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APPROXIMATE WEAK AND PROPER SUBDIFFERENTIALS OF THE DIFFERENCE OF TWO VECTOR CONVEX MAPPINGS AND APPLICATIONS

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Abstract. In this paper, we provide a general formula concerning the weak and proper approximate subdifferentials of the difference of two vector convex mappings (DC) in terms of the star difference. This formula is applied to establish necessary and sufficient approximate optimality conditions, characterizing weakly and properly approximate efficient solutions for a constrained DC programming problem and a constrained multiobjective fractional programming problem.
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1. Introduction

DC programming problems are classified as a type of nonconvex optimization problems that play an interesting and important role in real world problems due to their algorithmic aspects and abundance of applications; see, e.g., [1, 2, 3, 4, 5] and the references therein. DC vector optimization problems recently attracted a great deal of attention and numerous results on the analysis and algorithms were obtained; see, e.g., [6, 7, 8, 9, 10, 11, 12, 13, 14] and the references therein. In [6], Gadhi et al. established sufficient optimality conditions for a weak Pareto minimal solution of DC vector optimization problems in an ordered Banach space. In [7], Guo et al. gave sufficient optimality conditions for an approximate weak Pareto minimal solution of DC vector optimization problem by using the concept of approximate pseudo-dissipativity. In [8], Taa derived optimality conditions for DC vector optimization problems in terms of Lagrange-Fritz-Joh and Lagrange-Karush-Kuhn-Tucker multipliers rules.

This paper is motivated by the recent result developed by Ammar et al. [15]. They discussed the calculus rule for the strong approximate subdifferential of the difference of two convex vector mappings defined in a locally convex topological vector space and considered their applications to DC vector programming problems. They obtained, under the concept of the regular

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subdifferentiability (see [16]), the following formula

$$\partial_{\varepsilon}^{s}(K_1 - K_2)(x_0) = \bigcap_{\mu \in W_+} \left\{ A \in L(X,W) : A + \partial_{\mu}^{s} K_2(x_0) \subseteq \partial_{\mu+\varepsilon}^{s} K_1(x_0) \right\}, \tag{1.1}$$

where $K_1, K_2 : X \longrightarrow W \cup \{+\infty_W\}$ are two convex mappings, $\partial_{\mu}^{s} K_1$ is the strong approximate subdifferential at x_0, X and W are real Hausdorff locally convex topological vector spaces, W_+ is a convex cone inducing a partial preorder in W, and ε and μ are the elements of W_+.

Our goal is to extend formula (1.1) for the approximate weak and proper Pareto subdifferentials by using the scalarization process and the regular subdifferentiability. Our paper is organized as follows. In Section 2, we recall some notions and give some preliminary results, used in what follows. In Section 3, we develop the formula concerning the approximate weak and proper Pareto subdifferentials for the difference of two vector convex mappings. In Section 4, we establish the Pareto approximate optimality conditions of a constrained DC programming problem. In Section 5, the last section, we derive the Pareto approximate optimality conditions for a multiobjective fractional programming problem.

2. Preliminaries

In this section, we give some basic definitions and results. Let X, W, and Z be real separated topological vector spaces whose continuous dual spaces are denoted by X^*, W^*, and Z^*. Throughout this paper, we denote by $L(X,W)$ the set of all continuous linear operators from X into Y. Let W_+ be a convex cone of W with $\mathrm{int}W_+ \neq \emptyset$. The subset $l(W_+) := W_+ \cap -W_+$ is the lineality of W_+. If it is null, then W_+ is said to be pointed. For any $w_1, w_2 \in W$, the cone W_+ induces the following preorder relations

$$\begin{aligned}
w_1 &\leq_{W_+} w_2 \Longleftrightarrow w_2 - w_1 \in W_+, \\
w_1 &<_{W_+} w_2 \Longleftrightarrow w_2 - w_1 \in \mathrm{int}W_+, \\
w_1 &\lneq_{W_+} w_2 \Longleftrightarrow w_2 - w_1 \in W_+ \backslash l(W_+).
\end{aligned}$$

To space W, we attach an abstract maximal element with respect to " $\leq_{W_+}$ ", denoted by $+\infty_W$, such that $w \leq_{W_+} +\infty_W$, for all $w \in W$ and $w + (+\infty_W) := (+\infty_W) + w := +\infty_W$ for all $w \in W \cup \{+\infty_W\}$ and $\eta \cdot (+\infty_W) := +\infty_W$ for all $\eta \in \mathbb{R}_+$. The polar cone W_+^* and the strict polar cone $(W_+^*)^\circ$ of W_+ are defined, respectively, as

$$W_+^* := \{w^* \in W^* \ : \ w^*(W_+) \subseteq \mathbb{R}_+\}$$

and

$$(W_+^*)^\circ := \{w^* \in W^* \ : \ w^*(W_+ \backslash l(W_+)) \subseteq \mathbb{R}_+ \backslash \{0\}\}.$$

Clearly

$$(W_+^*)^\circ \subseteq W_+^* \backslash \{0\}. \tag{2.1}$$

A mapping $K_1 : X \longrightarrow W \cup \{+\infty_W\}$ is said to be

- W_+- convex if, for any $\beta \in [0,1]$ and any $u_1, u_2 \in X$,
$$K_1(\beta u_1 + (1-\beta)u_2) \leq_{W_+} \beta K_1(u_1) + (1-\beta) K_1(u_2),$$
- proper if $\mathrm{dom} K_1 := \{x \in X : K_1(x) \in W\} \neq \emptyset$,
- star W_+-lower semicontinuous if $w^* \circ K_1$ is lower semicontinuous for any $w^* \in W_+^*$.

In the sequel, $\Gamma(X,W_+)$ stands for the set of proper W_+-convex mappings from X to W and $\Gamma_0(X,W_+)$ for the set of star W_+-lower semicontinuous mappings in $\Gamma(X,W_+)$, while $\Gamma_0(X,\mathbb{R}_+)$ reduces to $\Gamma_0(X)$, the set of proper convex and lower semicontinuous functionals from X to $\mathbb{R}$. Let " $\leq_{Z_+}$ " be a partial preorder on Z induced by a nonempty convex cone $Z_+ \subset Z$. We say that a mapping $K_2 : Z \longrightarrow W \cup \{+\infty_W\}$ is said to be (Z_+,W_+)-nondecreasing if, for any $z_1, z_2 \in Z$

$$z_1 \leq_{Z_+} z_2 \Longrightarrow K_2(z_1) \leq_{W_+} K_2(z_2).$$

If $K_3 : X \longrightarrow Z \cup \{+\infty_Z\}$, then the composed mapping $K_2 \circ K_3 : X \longrightarrow W \cup \{+\infty_W\}$ is defined by

$$(K_2 \circ K_3)(x) := \begin{cases} K_2(K_3(x)) & \text{if } x \in \operatorname{dom} K_3, \\ +\infty_W & \text{otherwise.} \end{cases}$$

We can easily observe that if K_2 is W_+-convex, (Z_+,Y_+)-nondecreasing, and K_3 is Z_+-convex, then $K_2 \circ K_3$ is W_+-convex.

