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SENSITIVITY ANALYSIS OF AN OPTIMAL CONTROL PROBLEM UNDER LIPSCHITZIAN PERTURBATIONS

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Abstract. In this paper, we study the quantitative stability of an optimal control problem with respect to parametric perturbations. We essentially obtain two equivalent conclusions for the stability of this problem by using two independent methods. The first one makes recourse to standard computations based on the famous Gronwall Lemma while our second method employees rather stability of fixed points trough the celebrated Lim's Lemma for which we construct a suitable contracting set-valued mapping over a larger functional space than the one of continuous functions adopted in the close previous works. The second method allows us to introduce a further concept of approximate solutions regarded as approximate values of the optimal control for which we prove similar stability properties as in the case of exact solutions.

Keywords. Cauchy problem; Exact solutions; Initial-value problem; Parametric perturbation; Quantitative stability.

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1. Introduction

Quantitative stability of mathematical programming problems is still considered as one of the mostly important themes of the optimization field, which is justified by its interest in various domains of real applications. For comprehensive discussions on this topic, we refer for example to [1]-[9] and the references therein. In this paper, we aim at establishing a stability for typical optimal control problem under a constraint given by a first order dynamic linear system whose state depends on the control, which is, to the best of our knowledge, not yet covered by the previous results in the literature. Our key idea in this work exploits basic ingredients of the paper [10] on the stability of parametric ordinary differential equations based on a careful use the standard Gronwall Lemma. This gives us meaningful stability properties for the optimal control problem considered here by measuring the distance between solutions corresponding to data under perturbations. Furthermore, our viewpoint on the sensitivity analysis of the problem under consideration provides also stability estimates for the corresponding optimal values of the cost functional. The obtained quantitative stability regarding unique exact solutions is thereafter

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extended to approximate solutions by employing rather the fixed point approach for which we construct a suitable contracting set-valued mapping.

To introduce the problem, we first recall the notation of the space $\mathscr{M}_n(\mathbb{R})$ of square matrices of size n equipped with the norm associated to the Euclidean norm given for a matrix $A \in \mathscr{M}_n(\mathbb{R})$ by $\|A\| = \sqrt{\rho({}^tAA)}$, where $\rho(B)$ stands for the spectral radius of a matrix B. Let J be the following cost functional:

$$J(u) = \frac{1}{2}x^T(T)Gx(T) + \frac{1}{2}\int_0^T x^T(s)Qx(s)ds + \frac{1}{2}\int_0^T ru(s)^2 ds,$$

where Q and G are symmetric positive matrices in $\mathscr{M}_n(\mathbb{R})$, $r \in \mathbb{R}_+^*$, and x^T stands for the transpose of the vector x.

We consider the following optimal control problem: for $m, M \in \mathbb{R}$ $s.t$ $m < M$,

$$(\mathbf{OC}) \quad \begin{cases} \min J(u) \\ u \in U_{ad} = L^2\Big([0,T], V = [m,M]\Big) \end{cases} \tag{1.1}$$

subject to the constraint defined by the dynamic linear system given by:

$$\begin{cases} \dot{x}(t) = Ax(t) + u(t)Bx(t) \\ x(0) = x_0 \in \mathbb{R}^n \end{cases}, \tag{1.2}$$

where A and B are two matrices in $\mathscr{M}_n(\mathbb{R})$.

This optimal control problem was studied by Zerrik *et al* in [11]. They obtain the solutions of this problem via a fixed point method, that is, a fixed point of an appropriate contractive operator. Indeed, in [11], the optimal control problem in (1.1) was shown to admit a unique solution u^* given for every $t \in [0,T]$ by:

$$u^*(t) = \max(m, \min(M, -\frac{1}{r}p^{*T}(t)Bx^*(t))), \tag{1.3}$$

where x^* is the associated trajectory, the solution to system (1.2), and p^* is the associated adjoint vector solution to the following

$$\begin{cases} \dot{p}(t) = -(A + u(t)B)^T p(t) - Qx(t) \\ p(T) = Gx(T), \end{cases} \tag{1.4}$$

where the notation C^T stands for the transpose of a matrix C.

