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EXPLICIT ITERATIVE METHODS FOR THE SPLIT FEASIBILITY PROBLEM WITH MULTIPLE OUTPUT SETS

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Abstract. The purpose of this paper is to introduce some new explicit iterative methods for finding a solution of the split feasibility problem with multiple output sets. These methods are established by using the Tikhonov regularization method in real Hilbert spaces.

Keywords. Multiple output sets; Nonexpansive mapping; Regularization; Split feasibility problem.

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1. Introduction

Let H_1 and H_2 be two real Hilbert spaces. Let C and Q be nonempty, closed, and convex subsets of H_1 and H_2, respectively. Let $T: H_1 \to H_2$ be a bounded and linear operator. The *split convex feasibility problem* (SCFP, for short) is presented as follows:

$$\text{Find an element} \quad u^\star \in C \text{ such that } Tu^\star \in Q. \tag{1.1}$$

The SCFP was first introduced by Censor and Elfving [4] in order to model certain inverse problems. It plays an important role in medical image reconstruction and in signal processing; see, e.g., [1, 2]. Recently, various iterative algorithms were introduced for solving (1.1); see, e.g., [1, 2, 3, 5, 6, 9, 15, 16, 17, 19, 21, 24] and the references therein.

In 2010, Xu [19] introduced the following iterative method for solving Problem (1.1). For any $u_0 \in H$, he defined the sequence $\{u_n\}$ by

$$u_{n+1} = P_C[(1-t_n\varepsilon_n)u_n - \varepsilon_n T^*(I-P_Q)Tu_n], \ n \geq 0. \tag{1.2}$$

He proved that the sequence $\{u_n\}$ generated by (1.2) converges strongly to the minimum-norm solution to Problem (1.1) when $\{t_n\}$ and $\{\varepsilon_n\}$ satisfying the conditions below:

i) $t_n \to 0$ and $0 < \varepsilon_n < \dfrac{t_n}{\|T\|^2 + t_n}$;
ii) $\sum_{n=0}^\infty t_n\varepsilon_n = \infty$;
iii) $\dfrac{|\varepsilon_{n+1} - \varepsilon_n| + \varepsilon_n|t_{n+1} - t_n|}{t_n^2\varepsilon_n^2} \to 0$.

In 2012, Yao et al. [23] proved the strong convergence of iterative method (1.2) under the following conditions:

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i) $t_n \to 0$, and $\sum_{n=0}^{\infty} t_n = \infty$;

ii) $0 < \varepsilon_n < \dfrac{2}{\|T\|^2 + 2t_n}$, $\inf_n \varepsilon_n > 0$ and $|\varepsilon_{n+1} - \varepsilon_n| \to 0$.

In 2020, Reich et al. [10] presented and studied the following split feasibility problem with multiple output sets in Hilbert spaces: Let H, H_i, $i = 1,2,\ldots,m$, be real Hilbert spaces, and let $T_i : H \to H_i$, $i = 1,2,\ldots,m$, be bounded linear operators. Let C and Q_i be nonempty, closed, and convex subsets of H and H_i, $i = 1,2,\ldots,m$, respectively. Suppose that $\Omega^{SFPMOS} = C \cap (\cap_{i=1}^{m} T_i^{-1}(Q_i)) \neq \emptyset$. They considered the following problem:

$$\text{Find an element } u^\star \in \Omega^{SFPMOS}, \tag{1.3}$$

that is, a point $u^\star \in C$ such that $T_i u^\star \in Q_i$ for all $i = 1,2,\ldots,m$. In order to solve Problem (1.3), Reich et al. [10, 11] introduced some iterative methods which are based on the optimization approach. In 2022, Reich and Tuyen [12] proposed and studied the strong convergence of the following iterative scheme. Take any $u_0 \in H$ and define the sequence $\{u_n\}$ by

$$u_{n+1} = u_n - \varepsilon_n(F(u_n) + t_n U(u_n)),\ n \geq 0, \tag{1.4}$$

where $F = I - P_C + \sum_{i=1}^{m} T_i^*(I - P_{Q_i})T_i$ and $U : H \to H$ is L_U-Lipschitz and γ_U-strongly monotone. They proved that the sequence $\{u_n\}$ defined by (1.4) converges strongly to a solution of Problem (1.3) when the parameters control satisfy the following conditions:

i) $\lim_{n\to\infty} t_n = 0$, $\{t_n\} \subset (0, (\gamma_U - \varepsilon_n K L_U)/\varepsilon_n L_U^2)$, where $K = 1 + \sum_{i=1}^{m} \|T_i\|^2$;

ii) $\{\varepsilon_n\} \subset (0, \gamma/2KL_U)$ and $\sum_{n=1}^{\infty} t_n \varepsilon_n = \infty$;

iii) $\lim_{n\to\infty} \varepsilon_n / t_n = 0$;

iv) $\lim_{n\to\infty} \dfrac{|t_{n+1} - t_n|}{t_n \varepsilon_n} = 0$.

In this paper, we analyze and establish the strong convergence of iterative scheme (1.4) based on some conditions which are simpler than the conditions of Reich and Tuyen in [12]. We introduce several relaxed iterative methods for solving Problem (1.3) in the case where C and Q_i, $i = 1,2,\ldots,m$, are sublevel sets of convex functions. Two numerical examples are also presented to illustrate proposed methods.

2. Preliminaries

Let H be a real Hilbert space. We denote by $\langle u, v\rangle$ the inner product of two elements u, v in H. The induced norm is denoted by $\|\cdot\|$, that is, $\|u\| = \sqrt{\langle u, u\rangle}$ for all $u \in H$.