Given a mapping $K_1 : X \supseteq S \longrightarrow W \cup \{+\infty_W\}$ and $\varepsilon \in W$, we consider the following constrained vector optimization problem

$$(P) \qquad \min_{x \in S} K_1(x).$$

Let $x_0 \in S \cap \operatorname{dom} K_1$. Then x_0 is said to be

- a strongly ε-efficient (ε-optimal) solution if, $\forall x \in S$, $K_1(x) \geq_{W_+} K_1(x_0) - \varepsilon$,
- a weakly ε-efficient solution if $\nexists x \in S$ such that $K_1(x) <_{W_+} K_1(x_0) - \varepsilon$,
- a properly ε-efficient solution if $\exists \hat{W}_+ \subsetneq W$, a convex cone, such that $W_+ \backslash l(W_+) \subseteq \operatorname{int} \hat{W}_+$ and $\nexists x \in S$ such that $K_1(x) \lneq_{\hat{W}_+} K_1(x_0) - \varepsilon$.

The sets of strongly, weakly and properly ε-efficient solutions are denoted, respectively, by $E^s_\varepsilon(K_1,S,W_+)$, $E^w_\varepsilon(K_1,S,W_+)$, and $E^p_\varepsilon(K_1,S,W_+)$. Note that

$$E^p_\varepsilon(K_1,S,W_+) \subseteq E^w_\varepsilon(K_1,S,W_+).$$

The above definitions give important information about ε, and we can easily see if $E^\sigma_\varepsilon(K_1,S,W_+) \neq \emptyset$, then $\varepsilon \not<^\sigma_{W_+} 0$, where

$$\varepsilon \not<^\sigma_{W_+} 0 \Longleftrightarrow \begin{cases} \varepsilon \notin -\operatorname{int} W_+ & \text{if} \quad \sigma = w, \\ \\ \varepsilon \notin -W_+ \backslash l(W_+) & \text{if} \quad \sigma = p. \end{cases}$$

The ε-subdifferential of K_1 at $x_0 \in \operatorname{dom} K_1$ can be defined according to the different concepts of Pareto ε-solutions with respect to $\sigma \in \{s,w,p\}$ as follows

$$\partial^\sigma_\varepsilon K_1(x_0) := \{A \in L(X,W) : x_0 \in E^\sigma_\varepsilon(K_1 - A, X, W_+)\},$$

i.e.,

- $\partial^s_\varepsilon K_1(x_0) = \{A \in L(X,W) : \ \forall x \in X, K_1(x) - K_1(x_0) \geq_{W_+} A(x-x_0) - \varepsilon\}$,
- $\partial^w_\varepsilon K_1(x_0) = \{A \in L(X,W) : \ \nexists x \in X, K_1(x) - K_1(x_0) <_{W_+} A(x-x_0) - \varepsilon\}$,
- $\partial^p_\varepsilon K_1(x_0) = \{A \in L(X,W) : \ \exists \hat{W}_+ \subsetneq W$ a convex cone such that $W_+ \backslash l(W_+) \subseteq \operatorname{int} \hat{W}_+, \ \nexists x \in X, \ K_1(x) - K_1(x_0) \lneq_{\hat{W}_+} A(x-x_0) - \varepsilon\}$,

If $\varepsilon = 0_W$, thn $\partial_0^\sigma K_1(x_0) := \partial^\sigma K_1(x_0)$ is the exact Pareto subdifferential (see [16]), for any $\sigma \in \{s, w, p\}$. For simplicity, we consider the following notation

$$W_+^\sigma := \begin{cases} W_+^* \setminus \{0\} & \text{if} \quad \sigma = w, \\ \\ (W_+^*)^\circ & \text{if} \quad \sigma = p. \end{cases}$$

For any subset $S \subset X$, the vector indicator mapping $\delta_S^v : X \longrightarrow W \cup \{+\infty_W\}$ of S is defined by

$$\delta_S^v(x) := \begin{cases} 0, & \text{if } x \in S, \\ +\infty_W, & \text{otherwise.} \end{cases}$$

When $Y = \mathbb{R}$, the scalar indicator function is denoted by δ_S. The vector indicator mapping δ_S^v appears to possess properties such as scalar indicator function δ_S. It is easy to verify that $w^* \circ \delta_S^v = \delta_S$ for all $w^* \in W_+^\sigma$. For any $\eta \geq 0$, the vector ε-normal set to S at $x_0 \in S$ is defined as the strong Pareto ε-subdifferential of the indicator mapping δ_S^v at x_0 i.e.,

$$N_\varepsilon^v(S, x_0) := \partial_\varepsilon^s \delta_S^v(x_0) = \{A \in L(X, W) : A(x - x_0) \leq_{W_+} \varepsilon, \ \forall x \in S\}.$$

Following [17], for any $\lambda \geq 0$, a mapping $K_1 : X \to W \cup \{+\infty_W\}$ is said to be σ-regular λ-subdifferentiable at $x_0 \in \mathrm{dom} K_1$ with $\sigma \in \{p, w\}$ if

$$\partial_\lambda(w^* \circ K_1)(x_0) = \bigcup_{\substack{\varepsilon \in W_+^\lambda \\ \langle w^*, \varepsilon \rangle = \lambda}} w^* \circ \partial_\varepsilon^s K_1(x_0), \ \forall w^* \in W_+^\sigma,$$

where $W_+^0 = \{0_Y\}$ and $W_+^\lambda = W_+$ if $\lambda > 0$, and $w^* \circ \partial_\varepsilon^s K_1(x_0) := \{w^* \circ A : A \in \partial_\varepsilon^s K_1(x_0)\}$.

In the sequel, we need the following theorems. The first characterizes the approximate σ-subdifferential for $\sigma \in \{w, p\}$ and the second one gives a formula on the approximate subdifferential of the difference of two convex real functions.

Theorem 2.1. *([17]) Let $K_1 : X \to W \cup \{+\infty_W\}$ and $x_0 \in domK_1$. Then, for $\sigma \in \{p, w\}$,*

$$\partial_\varepsilon^\sigma K_1(x_0) \supseteq \bigcup_{w^* \in W_+^\sigma} \{A \in L(X, W) : w^* \circ A \in \partial_{\langle w^*, \varepsilon \rangle}(w^* \circ K_1)(x_0)\}, \ \forall \varepsilon \not<_{W_+}^\sigma 0,$$

with equality if K_1 is W_+-convex and W_+ is pointed as $\sigma = p$.

Theorem 2.2. *([18]) Let $K_1, K_2 : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be two functions, $x_0 \in \mathrm{dom}\, K_1 \cap \mathrm{dom}\, K_2$ and $\alpha \geq 0$. If X is locally convex and $K_1, K_2 \in \Gamma_0(X)$, then*

$$\partial_\alpha(K_1 - K_2)(x_0) = \bigcap_{\beta \geq 0} \left\{\partial_{\beta+\alpha} K_1(x_0) \stackrel{*}{-} \partial_\beta K_2(x_0)\right\},$$

where $\partial_{\beta+\alpha} K_1(x_0) \stackrel{*}{-} \partial_\beta K_2(x_0) := \left\{x^* \in X^* : x^* + \partial_\beta K_2(x_0) \subseteq \partial_{\beta+\alpha} K_1(x_0)\right\}$ is the set of star difference between $\partial_{\beta+\alpha} K_1(x_0)$ and $\partial_\beta K_2(x_0)$.

3. Approximate Weak and Proper Subdifferentials of the Difference of Two Vector Convex Mappings

In this section, we present our main result concerning the approximate weak and proper subdifferentials for the difference of two vector convex mappings.