In this paper, we deal with a perturbed form of the problem studied in [11]. Precisely, we consider that both of the cost functional and the control are subject to a parametric perturbation denoted by λ with a reference value $\overline{\lambda}$ in a subset of a normed space, denoted by $\Lambda(\overline{\lambda})$. Accordingly, the perturbed format of system (1.2) is as follows:

$$\begin{cases} \dot{x}_\lambda(t) = Ax_\lambda(t) + u_\lambda(t)Bx_\lambda(t) \\ x_\lambda(0) = x_\lambda^0 \in \mathbb{R}^n \end{cases}. \tag{1.5}$$

The parametric version of the optimal control (**OC**) is in turn as follows:

$$(\mathbf{OC})_\lambda \quad \begin{cases} \min J_\lambda(u_\lambda) \\ u_\lambda \in U_{ad} \end{cases}, \tag{1.6}$$

where J_λ is defined by

$$J_\lambda(u_\lambda) = \frac{1}{2}x_\lambda^T(T)Gx_\lambda(T) + \frac{1}{2}\int_0^T x_\lambda^T(s)Qx_\lambda(s)ds + \frac{1}{2}\int_0^T ru_\lambda(s)^2 ds.$$

Notation:

- For the initial value $\overline{\lambda}$ of parameter λ we write: $x_{\overline{\lambda}} = x$, $u_{\overline{\lambda}} = u$, and $x^0_{\overline{\lambda}} = x_0$.
- For each $\lambda \in \Lambda(\lambda)$, we denote by u^*_λ the unique solution to problem (1.6).

Therefore, similarly to (1.3), for each $\lambda \in \Lambda(\lambda)$, we have

$$u^*_\lambda(t) = \max(m, \min(M, -\frac{1}{r}p^{*\,T}_\lambda(t)Bx^*_\lambda)),$$

where x^*_λ is the associated trajectory, the solution to (1.5), and p^*_λ is the associated adjoint vector solution to

$$\begin{cases} \dot{p}_\lambda(t) = -(A + u_\lambda(t)B)^T p_\lambda(t) - Qx_\lambda(t) \\ p_\lambda(T) = Gx_\lambda(T). \end{cases} \tag{1.7}$$

This paper is organized as follows. In Section 2, we recall the necessary preliminaries related to the optimal control problem. This section also includes basic results of stability of fixed points regarding the so-called Lim's Lemma in both of its exact and approximate versions. Section 3 is devoted to our stability results based on direct computations with the help of Gronwall Lemma. First, with respect to parametric perturbations, we present a stability result for the system state under the assumption that the control is Lipschitzian in λ, and we give a stability result for the associated adjoint state. Consequently, we are able to deduce the stability of the optimal control and optimal value in Theorem 3.2. In Section 4, by perturbing the data of the constraint, in addition to the perturbation of the cost functional, we obtain quantitative stability results wherein the key idea is the celebrated Lim's Lemma as an alternative of Gronwall techniques. In Section 5, we introduce the notion of the ε-optimal control and involve the approximate version of Lim's Lemma established very recently by M. Ait Mansour *el al* in [12], which gives us a nice perspective for the stability analysis of approximate solutions to our optimal control problem.

2. PRELIMINARIES

2.1. **Exact and approximate versions of Lim's Lemma.** Let X and Y be two normed vector spaces whose norm is denoted by $\|.\|$. For any nonempty subset A of X and any point $x \in X$, $d(x,A) = \inf\{\|x-y\| : y \in A\}$ stands for the distance from x to A whereas $d(x,\emptyset) = \infty$. If B is another nonempty subset of X, $e(A,B)$ denotes the excess of A on B given by $e(A,B) = \sup\{d(a,B) : a \in A\}$. We adopt the convention $e(\emptyset,A) = 0$ for any subset $\emptyset \neq A \subset X$ and $e(A,\emptyset) = +\infty$. The *extended Hausdorff distance* between two subsets A and B of X is given by $h(A,B) = \max\{e(A,B), e(B,A)\}$.

Notice that the word "extended" refers to the possibility of the distance to be ∞. The minimal distance between two nonempty subsets A, B of X is denoted and given by

$$d(A,B) = \inf\{\|x-y\| : (x,y) \in A \times B\}.$$

When one of the sets A and B is empty, we set $d(A,B) = h(A,B) = +\infty$. For a given map $\Phi : X \rightrightarrows X$, for every $\varepsilon \geq 0$, we consider the notation $\varepsilon\text{-Fix}(\Phi) := \{x \in X \mid d(x,\Phi(x)) \leq \varepsilon\}$

to refer to the set of ε-approximate fixed points of Φ while we write $\text{Fix}(\Phi)$ to stand for fixed points of Φ i.e., $x \in \text{Fix}(\Phi)$ if and only if $x \in \Phi(x)$.