Let C be a nonempty, closed, and convex subset of H. It is known that, for each $u \in H$, there exists a unique point $P_C u \in C$ such that

$$\|u - P_C u\| = \inf_{v\in C} \|u - v\|. \tag{2.1}$$

The mapping $P_C : H \to C$ defined by (2.1) is called the *metric projection* of H onto C. We also recall (see, e.g., [8, Section 3]) that

$$\langle u - P_C u, v - P_C u\rangle \leq 0,\ \ \forall u \in H,\ \forall v \in C. \tag{2.2}$$

Let $S, A : H \to H$ are two operators from H into itself.

i) S is L_S *Lipschitz* if there exists a positive real number $L_S > 0$ such that

$$\|S(u) - S(v)\| \leq L_S \|u - v\|$$

for all $u, v \in H$. If $L_S = 1$, then we say that S is *nonexpansive*. In addition, if $L_U \in [0,1)$, then S is called a *strict contraction*;

ii) S is *firmly nonexpansive* if $2S - I$ is nonexpansive, which is equivalent to $S = (I+U)/2$, where $U: H \to H$ is nonexpansive;

iii) S is *averaged* if $S = (1-t)I + tU$, where $t \in (0,1)$ and $U: H \to H$ is nonexpansive. In this case, we say that S is t-averaged.

iv) A is *monotone* if $\langle u-v, A(u)-A(v)\rangle \geq 0$ for all $u, v \in H$;

v) A is β_A-*strongly monotone* with $\beta_A > 0$ if $\langle u-v, A(u)-A(v)\rangle \geq \beta_A \|u-v\|^2$ for all $u, v \in H$;

vi) A is γ_A-*co-coercive* if $\langle u-v, A(u)-A(v)\rangle \geq \gamma_A \|A(u)-A(v)\|^2$ for all $u, v \in H$.

We also need the following lemmas for our main results of this paper.

Lemma 2.1. (see [10]) *Let H be a real Hilbert space. Let C be a nonempty, closed, and convex subset of H. Then, for all $u, v \in H$,*

i) $\langle u-v, P_C u - P_C v\rangle \geq \|P_C u - P_C v\|^2$;

ii) $\langle u-v, (I-P_C)u - (I-P_C)v\rangle \geq \|(I-P_C)u - (I-P_C)v\|^2$.

Remark 2.1. It follows from Lemma 2.1 that $I - P_C$ is a nonexpansive mapping.

Lemma 2.2. (see [2, 20]) *The following statements hold:*

i) *If A is γ_A-co-coercive, then εA is γ_A/ε-co-coercive.*

ii) *S is averaged if and only if the component $I-S$ is γ-co-coercive with $\gamma > 1/2$. More precisely, for $t \in (0,1)$, S is t-averaged if and only if $I-S$ is $1/2t$-co-coercive.*

Lemma 2.3. [7] *Let T be a nonexpansive self-mapping of a closed and convex subset C of a Hilbert space H. Then $I-T$ is demiclosed, that is, whenever $\{u_n\}$ is a sequence in C which weakly converges to some $u \in C$ and the sequence $\{(I-T)(u_n)\}$ strongly converges to some v, it follows that $(I-T)(u) = v$.*

Lemma 2.4. [14] *Let $\{a_n\}$ and $\{b_n\}$ be bounded sequences in a Hilbert space H, and let $\{t_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Let $a_{n+1} = (1-\beta_n)b_n + \beta_n a_n$ for all $n \geq 0$ and $\limsup_{n\to\infty}(\|b_{n+1} - b_n\| - \|a_{n+1} - a_n\|) \leq 0$. Then $\lim_{n\to\infty} \|a_n - b_n\| = 0$.*

Lemma 2.5. [18] *Let $\{\Gamma_n\}$ be a sequence of nonnegative numbers, $\{b_n\}$ be a sequence in $(0,1)$, and $\{c_n\}$ a sequence of real numbers satisfying the following two conditions:*

i) $\Gamma_{n+1} \leq (1-b_n)\Gamma_n + b_n c_n$;

ii) $\sum_{n=0}^{\infty} b_n = \infty$, $\limsup_{n\to\infty} c_n \leq 0$.

Then $\lim_{n\to\infty} \Gamma_n = 0$.

3. Main Results

Consider Problem (1.3), and let $\Psi: H \to \mathbb{R}$ be defined by

$$\Psi(u) := \frac{1}{2}\|(I-P_C)u\|^2 + \frac{1}{2}\sum_{i=1}^{m} \|(I-P_{Q_i})T_i u\|^2$$

for all $u \in H$.

It is not difficult to see that Ψ is a convex, continuous, and proper function. Indeed, it is easy to see that Ψ is a continuous and proper function. We now prove that Ψ is a convex function. To do this, we first show that function $d_C(u) = \|(I-P_C)u\|$ for all $u \in H$, is a convex function. For every $x, y \in H$, it follows from the definition of $d_C(x)$ and $d_C(y)$ that there exist two sequences $\{x_n\}$ and $\{y_n\}$ in C such that $\|x-x_n\| \to d_C(x)$ and $\|y-y_n\| \to d_C(y)$, as $n \to \infty$. Since C is a convex set, $\lambda x_n + (1-\lambda)y_n \in C$ for all $\lambda \in [0,1]$ and for all $n \geq 1$. Thus

$$\begin{aligned} d_C(\lambda x + (1-\lambda)y) &= \inf_{z \in C} \|\lambda x + (1-\lambda)y - z\| \\ &\leq \|\lambda x + (1-\lambda)y - [\lambda x_n + (1-\lambda)y_n]\| \\ &\leq \lambda\|x - x_n\| + (1-\lambda)\|y - y_n\| \end{aligned}$$

for all $n \geq 1$. Letting $n \to \infty$, we obtain

$$d_C(\lambda x + (1-\lambda)y) \leq \lambda d_C(x) + (1-\lambda)d_C(y).$$

This shows that $d_C(u)$ is a convex function. Hence, $f(u) = d^2(u,C)/2$ is also a convex function (note that, the square of a nonnegative convex function is a convex function).

For each $i = 1,2,\ldots,m$ and for every $x, y \in H$, and for any $\lambda \in [0,1]$, it follows from the convexity of $\|(I-P_{Q_i})v\|^2$ on H_i that

$$\begin{aligned} \|(I-P_{Q_i})T_i[\lambda x + (1-\lambda)y]\|^2 &= \|(I-P_{Q_i})[\lambda T_i x + (1-\lambda)T_i y]\|^2 \\ &\leq \lambda\|(I-P_{Q_i})T_i x\|^2 + (1-\lambda)\|(I-P_{Q_i})T_i y\|^2, \end{aligned}$$

which implies that $\|(I-P_{Q_i})T_i u\|^2$ is a convex function on H. Thus we conclude that Ψ is a convex function.

