(y^{k+1})\\&+\frac{1}{2}d(y^{k+1})^TA_id(y^{k+1})-\xi_j(y^{k+1})\}\\&\geq\sum_{j\in\langle m\rangle}\sum_{i\in\langle p\rangle}\lambda_{ij}^k\{(A_iy^{k+1}+q_j)^Td(y^{k+1})+\frac{1}{2}d(y^{k+1})^TA_id(y^{k+1})\}\\&\geq\min_{d\in\mathbb{R}^n}\sum_{j\in\langle m\rangle}\sum_{i\in\langle p\rangle}\lambda_{ij}^k\{(A_iy^{k+1}+q_j)^Td+\frac{1}{2}d^TA_id\}.\end{aligned}$$

The definition of strong-convexity for $f_j(x,A_i)$ (Definition 2.3) yields that there exists a $\tau > 0$ such that

$$\Theta(y^{k+1}) \geq \min_{d\in\mathbb{R}^n}\sum_{j\in\langle m\rangle}\sum_{i\in\langle p\rangle}\lambda_{ij}^k\big((A_iy^{k+1}+q_j)^Td+\frac{\tau}{2}\|d\|^2\big). \tag{3.25}$$

Solving $\min\limits_{d\in\mathbb{R}^n}\sum\limits_{j\in\langle m\rangle}\sum\limits_{i\in\langle p\rangle}\lambda_{ij}^k\big((A_iy^{k+1}+q_j)^Td+\frac{\tau}{2}\|d\|^2\big)$, we obtain

$$\min_{d\in\mathbb{R}^n}\sum_{j\in\langle m\rangle}\sum_{i\in\langle p\rangle}\lambda_{ij}^k\big((A_iy^{k+1}+q_j)^Td+\frac{\tau}{2}\|d\|^2\big) = -\frac{1}{2\tau}\|\sum_{j\in\langle m\rangle}\sum_{i\in\langle p\rangle}\lambda_{ij}^k(A_iy^{k+1}+q_j)\|^2. \tag{3.26}$$

From (3.25) and (3.26), we have

$$\Theta(y^{k+1}) \geq -\frac{1}{2\tau}\|\sum_{j\in\langle m\rangle}\sum_{i\in\langle p\rangle}\lambda_{ij}^k(A_iy^{k+1}+q_j)\|^2. \tag{3.27}$$

Hence the lemma is proved. □

Theorem 3.6. *Assume that, for any $y \in \mathbb{R}^n$, $\|d(y)\|^2 \leq \frac{2}{\tau}\mid\Theta(y)\mid$, where $\tau > 0$ (given in Definition 2.3). Let y^0 be the initial point within a compact level-set of ξ, and the sequence generated by Algorithm 1 be as $\{y^k\}$. Then $\{y^k\}$ converges to a Pareto optimum point y^*. Furthermore, if $\alpha_k = 1$ for sufficiently large k, then the convergence rate of $\{y^k\}$ to y^* is superlinear.*

Proof. Given $\{y^k\}$, a sequence generated by Algorithm 1 with its initial point y^0 belonging to a compact level set of ξ, Theorem 3.5 guarantees the convergence of the sequence to y^*. Now, we aim to establish that $\{y^k\}$ converges to y^* with a superlinear rate. Since y^* is a RCP, according to the definition, there exists no descent direction at y^*, implying $d(y^*) = 0$. Since $\frac{1}{2}y^TA_iy + q_j^Ty + r_j$ is twice continuously differantiable for each $y \in \mathbb{R}^n$ and $i \in \langle p \rangle$. Thus

$$\begin{aligned}
&\frac{1}{2}(y^k + d^k)^T A_i(y^k + d^k) + q_j^T(y^k + d^k) + r_j \\
&\leq \frac{1}{2}y^{k^T}A_iy^k + q_j^Ty^k + r_j + (A_iy^k + q_j)^Td^k + \frac{1}{2}d^{k^T}A_id^k + \frac{\varepsilon}{2}\|d^k\|^2 \\
&\leq \max_{i\in\langle p\rangle}\{\frac{1}{2}y^{k^T}A_iy^k + q_j^Ty^k + r_j + (A_iy^k + q_j)^Td^k + \frac{1}{2}(d^k)^TA_i(d^k) + \frac{\varepsilon}{2}\|d^k\|^2\} \\
&\leq \max_{i\in A_i}\{\frac{1}{2}y^{k^T}A_iy^k + q_j^Ty^k + r_j\} \\
&+ \max_{j\in\langle m\rangle}\max_{i\in\langle p\rangle}\{\frac{1}{2}y^{k^T}A_iy^k + q_j^Ty^k + r_j + (A_iy^k + q_j)^Td^k + \frac{1}{2}(d^k)^TA_i(d^k) - \xi_j(y)\} + \frac{\varepsilon}{2}\|d^k\|^2.
\end{aligned}$$

Consequently, $\frac{1}{2}(y^k + d^k)^TA_i(y^k + d^k) + q_j^T(y^k + d^k) + r_j \leq \xi_j(y^k) + \Theta(y^k) + \frac{\varepsilon}{2}\|d^k\|^2$ holds for each A_i. Then, $\max_{i\in\langle p\rangle}\{\frac{1}{2}(y^k + d^k)^TA_i(y^k + d^k) + q_j^T(y^k + d^k) + r_j\} \leq \xi_j(y^k) + \Theta(y^k) + \frac{\varepsilon}{2}\|d^k\|^2$ and $\xi_j(y^k + d^k) - \xi_j(y^k) \leq \Theta(y^k) + \frac{\varepsilon}{2}\|d^k\|^2$. Note that

$$\begin{aligned}
\xi_j(y^k + d^k) - \xi_j(y^k) &\leq \sigma\Theta(y^k) + (1-\sigma)\Theta(y^k) + \frac{\varepsilon}{2}\|d^k\|^2 \\
&\leq \sigma\Theta(y^k) + (\varepsilon - \tau(1-\sigma))\frac{\|d^k\|^2}{2} \text{ for every } k \geq k_\varepsilon,
\end{aligned}$$

which inequality indicates that if $\varepsilon < \tau(1-\sigma)$, then, in Step 4 (Algorithm 1), $\alpha_k = 1$ is accepted for $k \geq k_\varepsilon$. For achieving superlinear convergence, we assume $\varepsilon < \tau(1-\sigma)$. Utilizing Taylor's first-order expansion of $\sum_{j\in\langle m\rangle}\sum_{i\in\langle p\rangle}\lambda_{ij}^k(A_iy^k + q_j)$ and (3.7), we conclude that, for any $k \geq k_\varepsilon$,

$$\begin{aligned}
&\|\sum_{j\in\langle m\rangle}\sum_{i\in\langle p\rangle}\lambda_{ij}^k(A_iy^k + q_j)\| \\
&\leq \|\sum_{j\in\langle m\rangle}\sum_{i\in\langle p\rangle}\lambda_{ij}^k(A_iy^k + q_j) - (\sum_{j\in\langle m\rangle}\sum_{i\in\langle p\rangle}\lambda_{ij}^k(A_iy^k + q_j) + \sum_{j\in\langle m\rangle}\sum_{i\in\langle p\rangle}\lambda_{ij}^kA_id^k)\| \\
&\leq \sum_{j\in\langle m\rangle}\sum_{i\in\langle p\rangle}\lambda_{ij}^k\|\varepsilon d^k\|.
\end{aligned}$$

