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ON APPROXIMATE POSITIVELY PROPERLY EFFICIENT SOLUTIONS IN NONSMOOTH SEMI-INFINITE MULTIOBJECTIVE OPTIMIZATION PROBLEMS WITH DATA UNCERTAINTY

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Abstract. In this paper, we exploit necessary/sufficient optimality conditions for ε-quasi positively properly efficient solutions of the semi-infinite multiobjective optimization problems with data uncertainty. We also consider Wolfe type dual problems/Mond–Weir type dual problems under the assumptions of generalized convexity. Finally, several illustrative examples are also provided.

Keywords. Duality theorem; Efficient solution; Generalized convexity; Optimality condition; Semi-infinite multiobjective optimization problem.

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1. INTRODUCTION

Semi-infinite multiobjective optimization problems find numerous applications in various fields, such as in engineering design, mathematical physics, robotics, optimal control, transportation problems, fuzzy sets, and cooperative games. Recently, optimality conditions and duality for semi-infinite multiobjective optimization problems have been considered by numerous researchers; see, e.g., [1, 2, 3, 4, 5, 6, 7] and the references therein. For semi-infinite multiobjective optimization problems with data uncertainty; we refer to, for example, [8, 9, 10, 11, 12] and the references therein. For isolated efficient solutions and properly efficient solutions for multiobjective optimization problems, we refer to [13, 14, 15, 16, 17, 18] and the references therein. In addition, optimality conditions and duality for isolated efficient solutions/properly efficient solutions of semi-infinite multiobjective optimization problems were studied in [19, 20, 21, 22, 23, 24, 25]. However, there are few results on isolated efficient solutions/properly efficient solutions for the semi-infinite multiobjective optimization problems with uncertainty data [26]. In addition, since sometimes exact solutions do not exist while the approximate ones do, even in the convex case (see [27, 28]), the study of approximate solutions becomes significant from both the theoretical aspect and computational applications. Optimality conditions and duality theorems for approximate solutions of a multiobjective optimization problems were studied in [29, 30, 31] and optimality conditions/duality theorems/saddle point theorems for approximate solutions of optimization problems with infinite constraints were

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given in [32, 33, 34, 35, 36, 37, 38, 39]. On the other hand, optimality conditions/duality theorems for approximate solutions of robust optimization problems with infinite constraints were obtained in [40, 41, 42]. However, to the best of our knowledge, up to now, there is no paper devoted to ε-quasi positively properly efficient solutions of semi-infinite multiobjective optimization problems with data uncertainty.

Inspired by the above observations, we provide some new results for optimality conditions and duality theorems for ε-quasi positively properly efficient solutions of semi-infinite multiobjective optimization problems with uncertainty data via the Mordukhovich subdifferential. Since the Mordukhovich subdifferential and the Mordukhovich normal cone might be nonconvex, the Mordukhovich subdifferential seems to be useful for deriving optimality conditions and duality for semi-infinite multiobjective optimization problems with uncertainty data. The rest of the paper is organized as follows. Section 2 presents essential mathematical tools, such as notations, definitions, and lemmas. In Section 3, we investigate optimality conditions for robust ε-quasi positively properly efficient solutions of semi-infinite multiobjective optimization problems. In Section 4, the last section, we study approximate Wolfe type dual problems/approximate Mond-Weir type dual problems with uncertain data.

2. Preliminaries

From now on, we use the standard notation of variational analysis in [43, 44] and all spaces under consideration are assumed to be Euclidean spaces $\mathbb{R}^n$ with $n \in \mathbb{N} := \{1, 2, \cdots\}$. The inner product and the norm in $\mathbb{R}^n$ are denoted by by $\langle \cdot, \cdot \rangle$ and $||\cdot||$, respectively. The closed unit ball in the dual space $\mathbb{R}^n$ is denoted by $\mathbb{B}_{\mathbb{R}^n}$. In addition, the topological closure and the topological interior of a set $D \subset \mathbb{R}^n$ are denoted by cl$D$ and int$D$. As usual, the polar cone of D is the set

$$D^\circ := \{x^* \in \mathbb{R}^n \mid \langle x^*, x\rangle \le 0, \forall x \in D\}. \tag{2.1}$$

Besides, the nonnegative (resp., nonpositive) orthant cone of Euclidean space $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+ = \{(x_1, \cdots, x_n) \mid x_i \ge 0, i = 1, \cdots, n\}$ (resp., $\mathbb{R}^n_-$) for $n \in \mathbb{N} := \{1, 2, \cdots\}$, while int$\mathbb{R}^n_+$ is borrowed to indicate the topological interior of $\mathbb{R}^n_+$.

Given a set-valued map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, we denote by

$$\operatorname*{Lim\,sup}_{x \to \bar{x}} G(x) := \{y \in \mathbb{R}^n \mid \exists \text{ sequence } x_k \to \bar{x} \text{ and } y_k \to y \text{ with } y_k \in G(x_k), \forall k \in \mathbb{N}\}$$

the sequential Painlevé-Kuratowski upper/outer limit of G as $x \to \bar{x}$.

Recall that a set $S \subset \mathbb{R}^n$ is said to be closed around $\bar{x} \in S$ if there exists a neighborhood U of $\bar{x}$ such that $S \cap \text{cl}U$ is closed. One says that S is locally closed if S is closed around x for every $x \in S$. Let $S \subset \mathbb{R}^n$ be closed around $\bar{x} \in S$. The Fréchet/regular normal cone to S at $\bar{x} \in S$ is defined by

$$\widehat{N}(\bar{x}; S) := \left\{ x^* \in \mathbb{R}^n \mid \limsup_{x \xrightarrow{S} \bar{x}} \frac{\langle x^*, x - \bar{x}\rangle}{\| x - \bar{x} \|} \le 0 \right\},$$

where $x \xrightarrow{S} \bar{x}$ means that $x \to \bar{x}$ with $x \in S$. If $\bar{x} \notin S$, we put $\widehat{N}(\bar{x}; S) := \emptyset$.

The Mordukhovich/limiting normal cone $N(\bar{x}; S)$ to S at $\bar{x} \in S \subset \mathbb{R}^n$ is obtained from Fréchet/regular normal cones by taking the sequential Painlevé Kuratowski upper limit as

$$N(\bar{x}; S) := \operatorname*{Lim\,sup}_{x \xrightarrow{S} \bar{x}} \widehat{N}(x; S).$$

If $\bar{x} \notin S$, we put $N(\bar{x};S) := \emptyset$. Specially, if S is convex, then $N(\bar{x};S) = \{x^* \in \mathbb{R}^n \mid \langle x^*, x-\bar{x}\rangle \leq 0, \forall x \in S\}$. Let $f : \mathbb{R}^n \to \bar{\mathbb{R}} := [-\infty, +\infty]$ be an extended real-valued function. The domain, graph, and epigraph of f are given by $\mathrm{dom} f := \{x \in \mathbb{R}^n \mid |f(x)| < +\infty\}$, $\mathrm{gph} f := \{(x,\mu) \in \mathbb{R}^n \times \mathbb{R} \mid \mu = f(x)\}$, and $\mathrm{epi} f := \{(x,\mu) \in \mathbb{R}^n \times \mathbb{R} \mid \mu \geq f(x)\}$, respectively. Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be finite at $\bar{x} \in \mathrm{dom} f$, then the Mordukhovich/limiting subdifferential of f at $\bar{x}$ is defined by

$$\partial f(\bar{x}) := \{x^* \in \mathbb{R}^n \mid (x^*, -1) \in N((\bar{x}, f(\bar{x})); \mathrm{epi} f)\}.$$

If $|f(\bar{x})| = +\infty$, then one puts $\partial f(\bar{x}) := \emptyset$. Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be finite at $\bar{x} \in \mathrm{dom} f$, then f is lower semi-continuous at $\bar{x}$ if $\liminf_{x \to \bar{x}} f(x) \geq f(\bar{x})$. Given $S \subset \mathbb{R}^n$, consider the indicator function $\delta(\cdot;S)$ defined by

$$\delta(x;S) := \begin{cases} 0, & \text{if } x \in S, \\ +\infty, & \text{otherwise.} \end{cases}$$

Furthermore, we have a relation between the Mordukhovich/limiting normal cone and the Mordukhovich/limiting subdifferential of the indicator function as $N(\bar{x};S) = \partial\delta(\bar{x};S)$ for all $\bar{x} \in S$. Let $f : \mathbb{R}^n \to \mathbb{R}$. We say that f is locally Lipschitz at $\bar{x} \in \mathbb{R}^n$ if there exist a positive constant $L > 0$ and a neighborhood U of $\bar{x}$ such that $|f(x_1) - f(x_2)| \leq L||x_1 - x_2||$ for all $x_1, x_2 \in U$. For a function f is locally Lipschitz at $\bar{x}$ with $L > 0$, it implies that (see [43, Corollary 1.81] $||x^*|| \leq L, \forall x^* \in \partial f(\bar{x})$.

Lemma 2.1. [43, Proposition 1.114] *Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ be finite at $\bar{x} \in \mathbb{R}^n$. If $\bar{x}$ is a local minimizer of f, then $0 \in \partial f(\bar{x})$.*

Lemma 2.2. [44, Corollary 2.21] *Let $f_k : \mathbb{R}^n \to \bar{\mathbb{R}}, k = 1, \cdots, m$ (with $m \geq 2$) be lower semi-continuous around $\bar{x} \in \mathbb{R}^n$, and let all but one of these functions be Lipschitz continuous around $\bar{x}$. Then $\partial(f_1 + \cdots + f_m)(\bar{x}) \subset \partial f_1(\bar{x}) + \cdots + \partial f_m(\bar{x})$.*

Let T be a nonempty infinite index set, and let $\mathbb{R}^{(T)}$ be the linear space given below

$$\mathbb{R}^{(T)} := \{\lambda = (\lambda_t)_{t \in T} \mid \lambda_t = 0 \text{ for all } t \in T \text{ but only finitely many } \lambda_t \neq 0\}.$$

Let $\mathbb{R}^{(T)}_+$ be the positive cone in $\mathbb{R}^{(T)}$ defined by

$$\mathbb{R}^{(T)}_+ := \{\lambda = (\lambda_t)_{t \in T} \in \mathbb{R}^{(T)} \mid \lambda_t \geq 0 \text{ for all } t \in T\}.$$

With $\lambda \in \mathbb{R}^{(T)}$, its supporting set, $T(\lambda) := \{t \in T \mid \lambda_t \neq 0\}$, is a finite subset of T. Give $z = (z_t)_{t \in T} \subset Z$, where Z is a real linear space. Letting $\lambda = (\lambda_t)_{t \in T}$, we see that

$$\langle \lambda, z\rangle = \sum_{t \in T} \lambda_t z_t = \begin{cases} \sum_{t \in T(\lambda)} \lambda_t z_t, & \text{if } T(\lambda) \neq \emptyset, \\ 0, & \text{if } T(\lambda) = \emptyset. \end{cases}$$

In this paper, we consider the following multiobjective semi-infinite programming problems with uncertain data:

$$(\text{USIMP})_s \quad \begin{array}{l} \min f(x) := (p_1(x) - s_1 q_1(x), \cdots, p_m(x) - s_m q_m(x)) \\ \text{s.t. } x \in C := \{x \in \Omega \mid g_t(x, v_t) \leq 0, \forall t \in T\}, \end{array}$$

where T is a nonempty infinite index set, Ω is a nonempty and locally closed subset of $\mathbb{R}^n$, $\mathscr{V}_t \subseteq \mathbb{R}^q, t \in T$ are uncertain convex compact sets, and the functions $p_k, q_k : \mathbb{R}^n \to \mathbb{R}, k = 1, \cdots, m, g_t : \mathbb{R}^n \times \mathscr{V}_t \to \mathbb{R}, t \in T$ are locally Lipschitz functions. Let $p := (p_1, \cdots, p_m)$ and $q := (q_1, \cdots, q_m)$, $s := (s_1, \cdots, s_m) \in \mathbb{R}^m, f := (f_1, \cdots, f_m)$ with $f_k := p_k - s_k q_k, k = 1, \cdots, m$.