Theorem 3.1. *Let $K_1, K_2 : X \longrightarrow W \cup \{+\infty_W\}$ be two vector mappings, $x_0 \in \operatorname{dom} K_1 \cap \operatorname{dom} K_2$ and $\sigma \in \{p, w\}$ with W_+ being pointed as $\sigma = p$. Then,*

$$\partial_\varepsilon^\sigma (K_1 - K_2)(x_0) \subseteq \bigcap_{\mu \in W_+} \left\{ A \in L(X,W) : A + \partial_\mu^s K_2(x_0) \subseteq \partial_{\mu+\varepsilon}^\sigma K_1(x_0) \right\},\ \forall \varepsilon \not<_{W_+}^\sigma 0,$$

with equality if X is locally convex, $K_1, K_2 \in \Gamma_0(X, W_+)$, and K_2 is σ-regular λ-subdifferentiable at x_0 for any $\lambda \geq 0$.

Proof. Fist, let us prove

$$\partial_\varepsilon^\sigma (K_1 - K_2)(x_0) \subseteq \bigcap_{\mu \in W_+} \left\{ A \in L(X,W) : A + \partial_\mu^s K_2(x_0) \subseteq \partial_{\mu+\varepsilon}^\sigma K_1(x_0) \right\},\ \forall \varepsilon \not<_{W_+}^\sigma 0.$$

For the case $\sigma = w$, we let $A \in \partial_\varepsilon^w (K_1 - K_2)(x_0)$. That is, for all $x \in X$,

$$K_1(x) - K_2(x) - K_1(x_0) + K_2(x_0) - A(x - x_0) + \varepsilon \in (W \backslash -\operatorname{int} W_+). \tag{3.1}$$

Let $\mu \in W_+$ and $B \in \partial_\mu^s K_2(x_0)$. That is, for all $x \in X$,

$$K_2(x) - K_2(x_0) - B(x - x_0) + \mu \in W_+. \tag{3.2}$$

By summing term by term in inequalities (3.1) and (3.2), we obtain, for all $x \in X$,

$$K_1(x) - K_1(x_0) - (A + B)(x - x_0) + \varepsilon + \mu \in (W \backslash -\operatorname{int} W_+) + W_+.$$

Now, we need to show that $(W \backslash -\operatorname{int} W_+) + W_+ \subseteq (W \backslash -\operatorname{int} W_+)$. Let $u = u_1 + u_2$, with $u_1 \in (W \backslash -\operatorname{int} W_+)$ and $u_2 \in W_+$. We proceed by contradiction. If $u \notin (W \backslash -\operatorname{int} W_+)$, then $u_1 = u - u_2 \in -\operatorname{int} W_+ - W_+ \subseteq -\operatorname{int} W_+$, which contradicts $u_1 \in (W \backslash -\operatorname{int} W_+)$. Thus, for all $x \in X$,

$$K_1(x) - K_1(x_0) - (A + B)(x - x_0) + \varepsilon + \mu \in (W \backslash -\operatorname{int} W_+). \tag{3.3}$$

For the case $\sigma = p$, we let $A \in \partial_\varepsilon^p (K_1 - K_2)(x_0)$. Then there exists a convex cone $\widetilde{W}_+ \subsetneq W$ such that $W_+ \backslash \{0_W\} \subseteq \operatorname{int} \widetilde{W}_+$. For all $x \in X$,

$$K_1(x) - K_2(x) - K_1(x_0) + K_2(x_0) - A(x - x_0) + \varepsilon \in W \backslash (-\widetilde{W}_+ \backslash l(\widetilde{W}_+)).$$

Following the proof in the case $\sigma = w$, we see that, for all $x \in X$,

$$K_1(x) - K_1(x_0) - (A + B)(x - x_0) + \varepsilon + \mu \in W \backslash (-\widetilde{W}_+ \backslash l(\widetilde{W}_+)) + W_+.$$

We claim that $W \backslash (-\widetilde{W}_+ \backslash l(\widetilde{W}_+)) + W_+ \subseteq W \backslash (-\widetilde{W}_+ \backslash l(\widetilde{W}_+))$. Indeed, let $u = u_1 + u_2$ with $u_1 \in W \backslash (-\widetilde{W}_+ \backslash l(\widetilde{W}_+))$ and $u_2 \in W_+$. If $u_2 = 0_W$, then $u \in W \backslash (-\widetilde{W}_+ \backslash l(\widetilde{W}_+))$. Otherwise, if $u_2 \in W_+ \backslash \{0_W\} \subseteq \operatorname{int} \widetilde{W}_+ \subseteq \widetilde{W}_+ \backslash l(\widetilde{W}_+)$, by assuming that $u \notin W \backslash (-\widetilde{W}_+ \backslash l(\widetilde{W}_+))$, we obtain $u_1 = u - u_2 \in -\widetilde{W}_+ \backslash l(\widetilde{W}_+) - \widetilde{W}_+ \backslash l(\widetilde{W}_+) \subseteq -\widetilde{W}_+ \backslash l(\widetilde{W}_+)$ which contradicts the fact that $u_1 \in W \backslash (-\widetilde{W}_+ \backslash l(\widetilde{W}_+))$. This yields that, for all $x \in X$,

$$K_1(x) - K_1(x_0) - (A + B)(x - x_0) + \varepsilon + \mu \in W \backslash (-\widetilde{W}_+ \backslash l(\widetilde{W}_+)). \tag{3.4}$$

Thus, from (3.3) and (3.4), we have, for any $\mu \in W_+$,

$$A + B \in \partial_{\varepsilon+\mu}^\sigma (K_1)(x_0), \text{ for all } B \in \partial_\mu^s K_2(x_0),$$

which implies, for any $\mu \in W_+$,

$$A \in \left\{ A \in L(X,W) : A + \partial_\mu^s K_2(x_0) \subseteq \partial_{\mu+\varepsilon}^\sigma K_1(x_0) \right\},$$

that is,

$$A \in \bigcap_{\mu \in W_+} \left\{ A \in L(X,W) : A + \partial^s_\mu K_2(x_0) \subseteq \partial^\sigma_{\mu+\varepsilon} K_1(x_0) \right\}.$$

Conversely, letting

$$A \in \bigcap_{\mu \in W_+} \left\{ A \in L(X,W) : A + \partial^s_\mu K_2(x_0) \subseteq \partial^\sigma_{\mu+\varepsilon} K_1(x_0) \right\},$$

for all $\mu \in W_+$, we have $A + \partial^s_\mu K_2(x_0) \subseteq \partial^\sigma_{\mu+\varepsilon} K_1(x_0)$, that is, $A + B \in \partial^\sigma_{\mu+\varepsilon} K_1(x_0)$ for all $B \in \partial^s_\mu K_2(x_0)$. Following Theorem 2.1, we see that there exists some $w^* \in W^\sigma_+$ such that

$$w^* \circ (A+B) = w^* \circ A + w^* \circ B \in \partial_{\langle w^*, \mu+\varepsilon \rangle} (w^* \circ K_1)(x_0), \forall B \in \partial^s_\mu K_2(x_0),$$

that is,

$$w^* \circ A + w^* \circ \partial^s_\mu K_2(x_0) \subseteq \partial_{\langle w^*, \mu+\varepsilon \rangle} (w^* \circ K_1)(x_0), \tag{3.5}$$

Let $\vartheta \in (\mathrm{int} W_+) \cup \{0_W\}$ as $\langle w^*, \vartheta \rangle \geq 0$ and K_2 be σ-regular $\langle w^*, \vartheta \rangle$-subdifferentiable at x_0. Then

$$\partial_{\langle w^*, \vartheta \rangle} (w^* \circ K_2)(x_0) = \bigcup_{\substack{\mu \in W_+^{\langle w^*, \vartheta \rangle} \\ \langle w^*, \mu \rangle = \langle w^*, \vartheta \rangle}} w^* \circ \partial^s_\mu K_2(x_0), \tag{3.6}$$

where

$$W_+^{\langle w^*, \vartheta \rangle} := \begin{cases} 0_W, & \text{if } \vartheta = 0_W, \\ W_+, & \text{if } \vartheta \in \mathrm{int}\, W_+. \end{cases}$$