Thus $\|\sum_{j\in\langle m\rangle}\sum\_i \in \langle p\rangle\lambda_{ij}^k(A_iy^k + q_j)\| \leq \varepsilon\|d^k\|$, which together with (3.27) yields $\Theta(y^{k+1}) \geq -\frac{1}{2\tau}(\varepsilon\|d^k\|)^2$, and $|\ \Theta(y^{k+1})\ | \leq \frac{1}{2\tau}(\varepsilon\|d^k\|)^2$. As $\alpha_k = 1$, we have $y^{k+1} = y^k + d^k$. It follows that $\|y^{k+1} - y^{k+2}\| = \|d^{k+1}\| \leq \frac{\varepsilon}{\tau}\|d^k\|$. If $k \geq k_\varepsilon$ and $j \geq 1$, then

$$\|y^{k+j} - y^{k+j+1}\| \leq \left(\frac{\varepsilon}{\tau}\right)^j\|y^k - y^{k+1}\|. \tag{3.28}$$

To establish the superlinear convergence rate, we consider $\zeta \in (0,1)$ and define

$$\varepsilon^* = \min\left\{1-\sigma, \frac{\zeta}{1+2\zeta}\right\}\tau.$$

If $\varepsilon < \varepsilon^*$ and $k \geq k_\varepsilon$, then we obtain by (3.28) and the convergence of $\{y^k\}$ (to y^*) that

$$\|y^* - y^{k+1}\| \leq \sum_{j=1}^{\infty} \|y^{k+j} - y^{k+j+1}\| \leq \sum_{j=1}^{\infty} \left(\frac{\zeta}{1+2\zeta}\right)^j \|y^k - y^{k+1}\|.$$

Thus $\|y^* - y^{k+1}\| \leq \frac{\zeta}{1+\zeta}\|y^k - y^{k+1}\|$ and

$$\|y^* - y^k\| \geq \|y^k - y^{k+1}\| - \|y^{k+1} - y^*\| \geq \|y^k - y^{k+1}\| - \frac{\zeta}{1+\zeta}\|y^k - y^{k+1}\|$$

which implies $\|y^* - y^k\| \geq \frac{1}{1+\zeta}\|y^k - y^{k+1}\|$. If $\varepsilon < \varepsilon^*$ and $k \geq k_\varepsilon$, then $\|y^* - y^{k+1}\| \leq \zeta\|y^* - y^k\|$, and $\frac{\|y^* - y^{k+1}\|}{\|y^* - y^k\|} \leq \zeta$. As ζ varies arbitrarily in $(0,1)$, we see that the quotient approaches zero. This concludes the proof. □

The following theorem establishes the quadratic convergence of the sequence generated by Algorithm 1.

Theorem 3.7. *Let $y^0 \in \mathbb{R}^n$ be the initial point in a compact level set of ξ, and $\{y^k\}$ be a sequence generated by Algorithm 1. If the second derivative of $\frac{1}{2}y^T A_i y + q_j^T y + r_j$ is Lipschitz continuous on $\mathbb{R}^n$ for each $i \in \langle p \rangle$ and $j \in \langle m \rangle$, then $\{y^k\}$ converges to y^* with a quadratic rate.*

Proof. Since the second derivative of $\frac{1}{2}y^T A_i y + q_j^T y + r_j$ is Lipschitz continuous on $\mathbb{R}^n$ for each $i \in \langle p \rangle$ and $j \in \langle m \rangle$, it follows that $\sum_{j\in\langle m\rangle}\sum_{i\in\langle p\rangle} \lambda_{ij}^k A_i$ is also Lipschitz continuous. If $\alpha_k = 1$ for sufficiently large k, then $y^{k+1} - y^k = d^k$,

$$\Big\|\sum_{j\in\langle m\rangle}\sum_{i\in\langle p\rangle} \lambda_{ij}^k (Ay + q_j)\Big\| \leq \sum_{j\in\langle m\rangle}\sum_{i\in\langle p\rangle} \lambda_{ij}^k \frac{\bar{L}}{2}\|d^k\|^2 \leq \frac{\bar{L}}{2}\|d^k\|^2,$$

where $\bar{L}$ is Lipschitz constant. Thus $\sum_{j\in\langle m\rangle}\sum_{i\in\langle p\rangle} \lambda_{ij}^k (Ay + q_j) \leq \frac{\bar{L}}{2}\|d^k\|^2$. According to Lemma 3.3, it follows that $|\,\Theta(y^{k+1})\,| \leq \frac{1}{2\tau}(\frac{\bar{L}}{2}\|d^k\|^2)^2$. Observe that $\|d(y^{k+1})\|^2 \leq \frac{2}{\tau}\,|\,\Theta(y^{k+1})\,|$. Hence, we have $\|d(y^{k+1})\| \leq \frac{\bar{L}}{2\tau}\|d^k\|^2$ for k large enough. Since y^k converges to y^* with a superlinear rate, there exists k_0 such that, for $k \geq k_0$, $\|y^* - y^{k+1}\| \leq \zeta\|y^* - y^k\|$. We also have

$$(1-\zeta)\|y^* - y^s\| \leq \|y^s - y^{s+1}\|. \tag{3.29}$$

and

$$\|y^s - y^{s+1}\| \leq (1+\zeta)\|y^* - y^s\|. \tag{3.30}$$

They imply that $(1-\zeta)\|y^* - y^s\| \leq \|y^s - y^{s+1}\| \leq (1+\zeta)\|y^* - y^s\|$. For $s = k \geq k_0$, by (3.30), we have $\|y^k - y^{k+1}\| \leq (1+\zeta)\|y^* - y^k\|$. Using (3.29) for $s = k+1$, we have $(1-\zeta)\|y^* - y^{k+1}\| \leq \|y^{k+1} - y^{k+2}\| = \|d^{k+1}\|$. Thus $(1-\zeta)\|y^* - y^{k+1}\| \leq \frac{\bar{L}}{\sqrt{2}\tau}(1+\zeta)^2\|y^* - y^k\|^2$. Since $\zeta \in (0,1)$ is arbitrary, one sees that $\{y^k\}$ converges to y^* quadratically. □

3.6. **Numerical examples.** We now apply Algorithm 1 to a number of numerical examples (Test problems given in Table 1 and explicit forms of each test problems are provided in Appendix A) and then compare with the weighted sum method by using an approximation of the non dominated Pareto front. We developed a *Python* code for Algorithm 1 and solved the subproblem P(y)by using *cvxopt.solvers.qp*. Since the subpproblem is a convex quadratic programming problem, it can be solved by using this solver. In our computations, we set the stopping criteria as $\|d^k\| < 10^{-4}$ or $\alpha_k < 10^{-5}$ or maximum 5000 number of iterations.

To obtain a set of efficient (Pareto optimal) solutions for a multiobjective function, we employ the multi-start technique to generate a well-distributed approximation of the Pareto front. Initially, we select 100 uniformly distributed random points within the range defined by lb and ub (where lb and ub are vectors in $\mathbb{R}^n$, and $lb < ub$), and then apply Algorithm 1 separately to each point. The resultant non-dominated set of RCPs derived from this process forms an approximate Pareto front. To assess the effectiveness of this approach, we compare the approximate Pareto fronts obtained via Algorithm 1 with those acquired through the weighted sum method across various examples.