We introduce the robust counterpart of the problem $(\text{USIMP})_s$, namely

$$(\text{RUSIMP})_s \quad \begin{array}{l} \min f(x) := (p_1(x) - s_1 q_1(x), \cdots, p_m(x) - s_m q_m(x)) \\ \text{s.t. } x \in C := \{x \in \Omega \mid g_t(x, v_t) \le 0, \forall v_t \in \mathscr{V}_t, t \in T\}. \end{array}$$

Now, we consider the following relation $\star$, which plays a key role in this paper

$$d = a \star b \Leftrightarrow d = (a_1 b_1, \cdots, a_m b_m)$$

for every $a = (a_1, \cdots, a_m) \in \mathbb{R}^m$ and $b = (b_1, \cdots, b_m) \in \mathbb{R}^m$. We can rewrite problem $(\text{RUSIMP})_s$ as follows:

$$(\text{RUSIMP})_s \quad \begin{array}{l} \min f(x) := p(x) - s \star q(x) \\ \text{s.t. } x \in C := \{x \in \Omega \mid g_t(x, v_t) \le 0, \forall v_t \in \mathscr{V}_t, t \in T\}. \end{array}$$

3. Optimality Conditions

First, we introduce the concept of the robust ε-quasi positively properly efficient solution for problem $(\text{USIMP})_s$ as follows.

Definition 3.1. Let $\varepsilon := (\varepsilon_1, \cdots, \varepsilon_m) \in \mathbb{R}^m_+ \backslash \{0\}$. A point $\bar{x} \in C$ is said to be

(i) a robust ε-quasi efficient solution to problem $(\text{USIMP})_s$ if

$$(p(x) - s \star p(x)) - (p(\bar{x}) - s \star q(\bar{x})) + \varepsilon ||x - \bar{x}|| \notin -\mathbb{R}^m_+ \backslash \{0\}, \forall x \in C.$$

(ii) a robust ε-quasi positively properly efficient solution to problem $(\text{USIMP})_s$ if there exists $\beta := (\beta_1, \cdots, \beta_m) \in \text{int}\mathbb{R}^m_+$ such that

$$\langle \beta, (p(x) - s \star p(x)) + \varepsilon ||x - \bar{x}|| \rangle \ge \langle \beta, (p(\bar{x}) - s \star p(\bar{x})) \rangle, \forall x \in C.$$

Now, we propose a constraint qualification as follows.

Definition 3.2. Let $\bar{x} \in C$. We say that the following robust constraint qualification (RCQ) is satisfied at $\bar{x}$ if

$$N(\bar{x}; C) \subseteq \bigcup_{\substack{\lambda \in A(\bar{x}) \\ v_t \in \mathscr{V}_t}} \left[\sum_{t \in T} \lambda_t \partial_x g_t(\bar{x}, v_t) \right] + N(\bar{x}; \Omega),$$

where

$$A(\bar{x}) := \{\lambda \in \mathbb{R}^{(T)}_+ \mid \lambda_t g_t(\bar{x}, v_t) = 0, \forall v_t \in \mathscr{V}_t, t \in T\} \tag{3.1}$$

is set of active constraint multipliers at $\bar{x} \in \Omega$.

Remark 3.1. It is worth to observe here that the condition (RCQ) in Definition 3.2 is an extension of the Definition 3.2 in [1, 19].

Now, we propose a necessary optimality condition for a robust ε-quasi positively properly efficient solution to the problem $(\text{USIMP})_s$ under the condition (RCQ).

Theorem 3.1. *Let $\bar{x} \in C$ be a robust ε-quasi positively properly efficient solution to problem $(\text{USIMP})_s$. Suppose that the condition (RCQ) at $\bar{x}$ holds. Then, there exist $\beta \in \text{int}\mathbb{R}^m_+, v_t \in \mathscr{V}_t, t \in T$ and $\lambda \in A(\bar{x})$ defined in (3.1) such that*

$$0 \in \sum_{k=1}^m \beta_k [\partial p_k(\bar{x}) - s_k \partial q_k(\bar{x})] + \sum_{t \in T} \lambda_t \partial_x g_t(\bar{x}, v_t) + \sum_{k=1}^m \beta_k \varepsilon_k \mathbb{B}_{\mathbb{R}^n} + N(\bar{x}; \Omega).$$

Proof. Let $\bar{x} \in C$ be a robust ε-quasi positively properly efficient solution to problem $(\text{USIMP})_s$. Then there exists $\beta := (\beta_1, \cdots, \beta_m) \in \text{int}\mathbb{R}^m_+$ such that

$$\langle \beta, p(x) - s \star q(x) + \varepsilon ||x - \bar{x}|| \rangle \geq \langle \beta, p(\bar{x}) - s \star q(\bar{x}) \rangle, \forall x \in C.$$

Then, for all $x \in C$,

$$\sum_{k=1}^{m} \beta_k [p_k(x) - s_k q_k(x)] + \sum_{k=1}^{m} \beta_k \varepsilon_k ||x - \bar{x}|| \geq \sum_{k=1}^{m} \beta_k [p_k(\bar{x}) - s_k q_k(\bar{x})]. \tag{3.2}$$

For any $x \in \mathbb{R}^n$, set

$$\Phi(x) := \sum_{k=1}^{m} \beta_k [p_k(x) - s_k q_k(x)] + \sum_{k=1}^{m} \beta_k \varepsilon_k ||x - \bar{x}||.$$

From (3.2), we deduce that $\bar{x}$ is a robust minimizer of the following scalar optimization problem

$$\min_{x \in C} \Phi(x).$$

It follows that $\bar{x}$ is a robust minimizer of the following unconstrained optimization problem

$$\min_{x \in \mathbb{R}^n} \{\Phi(x) + \delta(x; C)\}.$$

We deduce from Lemma 2.1 that

$$0 \in \partial (\Phi + \delta(\cdot; C)) (\bar{x}.) \tag{3.3}$$

Since function Φ is Lipschitz continuous around $\bar{x}$ and the function $\delta(\cdot; C)$ is lower semi-continuous around $\bar{x}$, we deduce from $\partial \delta(\bar{x}; C) = N(\bar{x}; C)$, (3.3) and Lemma 2.2 that

$$0 \in \partial \Phi(\bar{x}) + \partial \delta(\bar{x}; C) = \partial \Phi(\bar{x}) + N(\bar{x}; C). \tag{3.4}$$

Then we have

$$\partial \Phi(\bar{x}) = \partial \left[\sum_{k=1}^{m} \beta_k (p_k(\cdot) - s_k q_k(\cdot)) \right] (\bar{x}) = \sum_{k=1}^{m} \beta_k [\partial p_k(\bar{x}) - s_k \partial q_k(\bar{x})]. \tag{3.5}$$

Note further that we have $\partial(||\cdot - \bar{x}||)(\bar{x}) = \mathbb{B}_{\mathbb{R}^n}$. Because condition (RCQ) holds at $\bar{x} \in C$, so one has

$$N(\bar{x}; C) \subseteq \bigcup_{\substack{\lambda \in A(\bar{x}) \\ v_t \in \mathscr{V}_t}} \left[\sum_{t \in T} \lambda_t \partial_x g_t(\bar{x}, v_t) \right] + N(\bar{x}; \Omega), \tag{3.6}$$

where

$$A(\bar{x}) := \{\lambda \in \mathbb{R}^{(T)}_+ \mid \lambda_t g_t(\bar{x}, v_t) = 0, \forall v_t \in \mathscr{V}_t, t \in T\}.$$

It yields from (3.4)-(3.6) that

$$0 \in \sum_{k=1}^{m} \beta_k [\partial p_k(\bar{x}) - s_k \partial q_k(\bar{x})] + \bigcup_{\substack{\lambda \in A(\bar{x}) \\ v_t \in \mathscr{V}_t}} \left[\sum_{t \in T} \lambda_t \partial_x g_t(\bar{x}, v_t) \right] + \sum_{k=1}^{m} \beta_k \varepsilon_k \mathbb{B}_{\mathbb{R}^n} + N(\bar{x}; \Omega).$$

Therefore, it is clear that there exist $\beta := (\beta_1, \cdots, \beta_m) \in \text{int}\mathbb{R}^m_+, v_t \in \mathscr{V}_t, t \in T$ and $\lambda \in A(\bar{x})$ defined in (3.1) such that

$$0 \in \sum_{k=1}^{m} \beta_k [\partial p_k(\bar{x}) - s_k \partial q_k(\bar{x})] + \sum_{t \in T} \lambda_t \partial_x g_t(\bar{x}, v_t) + \sum_{k=1}^{m} \beta_k \varepsilon_k \mathbb{B}_{\mathbb{R}^n} + N(\bar{x}; \Omega).$$

The proof is complete. □

The following simple example shows that condition (RCQ) is essential in Theorem 3.1.