Theorem 2.1 ([12, Theorem 15]). *Let X be a metric space, and let $T_1 : X \rightrightarrows X$ and $T_2 : X \rightrightarrows X$. Suppose that both T_1 and T_2 are Lipschitz continuous on X with the same Lipschitz constant $\lambda \in [0,1)$. Then, for every $\varepsilon > 0$, the set of ε-approximate fixed points of T_i, $i = 1,2$, is nonempty, i.e., $\varepsilon\text{-Fix}(T_i) \neq \emptyset$, and moreover*

$$h(\varepsilon\text{-Fix}(T_1), \varepsilon\text{-Fix}(T_2)) \leq \frac{\varepsilon}{1-\lambda} + \frac{1}{1-\lambda} \sup_{x\in X} h(T_1(x), T_2(x)).$$

If X is in addition complete, then Theorem 2.1 implies Lim's Lemma.

Theorem 2.2 ([13, Lemma 1]). *Let X be a complete metric space, and let T_1 and T_2 map X into the family of nonempty and closed subsets of X. Suppose that both T_1 and T_2 are Lipschitz continuous on X with the same Lipschitz constant $\lambda \in [0,1)$. Then*

$$h(\text{Fix}(T_1), \text{Fix}(T_2)) \leq \frac{1}{1-\lambda} \sup_{x\in X} h(T_1(x), T_2(x)).$$

2.2. **The perturbed Cauchy problem.** The Euclidian space $\mathbb{R}^n$ is equipped with the Euclidien norm $\|.\|$. For a given point $x_0 \in \mathbb{R}^n$ and a nonnegative real-number $r > 0$, we denote by $B(x_0, r)$ the ball with center x_0 and radius r and consider a real-valued function $f : [0,T] \times B(x_0, r) \longrightarrow \mathbb{R}^n$, T being a nonnegative real number and standing for the final time of the interval of interest. Then, the corresponding Cauchy problem of the initial-value ordinary differential equation associated with these data is as follows:

$$S(f, x_0) \begin{cases} x'(t) = f(t, x(t)), \text{ for } a.e\ t \in [0,T] \\ x(0) = x_0. \end{cases}$$

We consider perturbed formats of system $S(f, x_0)$, which involves an external parameter λ that belongs to another space $\left(\Lambda(\overline{\lambda}), |.|\right)$. Precisely, The parametric Cauchy problem under consideration is as follows:

$$S(f_\lambda, x_\lambda^0) \begin{cases} x_\lambda'(t) = f_\lambda(t, x_\lambda(t)) \\ x_\lambda(0) = x_\lambda^0, \end{cases}$$

where $f_\lambda : [0,T] \times B(x_0, r) \longrightarrow \mathbb{R}^n$. The initial value of the parameter λ is denoted by $\overline{\lambda}$: $f_{\overline{\lambda}} = f$, $x_{\overline{\lambda}} = x$, and $x_{\overline{\lambda}}(0) = x_0$.

Using direct computations based on the famous Gronwall Lemma, the author of [10] proved the following result.

Theorem 2.3 ([10, Theorem 2]). *Assume that, for some $L > 0$, $L' > 0$, the following conditions hold:*

(h_1) *f_λ is L-Lipschitz continuous w.r to x, uniformly in t and λ;*
(h_2) *f_λ is L'-Lipschitz continuous w.r to λ, uniformly in t and x.*

Then, for all $t \in [0,T]$, the following estimate is satisfied

$$\|x(t) - x_\lambda(t)\| \leq e^{Lt}\|x_{\lambda_0} - x_0\| + \frac{L'}{L}(e^{Lt} - 1)\|\lambda - \overline{\lambda}\|.$$

3. Main Results

3.1. **Optimal control under Lipschitzian perturbations.** In view of the Gronwall's Lemma, one can easily observe that the trajectory of system (1.5) is uniformly bounded in t and u, so we assume the existence of a positive real d such that any trajectory is uniformly bounded with respect to λ, t, and u:

$$\forall \lambda \in \Lambda(\overline{\lambda}),\ \forall t \in [0,T],\ \forall u_\lambda \in U_{ad}, \quad \|x_\lambda(t)\| \leq d. \tag{3.1}$$

For the sake of simplicity, we define the following constants

- $\tilde{M} = \max(|m|, |M|)$;
- $\alpha = \|A\| + \tilde{M}\|B\|$;
- $\beta = Ld\|B\|$.