Let $f_i(u) = \|(I-P_{Q_i})T_i u\|^2/2$ for all $u \in H$. We next prove that $\nabla f_i(u) = T_i^*(I-P_{Q_i})T_i u$. Indeed, we take any point $x_0 \in H$ and letting $v = T_i^*(I-P_{Q_i})T_i x_0$. For every $h \in H$, we have

$$\begin{aligned} &f_i(x_0+h) - f_i(x_0) - \langle v, h\rangle \\ &\quad= \frac{1}{2}\|(I-P_{Q_i})T_i(x_0+h)\|^2 - \frac{1}{2}\|(I-P_{Q_i})T_i x_0\|^2 - \langle v, h\rangle \\ &\quad= \frac{1}{2}(\|T_i(x_0+h) - P_{Q_i}T_i(x_0+h)\|^2 - \|(I-P_{Q_i})T_i x_0\|^2) - \langle v, h\rangle \\ &\quad\leq \frac{1}{2}(\|T_i(x_0+h) - P_{Q_i}T_i x_0\|^2 - \|(I-P_{Q_i})T_i x_0\|^2) - \langle v, h\rangle \\ &\quad= \frac{1}{2}(\|(I-P_{Q_i}T_i)x_0 + T_i h\|^2 - \|(I-P_{Q_i})T_i x_0\|^2) - \langle v, h\rangle \\ &\quad= \frac{1}{2}(\|(I-P_{Q_i}T_i)x_0\|^2 + \|T_i h\|^2 - \|(I-P_{Q_i})T_i x_0\|^2) \\ &\qquad+ \langle (I-P_{Q_i}T_i)x_0, T_i h\rangle - \langle v, h\rangle \\ &\quad= \frac{1}{2}\|T_i h\|^2 + \langle T_i^*(I-P_{Q_i}T_i)x_0, h\rangle - \langle v, h\rangle \\ &\quad\leq \frac{1}{2}\|T_i\|^2\|h\|^2. \end{aligned}$$

Similarly, we also have

$$f_i(x_0) - f_i(x_0+h) + \langle v, h\rangle \leq \frac{1}{2}\|T_i\|^2\|h\|^2.$$

Combining the two above inequalities, we see that

$$\frac{|f_i(x_0+h)-f_i(x_0)-\langle v,h\rangle|}{\|h\|}\le\frac{1}{2}\|T_i\|^2\|h\|\to 0,$$

as $\|h\|\to 0$. Hence, $\nabla f_i(x_0)=T_i^*(I-P_{Q_i})T_ix_0$. Then we infer that Ψ is a Fréchet differentiable function and

$$\nabla\Psi(u)=(I-P_C)u+\sum_{i=1}^{m}T_i^*(I-P_{Q_i})T_iu$$

for all $u\in H$.

By Rockafellar's theorem [13], $F:=\nabla\Psi$ is a maximal monotone operator. Moreover, a point $u^\star\in H$ is a solution to Problem (1.3) if and only if $u^\star$ is a minimum point of Ψ. This is equivalent to

$$F(u^\star)=(I-P_C)u^\star+\sum_{i=1}^{m}T_i^*(I-P_{Q_i})T_iu^\star=0. \tag{3.1}$$

We first consider the following Tikhonov regularization method

$$\min_{u\in H}\Big\{\Psi(u)+\frac{t}{2}\|u\|^2\Big\},$$

where $t>0$. We see that

$$\nabla\Big(\Psi(u)+\frac{t}{2}\|u\|^2\Big)=F(u)+tu,$$

for all $u\in H$. Thus, in this case, we study and establish the convergence of the following explicit iterative method: For any $u_0\in H$, construct the sequence $\{u_n\}$ by

$$u_{n+1}=u_n-\varepsilon_n[F(u_n)+t_nu_n],\ n\ge 0, \tag{3.2}$$

where $\{t_n\}\subset(0,1)$ and $\{\varepsilon_n\}$ is a sequence of real numbers. Note that the sequence $\{u_n\}$ generated by (3.2) can be rewritten in the following form

$$u_{n+1}=(1-t_n\varepsilon_n)u_n-\varepsilon_nF(u_n),\ n\ge 0. \tag{3.3}$$

In order to establish the strong convergence of the iterative method (3.3), we first introduce the following proposition.

Proposition 3.1. *The mapping F is γ_F-co-coercive with $\gamma_F=1/(1+\sum_{i=1}^m\|T_i\|^2)$.*

Proof. For any $u,v\in H$, it follows from Lemma 2.1 ii) that

$$\begin{aligned}\langle u-v,F(u)-F(v)\rangle &=\langle u-v,(I-P_C)u-(I-P_C)v\rangle\\ &\quad+\sum_{i=1}^{m}\langle u-v,T_i^*(I-P_{Q_i})T_iu-T_i^*(I-P_{Q_i})T_iv\rangle\\ &=\langle u-v,(I-P_C)u-(I-P_C)v\rangle\\ &\quad+\sum_{i=1}^{m}\langle T_iu-T_iv,(I-P_{Q_i})T_iu-(I-P_{Q_i})T_iv\rangle\\ &\ge\|(I-P_C)u-(I-P_C)v\|^2\\ &\quad+\sum_{i=1}^{m}\|(I-P_{Q_i})T_iu-(I-P_{Q_i})T_iv\|^2.\end{aligned} \tag{3.4}$$

We also have

$$
\begin{aligned}
&\|F(u)-F(v)\|^2\\
&=\|(I-P_C)u-(I-P_C)v+\sum_{i=1}^{m}T_i^*(I-P_{Q_i})T_iu-T_i^*(I-P_{Q_i})T_iv\|^2\\
&\leq[\|(I-P_C)u-(I-P_C)v\|+\sum_{i=1}^{m}\|T_i^*(I-P_{Q_i})T_iu-T_i^*(I-P_{Q_i})T_iv\|]^2\\
&\leq[\|(I-P_C)u-(I-P_C)v\|+\sum_{i=1}^{m}\|T_i\|\|(I-P_{Q_i})T_iu-(I-P_{Q_i})T_iv\|]^2\\
&\leq(1+\sum_{i=1}^{m}\|T_i\|^2)[\|(I-P_C)u-(I-P_C)v\|^2 \qquad (3.5)\\
&\quad+\sum_{i=1}^{m}\|(I-P_{Q_i})T_iu-(I-P_{Q_i})T_iv\|^2].
\end{aligned}
$$

From (3.4) and (3.5), we obtain that

$$\langle u-v,F(u)-F(v)\rangle\geq\frac{1}{1+\sum_{i=1}^{m}\|T_i\|^2}\|F(u)-F(v)\|^2,$$

for all $u,v\in H$, that is, F is γ_F-co-coercive with $\gamma_F=1/(1+\sum_{i=1}^{m}\|T_i\|^2)$. This completes the proof. □

The following proposition is an important result that is needed to prove the strong convergence of iterative method (3.2).