From relation (3.5), we deduce that, for all $\vartheta \in (\mathrm{int} W_+) \cup \{0_W\}$,

$$w^* \circ A + \bigcup_{\substack{\mu \in W_+^{\langle w^*, \vartheta \rangle} \\ \langle w^*, \mu \rangle = \langle w^*, \vartheta \rangle}} w^* \circ \partial^s_\mu K_2(x_0) \subseteq \bigcup_{\substack{\mu \in W_+^{\langle w^*, \vartheta \rangle} \\ \langle w^*, \mu \rangle = \langle w^*, \vartheta \rangle}} \partial_{\langle w^*, \mu \rangle + \langle w^*, \varepsilon \rangle} (w^* \circ K_1)(x_0),$$

that is,

$$w^* \circ A + \bigcup_{\substack{\mu \in W_+^{\langle w^*, \vartheta \rangle} \\ \langle w^*, \mu \rangle = \langle w^*, \vartheta \rangle}} w^* \circ \partial^s_\mu K_2(x_0) \subseteq \partial_{\langle w^*, \vartheta \rangle + \langle w^*, \varepsilon \rangle} (w^* \circ K_1)(x_0). \tag{3.7}$$

Combining (3.6) and (3.7), we obtain

$$w^* \circ A + \partial_{\langle w^*, \vartheta \rangle} (w^* \circ K_2)(x_0) \subseteq \partial_{\langle w^*, \vartheta \rangle + \langle w^*, \varepsilon \rangle} (w^* \circ K_1)(x_0),$$

that is,

$$w^* \circ A \in \partial_{\langle w^*, \vartheta \rangle + \langle w^*, \varepsilon \rangle} (w^* \circ K_1)(x_0) \stackrel{*}{-} \partial_{\langle w^*, \vartheta \rangle} (w^* \circ K_2)(x_0). \tag{3.8}$$

Let us prove $\mathbb{R}_+ = \{ \langle w^*, \vartheta \rangle, \vartheta \in (\text{ int } W_+) \cup \{0_W\} \}, \ \forall w^* \in W^\sigma_+$.

In fact, we start with the case $\sigma = w$. For the first inclusion $\{ \langle w^*, \vartheta \rangle, \vartheta \in (\text{ int } W_+) \cup \{0_W\} \} \subseteq \mathbb{R}_+$ is obviously, for any $w^* \in W^*_+ \backslash \{0\}$. For the reverse inclusion, let $\gamma \in \mathbb{R}_+$. If $\gamma = 0$, we have $0 = \langle w^*, 0_W \rangle$. Otherwise, if $\gamma > 0$, by virtue of [16, Proposition 2.1], we find the existence of $\tilde{w} \in \mathrm{int} W_+$ such that $\langle w^*, \tilde{w} \rangle = 1$. We can write $\gamma = \langle w^*, \gamma \tilde{w} \rangle$, with $\gamma \tilde{w} \in \mathrm{int} W_+$. Conclusion, $\mathbb{R}_+ = \{ \langle w^*, \vartheta \rangle, \vartheta \in (\text{ int } W_+) \cup \{0_W\} \}$ for any $w^* \in W^*_+ \backslash \{0\}$. For the other case $\sigma = p$, the same result can be obtained from the first case $\sigma = w$ by using (2.1) only.

Now, we can write (3.8) equivalently as

$$w^* \circ A \in \partial_{\langle w^*, \varepsilon \rangle + \gamma} (w^* \circ K_1)(x_0) \stackrel{*}{-} \partial_\gamma (w^* \circ K_2)(x_0), \ \forall \gamma \geq 0,$$

which yields

$$w^* \circ A \in \bigcap_{\gamma \geq 0} \left\{ \partial_{\langle w^*, \varepsilon \rangle + \gamma} \left(w^* \circ K_1 \right) (x_0) \stackrel{*}{-} \partial_{\gamma} \left(w^* \circ K_2 \right) (x_0) \right\}.$$

Since $K_1, K_2 \in \Gamma_0(X, W_+)$, then $w^* \circ K_1, w^* \circ K_2 \in \Gamma_0(X)$. As the space X is locally convex, we obtain from Theorem 2.2 that

$$w^* \circ A \in \partial_{\langle w^*, \varepsilon \rangle}(w^* \circ K_1 - w^* \circ K_2)(x_0) = \partial_{\langle w^*, \varepsilon \rangle}(w^* \circ (K_1 - K_2))(x_0),$$

which yields by applying the scalarization Theorem 2.1 $A \in \partial^{\sigma}_{\varepsilon}(K_1 - K_2)(x_0)$. The proof is complete. □

In particular case, when $\varepsilon = 0_{W_+}$, we obtain the following corollary.

Corollary 3.1. *Let $K_1, K_2 : X \longrightarrow W \cup \{+\infty_W\}$ be two vector mappings, $x_0 \in \operatorname{dom} K_1 \cap \operatorname{dom} K_2$ and $\sigma \in \{p, w\}$ with W_+ being pointed as $\sigma = p$. Then*

$$\partial^{\sigma}(K_1 - K_2)(x_0) \subseteq \bigcap_{\mu \in W_+} \left\{ A \in L(X, W) : A + \partial^s_{\mu} K_2(x_0) \subseteq \partial^{\sigma}_{\mu} K_1(x_0) \right\},$$

with equality if X is locally convex, $K_1, K_2 \in \Gamma_0(X, W_+)$ and K_2 is σ-regular λ-subdifferentiable at x_0 for any $\lambda \geq 0$.

4. Pareto Approximate Optimality Conditions of a Constrained DC Programming Problem

In this section, we consider the following constrained DC programming problem

$$(Q_1) \quad \begin{cases} \min(F(x) - G(x)) \\ x \in S, \end{cases}$$

where S is a nonempty convex subset of X and $F, G \in \Gamma(X, W_+)$. By using the vector indicator mapping δ^v_S, we transform equivalently the problem (Q_1) to the unconstrained problem

$$\begin{cases} \min \left(F(x) + \delta^v_S(x) - G(x) \right) \\ x \in X. \end{cases}$$

The following Theorem is helpful in the sequel.

Theorem 4.1. *([17]) Let $K_1,\ K_2 : X \to W \cup \{+\infty_W\}$ and $\sigma \in \{p, w\}$ with W_+ be pointed as $\sigma = p$. Assume that K_2 is σ-regular λ-subdifferentiable at $x_0 \in domK_1 \cap domK_2$ for any $\lambda \geq 0$, and one of the following two qualification conditions is satisfied*

$$(MR)_1 \begin{cases} K_1, K_2 \in \Gamma(X, W_+),\ X \textit{ locally convex}, \\ \exists \bar{x} \in \operatorname{dom} K_1 \cap \operatorname{dom} K_2 \textit{ s.t. } K_1 \textit{ or } K_2 \textit{ is continuous at } \bar{x}. \end{cases}$$

$$(AB)_1 \begin{cases} K_1, K_2 \in \Gamma_0(X, W_+),\ X \textit{ Fréchet space}, \\ \mathbb{R}_+[\operatorname{dom} K_1 - \operatorname{dom} K_2] \textit{ is a closed vector subspace of } X. \end{cases}$$

Then, for all $\varepsilon \not<^{\sigma}_{W_+} 0$,

$$\partial^{\sigma}_{\varepsilon}(K_1 + K_2)(x_0) = \bigcup_{\substack{\varepsilon_1 \not<^{\sigma}_{W_+} 0,\ \varepsilon_2 \in W_+ \\ \varepsilon_2 = 0 \ if\ \varepsilon = 0, \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \partial^{\sigma}_{\varepsilon_1} K_1(x_0) + \partial^s_{\varepsilon_2} K_2(x_0).$$

We are now in a position to establish the optimality conditions characterizing completely an approximate weak and proper efficient solutions of problem (Q_1).