Example 3.1. Let $f:\mathbb{R}\to\mathbb{R}^2$ be defined by $f(x)=(p_1(x)-s_1q_1(x),p_2(x)-s_2q_2(x))$ with $p_1(x)=p_2(x)=x+1$ and $q_1(x)=q_2(x)=x^2+1, x\in\mathbb{R}$. Take $T=[0,1], \mathscr{V}_t=[2-t,2+t], t\in T$ and let $g_t:\mathbb{R}\times\mathscr{V}_t\to\mathbb{R}$ be given by $g_t(x,v_t)=-v_tx, x\in\mathbb{R},\ v_t\in\mathscr{V}_t,\ t\in T$. We consider the problem $(\text{USIMP})_s$ with $m=2$ and $\Omega=(-\infty,0]\subset\mathbb{R}$. By simple computation, one has $C=\{0\}$. Now, take $\bar{x}=0\in C$ and $s_1=s_2=0, (\beta_1,\beta_2)\in\text{int}\mathbb{R}^2_+, \beta_1+\beta_2=1, \varepsilon_1=\varepsilon_2=\frac{1}{2}$. Then, it is easy to see that $\bar{x}$ is a robust ε-quasi positively properly efficient solution to problem $(\text{USIMP})_s$. Indeed, we have, for all $x\in C$,

$$\sum_{k=1}^{2}\beta_k\left(p_k(x)-s_kq_k(x)\right)+\sum_{k=1}^{2}\beta_k\varepsilon_k||x-\bar{x}||=x+1+\frac{1}{2}|x|\geq 1=\sum_{k=1}^{2}\beta_k\left(p_k(\bar{x})-s_kq_k(\bar{x})\right).$$

On the other hand, taking $\bar{x}=0, s_1=s_2=0, (\beta_1,\beta_2)\in\text{int}\mathbb{R}^2_+, \beta_1+\beta_2=1, \varepsilon_1=\varepsilon_2=\frac{1}{2}$, $\mathbb{B}_{\mathbb{R}}=[-1,1], \lambda_t=0, t\in T$, we have $N(\bar{x};\Omega)=N(\bar{x};(-\infty,0])=[0,+\infty)$ and $\partial p_k(\bar{x})=\{0\}, \partial q_k(\bar{x})=\{0\}, k=1,2, \partial_x g_t(\bar{x},v_t)=\{-v_t\}, v_t\in\mathscr{V}_t, t\in T$. It is easy to see that

$$\begin{aligned}&0\notin\{1\}+\left[-\tfrac{1}{2},\tfrac{1}{2}\right]+[0,+\infty)\\&=\sum_{k=1}^{2}\beta_k\left[\partial p_k(\bar{x})-s_k\partial q_k(\bar{x})\right]+\sum_{t\in T}\lambda_t\partial_x g_t(\bar{x},v_t)+\sum_{k=1}^{2}\beta_k\varepsilon_k\mathbb{B}_{\mathbb{R}}+N(\bar{x};\Omega),\end{aligned}$$

$\lambda_t g_t(\bar{x},v_t)=0, v_t\in\mathscr{V}_t, t\in T$. The reason is that condition (RCQ) is not satisfied at $\bar{x}=0$. Indeed, one has

$$\bigcup_{\substack{\lambda\in A(\bar{x})\\ v_t\in\mathscr{V}_t}}\left[\sum_{t\in T}\lambda_t\partial_x g_t(\bar{x},v_t)\right]+N(\bar{x};\Omega)=[0,+\infty)$$

and $N(\bar{x};C)=N(\bar{x};\{0\})=\mathbb{R}$. Hence condition (RCQ) is not satisfied at $\bar{x}=0$.

Remark 3.2. Theorem 3.1 improves [13, Theorem 3.9], [19, Theorem 3.3], and [26, Theorem 2].

Now, we introduce a concept of the robust ε-quasi (KKT) condition for problem $(\text{USIMP})_s$.

Definition 3.3. A point $\bar{x}\in C$ is said to satisfy the robust ε-quasi (KKT) condition with respect to problem $(\text{USIMP})_s$ if there exist $\beta\in\text{int}\mathbb{R}^m_+, v_t\in\mathscr{V}_t, t\in T$ and $\lambda\in A(\bar{x})$ defined in (3.1) such that

$$0\in\sum_{k=1}^{m}\beta_k\left[\partial p_k(\bar{x})-s_k\partial q_k(\bar{x})\right]+\sum_{t\in T}\lambda_t\partial_x g_t(\bar{x},v_t)+\sum_{k=1}^{m}\beta_k\varepsilon_k\mathbb{B}_{\mathbb{R}^n}+N(\bar{x};\Omega).$$

The following simple example proves that a point satisfying the robust ε-quasi (KKT) condition is not necessarily a robust ε-quasi positively properly efficient solution to problem $(\text{USIMP})_s$ even in the smooth case.

Example 3.2. Let $f:\mathbb{R}\to\mathbb{R}^2$ be defined by $f(x)=(p_1(x)-s_1q_1(x),p_2(x)-s_2q_2(x))$ with $p_1(x)=p_2(x)=x^3$ and $q_1(x)=q_2(x)=x^2+1, x\in\mathbb{R}$. Take $T=[0,1], \mathscr{V}_t=[2-t,2+t], t\in T$ and let $g_t:\mathbb{R}\times\mathscr{V}_t\to\mathbb{R}$ be given by $g_t(x,v_t)=tx^2+2v_tx, x\in\mathbb{R},\ v_t\in\mathscr{V}_t,\ t\in T$. We consider the problem $(\text{USIMP})_s$ with $m=2$ and $\Omega=(-\infty,0]\subset\mathbb{R}$. By simple computation, one has $C=[-2,0]$. By choosing $\bar{x}=0\in C$, we have $N(\bar{x};\Omega)=N(\bar{x};(-\infty,0])=[0,+\infty)$ and $\partial p_k(\bar{x})=\{0\}, \partial q_k(\bar{x})=\{0\}, k=1,2,\ \partial_x g_t(\bar{x},v_t)=\{2v_t\}, v_t\in\mathscr{V}_t, t\in T$. On the other hand, taking $s_1=$

$s_2=0,\beta=(\beta_1,\beta_2)\in \text{int}\mathbb{R}^2_+,\beta_1+\beta_2=1,\varepsilon_1=\varepsilon_2=\frac{1}{2},\lambda_t=0,t\in T,\mathbb{B}_{\mathbb{R}}=[-1,1]$, it is easy to see that

$$\begin{aligned}&0\in\left[-\tfrac{1}{2},\tfrac{1}{2}\right]+[0,+\infty)\\&=\sum_{k=1}^{2}\beta_k\left[\partial p_k(\bar{x})-s_k\partial q_k(\bar{x})\right]+\sum_{t\in T}\lambda_t\partial_x g_t(\bar{x},v_t)+\sum_{k=1}^{2}\beta_k\varepsilon_k\mathbb{B}_{\mathbb{R}}+N(\bar{x};\Omega),\end{aligned}$$

$\lambda_t g_t(\bar{x},v_t)=0,v_t\in\mathscr{V}_t,t\in T$. Thus, the robust ε-quasi (KKT) condition is satisfied at $\bar{x}=0$. However, $\bar{x}=0\in C$ is not a robust ε-quasi positively properly efficient solution to problem $(\text{USIMP})_s$. To see this, we can choose $x=-2\in C$ and $\beta=(\beta_1,\beta_2)\in\text{int}\mathbb{R}^2_+,\beta_1+\beta_2=1,\varepsilon_1=\varepsilon_2=\frac{1}{2},s_1=s_2=0$. Then, it is easy to see that

$$\sum_{k=1}^{2}\beta_k\left[p_k(x)-s_kq_k(x)\right]+\sum_{k=1}^{2}\beta_k\varepsilon_k||x-\bar{x}||=-7<0=\sum_{k=1}^{2}\beta_k\left[p_k(\bar{x})-s_kq_k(\bar{x})\right].$$

Before we discuss sufficient condition for a robust ε-quasi positively properly efficient solution of problem $(\text{USIMP})_s$, we introduce the concepts of convexity, which are inspired by [26].

Definition 3.4. The locally Lipschitz functions $g_t:\mathbb{R}^n\times\mathscr{V}_t\to\mathbb{R},t\in T$ are said to be quasi-convex on Ω at $\bar{x}\in\Omega$ if, for all $x\in\Omega$, $g_t(x,v_t)\le g_t(\bar{x},v_t)\Rightarrow\langle x_t^*,x-\bar{x}\rangle\le 0$ for all $x_t^*\in\partial_x g_t(\bar{x},v_t),v_t\in\mathscr{V}_t,t\in T$.

Definition 3.5. We say that $p-s\star q$ is ε-quasi pseudo-convex on Ω at $\bar{x}\in\Omega$ if, for all $x\in\Omega$, there exist $x_k^*\in\partial p_k(\bar{x}),z_k^*\in\partial q_k(\bar{x}),k=1,\cdots,m$ such that

$$\begin{aligned}&\left\langle x_k^*-s_kz_k^*,x-\bar{x}\right\rangle+\varepsilon_k||x-\bar{x}||\ge 0\\&\Rightarrow p_k(x)-s_kq_k(x)+\varepsilon_k||x-\bar{x}||\ge p_k(\bar{x})-s_kq_k(\bar{x}),k=1,\cdots,m.\end{aligned}$$

Now, we give a sufficient condition for a robust ε-quasi positively properly efficient solution of problem $(\text{USIMP})_s$.

Theorem 3.2. *Assume that Ω is a convex set and $\bar{x}\in C$ satisfies ε-quasi robust* (KKT) *condition. If $p-s\star q$ is ε-quasi pseudo-convex on Ω at $\bar{x}$, and $g_t,t\in T$ are quasi-convex on Ω at $\bar{x}$, then $\bar{x}\in C$ is a robust ε-quasi positively properly efficient solution to problem* $(\text{USIMP})_s$.

Proof. Since $\bar{x}\in C$ satisfies robust ε-quasi (KKT) condition, then there exist $\beta:=(\beta_1,\cdots,\beta_m)\in\text{int}\mathbb{R}^m_+,\lambda\in\mathbb{R}^{(T)}_+$, $x_k^*\in\partial p_k(\bar{x}),z_k^*\in\partial q_k(\bar{x}),k=1,\cdots,m,x_t^*\in\partial_x g_t(\bar{x},v_t),v_t\in\mathscr{V}_t,t\in T$, and $b^*\in\mathbb{B}_{\mathbb{R}^n},w^*\in N(\bar{x};\Omega)$ such that

$$\sum_{k=1}^{m}\beta_k\left(x_k^*-s_kz_k^*\right)+\sum_{t\in T}\lambda_tx_t^*+\sum_{k=1}^{m}\beta_k\varepsilon_kb^*+w^*=0 \tag{3.7}$$

and

$$\lambda_tg_t(\bar{x},v_t)=0,\forall t\in T. \tag{3.8}$$

It follows from (3.7) that (for such $x\in C$)

$$\begin{aligned}&\left\langle\sum_{k=1}^{m}\beta_k\left(x_k^*-s_kz_k^*\right),x-\bar{x}\right\rangle+\left\langle\sum_{t\in T}\lambda_tx_t^*,x-\bar{x}\right\rangle+\left\langle\sum_{k=1}^{m}\beta_k\varepsilon_kb^*,x-\bar{x}\right\rangle\\&+\langle w^*,x-\bar{x}\rangle=0.\end{aligned} \tag{3.9}$$

Since Ω is a convex set and $w^* \in N(\bar{x};\Omega)$, it follows that, for any $x \in \Omega$, $\langle w^*, x-\bar{x}\rangle \leq 0$. Now, taking arbitrarily $x \in C$, we see that there exists $b^* \in \mathbb{R}_{\mathbb{R}^n}$ such that $||x-\bar{x}|| = \langle b^*, x-\bar{x}\rangle$. From (3.9), it follows that

$$\left\langle \sum_{k=1}^{m} \beta_k \left(x_k^* - s_k z_k^*\right), x-\bar{x} \right\rangle + \left\langle \sum_{t\in T} \lambda_t x_t^*, x-\bar{x} \right\rangle + \sum_{k=1}^{m} \beta_k \varepsilon_k ||x-\bar{x}|| \geq 0,$$

which means that

$$\left\langle \sum_{k=1}^{m} \beta_k \left(x_k^* - s_k z_k^*\right), x-\bar{x} \right\rangle + \sum_{k=1}^{m} \beta_k \varepsilon_k ||x-\bar{x}|| \geq -\left\langle \sum_{t\in T} \lambda_t x_t^*, x-\bar{x} \right\rangle. \tag{3.10}$$