In the weighted sum method (WSM), we address the following single-objective optimization problem

$$\min_{x\in\mathbb{R}^n} \left(w_1\xi_1(x) + w_2\xi_2(x) + \cdots + w_m\xi_m(x)\right),$$

where $(w_1, w_2, \ldots, w_m) = w > 0$, by using the technique developed in [5] with initial approximation $x^0 = \frac{1}{2}(lb + ub)$. For bi-objective optimization problems, we considered weights $(1,0)$ and $(0,1)$, along with 98 random weights uniformly distributed in the square area $[0,1]\times[0,1]$. For three-objective optimization problems, we included weights $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, along with 97 random weights uniformly distributed in the cubic area $[0,1]\times[0,1]\times[0,1]$. In the state of P(y), we solved the following subproblem in the provided examples:

$$\begin{aligned}
&P(y): \min_{d,t}\ t\\
&\text{s. t. } \frac{1}{2}y^T A_i y + q_j^T y + r_j + (Ay + q_j)^T d + \frac{1}{2}d^T A_i d - \xi_j(y) \le t, \quad \forall i \in \langle p\rangle,\ j \in \langle m\rangle\\
&lb \le y + d \le ub.
\end{aligned}$$

In the subsequent example, we elucidate the steps of Algorithm 1 by utilizing a single initial approximation.

Example 3.1. (Bi-objective quadratic two-dimensional problem with an uncertainty set of two elements) Consider the UQMOP $Q(S) = \{Q(A_i) : i \in \langle p\rangle\}$ such that $Q(A_i) : \min_{y\in\mathbb{R}^2} f(y, A_i)$, where $f : \mathbb{R}^2 \times S \to \mathbb{R}^2$, $S = \left\{A_1 = \begin{pmatrix} 2 & 1\\ 1 & 2\end{pmatrix}, A_2 = \begin{pmatrix} 4 & 0\\ 0 & 2\end{pmatrix}\right\}$, $\langle p\rangle = \{1,2\}$, $f(y,A_i) = \begin{bmatrix} f_1(y,A_i)\\ f_2(y,A_i)\end{bmatrix}$, $f_j(y,A_i) = \frac{1}{2}y^T A_i y + q_j^T y + r_j$, where $q_1 = \begin{pmatrix}3\\0\end{pmatrix}$, $q_2 = \begin{pmatrix}0\\4\end{pmatrix}$, $r_1 = 2$ and $r_2 = 3$. The OWWC type robust counterpart to Q(S) is given by $RDP : \min_{y\in\mathbb{R}^n} \xi(y)$ such that $\xi(y) = \begin{bmatrix}\xi_1(y)\\ \xi_2(y)\end{bmatrix}$ and $\xi_j(y) = \max_{i\in\langle p\rangle}\{\frac{1}{2}y^T A_i y + q_j^T y + r_j\}$, $\ j = 1,\ 2$. Consider $y^0 = (2.42023589, 0.77521806)^T$. Then, $f_1(y^0, A_1) = 18.53352838$, $f_1(y^0, A_2) = 21.5767543$, $f_2(y^0, A_1) = 15.37369295$, and $f_2(y^0, A_2) = 18.41691886$. $\xi_1(y^0) = 21.5767543$, $\xi_2(y^0) = 18.41691886$ and $\xi(y^0) = (21.5767543, 18.41691886)^T$. Observe that $S_1(y^0) = \{A_2\}$ and $S_2(y^0) = \{A_2\}$. The solution of $P(y^0)$ is obtained as $d^0 = (-2.39793688,\ -1.79841356)^T$, and $t^0 = -15.45325308$. We see that $\alpha_0 = 1$ satisfies (3.20). Therefore, $y^1 = y^0 + \alpha_0 d^0 = (0.02229902, -1.0231955)^T$. One can observe that $\xi(y^1) = (3.11482057,\ -0.04485847)^T < \xi(y^0)$. Using the stopping criterion $\|d^k\| < 10^{-4}$, the final solution is obtained as $y^* = (0.02227754,\ -1.0232116)$ after 3 iterations. Certainly, y^* serves as an approximate RCP for RDP.

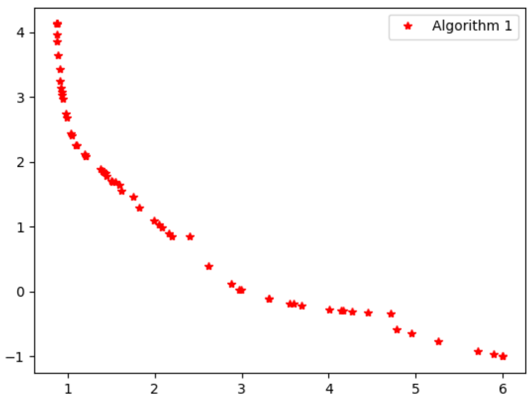
Next, we prove that $y^* = (0.02227754, -1.0232116)$ is a week POS of RDP. Observe that $f_1(y^*, A_1) = 3.08009894$, $f_1(y^*, A_2) = 3.11478718$ and $f_2(y^*, A_1) = -0.0795801$, $f_2(y^*, A_2) = -0.04489185$ and $\xi(y^*) = (3.11478718, -0.04489185)$. These imply $S_1(y^*) = \{A_2\}$ and $S_2(y^*) = \{A_2\}$. Hence,

$$\partial \xi_1(y^*) = conv\{(3.08911016, -2.0464232)^T\}$$

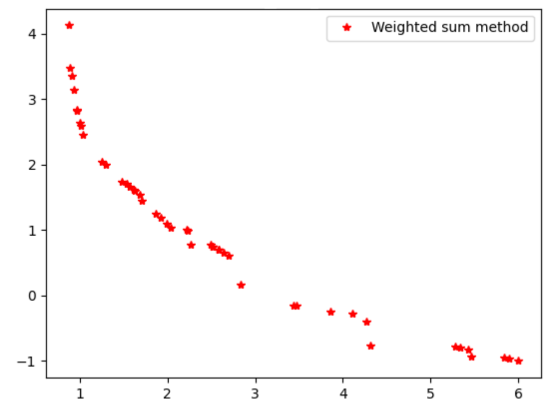
and

$$\partial \xi_2(y^*) = conv\{(0.08911016, 1.9535768)^T\}.$$

Therefore, 0 approximately lies within the convex combination of $\partial \xi_1(y^*)$ and $\partial \xi_2(y^*)$. According to Theorem 3.1, y^* qualifies as a WPOS for $\xi(y)$. Figure 1 illustrates a comparison of the



(A) Estimate the Pareto front of Example 3.1 (**TPN1**) utilizing NDM (Algorithm 1).



(B) Estimate the Pareto front of Example 3.1 (**TPN1**) utilizing WSM.

FIGURE 1. Depiction of approximate Pareto fronts for Example 3.1 (**TPN1**).

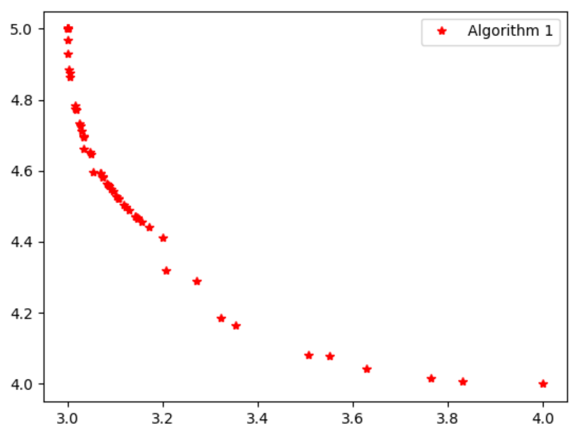
approximate Pareto front generated by 'NDM' with 'WSM' for Example 3.1 (**TPN1**).