Proposition 3.2. *If $t\in(0,1)$ and $\varepsilon\in(0,\frac{2\gamma_F}{1+2t\gamma_F})$, then $G^{t,\varepsilon}=(1-t\varepsilon)I-\varepsilon F$ is a strict contraction mapping with the coefficient $k=1-t\varepsilon$.*

Proof. For any $u,v\in H$, using Proposition 3.1, we have

$$
\begin{aligned}
\|G^{t,\varepsilon}(u)-G^{t,\varepsilon}(v)\|^2&=\|(1-t\varepsilon)(u-v)-\varepsilon(F(u)-F(v))\|^2\\
&=(1-t\varepsilon)^2\|u-v\|^2-2(1-t\varepsilon)\varepsilon\langle u-v,F(u)-F(v)\rangle\\
&\quad+\varepsilon^2\|F(u)-F(v)\|^2\\
&\leq(1-t\varepsilon)^2\|u-v\|^2-\varepsilon[2(1-t\varepsilon)\gamma_F-\varepsilon]\|F(u)-F(v)\|^2
\end{aligned}
$$

It follows from $\varepsilon\in(0,\frac{2\gamma_F}{1+2t\gamma_F})$ that $2(1-t\varepsilon)\gamma_F-\varepsilon>0$. Thus

$$\|G^{t,\varepsilon}(u)-G^{t,\varepsilon}(v)\|\leq(1-t\varepsilon)\|u-v\|,$$

for all $u,v\in H$. This completes the proof. □

Theorem 3.1. *Let $\{t_n\}$ and $\{\varepsilon_n\}$ be two positive sequences such that $\{t_n\}\subset(0,1)$ and $\{\varepsilon_n\}\subset[a,b]\subset(0,\frac{2\gamma_F}{1+2t_n\gamma_F})$. Let $\{t_n\}$ satisfy the following conditions*

$$t_n\to 0,\ \sum_{n=0}^{\infty}t_n=\infty.$$

Then the sequence $\{u_n\}$ generated by (3.2) converges strongly to an element $u^\star = P_{\Omega^{SFPMOS}}0$, as $n \to \infty$.

Proof. We first show that $\{u_n\}$ is bounded. Put $u^\star = P_{\Omega^{SFPMOS}}0$. It follows from (3.1) and Proposition 3.2 that

$$\begin{aligned}\|u_{n+1}-u^\star\| &= \|G^{t_n\varepsilon_n}(u_n)-G^{t_n\varepsilon_n}(u^\star)-t_n\varepsilon_n u^\star\|\\ &\le (1-t_n\varepsilon_n)\|u_n-u^\star\|+t_n\varepsilon_n\|u^\star\|\\ &\le \max\{\|u_n-u^\star\|,\|u^\star\|\}\\ &\ \ \vdots\\ &\le \max\{\|u_0-u^\star\|,\|u^\star\|\},\end{aligned}$$

which implies that $\{u_n\}$ is bounded.

We now prove that $\|u_{n+1}-u_n\| \to 0$. Indeed, since F is γ_F-co-coercive, then εF is γ_F/ε_n-co-coercive. We note that $\gamma_F/\varepsilon_n > 1/2$. Thus, from Lemma 2.2, we deduce that $I-\varepsilon_n F$ is $\varepsilon_n/2\gamma_F$ averaged, which implies that there exists a nonexpansive mapping S such that

$$I-\varepsilon_n F = (1-\frac{\varepsilon_n}{2\gamma_F})I+\frac{\varepsilon_n}{2\gamma_F}S.$$

Thus, we can rewrite (3.3) in the following form

$$\begin{aligned}u_{n+1} &= (I-\varepsilon_n F)(u_n)-t_n\varepsilon_n u_n\\ &= [(1-\frac{\varepsilon_n}{2\gamma_F})I+\frac{\varepsilon_n}{2\gamma_F}S](u_n)-t_n\varepsilon_n u_n\\ &= \beta_n u_n+(1-\beta_n)v_n,\end{aligned}$$

where $v_n = S(u_n)-2\gamma_F t_n u_n$ and $\beta_n = 1-\dfrac{\varepsilon_n}{2\gamma_F}$.

Next, we have

$$\begin{aligned}\|v_{n+1}-v_n\| &= \|S(u_{n+1})-2\gamma_F t_{n+1}u_{n+1}-S(u_n)+2\gamma_F t_n u_n\|\\ &\le \|S(u_{n+1})-S(u_n)\|+2\gamma_F\|t_{n+1}u_{n+1}-t_n u_n\|\\ &\le \|u_{n+1}-u_n\|+2\gamma_F(t_{n+1}\|u_{n+1}-u_n\|+|t_{n+1}-t_n|\|u_n\|).\end{aligned}$$

It follows from the boundedness of the sequence $\{u_n\}$ and $t_n \to 0$ that

$$\limsup_{n\to\infty}(\|v_{n+1}-v_n\|-\|u_{n+1}-u_n\|) \le 0.$$

Since $\{\varepsilon_n\} \subset [a,b] \subset (0,\dfrac{2\gamma_F}{1+2t_n\gamma_F})$, then $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$. Thus, from Lemma 2.4, we see that $\|u_n-v_n\| \to 0$. Hence,

$$\|u_{n+1}-u_n\| = \frac{\varepsilon_n}{2\gamma_F}\|u_n-v_n\| \to 0. \tag{3.6}$$

This together with (3.3) and the condition $t_n \to 0$ obtains $\|F(u_n)\| \to 0$. Observe that

$$\begin{aligned}\langle u_n-u^\star,F(u_n)\rangle &= \langle u_n-u^\star,F(u_n)-F(u^\star)\rangle\\ &\ge \|(I-P_C)u_n\|^2+\sum_{i=1}^m\|(I-P_{Q_i})T_iu_n\|^2,\end{aligned}$$

which together with the boundedness of $\{u_n\}$ and $\|F(u_n)\| \to 0$ deduces that

$$\|(I-P_C)u_n\| \to 0, \tag{3.7}$$

$$\|(I-P_{Q_i})T_iu_n\| \to 0, \tag{3.8}$$

for all $i = 1,2,\dots,m$.