Note that, for any $x \in C$, $\lambda_t g_t(x,v_t) \leq 0$ for any $v_t \in \mathscr{V}_t, t \in T$. It follows from (3.8) that $\lambda_t g_t(x,v_t) \leq 0 = \lambda_t g_t(\bar{x},v_t)$. By g_t is quasi-convex on Ω at $\bar{x}$ and $x_t^* \in \partial_x g_t(\bar{x},v_t), v_t \in \mathscr{V}_t, t \in T$, we obtain $\langle \lambda_t x_t^*, x-\bar{x}\rangle \leq 0$ for all $t \in T$. Thus it is easy to yield that $\langle \sum_{t\in T} \lambda_t x_t^*, x-\bar{x}\rangle \leq 0$, which together with (3.10) yileds that

$$\left\langle \sum_{k=1}^{m} \beta_k (x_k^* - s_k z_k^*), x-\bar{x} \right\rangle + \sum_{k=1}^{m} \beta_k \varepsilon_k ||x-\bar{x}|| \geq 0.$$

Since $p - s \star q$ is ε-quasi pseudo-convex on Ω at $\bar{x}$, it follows that

$$\sum_{k=1}^{m} \beta_k \left(p_k(x) - s_k q_k(x)\right) + \sum_{k=1}^{m} \beta_k \varepsilon_k ||x-\bar{x}|| \geq \sum_{k=1}^{m} \beta_k \left(p_k(\bar{x}) - s_k q_k(\bar{x})\right).$$

This follows that there exists $\beta := (\beta_1, \cdots, \beta_m) \in \text{int}\mathbb{R}_+^m$ such that

$$\langle \beta, (p(x) - s \star p(x)) + \varepsilon ||x-\bar{x}||\rangle \geq \langle \beta, (p(\bar{x}) - s \star p(\bar{x}))\rangle, \forall x \in C.$$

Therefore, $\bar{x}$ is a robust ε-quasi positively properly efficient solution to problem (USIMP)$_s$. □

Now, we present an example to show the importance of the ε-quasi pseudo-convexity in Theorem 3.2.

Example 3.3. Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f(x) = (p_1(x) - s_1 q_1(x), p_2(x) - s_2 q_2(x))$ with $p_1(x) = p_2(x) = x^3 - 1$ and $q_1(x) = q_2(x) = x^4 + 1, x \in \mathbb{R}$. Take $T = [0,1], \mathscr{V}_t = [2-t, 2+t], t \in T$ and let $g_t : \mathbb{R} \times \mathscr{V}_t \to \mathbb{R}$ be given by $g_t(x,v_t) = -v_t x^2, x \in \mathbb{R}$, $v_t \in \mathscr{V}_t$, $t \in T$. We consider the problem (USIMP)$_s$ with $m = 2$ and $\Omega = (-\infty, 0] \subset \mathbb{R}$. By selecting $\bar{x} = 0 \in \Omega$ and by simple computation, one has $C = (-\infty, 0]$, $N(\bar{x};\Omega) = N(\bar{x};(-\infty,0]) = [0,+\infty)$,

$$\partial p_k(\bar{x}) = \{0\}, \partial q_k(\bar{x}) = \{0\}, k = 1,2 \text{ and } \partial_x g_t(\bar{x},v_t) = \{0\}, v_t \in \mathscr{V}_t, t \in T.$$

It is easy to follow that $g_t, t \in T$ is quasi-convex on Ω at $\bar{x}$. Indeed, one has

$$g_t(x,v_t) = -v_t x^2 \leq 0 = g_t(\bar{x},v_t), v_t \in \mathscr{V}_t, t \in T$$

$$\Rightarrow \langle x_t^*, x-\bar{x}\rangle = 0 \leq 0, \forall x_t^* \in \partial_x g_t(\bar{x},v_t) = \{0\}, \forall x \in \Omega, t \in T.$$

It is it easy to imply that $\bar{x} = 0$ satisfies the robust $\varepsilon-$quasi (KKT) condition. Indeed, let us select $s_1 = s_2 = 0, \varepsilon_1 = \varepsilon_2 = \frac{1}{2}, \mathbb{B}_{\mathbb{R}} = [-1,1], \beta = (\beta_1, \beta_2) \in \text{int}\mathbb{R}_+^2$ with $\beta_1 + \beta_2 = 1$, one has

$$0 \in \left[-\frac{1}{2}, \frac{1}{2}\right] + [0,+\infty) = \sum_{k=1}^{2} \beta_k \left[\partial p_k(\bar{x}) - s_k \partial q_k(\bar{x})\right] + \sum_{t\in T} \lambda_t \partial_x g_t(\bar{x},v_t) + \sum_{k=1}^{2} \beta_k \varepsilon_k \mathbb{B}_{\mathbb{R}} + N(\bar{x};\Omega),$$

$\lambda_t g_t(\bar{x}, v_t) = 0, v_t \in \mathcal{V}_t, t \in T$. However, $\bar{x}$ is not a robust ε-quasi positively properly efficient of problem $(\text{USIMP})_s$. In order to see this, taking $s_1 = s_2 = 0, \varepsilon_1 = \varepsilon_2 = \frac{1}{2}, \hat{x} = -2 \in C$ and $\beta = (\beta_1, \beta_2) \in \text{int}\mathbb{R}^2_+$ with $\beta_1 + \beta_2 = 1$, we have

$$\sum_{k=1}^{2} \beta_k [p_k(\hat{x}) - s_k q_k(\hat{x})] + \sum_{k=1}^{2} \beta_k \varepsilon_k ||\hat{x} - \bar{x}|| = -8 < -1 = \sum_{k=1}^{2} \beta_k [p_k(\bar{x}) - s_k q_k(\bar{x})].$$

The reason is that $p - s \star q$ is not ε-quasi pseudo-convex on Ω at $\bar{x} = 0$. Indeed, taking $s_1 = s_2 = 0, \varepsilon_1 = \varepsilon_2 = \frac{1}{2}, x = -3 \in \Omega, x_k^* \in \partial p_k(\bar{x}) = \{0\}$ and $z_k^* \in \partial q_k(\bar{x}) = \{0\}, k = 1, 2$, we see that

$$p_k(x) - s_k q_k(x) + \varepsilon_k ||x - \bar{x}|| = x^3 - 1 + \frac{1}{2}|x| = -\frac{53}{2} < -1 = p_k(\bar{x}) - s_k q_k(\bar{x}), k = 1, 2.$$

However, $\langle x_k^* - s_k z_k^*, x - \bar{x} \rangle + \varepsilon_k ||x - \bar{x}|| = \frac{3}{2} > 0, k = 1, 2.$

Remark 3.3. Theorem 3.2 improves [26, Theorem 3].

4. Duality Theorems

4.1. **Wolfe type duality.** In this section, we consider the dual problem of the problem $(\text{USIMP})_s$ in the Wolfe type.

For $x \in \mathbb{R}^n, \beta := (\beta_1, \cdots, \beta_m) \in \text{int}\mathbb{R}^m_+$ with $\sum_{k=1}^m \beta_k = 1, v_t \in \mathcal{V}_t, t \in T$ and $\lambda \in \mathbb{R}^{(T)}_+, p := (p_1, \cdots, p_m), q := (q_1, \cdots, q_m), g_T := (g_t)_{t \in T}, s := (s_1, \cdots, s_m) \in \mathbb{R}^m_-$, Ω a nonempty locally closed subset of $\mathbb{R}^n$, let us denote a vector function $L^s := (L_1^{s_1}, \cdots, L_m^{s_m})$ by

$$L^s(x, v_t, \beta, \lambda) := p(x) - s \star q(x) + \sum_{t \in T} \lambda_t g_t(x, v_t) e,$$

where $e := (1, \cdots, 1) \in \mathbb{R}^m$. For $\varepsilon := (\varepsilon_1, \cdots, \varepsilon_m) \in \mathbb{R}^m_+ \backslash \{0\}$, we consider the Wolfe type dual problem $(\text{UWD})_s$ with respect to the primal problem $(\text{USIMP})_s$ as follows:

$$(\text{UWD})_s \quad \begin{cases} \max & L^s(y, v_t, \beta, \lambda) \\ \text{s.t.} & 0 \in \sum_{k=1}^m \beta_k (\partial p_k(y) - s_k \partial q_k(y)) + \sum_{t \in T} \lambda_t \partial_x g_t(y, v_t) \\ & + \sum_{k=1}^m \beta_k \varepsilon_k \mathbb{B}_{\mathbb{R}^n} + N(y; \Omega), \\ & y \in \Omega, \lambda \in \mathbb{R}^{(T)}_+, \beta \in \text{int}\mathbb{R}^m_+, \sum_{k=1}^m \beta_k = 1. \end{cases}$$

The optimistic counterpart, say $(\text{OUWD})_s$ of problem $(\text{UWD})_s$ (also known as a Wolfe type optimistic dual optimization problem) is a deterministic maximization problem given by

$$(\text{OUWD})_s \quad \begin{cases} \max & L^s(y, v_t, \beta, \lambda) \\ \text{s.t.} & 0 \in \sum_{k=1}^m \beta_k (\partial p_k(y) - s_k \partial q_k(y)) + \sum_{t \in T} \lambda_t \partial_x g_t(y, v_t) \\ & + \sum_{k=1}^m \beta_k \varepsilon_k \mathbb{B}_{\mathbb{R}^n} + N(y; \Omega), \\ & y \in \Omega, \lambda \in \mathbb{R}^{(T)}_+, \beta \in \text{int}\mathbb{R}^m_+, \sum_{k=1}^m \beta_k = 1, \forall v_t \in \mathcal{V}_t, t \in T. \end{cases}$$

The feasible set of problem (OUWD)$_s$ is defined by

$$C_{\text{OUWD}} := \{(y, v_t, \beta, \lambda) \in \Omega \times \mathscr{V}_t \times \text{int}\mathbb{R}^m_+ \times \mathbb{R}^{(T)}_+ \mid 0 \in \sum_{k=1}^m \beta_k \left(\partial p_k(y) - s_k \partial q_k(y)\right)$$
$$+ \sum_{t\in T} \lambda_t \partial_x g_t(y, v_t) + \sum_{k=1}^m \beta_k \varepsilon_k \mathbb{B}_{\mathbb{R}^n} + N(y;\Omega), \sum_{k=1}^m \beta_k = 1\}.$$

In what follows, we use the following notation for convenience:

$$u \preceq v \Leftrightarrow u - v \in -\mathbb{R}^m_+ \backslash \{0\}, \ \ u \npreceq v \text{ is the negation of } u \preceq v.$$

Now, we introduce the definition of the robust ε-quasi efficient solution / the robust ε-quasi positively properly efficient solution for problem (OUWD)$_s$.