Test Problems. A set of test problems including above Example 3.1 is constructed and executed by using both the methods. Details of these test problems are given in Table 1. In this table, m and n represent number of objective functions and dimension of decision variable (x). Also, the convex and non-convex problems are denoted by 'Y' and 'N', respectively in the following table. One can observe that we considered 2 or 3 objective 1, 2, and 3 dimensional problems in our execution. Explicit forms of each test problems are provided in Appendix A.

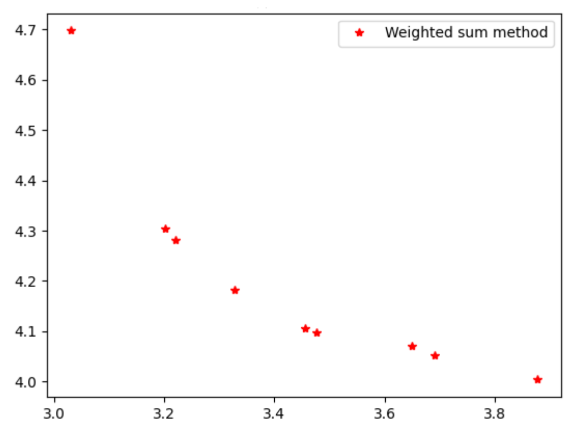
The functions in Test problems **TPN1** (Example 3.1), **TPN2**, **TPN3**, **TPN4**, **TPN5**, **TPN6**, and **TPN7** are convex due to the consideration of a parameter uncertainty set containing symmetric positive definite matrices. Algorithm 1 can generate good approximation Pareto fronts for these examples, as depicted in the corresponding figures. Also, **TPN8**, **TPN9**, and **TPN10** have nonconvex objective functions because their uncertainty sets may not include positive definite or semi-positive definite matrices. In these cases, the WSM fails to produce approximation Pareto fronts, while Algorithm 1 succeeds. Algorithm 1 eliminates the difficulty of choosing pre-order weights in the WSM, enabling us to find good approximate Pareto fronts without pre-order weight selection. Also, Theorem 2.1 shows that the POS to RDP is the robust POS to Q(S). Our primary goal is accomplished by utilizing RDP and Algorithm 1 to discover the approximate

TABLE 1. Details of test problems.

Problem	(m,n,p)	lb^T	ub^T	Convex
TPN1	(2,2,2)	$(-4,-4)^T$	$(4,4)^T$	Y
TPN2	(2,2,2)	$(-3,-4)^T$	$(4,4)^T$	Y
TPN3	(3,3,3)	$(1,-2,0)^T$	$(3.5,2,1)^T$	Y
TPN4	(3,3,3)	$(-1,-3,-1)^T$	$(3.5,3,1)^T$	Y
TPN5	(2,2,2)	$(-5,-5)^T$	$(4,4)^T$	Y
TPN6	(3,3,3)	$(1,-2,0)^T$	$(4,2,1)^T$	Y
TPN7	(3,3,3)	$(-1,-2,0)^T$	$(3.5,3,1)^T$	Y
TPN8	(3,2,3)	-5	5	N
TPN9	(2,1,2)	$(-1,-1)^T$	$(0,0)^T$	N
TPN10	(2,2,2)	$(-2,-2)^T$	$(5,5)^T$	N



(A) Estimate the Pareto front of **TPN2** utilizing NDM (Algorithm 1).



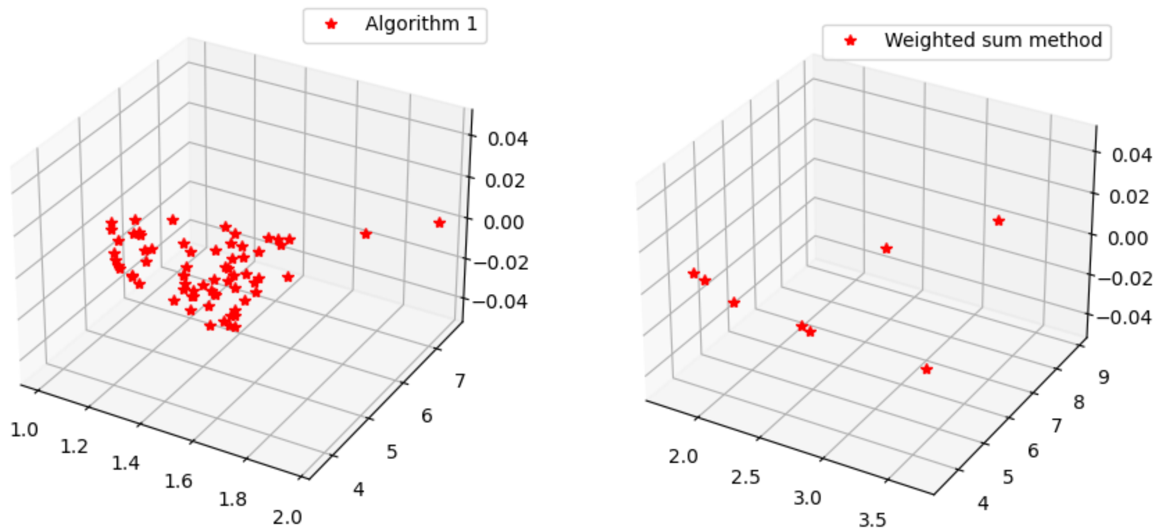
(B) Estimate the Pareto front of **TPN2** utilizing WSM.

FIGURE 2. Comparison of approximate Pareto fronts for **TPN2**.

Pareto front for the multi-objective optimization problem Q(S) with finite parameter uncertainty set.

4. CONCLUSIONS

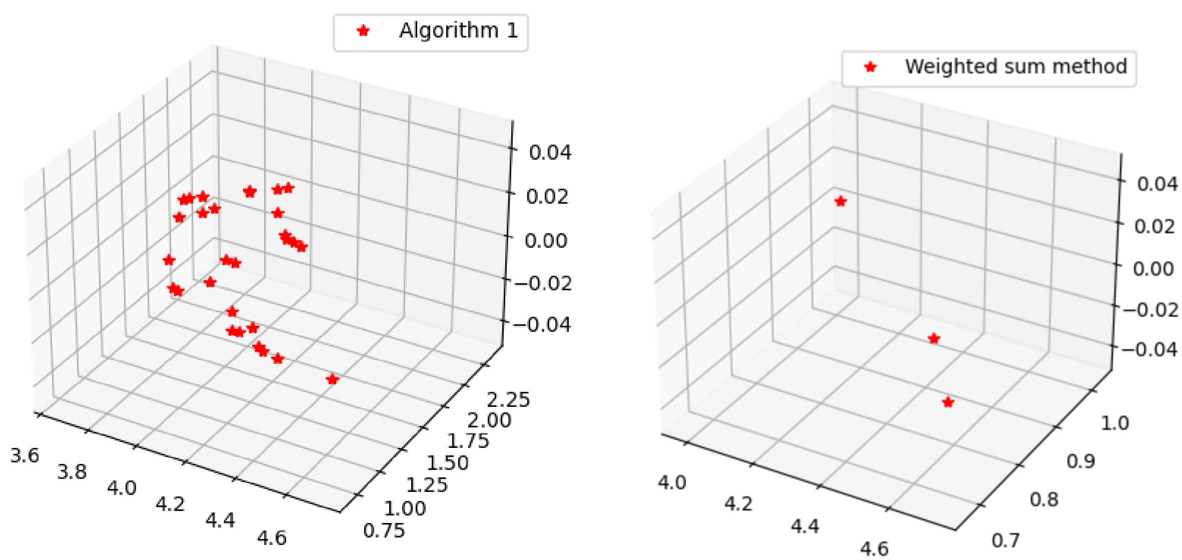
We presented a solution to the unconstrained uncertain quadratic multiobjective optimization problem, namely Q(S) (defined in 2.1). To solve Q(S), we used the objective-wise worst-case cost type robust counterpart, RDP (defined in (2.2)). As stated in Theorem 2.1, the POS of RDP is the robust POS to Q(S), so solving RDP rather than Q(S) was easier. RDP, on the other hand, is a deterministic nonsmooth quadratic multiobjective optimization problem. Our goal was to find a RCP for RDP that is both a POS for RDP and a robust POS for $Q(S)$. To achieve this, we developed a NDA (Algorithm 1) for RDP, which involves solving a subproblem to obtain the descent direction and implementing an Armijo-type line search technique. By using this algorithm, we obtained a sequence that converges to a RCP, and under certain assumptions, we proved that