We next prove that the weak cluster point set of $\{u_n\}$ is contained in Ω^{SFPMOS}. Indeed, suppose that p is an arbitrary weak cluster point of $\{u_n\}$. There exits a subsequence $\{u_{k_n}\}$ of $\{u_n\}$ such that $u_{k_n} \rightharpoonup p$, as $n \to \infty$. Since T_i is a bounded linear operator, one has $T_iu_{k_n} \rightharpoonup T_ip$ for each $i = 1,2,\dots,m$. Thus, applying Lemma 2.3 and using (3.7)–(3.8), we obtain that $p \in \Omega^{SFPMOS}$, as claimed.

Finally, we prove that $u_n \to u^\star$, as $n \to \infty$. Note that

$$\begin{aligned}
\|u_{n+1}-u^\star\|^2 &= \langle G^{t_n,\varepsilon_n}(u_n) - G^{t_n,\varepsilon_n}(u^\star) + G^{t_n,\varepsilon_n}(u^\star) - u^*, u_{n+1}-u^\star\rangle \\
&= \langle G^{t_n,\varepsilon_n}(u_n) - G^{t_n,\varepsilon_n}(u^\star), u_{n+1}-u^\star\rangle + \langle G^{t_n,\varepsilon_n}(u^\star) - u^*, u_{n+1}-u^\star\rangle \\
&= \langle G^{t_n,\varepsilon_n}(u_n) - G^{t_n,\varepsilon_n}(u^\star), u_{n+1}-u^\star\rangle - t_n\varepsilon_n\langle u^\star, u_{n+1}-u^\star\rangle \\
&\le \|G^{t_n,\varepsilon_n}(u_n) - G^{t_n,\varepsilon_n}(u^\star)\|\|u_{n+1}-u^\star\| - t_n\varepsilon_n\langle u^\star, u_{n+1}-u^\star\rangle \\
&\le (1-t_n\varepsilon_n)\|u_n-u^\star\|\|u_{n+1}-u^\star\| - t_n\varepsilon_n\langle u^\star, u_{n+1}-u^\star\rangle \\
&\le (1-t_n\varepsilon_n)\frac{\|u_n-u^\star\|^2+\|u_{n+1}-u^\star\|^2}{2} - t_n\varepsilon_n\langle u^\star, u_{n+1}-u^\star\rangle \\
&\le \frac{1-t_n\varepsilon_n}{2}\|u_n-u^\star\|^2 + \frac{1}{2}\|u_{n+1}-u^\star\|^2 - t_n\varepsilon_n\langle u^\star, u_{n+1}-u^\star\rangle,
\end{aligned}$$

which implies that

$$\|u_{n+1}-u^\star\|^2 \le (1-t_n\varepsilon_n)\|u_n-u^\star\|^2 - 2t_n\varepsilon_n\langle u^\star, u_{n+1}-u^\star\rangle. \tag{3.9}$$

Putting $\Gamma_n = \|u_n-u^\star\|^2$, $b_n = t_n\varepsilon_n$ and $c_n = 2\langle u^\star, u^\star - u_{n+1}\rangle$, we can rewrite (3.9) in the following form $\Gamma_{n+1} \le (1-b_n)\Gamma_n + b_nc_n$. Suppose that $\{u_{l_n}\}$ is a subsequence of the sequence $\{u_n\}$ such that

$$\limsup_{n\to\infty}\langle u^\star, u^\star - u_n\rangle = \lim_{n\to\infty}\langle u^\star, u^\star - u_{l_n}\rangle. \tag{3.10}$$

Since $\{u_n\}$ is bounded, we see that there exists a subsequence $\{u_{k_{l_n}}\}$ of $\{u_{l_n}\}$ such that $u_{k_{l_n}} \rightharpoonup q$, as $n \to \infty$. We may assume without loss of generality that $u_{l_n} \rightharpoonup q$ as $n \to \infty$. From the proof above, we have that $q \in \Omega^{SFPMOS}$. It follows from (2.2) and (3.10) that

$$\limsup_{n\to\infty}\langle u^\star, u^\star - u_n\rangle = \langle u^\star, u^\star - q\rangle = \langle 0 - P_{\Omega^{SFPMOS}}0, q - P_{\Omega^{SFPMOS}}0\rangle \le 0. \tag{3.11}$$

In view of $\|u_{n+1}-u_n\| \to 0$, we imply that $\limsup_{n\to\infty} c_n \le 0$. Furthermore, it is easy to see that $\sum_{n=0}^{\infty} t_n\varepsilon_n = \infty$. Hence, all the conditions of Lemma 2.5 are satisfied. Therefore, we deduce that $\Gamma_n \to 0$, that is, $u_n \to u^\star$ as $n \to \infty$, as asserted. This completes the proof. □

Next, we consider the Tikhonov regularization method

$$\min_{u\in H}\left\{\Psi(u) + \frac{t}{2}\|u-\bar{u}\|^2\right\}.$$

Using similar arguments as above, we can easily prove the following theorem.

Theorem 3.2. *Let $\{t_n\}$ and $\{\varepsilon_n\}$ be two positive sequences such that $\{t_n\} \subset (0,1)$ and $\{\varepsilon_n\} \subset [a,b] \subset (0, \dfrac{2\gamma_F}{1+2t_n\gamma_F})$. For any $\bar{u} \in H$, let $\{u_n\}$ be a sequence defined by $u_0 \in H$ and*

$$u_{n+1} = u_n - \varepsilon_n[F(u_n) + t_n(u_n - \bar{u})],\ n \geq 0. \tag{3.12}$$

Let $\{t_n\}$ satisfy $t_n \to 0$ and $\sum_{n=0}^{\infty} t_n = \infty$. Then $\{u_n\}$ converges strongly to $P_{\Omega^{SFPMOS}}\bar{u}$, as $n \to \infty$.