Definition 4.1. A point $(\bar{y}, \bar{v}_t, \bar{\beta}, \bar{\lambda}) \in C_{\text{OUWD}}$ is said to be

(i) a robust ε-quasi efficient solution to problem (OUWD)$_s$ if

$$L^s(y, v_t, \beta, \lambda) - L^s(\bar{y}, \bar{v}_t, \bar{\beta}, \bar{\lambda}) - \varepsilon\|\bar{y} - y\| \notin \mathbb{R}^m_+ \backslash \{0\}, \forall (y, v_t, \beta, \lambda) \in C_{\text{OUWD}}.$$

(ii) a robust ε-quasi positively properly efficient solution to problem (OUWD) if there exists $\theta := (\theta_1, \cdots, \theta_m) \in -\text{int}\mathbb{R}^m_+$ such that

$$\langle \theta, L^s(y, v_t, \beta, \lambda) + \varepsilon\|x - \bar{x}\| \rangle \geq \left\langle \theta, L^s(\bar{y}, \bar{v}_t, \bar{\beta}, \bar{\lambda}) \right\rangle, \forall (y, v_t, \beta, \lambda) \in C_{\text{OUWD}}.$$

Motivated by the definition of generalized convexity due to [42], we introduce the concepts as follows:

Definition 4.2. We say that $(p - s \star q, g_T)$ is generalized convex on Ω at $\bar{x} \in \Omega$ if, for any $x \in \Omega$, $x^*_k \in \partial p_k(\bar{x}), z^*_k \in \partial q_k(\bar{x}), k = 1, \cdots, m$ and $x^*_t \in \partial_x g_t(\bar{x}, v_t), v_t \in \mathscr{V}_t, t \in T$, there exists $w \in N(\bar{x};\Omega)^\circ$ such that $(p_k(x) - s_k q_k(x)) - (p_k(\bar{x}) - s_k q_k(\bar{x})) \geq \left\langle x^*_k - s_k z^*_k, w \right\rangle$, $k = 1, \cdots, m$, $g_t(x) - g_t(\bar{x}) \geq \langle x^*_t, w \rangle$ for all $t \in T$ and $\langle b^*, w \rangle \leq \|x - \bar{x}\|$ for all $b^* \in \mathbb{B}_{\mathbb{R}^n}$.

Remark 4.1. Note that if Ω is a convex set and $p_k, q_k, k = 1, \cdots, m, g_t, t \in T$ are convex functions, then $(p - s \star q, g_T)$ is generalized convex on Ω at any $\bar{x} \in \Omega$ with $w := x - \bar{x}$ for each $x \in \Omega$. In addition, by a similar argument in [42, Example 3], we can prove that the class of generalized convex functions is properly larger than the one of convex functions.

Theorem 4.1. **(ε-quasi weak robust duality)** *Suppose that* $x \in C$ *and* $(y, v_t, \beta, \lambda) \in C_{\text{OUWD}}$. *If* $(p - s \star q, g_T)$ *is generalized convex on* Ω *at* y, *then* $p(x) - s \star q(x) \npreceq L^s(y, v_t, \beta, \lambda) - \varepsilon\|x - y\|$.

Proof. Since $(y, v_t, \beta, \lambda) \in C_{\text{OUWD}}$, there exist $x^*_k \in \partial p_k(y), z^*_k \in \partial q_k(y), k = 1, \cdots, m$, $\beta \in \text{int}\mathbb{R}^m_+$ with $\sum_{k=1}^m \beta_k = 1$ and $x^*_t \in \partial_x g_t(y, v_t), v_t \in \mathscr{V}_t, t \in T$, $\lambda \in \mathbb{R}^{(T)}_+$, as well as $b^* \in \mathbb{B}_{\mathbb{R}^n}$ such that

$$-\left(\sum_{k=1}^m \beta_k (x^*_k - s_k z^*_k) + \sum_{t\in T} \lambda_t x^*_t + \sum_{k=1}^m \beta_k \varepsilon_k b^*\right) \in N(y;\Omega). \tag{4.1}$$

Let $x \in C$. Suppose on contrary that $p(x) - s \star q(x) \preceq L^s(y, v_t, \beta, \lambda) - \varepsilon\|x - y\|$, which together with $\beta := (\beta_1, \cdots, \beta_m) \in \text{int}\mathbb{R}^m_+$ yield that $\langle \beta, p(x) - s \star q(x) - L^s(y, v_t, \beta, \lambda) + \varepsilon\|x - y\| \rangle < 0$. Thus

$$\sum_{k=1}^m \beta_k \left[(p_k(x) - s_k q_k(x)) - (p_k(y) - s_k q_k(y))\right] + \sum_{k=1}^m \beta_k \varepsilon_k \|x - y\| - \sum_{t\in T} \lambda_t g_t(y, v_t) < 0. \tag{4.2}$$

By the definition of polar cone (2.1), the generalized convexity of $(p-s\star q, g_T)$ on Ω at y and (4.1), for such x, there exists $w \in N(y;\Omega)^\circ$ such that

$$\begin{aligned} 0 &\leq \sum_{k=1}^{m} \beta_k \langle x_k^* - s_k z_k^*, w\rangle + \sum_{t\in T} \lambda_t \langle x_t^*, w\rangle + \sum_{k=1}^{m} \beta_k \varepsilon_k \langle b^*, w\rangle \\ &\leq \sum_{k=1}^{m} \beta_k [(p_k(x) - s_k q_k(x)) - (p_k(y) - s_k q_k(y))] + \sum_{t\in T} \lambda_t [g_t(x,v_t) - g_t(y,v_t)] \\ &+ \sum_{k=1}^{m} \beta_k \varepsilon_k ||x-y||. \end{aligned} \tag{4.3}$$

By $x \in C$, it is obvious that $g_t(x,v_t) \leq 0, \forall v_t \in \mathscr{V}_t, t \in T$. Thus $\sum_{t\in T} \lambda_t g_t(x,v_t) \leq 0$. From (4.3), we can assert that

$$\textstyle\sum_{k=1}^{m} \beta_k [(p_k(x) - s_k q_k(x)) - (p_k(y) - s_k q_k(y))] + \sum_{k=1}^{m} \beta_k \varepsilon_k ||x-y|| - \displaystyle\sum_{t\in T} \lambda_t g_t(y,v_t) \geq 0,$$

which contradicts (4.2). The proof is complete. □

The following example demonstrates that the generalized convexity of $(p - s\star q, g_T)$ imposed in Theorem 4.1 is essential.

Example 4.1. Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f(x) = (p_1(x) - s_1 q_1(x), p_2(x) - s_2 q_2(x))$ with $s = (s_1, s_2) = (-1,-1)$, and $p_1(x) = p_2(x) = x^5$ and $q_1(x) = q_2(x) = x^4 + 1, x \in \mathbb{R}$. Take $T = [0,1], \mathscr{V}_t = [2-t, 2+t], t \in T$ and let $g_t : \mathbb{R} \times \mathscr{V}_t \to \mathbb{R}$ be given by $g_t(x,v_t) = tx^2 + 3v_t x$, $x \in \mathbb{R}$, $v_t \in \mathscr{V}_t$, $t \in T$. We consider the problem $(\text{USIMP})_s$ with $m = 2$ and $\Omega = (-\infty, 0] \subset \mathbb{R}$. By simple computation, one has $C = [-3,0]$. Now, consider the dual problem $(\text{OUWD})_s$. By choosing $\bar{y} = 0 \in \Omega, \bar{\lambda}_t = 0, \bar{v}_t \in \mathscr{V}_t, t \in T, \bar{\beta} = (\bar{\beta}_1, \bar{\beta}_2) \in \text{int}\mathbb{R}^2_+$ with $\bar{\beta}_1 + \bar{\beta}_2 = 1, \varepsilon_1 = \varepsilon_2 = \frac{1}{2}, \mathbb{B}_{\mathbb{R}} = [-1,1]$, we have $N(\bar{y};\Omega) = N(\bar{y};(-\infty,0]) = [0,+\infty)$ and $\partial p_k(\bar{y}) = \{0\}, \partial q_k(\bar{y}) = \{0\}, k = 1,2, \partial_x g_t(\bar{y},\bar{v}_t) = \{3\bar{v}_t\}, \bar{v}_t \in \mathscr{V}_t, t \in T$. It is easy to see that

$$0 \in \left[-\frac{1}{2}, \frac{1}{2}\right] + [0,+\infty) = \sum_{k=1}^{2} \bar{\beta}_k [\partial p_k(\bar{y}) - s_k \partial q_k(\bar{y})] + \sum_{t\in T} \bar{\lambda}_t \partial_x g_t(\bar{y},\bar{v}_t) + \sum_{k=1}^{2} \bar{\beta}_k \varepsilon_k \mathbb{B}_{\mathbb{R}} + N(\bar{y};\Omega),$$

where $\sum_{k=1}^{m} \bar{\beta}_k = 1$. Thus $(\bar{y}, \bar{v}_t, \bar{\beta}, \bar{\lambda}) \in C_{\text{OUWD}}$. However, by choosing $\bar{x} = -2 \in C = (-\infty, 0]$, we see that

$$\begin{aligned} p(\bar{x}) - s\star q(\bar{x}) &= (p_1(\bar{x}) - s_1 q_1(\bar{x}), p_2(\bar{x}) - s_2 q_2(\bar{x})) \\ &= (-15,-15) \\ &\preceq (0,0) \\ &= (L_1^{s_1}(\bar{y},\bar{v}_t,\bar{\beta},\bar{\lambda}) - \varepsilon_1 ||\bar{x}-\bar{y}||, L_2^{s_2}(\bar{y},\bar{v}_t,\bar{\beta},\bar{\lambda}) - \varepsilon_2 ||\bar{x}-\bar{y}||) \\ &= L^s(\bar{y},\bar{v}_t,\bar{\beta},\bar{\lambda}) - \varepsilon ||\bar{x}-\bar{y}||. \end{aligned}$$

The reason is that $(p - s\star q, g_T)$ is not generalized convex on Ω at $\bar{y} \in \Omega$. To see this, we can choose $y = -3 \in \Omega$ and $x_k^* \in \partial p_k(\bar{y}) = \{0\}, z_k^* \in \partial q_k(\bar{y}) = \{0\}, k = 1,2$. Then, it is easy to see that $N(\bar{y};\Omega)^\circ = N(\bar{y};(-\infty,0])^\circ = (-\infty,0]$ and

$$p_k(y) - s_k q_k(y) - [p_k(\bar{y}) - s_k q_k(\bar{y})] = -162 < 0 = \langle x_k^* - s_k z_k^*, w\rangle, \forall w \in N(\bar{y};\Omega)^\circ, k = 1,2.$$

Remark 4.2. Theorem 4.1 improves [13, Theorem 4.1] and [19, Theorem 4.1].

Theorem 4.2. **($\varepsilon-$quasi strong robust duality)** *Let $\bar{x} \in C$ be a robust ε-quasi positively properly efficient solution to problem* $(\text{USIMP})_s$ *such that condition* (RCQ) *is satisfied at $\bar{x}$. Then* *there exists* $(\bar{v}_t, \bar{\beta}, \bar{\lambda}) \in \mathcal{V}_t \times \mathrm{int}\mathbb{R}^m_+ \times \mathbb{R}^{(T)}_+$ *such that* $(\bar{x}, \bar{v}_t, \bar{\beta}, \bar{\lambda}) \in C_{\mathrm{OUWD}}$ *and* $p(\bar{x}) - s \star q(\bar{x}) = L^s(\bar{x}, \bar{v}_t, \bar{\beta}, \bar{\lambda})$. *If, in addition,* $(p - s\star q, g_T)$ *is generalized convex on* Ω *at* $y \in \Omega$*, then* $(\bar{x}, \bar{v}_t, \bar{\beta}, \bar{\lambda})$ *is a robust* ε*-quasi efficient solution to problem* $(\mathrm{OUWD})_s$.