(A) Estimate the Pareto front of **TPN3** utilizing NDM (Algorithm 1).

(B) Estimate the Pareto front of **TPN3** utilizing WSM.

FIGURE 3. Depiction of approximate Pareto fronts for **TPN3**.

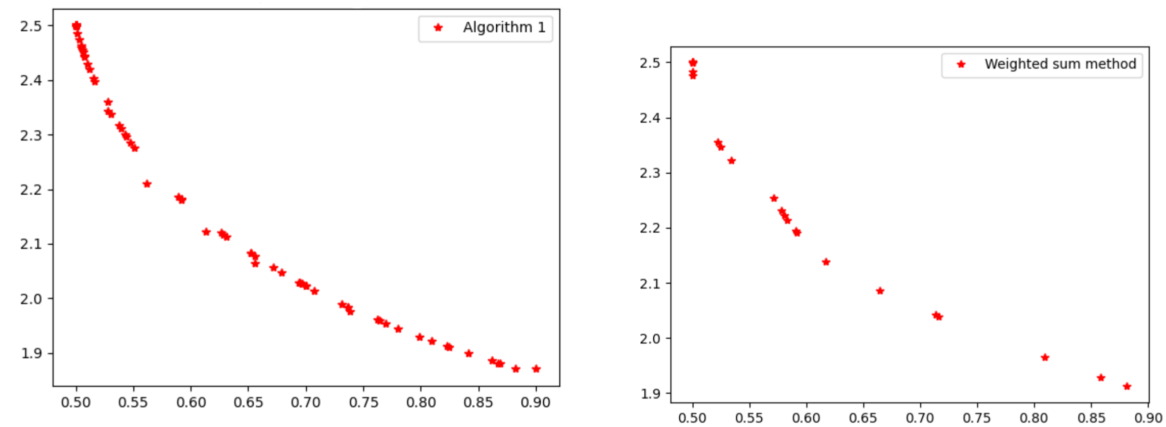


(A) Estimate the Pareto front of **TPN4** utilizing NDM (Algorithm 1).

(B) Estimate the Pareto front of **TPN4** utilizing WSM.

FIGURE 4. Depiction of approximate Pareto fronts for **TPN4**.

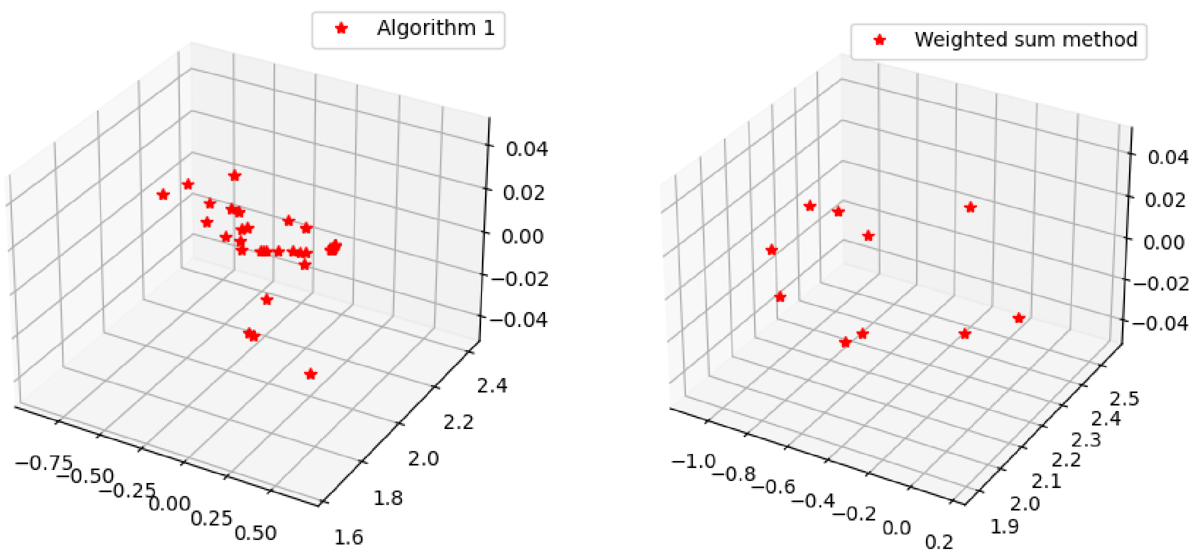
the convergence rate is superlinear or quadratic. At the end of the paper, we also provided some test problems to verify the effectiveness of NDA, which shows that our algorithm can generate good approximate Pareto fronts without the need for predetermined weighting factors or any other kind of predetermined ranking or ordering information for objective functions. Therefore,



(A) Estimate the Pareto front of **TPN5** utilizing NDM (Algorithm 1).

(B) Estimate the Pareto front of **TPN5** utilizing WSM.

FIGURE 5. Depiction of approximate Pareto fronts for **TPN5**.

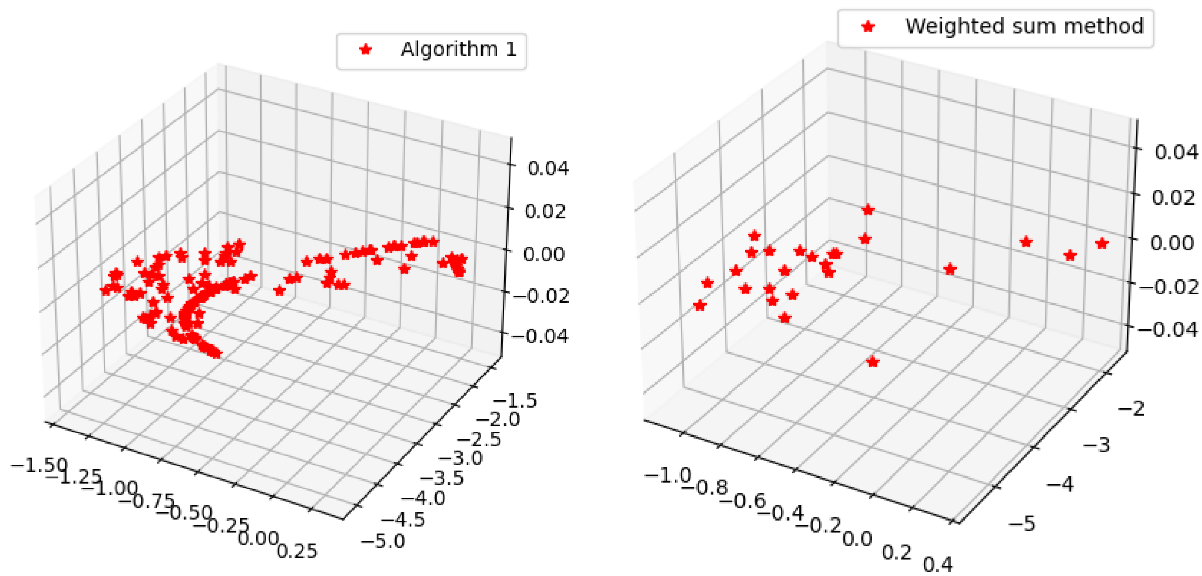


(A) Estimate the Pareto front of **TPN6** utilizing NDM (Algorithm 1).