Theorem 4.2. *Let $F, G : X \longrightarrow W \cup \{+\infty_W\}$, S be a nonempty convex closed in X and $\sigma \in \{w, p\}$ with W_+ being pointed as $\sigma = p$. Assume that $G \in \Gamma_0(X, W_+)$ and is σ-regular λ-subdifferentiable at $x_0 \in \operatorname{dom} F \cap \operatorname{dom} G \cap S$ for any $\lambda \geq 0$, and one of the following two qualification conditions is satisfied,*

$$(MR)_2 \begin{cases} F \in \Gamma_0(X, W_+), \ X \text{ locally convex,} \\ \operatorname{dom} F \cap \operatorname{int}(S) \neq \emptyset \text{ or } F \text{ is continuous at some point of } \operatorname{dom} F \cap S. \end{cases}$$

$$(AB)_2 \begin{cases} F \in \Gamma_0(X, W_+), \ X \text{ Fréchet space,} \\ \mathbb{R}_+[\operatorname{dom} F - S] \text{ is a closed vector subspace of } X. \end{cases}$$

Then, x_0 is an ε-σ-efficient solution of (Q_1) if and only if, for all $\varepsilon \not<^{\sigma}_{W_+} 0$,

$$\partial^s_\mu G(x_0) \subseteq \bigcup_{\substack{\varepsilon_1 \not<^{\sigma}_{W_+} 0,\ \varepsilon_2 \in W_+ \\ \varepsilon_2 = 0 \ if \mu+\varepsilon=0, \\ \varepsilon_1+\varepsilon_2=\mu+\varepsilon}} \partial^\sigma_{\varepsilon_1} F(x_0) + N^v_{\varepsilon_2}(S, x_0), \ \forall \mu \in W_+.$$

Proof. Let $\varepsilon \not<^{\sigma}_{W_+} 0$. Then x_0 is an ε-σ-efficient solution to (Q_1) if and only if

$$0 \in \partial^\sigma_\varepsilon \left((F + \delta^v_S) - G \right)(x_0).$$

Since $F, \delta^v_S \in \Gamma_0(X, W_+)$, then $\left(F + \delta^v_S\right) \in \Gamma_0(X, W_+)$. As X is locally convex and G is σ-regular λ-subdifferentiable at x_0 for any $\lambda \geq 0$, by virtue of Theorem 3.1, one has

$$\partial^s_\mu G(x_0) \subseteq \partial^\sigma_{\mu+\varepsilon} (F + \delta^v_S)(x_0), \ \forall \mu \in W_+. \tag{4.1}$$

Following [16], the vector indicator mapping δ^v_S is continuous at x_0 if and only if $x_0 \in \operatorname{int}(S)$. Hence, by putting $K_1 := \delta^v_S$ and $K_2 := F$, we observe by means of the condition $(MR)_2$ or $(AB)_2$ that all the assumptions of Theorem 4.1 are satisfied. Then expression (4.2) becomes equivalent to

$$\partial^s_\mu G(x_0) \subseteq \bigcup_{\substack{\varepsilon_1 \not<^{\sigma}_{W_+} 0,\ \varepsilon_2 \in W_+ \\ \varepsilon_2 = 0 \ if \ \mu+\varepsilon=0, \\ \varepsilon_1+\varepsilon_2=\mu+\varepsilon}} \partial^\sigma_{\varepsilon_1} F(x_0) + N^v_{\varepsilon_2}(S, x_0), \forall \mu \in W_+,$$

which completes the proof. □

Remark 4.1.
- (i) If $S = X$, then condition $(MR)_2$ is satisfied. Furthermore, the statement (4.1) in the above proof can be written equivalently as $\partial^s_\mu G(x_0) \subseteq \partial^\sigma_{\mu+\varepsilon} F(x_0)$ for all $\mu \in W_+$ and $\varepsilon \not<^{\sigma}_{W_+} 0$.
- (ii) If $F = 0$, then inclusion (4.1) reduces to $\partial^s_\mu G(x_0) \subseteq \partial^\sigma_{\mu+\varepsilon} \delta^v_S(x_0)$ for all $\mu \in W_+$ and $\varepsilon \not<^{\sigma}_{W_+} 0$.

The following example explains how to apply Theorem 4.2 for the case $S = X$.

Example 4.1. Let $X = S := \mathbb{R}$, $\sigma = w$, and $W := \mathbb{R}^2$ be endowed with its natural order induced by the nonnegative orthant $W_+ := \mathbb{R}^2_+ = \{(v_1, v_2) \in \mathbb{R}^2, \ v_1, v_2 \geq 0\}$. Consider the following programming problem

$$(P)\quad \begin{cases} \min\left\{\Big([x]_+,0\Big)-\Big(1+x,\dfrac{x^2}{2}\Big)\right\} \\ x\in\mathbb{R}, \end{cases}$$

where $[x]_+ = \max(0,x)$ is the nonnegative part of the scalar x. Let $F(x) = \big(f_1(x), f_2(x)\big) = \big([x]_+,0\big)$, $G(x) = \big(g_1(x), g_2(x)\big) = \big(1+x,\frac{x^2}{2}\big)$, and $x_0 = 0$. Obviously, F and G are convex, and problem (P) becomes DC programming problem. It is easy to see that G satisfies condition (5.5). Thus G is w-regular λ-subdifferentiable at $x_0 = 0$ ($\lambda \geq 0$) and immediately we have, for all $\mu = (\mu_1,\mu_2) \in \mathbb{R}^2_+$ and $\eta = (\eta_1,\eta_2) \in \mathbb{R}^2_+$,

$$\partial^s_\mu G(x_0) = \partial_{\mu_1} g_1(x_0) \times \partial_{\mu_2} g_2(x_0) = \{1\} \times \left[-\sqrt{2\mu_2}, \sqrt{2\mu_2}\right],$$
$$\partial^s_\eta F(x_0) = \partial_{\eta_1} f_1(x_0) \times \{0\} = [0,1]\times\{0\}.$$

By taking $\varepsilon = (\frac{1}{2},\frac{1}{2})$ and according to [19, Theorem 4.2], we obtain

$$\begin{aligned}\partial^w_{\mu+\varepsilon} F(x_0) &= \bigcup_{(\eta_1,\eta_2)\in\mathbb{R}^2_+\cap(\mu+\varepsilon-\mathbb{R}\setminus -int\mathbb{R}^2_+)} \partial^s_{(\eta_1,\eta_2)} F(x_0) + Z_w\left(\mathbb{R},\mathbb{R}^2\right) \\ &= [0,1]\times\{0\} + Z_w\left(\mathbb{R},\mathbb{R}^2\right)\end{aligned}$$

where the set $Z_w\left(\mathbb{R},\mathbb{R}^2\right)$ of w-zerolike matrices can be given as

$$\begin{aligned} Z_w\left(\mathbb{R},\mathbb{R}^2\right) &= \left\{B\in\mathbb{R}^{1\times 2} : \exists v\in\mathbb{R}^2_+\setminus\{0\},\ B^T v = 0\right\} \\ &= \left\{B\in\mathbb{R}^{1\times 2} : \exists v\in\mathbb{R}^2_+,\ \|v\|_1 = 1,\ B^T v = 0\right\} \\ &= \left\{(x,y)\in\mathbb{R}^2 : \exists (v_1,v_2)\in\mathbb{R}^2_+, v_1+v_2 = 1,\ v_1 x + v_2 y = 0\right\} \\ &= \left\{(x,y)\in\mathbb{R}^2 : 0\in[x,y] \text{ or } 0\in[y,x]\right\} \\ &= (\mathbb{R}_-\times\mathbb{R}_+)\cup(\mathbb{R}_+\times\mathbb{R}_-). \end{aligned}$$

It is easy to check that $\partial^s_\mu G(x_0) \subseteq \partial^w_{\mu+\varepsilon} F(x_0)$ for all $\mu = (\mu_1,\mu_2)\in\mathbb{R}^2_+$. Thus, by Theorem 4.2, x_0 is a weakly ε-solution to problem (P).