Proof. According to Theorem 3.1, there exist $\beta := (\beta_1, \cdots, \beta_m) \in \mathrm{int}\mathbb{R}^m_+, v_t \in \mathcal{V}_t, t \in T$ and $\lambda \in A(\bar{x})$ defined in (3.1) such that

$$0 \in \sum_{k=1}^m \beta_k(\partial p_k(\bar{x}) - s_k \partial q_k(\bar{x})) + \sum_{t\in T} \lambda_t \partial_x g_t(\bar{x}, v_t) + \sum_{k=1}^m \beta_k \varepsilon_k \mathbb{B}_{\mathbb{R}^n} + N(\bar{x}; \Omega).$$

Putting

$$\bar{\beta}_k := \frac{\beta_k}{\sum\limits_{k=1}^m \beta_k}, k = 1, \cdots, m, \bar{\lambda}_t := \frac{\lambda_t}{\sum\limits_{k=1}^m \beta_k}, \bar{v}_t := \frac{v_t}{\sum\limits_{k=1}^m \beta_k}, t \in T,$$

one has $\bar{\beta} := (\bar{\beta}_1, \cdots, \bar{\beta}_m) \in \mathbb{R}^m_+$ with $\sum_{k=1}^m \bar{\beta}_k = 1$, $\bar{\lambda} := (\bar{\lambda}_t)_{t\in T} \in \mathbb{R}^{(T)}_+, \bar{v}_t \in \mathcal{V}_t, t \in T$. Furthermore, the assertion in (4.9) is also valid when β_k's, λ_t's, and v_t's are replaced by $\bar{\alpha}_k$'s, $\bar{\lambda}_t$'s, and $\bar{v}_t$'s, respectively. Thus $(\bar{x}, \bar{v}_t, \bar{\beta}, \bar{\lambda}) \in C_{\mathrm{OWMD}}$. Besides, since $\lambda \in A(\bar{x})$ is defined in (3.1), one has $\lambda_t g_t(\bar{x}, v_t) = 0$ for all $v_t \in \mathcal{V}_t, t \in T$, which implies that $\sum_{t\in T} \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) = 0$. Therefore, one has

$$p(\bar{x}) - s \star q(\bar{x}) = p(\bar{x}) - s \star q(\bar{x}) + \sum_{t\in T} \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) e = L^s(\bar{x}, \bar{v}_t, \bar{\beta}, \bar{\lambda}).$$

If $(p - s\star q, g_T)$ is generalized convex on Ω at any $y \in \Omega$, then we obtain by Theorem 4.1 that

$$L^s(\bar{x}, \bar{v}_t, \bar{\beta}, \bar{\lambda}) = p(\bar{x}) - s \star q(\bar{x}) \npreceq L^s(y, v_t, \beta, \lambda) - \varepsilon ||\bar{x} - y||,$$

for any $(y, v_t, \beta, \lambda) \in C_{\mathrm{OUWD}}$, which means that $(\bar{x}, \bar{v}_t, \bar{\beta}, \bar{\lambda})$ is a robust ε-quasi positively properly efficient solution to problem $(\mathrm{OUWD})_s$. The proof is complete. □

Note that the conclusion of Theorem 4.2 may fail to hold if condition (RCQ) is not satisfied. To see this, let us look back at Example 3.1.

Remark 4.3. Theorem 4.2 improves [13, Theorem 4.4] and [19, Theorem 4.2].

Theorem 4.3. **($\varepsilon-$quasi converse robust duality)** *Let* $(\bar{x}, \bar{v}_t, \bar{\beta}, \bar{\lambda}) \in C_{\mathrm{OUWD}}$ *such that* $p(\bar{x}) - s \star q(\bar{x}) = L^s(\bar{x}, \bar{v}_t, \bar{\beta}, \bar{\lambda})$. *If* $\bar{x} \in C$ *and* $(p - s\star q, g_T)$ *is generalized convex on* Ω *at* $\bar{x}$*, then* $\bar{x}$ *is a robust* ε*-quasi positively properly efficient solution to problem* $(\mathrm{USIMP})_s$.

Proof. Since $(\bar{x}, \bar{v}_t, \bar{\beta}, \bar{\lambda}) \in C_{\mathrm{OUWD}}$, we see that there exist $x^*_k \in \partial p_k(\bar{x}), z^*_k \in \partial q_k(\bar{x}), k = 1, \cdots, m$, $\bar{\beta} \in \mathrm{int}\mathbb{R}^m_+$ with $\sum_{k=1}^m \bar{\beta}_k = 1$ and $x^*_t \in \partial_x g_t(\bar{x}, \bar{v}_t), \bar{v}_t \in \mathcal{V}_t, t \in T$, $\bar{\lambda} := (\bar{\lambda}_t)_{t\in T} \in \mathbb{R}^{(T)}_+$, as well as $b^* \in \mathbb{B}_{\mathbb{R}^n}$ such that

$$-\left(\sum_{k=1}^m \bar{\beta}_k(x^*_k - s_k z^*_k) + \sum_{t\in T} \bar{\lambda}_t x^*_t + \sum_{k=1}^m \bar{\beta}_k \varepsilon_k b^*\right) \in N(\bar{x}; \Omega). \tag{4.4}$$

Suppose on contrary that $\bar{x} \in C$ is not a robust ε-quasi positively properly efficient solution to problem $(\mathrm{USIMP})_s$. It then follows that there exists $\hat{x} \in C$ satisfying

$$\sum_{k=1}^m \bar{\beta}_k \left[p_k(\hat{x}) - s_k q_k(\hat{x})\right] + \sum_{k=1}^m \bar{\beta}_k \varepsilon_k ||\hat{x} - x|| < \sum_{k=1}^m \bar{\beta}_k \left[p_k(\bar{x}) - s_k q_k(\bar{x})\right]. \tag{4.5}$$

By the generalized convexity of $(p-s\star q, g_T)$ on Ω at $\bar{x}$, for such $\hat{x}$, there exists $w\in N(\bar{x};\Omega)^\circ$ such that

$$(p_k(\hat{x})-s_kq_k(\hat{x}))-(p_k(\bar{x})-s_kq_k(\bar{x}))\geq \langle x_k^*-s_kz_k^*,w\rangle, k=1,\cdots,m \tag{4.6}$$

$$g_t(\hat{x},\bar{v}_t)-g_t(\bar{x},\bar{v}_t)\geq \langle x_t^*,w\rangle, \bar{v}_t\in\mathscr{V}_t, t\in T \tag{4.7}$$

and

$$\langle b^*,w\rangle\leq ||\hat{x}-\bar{x}||, b^*\in\mathbb{B}_{\mathbb{R}^n}. \tag{4.8}$$

Combining (4.5) and (4.6), one has

$$\sum_{k=1}^m\bar{\beta}_k\langle x_k^*-s_kz_k^*,w\rangle+\sum_{k=1}^m\bar{\beta}_k\varepsilon_k||\hat{x}-x||<0. \tag{4.9}$$

In addition, since $p(\bar{x})-s\star q(\bar{x})=L^s(\bar{x},\bar{v}_t,\bar{\beta},\bar{\lambda})=p(\bar{x})-s\star q(\bar{x})+\sum_{t\in T}\bar{\lambda}_tg_t(\bar{x},\bar{v}_t)e$, it follows that $\sum_{t\in T}\bar{\lambda}_tg_t(\bar{x},\bar{v}_t)e=(0,\cdots,0)\in\mathbb{R}^m$. Thus, one has $\sum_{t\in T}\bar{\lambda}_tg_t(\bar{x},\bar{v}_t)=0$. Note that, for any $\hat{x}\in C$, $\sum_{t\in T}\bar{\lambda}_tg_t(\hat{x},\bar{v}_t)\leq 0$. It follows that $\sum_{t\in T}\bar{\lambda}_tg_t(\hat{x},\bar{v}_t)\leq 0=\sum_{t\in T}\bar{\lambda}_tg_t(\bar{x},\bar{v}_t)$, which together with (4.7) implies that $\sum_{t\in T}\bar{\lambda}_t\langle x_t^*,w\rangle\leq 0$. Combining (4.8) and (4.9) yields

$$\sum_{k=1}^m\bar{\beta}_k\langle x_k^*-s_kz_k^*,w\rangle+\sum_{k=1}^m\bar{\beta}_k\varepsilon_k\langle b^*,w\rangle+\sum_{t\in T}\bar{\lambda}_t\langle x_t^*,w\rangle<0. \tag{4.10}$$

On the other hand, by the definition of polar cone (2.1), it yields from (4.4) and the relation $w\in N(\bar{x};\Omega)^\circ$ that

$$\sum_{k=1}^m\bar{\beta}_k\langle x_k^*-s_kz_k^*,w\rangle+\sum_{k=1}^m\bar{\beta}_k\varepsilon_k\langle b^*,w\rangle+\sum_{t\in T}\bar{\lambda}_t\langle x_t^*,w\rangle\geq 0,$$

which contradicts (4.10). The proof is complete. □

By virtue of Example 4.1, we see that the result of Theorem 4.3 may not be valid if the generalized convexity of $(p-s\star q,g_T)$ is not satisfied.

4.2. **Mond-Weir type duality.** In this section, we consider the dual problem of the problem (USIMP)$_s$ in the Mond-Weir type.

For $x\in\mathbb{R}^n,\beta:=(\beta_1,\cdots,\beta_m)\in\text{int}\mathbb{R}^m_+$ with $\sum_{k=1}^m\beta_k=1$ and $\lambda\in\mathbb{R}^{(T)}_+, v_t\in\mathscr{V}_t, t\in T, p:=(p_1,\cdots,p_m), q:=(q_1,\cdots,q_m), s:=(s_1,\cdots,s_m)\in\mathbb{R}^m_-, g_T:=(g_t)_{t\in T}$, Ω, a nonempty locally closed subset of $\mathbb{R}^n$, let us denote a vector function $L^s:=(L_1^{s_1},\cdots,L_m^{s_m})$ by $L^s(x,v_t,\beta,\lambda):=p(x)-s\star q(x)$. For $\varepsilon:=(\varepsilon_1,\cdots,\varepsilon_m)\in\mathbb{R}^m_+\backslash\{0\}$, we consider the Mond–Weir type dual problem (UMWD)$_s$ with respect to its primal problem (USIMP)$_s$ as follows:

$$(\text{UMWD})_s\quad\begin{cases}\max & L^s(y,v_t,\beta,\lambda)\\ \text{s.t.} & 0\in\sum_{k=1}^m\beta_k(\partial p_k(y)-s_k\partial q_k(y))+\sum_{t\in T}\lambda_t\partial_xg_t(y,v_t)\\ & +\sum_{k=1}^m\beta_k\varepsilon_k\mathbb{B}_{\mathbb{R}^n}+N(y;\Omega), \sum_{t\in T}\lambda_tg_t(y,v_t)\geq 0,\\ & y\in\Omega,\lambda\in\mathbb{R}^{(T)}_+,\beta\in\text{int}\mathbb{R}^m_+,\sum_{k=1}^m\beta_k=1.\end{cases}$$