(B) Estimate the Pareto front of **TPN6** utilizing WSM.

FIGURE 6. Depiction of approximate Pareto fronts for **TPN6**.

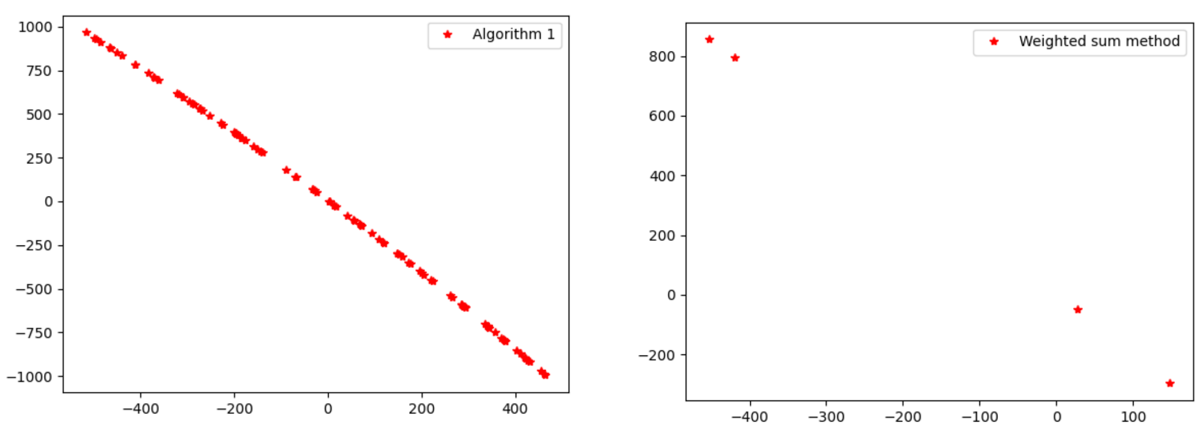
the difficulty of choosing the pre-order weight in the WSM can be eliminated with the use of our algorithm. Future work will be focused on the comparison of the spreading of approximate Pareto fronts by using performance profiles based on factors such as the Delta-spread metric and hypervolume metrics. In addition, we will justify computational efficiency by using the performance profiles based on the number of iterations and function evaluations.



(A) Estimate the Pareto front of **TPN7** utilizing NDM (Algorithm 1).

(B) Estimate the Pareto front of **TPN7** utilizing WSM.

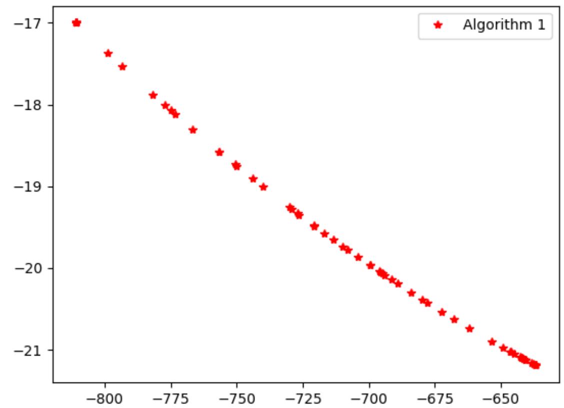
FIGURE 7. Depiction of approximate Pareto fronts for **TPN7**.



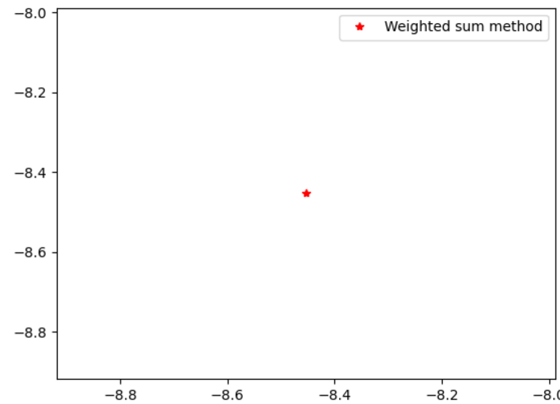
(A) Estimate the Pareto front of **TPN8** utilizing NDM (Algorithm 1).

(B) Estimate the Pareto front of **TPN8** utilizing WSM.

FIGURE 8. Depiction of approximate Pareto fronts for **TPN8**.

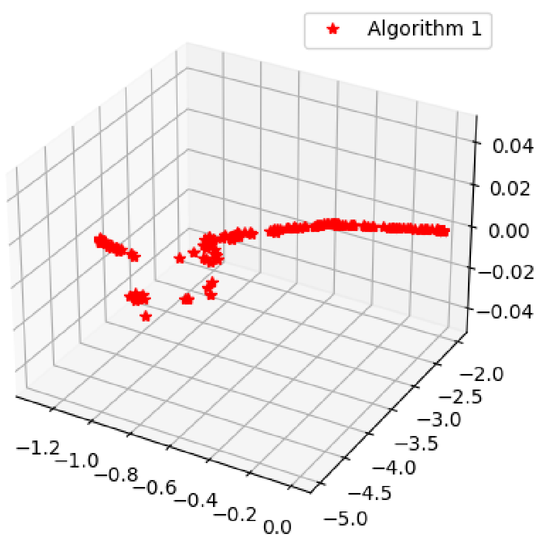


(A) Estimate the Pareto front of **TPN9** utilizing NDM (Algorithm 1).

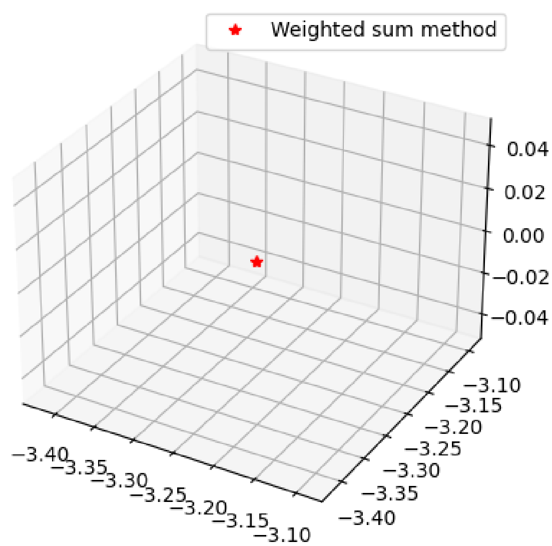


(B) Estimate the Pareto front of **TPN9** utilizing WSM.