In the sequel, we establish the σ-efficient optimality conditions in terms of approximate subdifferentials and the vector ε-normal set of the following constrained vector problem

$$(Q_2)\quad \begin{cases} \min(F(x)-G(x)) \\ H(x)\in -Z_+, \end{cases}$$

where $F,G\in\Gamma(X,W_+)$ and $H\in\Gamma(X,Z_+)$. The unconstrained problem below is equivalent to the problem (Q_2)

$$\begin{cases} \min\Big(F(x)+\delta^v_{-Z_+}\circ H(x)-G(x)\Big) \\ x\in X. \end{cases}$$

The following Theorem is needed.

Theorem 4.3. *([19]) Let $K_1 : X\to W\cup\{+\infty_W\}$, $K_3 : X\to Z\cup\{+\infty_Z\}$, $K_2 : Z\to W\cup\{+\infty_W\}$, $x_0\in domK_1\cap K_3^{-1}(domK_2)\cap domK_3$ and $\sigma\in\{p,w\}$ with W_+ being pointed as $\sigma = p$. Assume* *that K_2 is (Z_+,W_+)-nondecreasing on Z and σ-regular λ-subdifferentiable at $K_3(x_0)$ for any $\lambda \geq 0$, and one of the two following qualification conditions is satisfied*

$$(MR)_3 \left\{ \begin{array}{l} K_1 \in \Gamma(X,W_+),\ K_3 \in \Gamma(X,Z_+),\ K_2 \in \Gamma(Z,W_+),\ X \text{ and } Z \text{ locally convex}, \\ \exists \bar{x} \in \operatorname{dom} K_1 \cap \operatorname{dom} K_3 \text{ s.t. } K_2 \text{ is finite and continuous at } K_3(\bar{x}). \end{array} \right.$$

$$(AB)_3 \left\{ \begin{array}{l} K_1 \in \Gamma_0(X,W_+),\ K_3 \in \Gamma_0(X,Z_+),\ K_2 \in \Gamma_0(Z,W_+),\ X \text{ and } Z \text{ Fréchet spaces}, \\ \mathbb{R}_+[\operatorname{dom} K_2 - K_3(\operatorname{dom} K_1 \cap \operatorname{dom} K_3)] \text{ is a closed vector subspace of } Z. \end{array} \right.$$

Then, for all $\varepsilon \not<^{\sigma}_{W_+} 0$,

$$\partial^{\sigma}_{\varepsilon}(K_1+K_2\circ K_3)(x_0) = \bigcup_{\substack{\varepsilon_1 \not<^{\sigma}_{W_+} 0,\ \varepsilon_2 \in W_+ \\ \varepsilon_2 = 0 \ if \ \varepsilon = 0, \\ \varepsilon_1+\varepsilon_2=\varepsilon}} \left\{ \bigcup_{A \in \partial^{s}_{\varepsilon_2} K_2(K_3(x_0))} \partial^{\sigma}_{\varepsilon_1}(K_1+A\circ K_3)(x_0) \right\}.$$

Now, we are ready to state σ-efficient optimality conditions of the problem (Q_2).

Theorem 4.4. *Let $F,G : X \to W \cup \{+\infty_W\}$, $H : X \to Z \cup \{\infty_Z\}$ and Z_+ be nonempty convex closed in X and $\sigma \in \{p,w\}$ with W_+ being pointed as $\sigma = p$. Assume that $H^{-1}(-Z_+)$ is closed, $G \in \Gamma_0(X,Y_+)$ and is σ-regular λ-subdifferentiable at $x_0 \in domF \cap H^{-1}(-Z_+) \cap domH \cap domG$ for any $\lambda \geq 0$, and one of the two following qualification conditions is satisfied*

$$(MR)_4 \left\{ \begin{array}{l} F \in \Gamma_0(X,W_+),\ H \in \Gamma_0(X,Z_+),\ X \text{ and } Z \text{ locally convex}, \\ H(\operatorname{dom} F \cap \operatorname{dom} H) \cap \operatorname{int}(-Z_+) \neq \emptyset. \end{array} \right.$$

$$(AB)_4 \left\{ \begin{array}{l} F \in \Gamma_0(X,W_+),\ H \in \Gamma_0(X,Z_+),\ X \text{ and } Z \text{ Fréchet spaces}, \\ \mathbb{R}_+[Z_+ + H(\operatorname{dom} F \cap \operatorname{dom} H)] \text{ is a closed vector subspace of } Z. \end{array} \right.$$

Then, x_0 is an ε-σ-efficient solution to (Q_2) if and only if, for all $\varepsilon \not<^{\sigma}_{W_+} 0$.

$$\partial^{s}_{\mu} G(x_0) \subseteq \bigcup_{\substack{\varepsilon_1 \not<^{\sigma}_{W_+} 0,\ \varepsilon_2 \in W_+ \\ \varepsilon_2 = 0 \ if \ \mu+\varepsilon = 0, \\ \varepsilon_1+\varepsilon_2=\mu+\varepsilon}} \left\{ \bigcup_{A \in N^{v}_{\varepsilon_2}(-Z_+,H(x_0))} \partial^{\sigma}_{\varepsilon_1}(F+A\circ H)(x_0) \right\},\ \forall \mu \in W_+.$$

Proof. Let $\varepsilon \not<^{\sigma}_{W_+} 0$. Then x_0 is an ε-σ-efficient solution to (Q_2) if and only if

$$0 \in \partial^{\sigma}_{\varepsilon}\left(F + \delta^{v}_{-Z_+}\circ H - G\right)(x_0).$$

Recall that the vector indicator mapping $\delta^{v}_{-Z_+} : Z \to W \cup \{+\infty_W\}$ is (Z_+,W_+)-nondecreasing and W_+-convex (see [16]). Since H is Z_+-convex, then $\delta^{v}_{-Z_+}\circ H$ is W_+-convex. From the fact that $w^* \circ \delta^{v}_{-Z_+} \circ H = \delta_{-Z_+}\circ H$ for any $w^* \in W^{\sigma}_+$, it follows that

$$\begin{array}{rcl} \operatorname{Epi}\left(w^* \circ \delta^{v}_{-Z_+} \circ H\right) & = & \{(x,\beta) : H(x) \in -Z_+, \beta \in \mathbb{R}^+\}, \\ & = & H^{-1}(-Z_+) \times \mathbb{R}^+. \end{array}$$