The optimistic counterpart, say $(\text{OUMWD})_s$ of problem $(\text{UMWD})_s$ (also known as a Mond–Weir type optimistic dual optimization problem), is a deterministic maximization problem given by

$$(\text{OUMWD})_s \quad \begin{cases} \max & L^s(y,v_t,\beta,\lambda) \\ \text{s.t.} & 0 \in \sum_{k=1}^{m} \beta_k \left(\partial p_k(y) - s_k \partial q_k(y)\right) + \sum_{t\in T} \lambda_t \partial_x g_t(y,v_t) \\ & + \sum_{k=1}^{m} \beta_k \varepsilon_k \mathbb{B}_{\mathbb{R}^n} + N(y;\Omega), \\ & \sum_{t\in T} \lambda_t g_t(y,v_t) \geq 0, \forall v_t \in \mathscr{V}_t, t \in T, \\ & y \in \Omega, \lambda \in \mathbb{R}_+^{(T)}, \beta \in \text{int}\mathbb{R}_+^m, \sum_{k=1}^{m} \beta_k = 1. \end{cases}$$

The feasible set of problem $(\text{OUMWD})_s$ is defined by

$$C_{\text{OUMWD}} := \{(y,v_t,\beta,\lambda) \in \Omega \times \mathscr{V}_t \times \text{int}\mathbb{R}_+^m \times \mathbb{R}_+^{(T)} \mid 0 \in \sum_{k=1}^{m} \beta_k \left(\partial p_k(y) - s_k \partial q_k(y)\right)$$
$$+ \sum_{t\in T} \lambda_t \partial_x g_t(y,v_t) + \sum_{k=1}^{m} \beta_k \varepsilon_k \mathbb{B}_{\mathbb{R}^n} + N(y;\Omega), \sum_{t\in T} \lambda_t g_t(y,v_t) \geq 0, \sum_{k=1}^{m} \beta_k = 1\}.$$

In what follows, we use the following notation for convenience:

$$u \preceq v \Leftrightarrow u - v \in -\mathbb{R}_+^m \backslash \{0\}, \ \ u \not\preceq v \text{ is the negation of } u \preceq v.$$

Now, we introduce the definition of a robust ε-quasi efficient solution / a robust ε-quasi positively properly efficient solution to problem $(\text{OUMWD})_s$.

Definition 4.3. A point $(\bar{y},\bar{v}_t,\bar{\beta},\bar{\lambda}) \in C_{\text{OUMWD}}$ is said to be

(i) a robust ε-quasi efficient solution to problem $(\text{OUMWD})_s$ if

$$L^s(y,v_t,\beta,\lambda) - L^s(\bar{y},\bar{v}_t,\bar{\beta},\bar{\lambda}) - \varepsilon||\bar{y}-y|| \notin \mathbb{R}_+^m \backslash \{0\}, \forall (y,v_t,\beta,\lambda) \in C_{\text{OUMWD}}.$$

(ii) a robust positively properly efficient solution to problem $(\text{OUMWD})_s$ if there exists $\theta := (\theta_1,\cdots,\theta_m) \in -\text{int}\mathbb{R}_+^m$ such that

$$\langle \theta, L^s(y,v_t,\beta,\lambda) + \varepsilon||\bar{y}-y|| \rangle \geqslant \langle \theta, L^s(\bar{y},\bar{v}_t,\bar{\beta},\bar{\lambda}) \rangle, \forall (y,v_t,\beta,\lambda) \in C_{\text{OUMWD}}.$$

Motivated by the definition of generalized convexity due to [42], we introduce the concept as follows.

Definition 4.4. We say that $(p - s\star q, g_T)$ is ε-quasi pseudo-generalized convex on Ω at $\bar{x} \in \Omega$ if, for any $x \in \Omega$, $x_k^* \in \partial p_k(\bar{x}), z_k^* \in \partial q_k(\bar{x}), k = 1,\cdots,m$ and $x_t^* \in \partial_x g_t(\bar{x},v_t), v_t \in \mathscr{V}_t, t \in T$, there exists $w \in N(\bar{x};\Omega)^\circ$ such that

$$p_k(x) - s_k q_k(x) + \varepsilon_k||x-\bar{x}|| < p_k(\bar{x}) - s_k q_k(\bar{x}) \Rightarrow \langle x_k^* - s_k z_k^*, w\rangle + \varepsilon_k||x-\bar{x}|| < 0, k = 1,\cdots,m,$$

$$g_t(x,v_t) \leq g_t(\bar{x},v_t) \Rightarrow \langle x_t^*, w\rangle \leq 0 \text{ for all } t \in T, \text{ and } \langle b^*, w\rangle \leq ||x-\bar{x}|| \text{ for all } b^* \in \mathbb{B}_{\mathbb{R}^n}.$$

Remark 4.4. If $(p - s\star q, g_T)$ is ε-quasi generalized convex on Ω at any $\bar{x} \in \Omega$, then $(p - s\star q, g_T)$ is pseudo-generalized convex on Ω at $x \in \Omega$. In addition, by a similar argument in ([26, Example 6]), we can prove that the class of the ε-quasi pseudo-generalized convex functions is properly larger than the one of the generalized convex functions.

Theorem 4.4. **($\varepsilon-$quasi weak robust duality)** *Let $x \in C$ and $(y, v_t, \beta, \lambda) \in C_{\text{OUMWD}}$. If $(p - s\star q, g_T)$ is ε-quasi pseudo-generalized convex on Ω at y, then $p(x) - s\star q(x) \not\preceq L^s(y, v_t, \beta, \lambda) - \varepsilon||x-y||$.*

Proof. Since $(y, v_t, \beta, \lambda) \in C_{\text{OUMWD}}$, there exist $x_k^* \in \partial p_k(y), z_k^* \in \partial q_k(y), k = 1, \cdots, m, \beta \in \text{int}\mathbb{R}_+^m$ with $\sum_{k=1}^m \beta_k = 1$ and $x_t^* \in \partial_x g_t(y, v_t), v_t \in \mathscr{V}_t, t \in T$, $\lambda \in \mathbb{R}_+^{(T)}$, as well as $b^* \in \mathbb{B}_{\mathbb{R}^n}$ such that

$$-\left(\sum_{k=1}^m \beta_k(x_k^* - s_k z_k^*) + \sum_{t\in T} \lambda_t x_t^* + \sum_{k=1}^m \beta_k \varepsilon_k b^*\right) \in N(y;\Omega) \tag{4.11}$$

and

$$\sum_{t\in T} \lambda_t g_t(y, v_t) \geq 0. \tag{4.12}$$

Let $x \in C$. Suppose on contrary that $p(x) - s\star q(x) \preceq L^s(y, v_t, \beta, \lambda,) - \varepsilon||x-y||$. It follows from $\beta := (\beta_1, \cdots, \beta_m) \in \text{int}\mathbb{R}_+^m$ that $\langle \beta, p(x) - s\star q(x) - L^s(y, v_t, \beta, \lambda) + \varepsilon||x-y||\rangle < 0$. Therefore, one has

$$\langle \beta, (p(x) - s\star q(x)) - (p(y) - s\star q(y)) + \varepsilon||x-y||\rangle < 0,$$

which is equivavalent to the following inequality

$$\sum_{k=1}^m \beta_k\left[p_k(x) - s_k q_k(x)\right] + \sum_{k=1}^m \beta_k \varepsilon_k ||x-y|| < \sum_{k=1}^m \beta_k\left[p_k(y) - s_k q_k(y)\right]. \tag{4.13}$$

Note that, for $x \in C$, $g_t(x, v_t) \leq 0$ for any $v_t \in \mathscr{V}_t, t \in T$. It yields that $\sum_{t\in T} \lambda_t g_t(x, v_t) \leq 0$, which together with (4.12) implies that

$$\sum_{t\in T} \lambda_t g_t(x, v_t) \leq \sum_{t\in T} \lambda_t g_t(y, v_t). \tag{4.14}$$

By the ε-quasi pseudo-generalized convexity of $(p - s\star q, g_T)$ on Ω at $y \in \Omega$ and (4.13), (4.14) for such $x \in C \subseteq \Omega, x_k^* \in \partial p_k(y), z_k^* \in \partial q_k(y), k = 1, \cdots, m, x_t^* \in \partial_x g_t(y, v_t), v_t \in \mathscr{V}_t, t \in T$, we see that there exists $w \in N(y;\Omega)^\circ$ such that

$$\sum_{k=1}^m \beta_k \langle x_k^* - s_k z_k^*, w\rangle + \sum_{t\in T} \beta_k \varepsilon_k \langle b^*, w\rangle + \sum_{t\in T} \lambda_t \langle x_t^*, w\rangle < 0. \tag{4.15}$$

On the other hand, by the definition of polar cone (2.1), it yields from (4.11) and the relation $w \in N(y;\Omega)^\circ$ that

$$\sum_{k=1}^m \beta_k \langle x_k^* - s_k z_k^*, w\rangle + \sum_{t\in T} \beta_k \varepsilon_k \langle b^*, w\rangle + \sum_{t\in T} \lambda_t \langle x_t^*, w\rangle \geq 0,$$

which contradicts (4.15). The proof is complete. □

The following example demonstrates that the ε-quasi pseudo-generalized convexity of $(p - s\star q, g_T)$ imposed in the Theorem 4.4 is essential.

Example 4.2. Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f(x) = (p_1(x) - s_1 q_1(x), p_2(x) - s_2 q_2(x))$ with $s = (s_1, s_2) = (-1, -1)$ and

$$p_1(x) = p_2(x) = x^7, q_1(x) = q_2(x) = x^6 + 1, x \in \mathbb{R}.$$

Take $T=[0,1], \mathscr{V}_t=[2-t,2+t], t\in T$ and let $g_t:\mathbb{R}\times\mathscr{V}_t\to\mathbb{R}$ be given by

$$g_t(x,v_t)=-v_tx^2, x\in\mathbb{R}, v_t\in\mathscr{V}_t, t\in T.$$

We consider the problem (USIMP)$_s$ with $m=2$ and $\Omega=(-\infty,0]\subset\mathbb{R}$. Simple calculation yields $C=(-\infty,0]$. Now, we consider dual problem (OUMWD)$_s$. By choosing $\bar{y}=0\in\Omega, \varepsilon_1=\varepsilon_2=\frac{1}{3}, \bar{v}_t\in\mathscr{V}_t, \bar{\lambda}\in\mathbb{R}^{(T)}_+, \bar{\beta}=(\bar{\beta}_1,\bar{\beta}_2)\in\text{int}\mathbb{R}^2_+$ with $\bar{\beta}_1+\bar{\beta}_2=1$ and $B_{\mathbb{R}}=[-1,1]$, we have $N(\bar{y};\Omega)=N(\bar{y};(-\infty,0])=[0,+\infty)$ and $\partial p_k(\bar{y})=\{0\}, \partial q_k(\bar{y})=\{0\}, k=1,2, \partial_x g_t(\bar{y},\bar{v}_t)=\{0\}, \bar{v}_t\in\mathscr{V}_t, t\in T$. It is easy to see that

$$0\in\left[-\frac{1}{3},\frac{1}{3}\right]+[0,+\infty)=\sum_{k=1}^{2}\bar{\beta}_k\left[\partial p_k(\bar{y})-s_k\partial q_k(\bar{x})\right]+\sum_{t\in T}\bar{\lambda}_t\partial_x g_t(\bar{y},\bar{v}_t)+\sum_{k=1}^{2}\bar{\beta}_k\varepsilon_k\mathbb{B}_{\mathbb{R}}+N(\bar{y};\Omega),$$