FIGURE 9. Depiction of approximate Pareto fronts for **TPN9**.



(A) Estimate the Pareto front of **TPN10** utilizing NDM (Algorithm 1).



(B) Estimate the Pareto front of **TPN10** utilizing WSM.

FIGURE 10. Depiction of approximate Pareto fronts for **TPN10**.

Acknowledgments

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APPENDIX A. DETAILS OF TEST PROBLEMS

Appendix A: Details of test problems

(TPN1) UQMOP: $Q(S) = \{Q(A_i) : A_i \in S\}$ such that $Q(A_i) : \min_{x \in \mathbb{R}^2} f(x, A_i)$, where $f : \mathbb{R}^2 \times S \to \mathbb{R}^2$, $A_i \in S = \left\{ A_1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},\ A_2 = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \right\}$, $f(x, A_i) = \begin{bmatrix} f_1(x, A_i) \\ f_2(x, A_i) \end{bmatrix}$, where $f_j(x, A_i) = \frac{1}{2} x^T A_i x + q_j^T x + r_j,\ j = 1, 2,\ q_1 = \begin{pmatrix} 3 \\ 0 \end{pmatrix},\ q_2 = \begin{pmatrix} 0 \\ 4 \end{pmatrix},\ r_1 = 2$ and $r_2 = 3$. OWWC type robust counterpart to Q(S) is as follows: $RDP : \min_{x \in \mathbb{R}^n} \xi(x)$, where $\xi(x) = \begin{bmatrix} \xi_1(x) \\ \xi_2(x) \end{bmatrix}$ and $\xi_j(x) = \max_{A_i \in S} \{\frac{1}{2} x^T A_i x + q_j^T x + r_j\},\ j = 1,\ 2$.

(TPN2) UQMOP: $Q(S) = \{Q(A_i) : A_i \in S\}$ such that $Q(A_i) : \min_{x \in \mathbb{R}^2} f(x, A_i)$, where $f : \mathbb{R}^2 \times S \to \mathbb{R}^2$, $S = \left\{ A_1 = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix},\ A_2 = \begin{pmatrix} 2 & 3 \\ 3 & 8 \end{pmatrix} \right\}$, $f(x, A) = \begin{bmatrix} f_1(x, A) \\ f_2(x, A) \end{bmatrix}$, where $f_j(x, A_i) = \frac{1}{2} x^T A_i x + q_j^T x + r_j,\ j = 1, 2,\ q_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix},\ q_2 = \begin{pmatrix} 4 \\ 6 \end{pmatrix},\ r_1 = 4$ and $r_2 = 8$. OWWC type robust counterpart to Q(S) is as $RDP : \min_{x \in \mathbb{R}^2} \xi(x)$, where $\xi(x) = \begin{bmatrix} \xi_1(x) \\ \xi_2(x) \end{bmatrix}$ and $\xi_j(x) = \max_{A_i \in S} \{\frac{1}{2} x^T A x + q_j^T x + r_j\},\ j = 1,\ 2$.

(TPN3) UQMOP: $Q(S) = \{Q(A_i) : A_i \in S\}$ such that $Q(A_i) : \min_{x \in \mathbb{R}^3} f(x, A_i)$, where $f : \mathbb{R}^3 \times S \to \mathbb{R}^3$, and $S = \left\{ A_1 = \begin{pmatrix} 4 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 3 \end{pmatrix},\ A_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix},\ A_3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix} \right\}$, $f(x, A) = \begin{bmatrix} f_1(x, A_i) \\ f_2(x, A_i) \\ f_3(x, A_i) \end{bmatrix}$, where $f_j(x, A_i) = \frac{1}{2} x^T A_i x + q_j^T x + r_j,\ j = 1, 2, 3,\ q_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},\ q_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, $q_3 = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix},\ r_1 = 3,\ r_1 = 4$ and $r_2 = 5$. OWWC type robust counterpart to Q(S) is as $RDP : \min_{x \in \mathbb{R}^3} \xi(x)$, where $\xi(x) = \begin{bmatrix} \xi_1(x) \\ \xi_2(x) \\ \xi_3(x) \end{bmatrix}$ and $\xi_j(x) = \max_{A_i \in S} \{\frac{1}{2} x^T A_i x + q_j^T x + r_j\},\ \ j = 1,\ 2,\ 3$.

(TPN4) UQMOP: $Q(S) = \{Q(A_i) : A_i \in S\}$ such that $Q(A_i) : \min_{x \in \mathbb{R}^3} f(x, A_i)$, where $f : \mathbb{R}^3 \times S \to \mathbb{R}^3$,

$$S = \left\{ A_1 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 7 \end{pmatrix},\ A_2 = \begin{pmatrix} 8 & 0 & 4 \\ 0 & 2 & 3 \\ 0 & 3 & 5 \end{pmatrix},\ A_3 = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 10 \end{pmatrix} \right\},\ f(x, A_i) = \begin{bmatrix} f_1(x, A_i) \\ f_2(x, A_i) \\ f_3(x, A_i) \end{bmatrix},$$

where $f_j(x, A_i) = \frac{1}{2} x^T A_i x + q_j^T x + r_j\ q_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix},\ q_2 = 2q_1,\ q_3 = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix},\ r_1 = 5,\ r_2 =$

6, $r_3 = 7$. OWWC type robust counterpart to Q(S) is given by $RDP : \min_{x\in\mathbb{R}^3} \xi(x)$, where $\xi(x) = \begin{bmatrix}\xi_1(x)\\ \xi_2(x)\\ \xi_3(x)\end{bmatrix}$ and $\xi_j(x) = \max_{i\in\langle p\rangle} \frac{1}{2}x^T A_i x + q_j^T x + r_j$, $j = 1, 2, 3$.

(TPN5) UQMOP: $Q(S) = \{Q(A_i) : A_i \in S\}$ such that $Q(A_i) : \min_{x\in\mathbb{R}^2} f(x,A_i)$, where $f : \mathbb{R}^2 \times S \to \mathbb{R}^2$, $S = \left\{A_1 = \begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}, A_2 = \begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\right\}$, $f(x,A_i) = \begin{bmatrix}f_1(x,A_i)\\ f_2(x,A_i)\end{bmatrix}$, $f_j(x,A_i) = \frac{1}{2}x^T A_i x + q_j^T x + r_j$ $j = 1,2$, where $q_1 = \begin{pmatrix}1\\ 0\end{pmatrix}$,

$q_2 = \begin{pmatrix}0\\ 1\end{pmatrix}$, $r_1 = 1$ and $r_2 = 2$. OWWC type robust counterpart to Q(S) is given by $RDP : \min_{x\in\mathbb{R}^2} \xi(x)$,

$$\text{where } \xi(x) = \begin{bmatrix}\xi_1(x)\\ \xi_2(x)\end{bmatrix} \text{ and } \xi_j(x) = \max_{A_i\in S}\{\frac{1}{2}x^T A_i x + q_j^T x + r_j\}, \quad j = 1, 2.$$