Since $H^{-1}(-Z_+)$ is closed, we deduce that $\operatorname{Epi}\left(w^* \circ \delta^{v}_{-Z_+} \circ H\right)$ is closed, which yields that $\delta^{v}_{-Z_+}\circ H$ is star W_+-lower semicontinuous. According to Theorem 3.1, we have, for all $\mu \in W_+$,

$$\partial^{s}_{\mu} G(x_0) \subseteq \partial^{\sigma}_{\mu+\varepsilon}(F+\delta^{v}_{-Z_+}\circ H)(x_0). \tag{4.2}$$

Note that $\delta^v_{-Z_+}$ is σ-regular λ-subdifferentiable at $G(x_0)$ for any $\lambda \geq 0$ (see [17]). By taking $K_1 := F$, $K_2 := \delta^v_{-Z_+}$ and $K_3 := H$, we observe by means of $(MR)_4$ or $(AB)_4$ that all the hypotheses of Theorem 4.3 are satisfied. Thus inclusion (4.2) becomes equivalent to

$$\partial^s_\mu G(x_0) \subseteq \bigcup_{\substack{\varepsilon_1 \not<^{\sigma}_{W_+} 0,\ \varepsilon_2 \in W_+ \\ \varepsilon_2 = 0\ if\ \mu+\varepsilon=0, \\ \varepsilon_1+\varepsilon_2=\mu+\varepsilon}} \left\{ \bigcup_{A \in \partial^s_{\varepsilon_2} \delta^v_{-Z_+}(H(x_0))} \partial^\sigma_{\varepsilon_1}(F + A \circ H)(x_0) \right\},\ \forall \mu \in W_+,$$

i.e.,

$$\partial^s_\mu G(x_0) \subseteq \bigcup_{\substack{\varepsilon_1 \not<^{\sigma}_{W_+} 0,\ \varepsilon_2 \in W_+ \\ \varepsilon_2 = 0\ if\ \mu+\varepsilon=0, \\ \varepsilon_1+\varepsilon_2=\mu+\varepsilon}} \left\{ \bigcup_{A \in N^v_{\varepsilon_2}(-Z_+, H(x_0))} \partial^\sigma_{\varepsilon_1}(F + A \circ H)(x_0) \right\},\ \forall \mu \in W_+.$$

This completes the proof. □

5. The Application to a Multiobjective Fractional Programming Problem

In this section, by applying the previous results, we present weak and proper approximate optimality conditions for the following multiobjective fractional programming problem

$$(Q_3) \quad \begin{cases} \min \left\{ \dfrac{f_1(x)}{g_1(x)}, \dots, \dfrac{f_s(x)}{g_s(x)} \right\} \\ H(x) \in -Z_+, \end{cases}$$

where $f_j, g_j : X \longrightarrow \mathbb{R}$, $j = 1, \dots, s$, are proper and convex functions and $H : X \longrightarrow Z \cup \{+\infty_Z\}$ is a proper and Z_+-convex mapping. Moreover, we assume that $f_j(x) \geq 0$, for any $x \in H^{-1}(-Z_+)$ and $j \in \{1,...,s\}$ and the following additional hypothesis

$(\mathscr{H})$ $\quad \exists c_1, c_2 > 0$, such that $c_1 \leq g_j(x) \leq c_2$, for all $x \in H^{-1}(-Z_+)$ and $j \in \{1,...,s\}$.

The following notations are used in the sequel

$$\begin{aligned} &\varepsilon := (\varepsilon_1, \dots, \varepsilon_s), \\ &\varepsilon_0 := (\varepsilon_1 g_1(x_0), \dots, \varepsilon_s g_s(x_0)), \\ &v_j := \tfrac{f_j(x_0)}{g_j(x_0)} - \varepsilon_j \geq 0. \end{aligned}$$

If we endow the finite-dimensional space $W := \mathbb{R}^s$ with its natural order induced by the non-negative orthant $W_+ := \mathbb{R}^s_+ = \{(w_1, \dots, w_s) \in \mathbb{R}^s,\ w_j \geq 0,\ \forall j = 1, \dots, s\}$.

The following definitions can be found in [20, 21].

Definition 5.1. A point $x_0 \in H^{-1}(-Z_+)$ is said to be

- weakly ε-efficient solution of (Q_3) if there does not exist $x \in H^{-1}(-Z_+)$ such that
$$\frac{f_j(x)}{g_j(x)} < \frac{f_j(x_0)}{g_j(x_0)} - \varepsilon_j,\ \forall j \in \{1, \dots, s\}.$$
- ε-efficient solution of (Q_3) if there does not exist $x \in H^{-1}(-Z_+)$ such that
$$\frac{f_j(x)}{g_j(x)} \leq \frac{f_j(x_0)}{g_j(x_0)} - \varepsilon_j,\ \forall j \in \{1, \dots, s\},$$

with at least one strict inequality.

- properly ε-efficient solution of (Q_3) in Geoffrion's sense if it is ε-efficient of (Q_3) and there exists $\beta > 0$ such that, for each $i \in \{1,...,s\}$ and each $x \in H^{-1}(-Z_+)$ satisfying $\dfrac{f_i(x_0)}{g_i(x_0)} - \dfrac{f_i(x)}{g_i(x)} - \varepsilon_i > 0$, there exists an index $k \in \{1,...,s\}$ with $\frac{f_k(x)}{g_k(x)} - \frac{f_k(x_0)}{g_k(x_0)} + \varepsilon_k > 0$ and
$$\frac{\dfrac{f_i(x_0)}{g_i(x_0)} - \dfrac{f_i(x)}{g_i(x)} - \varepsilon_i}{\frac{f_k(x)}{g_k(x)} - \frac{f_k(x_0)}{g_k(x_0)} + \varepsilon_k} \le \beta.$$

By using a parametric approach, we can transform problem (Q_3) into a vector DC programming problem with the parametric $v := (v_1,...,v_s) \in \mathbb{R}^s_+$, defined as follows
$$(Q_v)\begin{cases} \min(F(x) - G(x)) \\ H(x) \in -Z_+, \end{cases}$$
where $F, G : X \longrightarrow \mathbb{R}^s$ are defined for any $x \in X$ by
$$F(x) := (f_1(x),...,f_s(x)),\ G(x) := (v_1 g_1(x),\dots,v_s g_s(x)).$$

Proposition 5.1. *([20]) A point $x_0 \in H^{-1}(-Z_+)$ is said to be a weakly ε-efficient solution of (Q_3) if and only if x_0 is a weakly ε_0-efficient solution of (Q_v).*

Lemma 5.1. *Let $x_0 \in H^{-1}(-Z_+)$. Then x_0 is a properly ε-efficient solution of (Q_3) if and only if x_0 is a properly ε_0-efficient solution of (Q_v).*