Lemma 3.1. *Assume that condition (3.1) is verified. If the control u_λ is L-Lipshitz in λ uniformly in t and x, then, for all $t \in [0,T]$,*

$$\|x_\lambda(t) - x(t)\| \leq e^{\alpha t}\|x_\lambda^0 - x_0\| + \frac{\beta}{\alpha}\left(e^{\alpha t} - 1\right)|\lambda - \overline{\lambda}|. \tag{3.2}$$

Proof. We consider the function $f_\lambda(t,x,u) = Ax(t) + u_\lambda Bx(t)$. Clearly, f_λ is α lipschitz in x uniformly in t, u, and λ. Since u_λ is L-lipshitz in λ uniformly in t and x, then f_λ is Lipschitz in λ with the same constant β. Finally, from Theorem 2.3, we obtain he required inequality. This completes the proof . □

In the following Lemma, we justify that the adjoint state given in (1.7) is uniformly bounded in λ.

Lemma 3.2. *Let (3.1) hold. Then, for all $t \in [0,T]$ and $\lambda \in \Lambda(\overline{\lambda})$, $\|p_\lambda(t)\| \leq d(\|G\| + T\|Q\|)e^{\alpha(T-t)}$.*

Proof. By integrating this system on the interval $[t,T]$ and by using the triangular inequality, we obtain

$$\begin{aligned}
\|p_\lambda(t)\| &\leq \|p_\lambda(T)\| + d\|Q\|(T-t) + \alpha\int_t^T \|p_\lambda(s)\|ds \\
&\leq \|Gx_\lambda(T)\| + Td\|Q\| + \alpha\int_t^T \|p_\lambda(s)\|ds \\
&\leq d\left(\|G\| + T\|Q\|\right) + \alpha\int_t^T \|p_\lambda(s)\|ds.
\end{aligned}$$

Therefore, the Gronwall Lemma leads to $\|p_\lambda(t)\| \leq d\left(\|G\| + T\|Q\|\right)e^{\alpha(T-t)}$. This completes the proof. □

Throughout the rest of this section, we use the notation $\delta := d\left(\|G\| + T\|Q\|\right)e^{\alpha T}$.

Lemma 3.3. *Let (3.1) hold, and let the control u_λ be L-lipshitz in λ uniformly in t and x. Then, for all $t \in [0,T]$ and all $\lambda \in \Lambda(\overline{\lambda})$,*

$$\int_t^T \|x_\lambda(s) - x(s)\|ds \leq \frac{e^{\alpha(T-t)} - 1}{\alpha}\left(\|x_\lambda(T) - x(T)\| + \beta T|\lambda - \overline{\lambda}|\right). \tag{3.3}$$

Proof. For all $t \in [0,T]$ and $\lambda \in \Lambda(\overline{\lambda})$, it is easy to see that

$$\begin{aligned}&\|x_\lambda(t)-x(t)\|\\&\leq \|x_\lambda(T)-x(T)\| + \int_t^T \|f(s,x(s)-f_\lambda(s,x(s)\|ds + \int_t^T \|f_\lambda(s,x(s)-f_\lambda(s,x_\lambda(s)\|ds.\end{aligned}$$

Since u_λ is L-Lipshitz in λ uniformly in t and x, it follows that f_λ is Lipschitz in λ with the same constant β. Then the previous inequality reduces to

$$\|x_\lambda(t)-x(t)\| \leq \|x_\lambda(T)-x(T)\| + \beta T|\lambda-\overline{\lambda}| + \alpha\int_t^T \|x_\lambda(s)-x(s)\|ds.$$

Thus, from the Gronwall Lemma, we obtain

$$\|x_\lambda(t)-x(t)\| \leq e^{\alpha(T-t)}\left(\|x_\lambda(T)-x(T)\| + \beta T|\lambda-\overline{\lambda}|\right).$$

By integrating this inequality on the interval $[t,T]$, we obtain the required inequality in (3.3). This completes the proof. □

In the following step, we consider the adjoint state of the constraint of our problem for which we state and prove a continuity result with respect to the final state and the parameter λ as follows.