$\sum_{t\in T}\bar{\lambda}_t g_t(\bar{y},\bar{v}_t)=0\geq 0$. Thus, $(\bar{y},\bar{v}_t,\bar{\beta},\bar{\lambda})\in C_{\text{OUMWD}}$. However, by choosing $\bar{x}=-2\in C=(-\infty,0]$, it follows that

$$\begin{aligned}p(\bar{x})-s\star q(\bar{x}) &=(p_1(\bar{x})-s_1q_1(\bar{x}),p_2(\bar{x})-s_2q_2(\bar{x}))\\ &=(-63,-63)\\ &\preceq\left(\tfrac{1}{3},\tfrac{1}{3}\right)\\ &=(L_1^{s_1}(\bar{y},\bar{v}_t,\bar{\beta},\bar{\lambda})-\varepsilon_1|\bar{x}-\bar{y}||,L_2^{s_2}(\bar{y},\bar{v}_t,\bar{\beta},\bar{\lambda})-\varepsilon_2|\bar{x}-\bar{y}||)\\ &=L^s(\bar{y},\bar{v}_t,\bar{\beta},\bar{\lambda})-\varepsilon|\bar{x}-\bar{y}||.\end{aligned}$$

The reason is that $(p-s\star q,g_T)$ is not $\varepsilon-$quasi pseudo-generalized convex on Ω at $\bar{y}=0$. To see this, we can choose $y=-3\in\Omega$ and $x_k^*\in\partial p_k(\bar{y})=\{0\}, z_k^*\in\partial q_k(\bar{y})=\{0\}, k=1,2, \varepsilon_1=\varepsilon_2=\frac{1}{3}$. Then, it is easy to see that $N(\bar{y};\Omega)^\circ=N(\bar{y};(-\infty,0])^\circ=(-\infty,0]$ and

$$p_k(y)-s_kq_k(y)+\varepsilon_k||y-\bar{y}||=-1456<1=p_k(\bar{y})-s_kq_k(\bar{y}), k=1,2.$$

However, $\langle x_k^*-s_kz_k^*,w\rangle+\varepsilon_k||y-\bar{y}||=1>0, \forall w\in N(\bar{y};\Omega)^\circ$, $k=1,2$.

Remark 4.5. Theorem 4.4 improves Theorem 4 in [26].

Theorem 4.5. **($\varepsilon-$quasi strong robust duality)** *Let $\bar{x}\in C$ be a robust ε-quasi positively properly efficient solution to problem* (USIMP)$_s$ *such that condition* (RCQ) *is satisfied at $\bar{x}$. Then there exists $(\bar{v}_t,\bar{\beta},\bar{\lambda})\in\mathscr{V}_t\times\text{int}\mathbb{R}^m_+\times\mathbb{R}^{(T)}_+$ such that $(\bar{x},\bar{v}_t,\bar{\beta},\bar{\lambda})\in C_{\text{OUMWD}}$ and $p(\bar{x})-s\star q(\bar{x})=L^s(\bar{x},\bar{v}_t,\bar{\beta},\bar{\lambda})$. If, in addition, $(p-s\star q,g_T)$ is ε-quasi pseudo-generalized convex on Ω at $y\in\Omega$, then $(\bar{x},\bar{v}_t,\bar{\beta},\bar{\lambda})$ is a robust ε-quasi efficient solution to problem* (OUMWD)$_s$.

Proof. According to Theorem 3.1, there exist $\beta:=(\beta_1,\cdots,\beta_m)\in\text{int}\mathbb{R}^m_+, v_t\in\mathscr{V}_t, t\in T$ and $\lambda\in A(\bar{x})$ defined in (3.1) such that

$$0\in\sum_{k=1}^{m}\beta_k(\partial p_k(\bar{x})-s_k\partial q_k(\bar{x}))+\sum_{t\in T}\lambda_t\partial_x g_t(\bar{x},v_t)+\sum_{k=1}^{m}\beta_k\varepsilon_k\mathbb{B}_{\mathbb{R}^n}+N(\bar{x};\Omega). \tag{4.16}$$

Putting

$$\bar{\beta}_k:=\frac{\beta_k}{\sum_{k=1}^m\beta_k}, k=1,\cdots,m, \bar{\lambda}_t:=\frac{\lambda_t}{\sum_{k=1}^m\beta_k}, \bar{v}_t:=\frac{v_t}{\sum_{k=1}^m\beta_k}, t\in T,$$

one has $\bar{\beta}:=(\bar{\beta}_1,\cdots,\bar{\beta}_m)\in\text{int}\mathbb{R}^m_+$ with $\sum_{k=1}^m\bar{\beta}_k=1$, $\bar{\lambda}:=(\bar{\lambda}_t)_{t\in T}\in\mathbb{R}^{(T)}_+, \bar{v}_t\in\mathscr{V}_t, t\in T$. Furthermore, the assertion in (4.16) is also valid when β_k's, λ_t's, and v_t's are replaced by $\bar{\beta}_k$'s,

$\bar{\lambda}_t$'s, and $\bar{v}_t$'s, respectively. Clearly, $(\bar{x}, \bar{v}_t, \bar{\beta}, \bar{\lambda}) \in C_{\text{OUMWD}}$. In view of $\lambda \in A(\bar{x})$, one has $\lambda_t g_t(\bar{x}, v_t) = 0, \forall v_t \in \mathcal{V}_t, t \in T$, which implies that $\sum_{t\in T} \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) = 0$. Thus

$$p(\bar{x}) - s \star q(\bar{x}) = L^s(\bar{x}, \bar{v}_t, \bar{\beta}, \bar{\lambda}).$$

If $(p - s\star q, g_T)$ is ε-quasi pseudo-generalized convex on Ω at any $y \in \Omega$, then we obtain by Theorem 4.4 that $L^s(\bar{x}, \bar{v}_t, \bar{\beta}, \bar{\lambda}) = p(\bar{x}) - s\star q(\bar{x}) \not\preceq L^s(y, v_t, \beta, \lambda) - \varepsilon||\bar{x} - y||$, for any $(y, v_t, \beta, \lambda) \in C_{\text{OUMWD}}$, that is, $(\bar{x}, \bar{v}_t, \bar{\beta}, \bar{\lambda})$ is a robust ε-quasi efficient solution to problem (OUMWD)$_s$. The proof is complete. □

Note that the conclusion of Theorem 4.5 may fail to hold if condition (RCQ) is not satisfied. To see this, let us recall Example 3.1.

Remark 4.6. Theorem 4.5 improves [26, Theorem 5].

Theorem 4.6. **($\varepsilon-$quasi converse robust duality)** *Let $(\bar{x}, \bar{v}_t, \bar{\beta}, \bar{\lambda}) \in C_{\text{OUMWD}}$. If $\bar{x} \in C$ and $(p - s\star q, g_T)$ is ε-quasi pseudo-generalized convex on Ω at $\bar{x}$, then $\bar{x}$ is a robust ε-quasi positively properly efficient solution to problem* (USIMP)$_s$.

Proof. Since $(\bar{x}, \bar{v}_t, \bar{\beta}, \bar{\lambda}) \in C_{\text{OUMWD}}$, there exist $x_k^* \in \partial p_k(\bar{x}), z_k^* \in \partial q_k(\bar{x}), k = 1, \cdots, m, \bar{\beta} \in \text{int}\mathbb{R}_+^m$ with $\sum_{k=1}^m \bar{\beta}_k = 1$ and $x_t^* \in \partial_x g_t(\bar{x}, \bar{v}_t), \bar{v}_t \in \mathcal{V}_t, t \in T$, $\bar{\lambda} := (\bar{\lambda}_t)_{t\in T} \in \mathbb{R}_+^{(T)}$, as well as $b^* \in \mathbb{B}_{\mathbb{R}^n}$ such that

$$-\left(\sum_{k=1}^m \bar{\beta}_k(x_k^* - s_k z_k^*) + \sum_{t\in T} \bar{\lambda}_t x_t^* + \sum_{k=1}^m \bar{\beta}_k \varepsilon_k b^*\right) \in N(\bar{x}; \Omega) \tag{4.17}$$

and

$$\sum_{t\in T} \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) \geq 0. \tag{4.18}$$

Suppose on contrary that $\bar{x} \in C$ is not a robust ε-quasi positively properly efficient solution to problem (USIMP)$_s$. For such $\bar{\beta} := (\bar{\beta}_1, \cdots, \bar{\beta}_m) \in \text{int}\mathbb{R}_+^m$, it then follows that there exists $\tilde{x} \in C$ satisfying

$$\sum_{k=1}^m \bar{\beta}_k [p_k(\tilde{x}) - s_k q_k(\tilde{x})] + \sum_{k=1}^m \bar{\beta}_k \varepsilon_k ||\tilde{x} - \bar{x}|| < \sum_{k=1}^m \bar{\beta}_k [p_k(\bar{x}) - s_k q_k(\bar{x})]. \tag{4.19}$$

Note that, for any $\tilde{x} \in C$, $g_t(\tilde{x}, \bar{v}_t) \leq 0$ for any $\bar{v}_t \in \mathcal{V}_t, t \in T$, which yields that $\sum_{t\in T} \bar{\lambda}_t g_t(\tilde{x}, \bar{v}_t) \leq 0$. From (4.18), we see that

$$\sum_{t\in T} \bar{\lambda}_t g_t(\tilde{x}, \bar{v}_t) \leq \sum_{t\in T} \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t). \tag{4.20}$$

By the ε-quasi pseudo-generalized convexity of $(p - s\star q, g_T)$ on Ω at $\bar{x} \in \Omega$ and (4.19), (4.20) for such $x \in C \subseteq \Omega, x_k^* \in \partial p_k(\bar{x}), z_k^* \in \partial q_k(\bar{x}), k = 1, \cdots, m, x_t^* \in \partial_x g_t(\bar{x}, \bar{v}_t), \bar{v}_t \in \mathcal{V}_t, t \in T$, there exists $w \in N(y; \Omega)^\circ$ such that

$$\sum_{k=1}^m \bar{\beta}_k \langle x_k^* - s_k z_k^*, w\rangle + \sum_{k=1}^m \bar{\beta}_k \varepsilon_k \langle b^*, w\rangle + \sum_{t\in T} \bar{\lambda}_t \langle x_t^*, w\rangle < 0. \tag{4.21}$$

Besides, by the definition of polar cone (2.1), it yields from (4.17) and the relation $w \in N(\bar{x}; \Omega)^\circ$ that $\sum_{k=1}^m \bar{\beta}_k \langle x_k^* - s_k z_k^*, w\rangle + \sum_{k=1}^m \bar{\beta}_k \varepsilon_k \langle b^*, w\rangle + \sum_{t\in T} \bar{\lambda}_t \langle x_t^*, w\rangle \geq 0$, which contradicts (4.21). The proof is complete. □

By virtue of Example 4.2, we see that the result of Theorem 4.6 may not be valid if the ε-quasi pseudo-generalized convexity of $(p-s\star q, g_T)$ is not satisfied.

Remark 4.7. Theorem 4.6 improves [26, Theorem 6].

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