Proof. Suppose that x_0 is a properly ε-efficient solution of (Q_3). By definition, x_0 is an ε-efficient solution of (Q_3) and there exists $\beta > 0$ such that, for each $i \in \{1,...,s\}$ and each $x \in H^{-1}(-Z_+)$ satisfying
$$\frac{f_i(x_0)}{g_i(x_0)} - \frac{f_i(x)}{g_i(x)} - \varepsilon_i > 0, \tag{5.1}$$
there exists an index $k \in \{1,...,s\}$ with
$$\frac{f_k(x)}{g_k(x)} - \frac{f_k(x_0)}{g_k(x_0)} + \varepsilon_k > 0, \tag{5.2}$$
and
$$\frac{\dfrac{f_i(x_0)}{g_i(x_0)} - \dfrac{f_i(x)}{g_i(x)} - \varepsilon_i}{\dfrac{f_k(x)}{g_k(x)} - \dfrac{f_k(x_0)}{g_k(x_0)} + \varepsilon_k} \le \beta. \tag{5.3}$$
Since x_0 is an ε-efficient solution to (Q_3), then, according to [20, Proposition 3.1], x_0 is an ε_0-efficient solution to (Q_v). Furthermore, putting $l_i(x) := f_i(x) - v_i g_i(x)$, we find from (5.1), (5.2), and the fact that $g_i(x) > 0$ that
$$\begin{cases} l_i(x_0) - l_i(x) - \varepsilon_i g_i(x_0) &= g_i(x)\left[\dfrac{f_i(x_0)}{g_i(x_0)} - \dfrac{f_i(x)}{g_i(x)} - \varepsilon_i\right] > 0, \\ l_k(x) - l_k(x_0) + \varepsilon_k g_k(x_0) &= g_k(x)\left[\dfrac{f_k(x)}{g_k(x)} - \dfrac{f_k(x_0)}{g_k(x_0)} + \varepsilon_k\right] > 0. \end{cases}$$

Clearly, condition (5.3) can be rewritten equivalently as

$$\frac{l_i(x_0)-l_i(x)-\varepsilon_i g_i(x_0)}{l_k(x)-l_k(x_0)+\varepsilon_k g_k(x_0)} = \frac{g_i(x)\left[\dfrac{f_i(x_0)}{g_i(x_0)}-\dfrac{f_i(x)}{g_i(x)}-\varepsilon_i\right]}{g_k(x)\left[\dfrac{f_k(x)}{g_k(x)}-\dfrac{f_k(x_0)}{g_k(x_0)}+\varepsilon_k\right]} \leq \beta\,\frac{g_i(x)}{g_k(x_0)}. \tag{5.4}$$

According to the assumption $(\mathscr{H})$, we see that (5.4) becomes

$$\frac{l_i(x_0)-l_i(x)-\varepsilon_i g_i(x_0)}{l_k(x)-l_k(x_0)+\varepsilon_k g_k(x_0)} \leq \beta\frac{c_1}{c_2}.$$

Thus x_0 is a properly ε_0-efficient solution to (Q_v). Similarly we prove the reciprocal implication. This completes the proof. □

Now, we present some necessary and sufficient approximate optimality conditions characterizing a weakly and properly ε-efficient solution for problem (Q_3).

Theorem 5.1. *Let $f_i, g_i : X \to \mathbb{R}\cup\{+\infty\}$, $H : X \to Z\cup\{\infty_Z\}$, $x_0 \in H^{-1}(-Z_+)$, Z_+ be nonempty convex closed in X, and $\sigma \in \{w,p\}$. Suppose that $H^{-1}(-Z_+)$ is closed, $g_i \in \Gamma_0(X)$ $(i=1,\dots,s)$, and there exists some $b \in \bigcap_{i=1}^{s}\operatorname{dom} g_i$ such that $(s-1)$ functions g_i are continuous at b. If assumption $(\mathscr{H})$ and one of the two following qualification conditions are satisfied*

$$(MR)_5\begin{cases} f_i \in \Gamma_0(X),\ H \in \Gamma_0(X,Z_+),\ X \text{ and } Z \text{ locally convex},\\ H\left(\displaystyle\bigcap_{i=1}^{s}\operatorname{dom} f_i \cap \operatorname{dom} H\right)\cap \operatorname{int}(-Z_+) \neq \emptyset.\end{cases}$$

$$(AB)_5\begin{cases} f_i \in \Gamma_0(X),\ H \in \Gamma_0(X,Z_+),\ X \text{ and } Z \text{ Fréchet spaces},\\ \mathbb{R}_+\left[Z_+ + H\left(\displaystyle\bigcap_{i=1}^{s}\operatorname{dom} f_i \cap \operatorname{dom} H\right)\right] \text{ is a closed vector subspace of } Z,\end{cases}$$

then x_0 is an ε-σ-efficient solution to (Q_3) if and only if, for all $\varepsilon \not<^{\sigma}_{\mathbb{R}^s_+} 0$ and $\mu = (\mu_1,\dots,\mu_s) \in \mathbb{R}^s_+$,

$$\begin{aligned}&\partial_{\mu_1}(v_1g_1)(x_0)\times\ldots\times\partial_{\mu_s}(v_sg_s)(x_0)\\ &\subseteq \bigcup_{\substack{\eta_1 \not<^{\sigma}_{\mathbb{R}^s_+} 0,\ \eta_2\in\mathbb{R}^s_+\\ \eta_2=0 \text{ if } \mu+\varepsilon_0=0,\\ \eta_1+\eta_2=\mu+\varepsilon_0}}\left\{\bigcup_{A\in N^{v}_{\eta_2}(-Z_+,H(x_0))}\partial^{\sigma}_{\eta_1}((f_1,\dots,f_s)+A\circ H)(x_0)\right\}.\end{aligned}$$

Proof. For $W=\mathbb{R}^s$ and $W_+=\mathbb{R}^s_+$, since $f_i, g_i \in \Gamma_0(X)$, we have $F, G \in \Gamma_0(X,\mathbb{R}^s_+)$. Let $\lambda \geq 0$. By virtue of [17], the λ-subdifferential σ-regularity of $G=(v_1g_1,\dots,v_sg_s)$ holds under the well-known Moreau-Rockafellar qualification condition

$$\begin{cases} g_i \in \Gamma_0(X),\ (i=1,\dots,s),\ X \text{ separated locally convex},\\ \exists b \in \bigcap_{i=1}^{s}\operatorname{dom} g_i \text{ such that } (s-1) \text{ functions } g_i \text{ are continuous at } b.\end{cases} \tag{5.5}$$

For our goal, this qualification condition is verified. By Proposition 5.1 and Lemma 5.1, x_0 is an ε-σ-efficient solution of (Q_3) if and only if x_0 is an ε_0-σ-efficient solution of (Q_v). Under

$(MR)_5$ or $(AB)_5$, we observe that all the hypotheses of Theorem 4.4 are satisfied, so

$$\partial_\mu^s G(x_0) \subseteq \bigcup_{\substack{\eta_1 \not<_{\mathbb{R}_+^s}^\sigma 0,\ \eta_2 \in \mathbb{R}_+^s \\ \eta_2 = 0\ if\ \mu+\varepsilon_0=0, \\ \eta_1+\eta_2=\mu+\varepsilon_0}} \left\{ \bigcup_{A \in N_{\eta_2}^v(-Z_+, H(x_0))} \partial_{\eta_1}^\sigma (F + A \circ H)(x_0) \right\}. \tag{5.6}$$

As $\partial_\mu^s G(x_0) = \partial_{\mu_1}(v_1 g_1)(x_0) \times \ldots \times \partial_{\mu_s}(v_s g_s)(x_0)$, we see that (5.6) becomes

$$\begin{aligned} &\partial_{\mu_1}(v_1 g_1)(x_0) \times \ldots \times \partial_{\mu_s}(v_s g_s)(x_0) \\ &\subseteq \bigcup_{\substack{\eta_1 \not<_{\mathbb{R}_+^s}^\sigma 0,\ \eta_2 \in \mathbb{R}_+^s \\ \eta_2 = 0\ if\ \mu+\varepsilon_0=0, \\ \eta_1+\eta_2=\mu+\varepsilon_0}} \left\{ \bigcup_{A \in N_{\eta_2}^v(-Z_+, H(x_0))} \partial_{\eta_1}^\sigma ((f_1, \ldots, f_s) + A \circ H)(x_0) \right\}. \end{aligned}$$

The proof is complete. □

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