Finally, we study the convergence of the sequence $\{w_n\}$ generated by the following scheme: For any $w_0 \in H$, we define sequence $\{w_n\}$ by

$$w_{n+1} = w_n - \varepsilon_n[F(w_n) + t_n U(w_n)],\ n \geq 0, \tag{3.13}$$

where $U:\ H \to H$ is L_U-Lipschitz and β_U-strongly monotone operator.

Theorem 3.3. *Let $\{t_n\}$ and $\{\varepsilon_n\}$ be two positive sequences such that $\{t_n\} \subset (0,1)$ and $\{\varepsilon_n\} \subset [a,b] \subset (0, \dfrac{2\gamma_F}{1+2t_n\gamma_F})$. Let $\{t_n\}$ satisfy the conditions: $t_n \to 0$ and $\sum_{n=0}^{\infty} t_n = \infty$. Then the sequence $\{w_n\}$ defined by (3.13) converges strongly to an element $p^\star \in \Omega^{SFPMOS}$ which is a unique solution to the following variational inequality*

$$\langle y - p^\star, U(p^\star)\rangle \geq 0,\ \forall y \in \Omega^{SFPMOS}. \tag{3.14}$$

Proof. Let μ be a positive number with $\mu \in (0, 2\beta_U/L_U^2)$. We write $t_n = \rho_n\mu$ with $\rho_n = t_n/\mu$. Since $t_n \to 0$, we may assume that $\rho_n < 1$ for all n. Since $\mu \in (0, 2\beta_U/L_U^2)$, we have that $I - \mu U$ is a strict contraction with the contraction coefficient $\tau = \sqrt{1-\mu(2\beta_U - \mu L_U^2)}$ (see [22]). Thus $P_{\Omega^{SFPMOS}}(I-\mu U)$ is also a strict contraction. Banach fixed point theorem guarantees that $P_{\Omega^{SFPMOS}}(I-\mu U)$ has a unique fixed point $p^\star$. It follows from (2.2) that $p^\star$ is a unique solution to variational inequality (3.14).

Let $\{u_n\}$ be a sequence defined by (3.12), where $\bar{u} = (I-\mu U)(p^\star)$ and t_n is replaced by ρ_n. From Theorem 3.2, we obtain that $u_n \to p^\star = P_{\Omega^{SFPMOS}}(I-\mu U)(p^\star)$, as $n \to \infty$. We now rewrite the formulas to define $\{u_n\}$ and $\{w_n\}$ in the following forms:

$$u_{n+1} = G^{\rho_n,\varepsilon_n}(u_n) + \rho_n\varepsilon_n(I-\mu U)(p^\star),$$
$$w_{n+1} = G^{\rho_n,\varepsilon_n}(w_n) + \rho_n\varepsilon_n(I-\mu U)(w_n).$$

Note that

$$\begin{aligned} \|w_{n+1} - u_{n+1}\| &\leq \|G^{\rho_n,\varepsilon_n}(w_n) - G^{\rho_n,\varepsilon_n}(u_n)\| + \rho_n\varepsilon_n\|(I-\mu U)(w_n) - (I-\mu U)(p^\star)\| \\ &\leq (1-\rho_n\varepsilon_n)\|w_n - u_n\| + \rho_n\varepsilon_n\tau\|w_n - p^\star\| \\ &\leq (1-\rho_n\varepsilon_n)\|w_n - u_n\| + \rho_n\varepsilon_n\tau(\|w_n - u_n\| + \|u_n - p^\star\|) \\ &= [1-(1-\tau)\rho_n\varepsilon_n]\|w_n - u_n\| + \rho_n\varepsilon_n\tau\|u_n - p^\star\|. \end{aligned} \tag{3.15}$$

Putting $\Gamma_n = \|w_n - u_n\|$, $b_n = (1-\tau)\rho_n\varepsilon_n$, and $c_n = \dfrac{\tau}{1-\tau}\|u_n - p^\star\|$ we can rewrite (3.15) in the form $\Gamma_{n+1} \leq (1-b_n)\Gamma_n + b_n c_n$. It is easy to see that $\sum_{n=0}^{\infty} b_n = \infty$ and $\lim_{n\to\infty} c_n = 0$. Thus all the conditions of Lemma 2.5 are satisfied. Therefore, we infer that $\Gamma_n \to 0$, that is, $\|w_n - u_n\| \to 0$. From $u_n \to p^\star$, we obtain that $w_n \to p^\star$. This complete the proof. □

Remark 3.1. We see that the strong convergence of e iterative method (3.14) is established based on simpler conditions than the results in [12]. In particular, when $m = 1$, Problem (1.3) becomes Problem (1.1). Thus our results are better than the result in [23] (we remove $|\varepsilon_{n+1}-\varepsilon_n|\to 0$).

4. Relaxed Iterative Methods

The relaxed iterative method for solving Problem (1.1) was first introduced and studied in [21]. We now study Problem (1.3) when C and Q_i, $i=1,2,\dots,m$, are sublevel sets of the lower semicontinuous convex functions $h:\ H\to\mathbb{R}$ and $h_i:\ H_i\to\mathbb{R}$, $i=1,2,\dots,m$, respectively. Suppose that

$$C=\{u\in H:\ h(u)\le 0\},$$
$$Q_i=\{u\in H:\ h_i(T_iu)\le 0\},\ i=1,2,\dots,m.$$

We assume that h and h_i, $i=1,2,\dots,m$, are subdifferentiable on H and that the subdifferentials ∂h and ∂h_i, $i=1,2,\dots,m$, are bounded (on bounded sets). Recall that the subdifferential of a convex function $\Xi:\ H\to\mathbb{R}$ is defined by

$$\partial\Xi(u):=\{\xi\in H:\ \Xi(w)-\Xi(u)\ge\langle w-u,\xi\rangle\ \ \forall w\in H\}.$$

For a given point $u_n\in H$, we define the subsets C_n and $Q_{i,n}$ by

$$C_n:=\{u\in H:\ h(u_n)\le\langle u_n-u,\xi_n\rangle\},$$
$$Q_{i,n}:=\{v\in H_i:\ h_i(T_iu_n)\le\langle T_iu_n-v,\eta_{i,n}\rangle\},\ i=1,2,\dots,m,$$

where $\xi_n\in\partial h(u_n)$ and $\eta_{i,n}\in\partial h_i(T_iu_n)$ for all $i=1,2,\dots,m$. The sets C_n and $Q_{i,n}$ are called the relaxed sets of C and Q_i, respectively. It is easy to see that C_n and $Q_{i,n}$ are half-spaces of H and H_i, respectively, and that $C\subset C_n$ and $Q_i\subset Q_{i,n}$ for all $i=1,2,\dots,m$.