Lemma 3.4. *Let (3.1) hold. If, moreover, the control u_λ is L-lipshitz in λ uniformly in t and x, then, for all $t \in [0,T]$ and $\lambda \in \Lambda(\overline{\lambda})$,*

$$\begin{aligned}&\|p_\lambda(t)-p(t)\|\\&\leq e^{\alpha T}\left(\|G\| + \|Q\|\frac{e^{\alpha T}-1}{\alpha}\right)\|x_\lambda(T)-x(T)\| + e^{\alpha T}\left(L\delta\|B\|T + \|Q\|\frac{e^{\alpha T}-1}{\alpha}\beta T\right)|\lambda-\overline{\lambda}|.\end{aligned}$$

Proof. By integrating the two corresponding systems over $[t,T]$, we are able to write

$$\begin{aligned}p(t)-p_\lambda(t) = p(T)-p_\lambda(T) &+ \int_t^T (A+u(s)B)^T(p(s)-p_\lambda(s))ds\\&+\int_t^T Q(x(s)-x_\lambda(s))ds + \int_t^T (u(s)-u_\lambda(s))B^T p_\lambda(s)ds.\end{aligned}$$

The fact that u_λ is L-Lipschitz in λ gives us

$$\begin{aligned}\|p(t)-p_\lambda(t)\| \leq \|p(T)-p_\lambda(T)\| &+ \alpha\int_t^T \|p(s)-p_\lambda(s)\|ds\\&+\|Q\|\int_t^T \|x(s)-x_\lambda(s)\|ds + L\delta\|B\|(T-t)|\lambda-\overline{\lambda}|.\end{aligned}$$

Accordingly, Lemma 3.3 leads to

$$\begin{aligned}&\|p(t)-p_\lambda(t)\|\\&\leq \|p(T)-p_\lambda(T)\| + L\delta\|B\|T|\lambda-\overline{\lambda}|\\&\quad+\|Q\|\left(\frac{e^{\alpha T}-1}{\alpha}\left(\|x_\lambda(T)-x(T)\| + \beta T\|\lambda-\overline{\lambda}\|\right)\right) + \alpha\int_t^T \|p(s)-p_\lambda(s)\|ds.\end{aligned}$$

Since $p_\lambda(T) = Gx_\lambda(T)$ for all $\lambda \in \Lambda(\overline{\lambda})$, then

$$\|p(t) - p_\lambda(t)\| \leq \left(\|G\| + \|Q\|\frac{e^{\alpha T}-1}{\alpha}\right) e^{\alpha T}\|x_\lambda(T) - x(T)\| + \left(L\delta\|B\|T + \|Q\|\frac{e^{\alpha T}-1}{\alpha}\beta T\right) e^{\alpha T}|\lambda - \overline{\lambda}|.$$

This completes the proof. □

The following Theorem shows the continuity of the optimal control with respect to the initial and final states and parameter λ.

Theorem 3.1. *Let (3.1) hold. If u_λ is L-lipshitz in λ uniformly in t and x, then, for all $t \in [0,T]$ and $\lambda \in \Lambda(\overline{\lambda})$, $|u^*_\lambda(t) - u^*(t)| \leq A_1\|x_\lambda(T) - x(T)\| + A_2\|x^0_\lambda - x_0\| + A_3|\lambda - \overline{\lambda}|$, where the constants A_i, for $i \in \{1,2,3\}$, are given by*

$$\begin{cases} A_1 = \frac{\|B\|\max(d,\delta)}{r}(\|G\| + \|Q\|\frac{e^{\alpha T}-1}{\alpha})e^{\alpha T} \\ A_2 = \frac{\|B\|\max(d,\delta)}{r}e^{\alpha T} \\ A_3 = \frac{\|B\|\max(d,\delta)}{r}\left(\frac{\beta}{\alpha}(e^{\alpha T}-1) + L\delta T\|B\| + \|Q\|\frac{e^{\alpha T}-1}{\alpha}\beta T\right). \end{cases}$$

Proof. Let $t \in [0,T]$ and $\lambda \in \Lambda(\overline{\lambda})$. It follows that

$$\begin{aligned} |u^*_\lambda(t) - u^*(t)| &\leq \frac{1}{r}|{p^*_\lambda}^T(t)Bx^*_\lambda(t) - {p^*}^T(t)Bx^*(t)| \\ &\leq \frac{1}{r}|\left(p^*_\lambda(t) - p^*(t)\right)^T Bx^*_\lambda(t) + {p^*}^T(t)B\left(x^*_\lambda(t) - x^*(t)\right)|. \end{aligned}$$

From the Cauchy-Schwarz inequality, we obtain

$$|u^*_\lambda(t) - u^*(t)| \leq \frac{\|B\|}{r}\left(\|p^*_\lambda(t) - p^*(t)\|\|x^*_\lambda(t)\| + \|p^*_\lambda(t)\|\|x^*_\lambda(t) - x^*(t)\|\right),$$

$$|u^*_\lambda(t) - u^*(t)| \leq \frac{d\|B\|}{r}\|p^*_\lambda(t) - p^*(t)\| + \frac{\delta\|B\|}{r}\|x^*_\lambda(t) - x^*(t)\|,$$

and

$$|u^*_\lambda(t) - u^*(t)| \leq \frac{\|B\|\max(d,\delta)}{r}\left(\|p^*_\lambda(t) - p^*(t)\| + \|x^*_\lambda(t) - x^*(t)\|\right).$$