(TPN6) UQMOP: $Q(S) = \{Q(A_i) : A_i \in S\}$ such that $Q(A_i) : \min\limits_{x\in\mathbb{R}^3} f(x,A_i)$, where $f : \mathbb{R}^3 \times S \to \mathbb{R}^3$, $S = \left\{A_1 = \begin{pmatrix}1 & 0 & 1\\ 0 & 2 & 0\\ 1 & 0 & 3\end{pmatrix}, A_2 = \begin{pmatrix}1 & 1 & 0\\ 1 & 3 & 0\\ 0 & 0 & 2\end{pmatrix}, A_3 = \begin{pmatrix}8 & 0 & 0\\ 0 & 4 & 1\\ 0 & 1 & 1\end{pmatrix}\right\}$, $f(x,A_i) = \begin{bmatrix}f_1(x,A_i)\\ f_2(x,A_i)\\ f_3(x,A_i)\end{bmatrix}$, where $f_j(x,A_i) = \frac{1}{2}x^T A_i x + q_j^T x + r_j$, $q_1 = \begin{pmatrix}1\\ 2\\ 3\end{pmatrix}$, $q_2 = \begin{pmatrix}2\\ 1\\ 0\end{pmatrix}$, $q_3 = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$, $r_1 = 1$, $r_2 = 2$ and $r_3 = 3$. OWWC type robust counterpart to Q(S) is given by

$RDP : \min_{x\in\mathbb{R}^3} \xi(x)$, where $\xi(x) = \begin{bmatrix}\xi_1(x)\\ \xi_2(x)\\ \xi_3(x)\end{bmatrix}$ and $\xi_j(x) = \max_{A_i\in S} \frac{1}{2}x^T A_i x + q_j^T x + r_j$, $j = 1, 2, 3$.

(TPN7) UQMOP: $Q(S) = \{Q(A_i) : A_i \in S\}$ such that $Q(A_i) : \min_{x\in\mathbb{R}^3} f(x,A_i)$. $f : \mathbb{R}^3 \times S \to \mathbb{R}^3$, $S = \left\{A_1 = \begin{pmatrix}1 & 1 & 2\\ 1 & 4 & 1\\ 2 & 1 & 5\end{pmatrix}, A_2 = \begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 0\\ 1 & 0 & 3\end{pmatrix}, A_3 = \begin{pmatrix}1 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 1\end{pmatrix}\right\}$. $f(x,A_i) = \begin{bmatrix}f_1(x,A_i)\\ f_2(x,A_i)\\ f_3(x,A_i)\end{bmatrix}$, where $f_j(x,A_i) = \frac{1}{2}x^T A_i x + q_j^T x + r_j$, $j = 1, 2, 3$, $q_1 = \begin{pmatrix}1\\ 2\\ 3\end{pmatrix}$, $q_2 = 2q_1$, $q_3 = \begin{pmatrix}2\\ 0\\ 1\end{pmatrix}$, $r_1 = r_2 = r_3 = 0$. OWWC type robust counterpart to Q(S) is given by $RDP : \min_{x\in\mathbb{R}^3} \xi(x)$, where $\xi(x) = \begin{bmatrix}\xi_1(x)\\ \xi_2(x)\\ \xi_3(x)\end{bmatrix}$ and $\xi_j(x) = \max\limits_{A_i\in S} \frac{1}{2}x^T A_i x + q_j^T x + r_j$, $j = 1, 2, 3$.

(TPN8) UQMOP: $Q(S) = \{Q(A_i) : A_i \in S\}$ such that $Q(A_i) : \min\limits_{x\in\mathbb{R}^2} f(x,A_i)$, where $f : \mathbb{R}^2 \times S \to \mathbb{R}$, $A_i \in S = \{A_1 = \begin{pmatrix}-2\end{pmatrix}, A_2 = \begin{pmatrix}-4\end{pmatrix}\}$, $f(x,A_i) = \begin{bmatrix}f_1(x,A_i)\\ f_2(x,A_i)\end{bmatrix}$, where $f_j(x,A_i) = \frac{1}{2}x^T A_i x +$

$q_j^T x + r_j$ $j = 1,2$, $q_1 = \begin{pmatrix}100\end{pmatrix}$, $q_2 = \begin{pmatrix}-200\end{pmatrix}$, $r_1 = 1$ and $r_2 = 4$. OWWC type robust counterpart to Q(S) is given by $RDP : \min_{x\in\mathbb{R}^n} \xi(x)$, where $\xi(x) = \begin{bmatrix}\xi_1(x)\\ \xi_2(x)\end{bmatrix}$ i.e., $\xi_j(x) = \max_{A_i\in S} \frac{1}{2}x^T A_i x + q_j^T x + r_j,\ \ j = 1,\ 2.$

(TPN9) UQMOP: $Q(S) = \{Q(A_i) : A_i \in S\}$ such that $Q(A_i) : \min_{x\in\mathbb{R}^2} f(x,A_i)$, where $f : \mathbb{R}^2 \times S \to \mathbb{R}^2$, $S = \left\{A_1 = \begin{pmatrix}1 & 1\\ 1 & -2\end{pmatrix},\ A_2 = \begin{pmatrix}-2 & 1\\ 1 & 3\end{pmatrix}\right\}$,

$$f(x,A) = \begin{bmatrix}f_1(x,A_i)\\ f_2(x,A_i)\end{bmatrix} = \begin{bmatrix}\frac{1}{2}x^T A_i x + q_1^T x + r_1\\ \frac{1}{2}x^T P_x + q_2^T x + r_2\end{bmatrix}$$

where $q_1 = \begin{pmatrix}1\\ 2\end{pmatrix}, q_2 = \begin{pmatrix}3\\ 2\end{pmatrix}, r_1 = 1$ and $r_2 = 3$. OWWC type robust counterpart to Q(S) is given by $RDP : \min_{x\in\mathbb{R}^n} \xi(x)$, where $\xi(x) = \begin{bmatrix}\xi_1(x)\\ \xi_2(x)\end{bmatrix}$, and $\xi_j(x) = \max_{A_i\in S} \frac{1}{2}x^T A_i x + q_j^T x + r_j,\ j = 1,\ 2.$

(TPN10) UQMOP: $Q(S) = \{Q(A_i) : A_i \in S\}$ such that $Q(A_i) : \min_{x\in\mathbb{R}^3} f(x,A_i)$, where $f : \mathbb{R}^3 \times S \to \mathbb{R}^3,\ S = \left\{A_1 = \begin{pmatrix}-1 & 0 & 1\\ 0 & 2 & 0\\ 1 & 0 & 3\end{pmatrix}, A_2 = \begin{pmatrix}1 & 1 & 0\\ 1 & 3 & 0\\ 0 & 0 & -2\end{pmatrix}\right\},\ f(x,A_i) = \begin{bmatrix}f_1(x,A_i)\\ f_2(x,A_i)\\ f_3(x,A_i)\end{bmatrix}$, $f_j(x,A_i) = \frac{1}{2}x^T A_i x + q_j^T x + r_j, j = 1,2,3,\ q_1 = \begin{pmatrix}1\\ 2\\ 3\end{pmatrix},\ q_2 = \begin{pmatrix}2\\ 1\\ 0\end{pmatrix},\ q_3 = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix},\ r_1 = 1,\ r_2 = 2$ and $r_2 = 3$. OWWC type robust counterpart to Q(S) is given by $RDP :$ $\min_{x\in\mathbb{R}^3} \xi(x)$, where $\xi(x) = \begin{bmatrix}\xi_1(x)\\ \xi_2(x)\\ \xi_3(x)\end{bmatrix}$ and $\xi_j(x) = \max_{A_i\in S} \frac{1}{2}x^T A_i x + q_j^T x + r_j,\ j = 1,\ 2,\ 3.$