It is known that, in the general case, it is not easy to calculate the projections P_Cx and $P_{Q_i}y$. Therefore we introduce two relaxed iterative methods corresponding to the proposed iterative methods, when P_C and P_{Q_i} are replaced by P_{C_n} and $P_{Q_{i,n}}$, respectively, which are defined as follows:

$$P_{C_n}u:=u-\max\left\{\frac{\langle u,\xi_n\rangle-\langle u_n,\xi_n\rangle+h(u_n)}{\|\xi_n\|^2},0\right\}\xi_n,$$
$$P_{Q_{i,n}}v:=v-\max\left\{\frac{\langle y,\eta_{i,n}\rangle-\langle T_iu_n,\eta_{i,n}\rangle+h_i(T_iu_n)}{\|\eta_{i,n}\|^2},0\right\}\eta_{i,n}.$$

By using a similar technique as in [12], we can easily prove the following theorem.

Theorem 4.1. *Let $\{t_n\}$ and $\{\varepsilon_n\}$ be two positive sequences such that $\{t_n\}\subset(0,1)$ and $\{\varepsilon_n\}\subset[a,b]\subset(0,\dfrac{2\gamma_F}{1+2t_n\gamma_F})$. Let μ be a positive real number with $\mu\in(0,2\beta_U/L_U^2)$. Suppose that the sequence $\{t_n\}$ satisfies the following conditions*

$$t_n\to 0,\ \sum_{n=0}^{\infty}t_n=\infty.$$

For any $w_0\in H$, let $\{w_n\}$ be the sequence defined by

$$w_{n+1}=w_n-\varepsilon_n[F_n(w_n)+t_nU(w_n)],\ n\ge 0,$$

where $F_n=I-P_{C_n}+\sum_{i=1}^m T_i^*(I-P_{Q_{i,n}})T_i$. Then $\{w_n\}$ converges strongly to an element $p^\star\in\Omega^{SFPMOS}$ which is a unique solution to the following variational inequality (3.14).

5. Numerical Experiments

In this section, our algorithms are implemented in MATLAB 14a running on the DESKTOP-9RLTPS0, Intel(R) Core(TM) i5-10210U CPU @ 1.60GHz with 2.11 GHz and 8GB RAM.

Example 5.1. Consider the following split feasibility with multiple output sets: Let C, Q_1, Q_2, and Q_3 be closed and convex subsets of $\mathbb{R}^5$, R^4, $\mathbb{R}^3$, and $\mathbb{R}^2$, respectively, which are defined by

$$\begin{aligned}
C &= \{(x_1,x_2,x_3,x_4,x_5) \in \mathbb{R}^5 : x_1 - x_2 + x_3 - x_4 + 2x_5 = 2\},\\
Q_1 &= \{(y_1,y_2,y_2,y_4) \in \mathbb{R}^4 :\ y_1 + y_2 - y_3 + y_4 = 1\},\\
Q_2 &= \{(z_1,z_2,z_3) \in \mathbb{R}^3 :\ z_1 - 2z_2 + z_3 = 0\},\\
Q_3 &= \{(v_1,v_2) \in \mathbb{R}^2 :\ v_1 - v_2 = 1\}.
\end{aligned}$$

The representing matrices of the transfer mappings $T_1 :\ \mathbb{R}^5 \to \mathbb{R}^4$, $T_2 :\ \mathbb{R}^5 \to \mathbb{R}^3$, and $T_3 :\ \mathbb{R}^5 \to \mathbb{R}^2$ are

$$T_1 = \begin{pmatrix} 1 & 2 & -1 & 1 & 0\\ 1 & 1 & 2 & -1 & 1\\ 2 & -1 & 1 & 2 & 0\\ 1 & -5 & 1 & 1 & -1 \end{pmatrix},$$

$$T_2 = \begin{pmatrix} 2 & 1 & -1 & 1 & 2\\ 1 & -1 & 2 & 1 & -1\\ 1 & -4 & 6 & 1 & -4 \end{pmatrix},$$

$$T_3 = \begin{pmatrix} 2 & 1 & 3 & 0 & -1\\ 1 & 2 & 3 & 0 & -1 \end{pmatrix}.$$

It is easy to check that

$$\Omega^{SFPMOS} := C \cap_{i=1}^{3} T_i^{-1}(Q_i) = \{(1+\xi, \xi, -1, -1, 0.5) :\ \xi \in \mathbb{R}\}.$$

We now test the convergence of the iterative methods (3.2) and (3.12). The parameters t_n and ε_n are chosen as follows:

$$t_n = (n+1)^{-0.595},\ \varepsilon_n = \begin{cases} \dfrac{2\gamma_F}{1+4\gamma_F} & \text{if } n \text{ even},\\ \dfrac{2\gamma_F}{1+8\gamma_F} & \text{if } n \text{ odd}, \end{cases}$$

for all $n \geq 0$. Note that, in this case, $|\varepsilon_{n+1} - \varepsilon_n| \nrightarrow 0$.

a) Applying the iterative method (3.2) with the initial point $u_0 = (1,2,3,4,5)$.

We see that $u^\star = (0.5, -0.5, -1, -1, 0.5)$ is the minimum norm solution to the problem. We use the condition $\sigma_n := \|u_n - u^\star\|^2 < \varepsilon$ to stop the iterative process, where ε is a given error. We obtain the following table of numerical results.

b) Applying the iterative method (3.12) with the initial point $u_0 = (1,2,3,4,5)$ and $\bar{u} = (2,1,-2,1,2)$.

We see that $p^\star = P_{\Omega^{SFPMOS}}\bar{u} = (2,1,-1,-1,0.5)$. Thus, in this case, we use the condition $\sigma_n := \|u_n - p^\star\|^2 < \varepsilon$ to stop the iterative process, where ε is a given error. We obtain the following table of numerical results.

ε	σ_n	n	Time (s)
10^{-2}	9.999885×10^{-3}	78256	2.510100
10^{-3}	9.999951×10^{-4}	180996	5.883531
10^{-4}	9.999973×10^{-5}	448854	14.459758

TABLE 1. Table of numerical results for the iterative method (3.2)

ε	σ_n	n	Time (s)
10^{-2}	9.998124×10^{-3}	7639	0.284367
10^{-3}	9.999802×10^{-4}	48086	1.781236
10^{-4}	9.999975×10^{-5}	335068	11.933642

TABLE 2. Table of numerical results for the iterative method (3.12)

The behavior of the function σ_n in Table 1 and Table 2 is presented in the figure below.

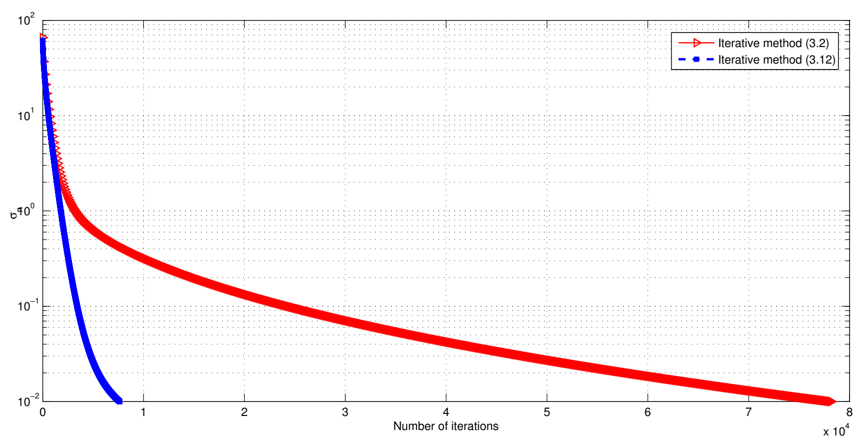

FIGURE 1. The behavior of σ_n with $\varepsilon = 10^{-2}$

Example 5.2. Suppose that $H = L^2[0,1]$ with the inner product $\langle x,y\rangle = \int_0^1 x(t)y(t)dt$ for all $x,y \in L^2[0,1]$ and the norm $\|x\| = \left(\int_0^1 x^2(t)dt\right)^{1/2}$ for all $x \in L^2[0,1]$. Consider the split feasibility problem with multiple output sets with the following data:

$$C = \{x \in L^2[0,1] : \langle a,x\rangle \le b\},$$
$$Q_i = \{y \in L^2[0,1] : \langle a_i,y\rangle \le b_i\},\ i = 1,2,\dots,100,$$

where $a(t) = t^2$, $b = 0.5$, $a_i(t) = \cos(it)+t$, and $b_i = 1/i$, for all $i = 1,2,\dots,100$. For each $i = 1,2,\dots,100$, let $T_i :\ L^2[0,1] \to L^2[0,1]$, be linear operator which is defined by $T_i x = ix$ for all $x \in L^2[0,1]$.

It is easy to see that the solution set $\Omega^{SFPMOS} \neq \emptyset$ because $u(t) = 0$ belongs to Ω^{SFPMOS}.

We now test the convergence of the iterative method (3.2) with the initial point $x_0(t) = \exp(t)$ and the parameter ε_n is chosen as in Example 5.1. In this case, we use the condition $\sigma_n :=$

$\|u_{n+1}-u_n\| < \varepsilon$ to stop the iterative process, where ε is a given error. Moreover, at nth iteration step, we also define the number D_n, which is determined by

$$D_n = \max\left\{ \langle a, u_n\rangle - b, \max_{i=1,2,\ldots,100} \{\langle a_i, T_i u_n\rangle - b_i\} \right\}.$$

Note that if $D_n \le 0$, then $u_n \in \Omega$. We obtain the following table of numerical results.

t_n	ε	σ_n	n	D_n	Time (s)
$t_n = 1/n^{0.25}$					
	10^{-4}	9.998781×10^{-5}	240	0.646375	0.225530
	10^{-5}	9.999578×10^{-6}	7680	0.194301	5.278357
	10^{-6}	9.999913×10^{-7}	29673	0.021703	19.577540
$t_n = 1/n^{0.85}$					
	10^{-4}	9.991735×10^{-5}	239	0.646896	0.222085
	10^{-5}	9.999562×10^{-6}	7674	0.196354	5.248964
	10^{-6}	9.999901×10^{-7}	29542	0.023546	19.285241

TABLE 3. Table of numerical results for the iterative method (3.2) for Example 5.2

The behavior of the function σ_n in Table 3 is presented in the figure below.

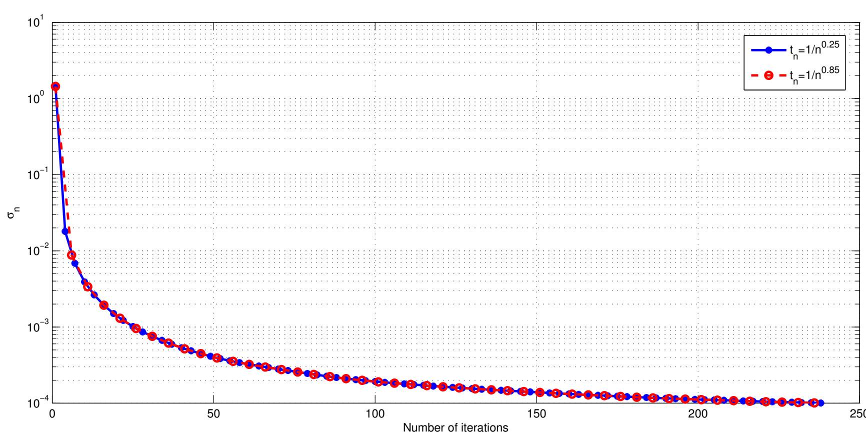

FIGURE 2. The behavior of σ_n with $\varepsilon = 10^{-4}$

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