Therefore, from Lemmas 3.1, 3.2, and 3.4, we deduce that

$$|u^*_\lambda(t) - u^*(t)| \leq A_1\|x_\lambda(T) - x(T)\| + A_2\|x^0_\lambda - x_0\| + A_3|\lambda - \overline{\lambda}|,$$

with

$$\begin{cases} A_1 = \frac{\|B\|\max(d,\delta)}{r}(\|G\| + \|Q\|\frac{e^{\alpha T}-1}{\alpha})e^{\alpha T} \\ A_2 = \frac{\|B\|\max(d,\delta)}{r}e^{\alpha T} \\ A_3 = \frac{\|B\|\max(d,\delta)}{r}\left(\frac{\beta}{\alpha}(e^{\alpha T}-1) + L\delta T\|B\| + \|Q\|\frac{e^{\alpha T}-1}{\alpha}\beta T\right). \end{cases}$$

This completes the proof. □

Corollary 3.1. *Let (3.1) hold. If, moreover, u_λ is L-Lipshitz in λ uniformly in t and x, then, for all $t \in [0,T]$ and $\lambda \in \Lambda(\overline{\lambda})$,*

$$|u^*_\lambda(t) - u^*(t)| \leq (A_1e^{\alpha T} + A_2)\|x^0_\lambda - x_0\| + (\frac{\beta A_1}{\alpha}(e^{\alpha T}-1) + A_3)|\lambda - \overline{\lambda}|,$$

where the constants A_i, for $i \in \{1,2,3\}$, are the same defined in Theorem 3.1.

This corollary is immediate result from Theorem 3.1 and estimate (3.2). Next, we give a stability estimate for the optimal value of our main problem.

Theorem 3.2. *Let (3.1) hold. If, moreover, u_λ is L-lipshitz in λ uniformly in t and x, then, for all $t \in [0,T]$ and $\lambda \in \Lambda(\overline{\lambda})$,*

$$\left|J_\lambda(u_\lambda^*) - J(u^*)\right| \leq B_1\|x_\lambda^*(T) - x^*(T)\| + B_2\|x_\lambda^0 - x_0\| + B_3|\lambda - \overline{\lambda}|,$$

with

$$\begin{cases} B_1 = d\|G\| + d\|Q\| + rT\tilde{M}A_1 \\ B_2 = d\|Q\| \left(\frac{e^{\alpha T}-1}{\alpha}\right) + rT\tilde{M}A_2 \\ B_3 = d\|Q\| + \left(\frac{\beta T(e^{\alpha T}-1)}{\alpha} + \frac{\beta}{\alpha}\left(\frac{e^{\alpha T}-1}{\alpha} - T\right)\right) + rT\tilde{M}A_3. \end{cases}$$

Proof. Clearly, we have

$$J_\lambda(u_\lambda^*) - J(u^*) = \frac{1}{2}\left(x_\lambda^T(T)Gx_\lambda(T) - x^T(T)Gx(T)\right) + \frac{1}{2}\int_0^T x_\lambda^T(s)Qx_\lambda(s) - x^T(s)Qx(s)ds$$
$$+ \frac{r}{2}\int_0^T u_\lambda^*(s)^2 - u^*(s)^2 ds.$$

Observe that

$$x_\lambda^T(T)Gx_\lambda(T) - x^T(T)Gx(T) = \left(x_\lambda(T) - x(T)\right)^T Gx_\lambda(T) + x^T(T)G\left(x_\lambda(T) - x(T)\right).$$

Then, the Cauchy-Schwartz inequality implies that

$$\left\|x_\lambda^T(T)Gx_\lambda(T) - x^T(T)Gx(T)\right\| \leq 2d\|G\|\|x_\lambda(T) - x(T)\|. \tag{3.4}$$

In the same way, we have

$$\left|\int_0^T x_\lambda^T(s)Qx_\lambda(s) - x^T(s)Qx(s)ds\right| \leq 2d\|Q\|\left(\int_0^t \|x_\lambda(s) - x(s)\|ds + \int_t^T \|x_\lambda(s) - x(s)\|ds\right).$$

By integrating (3.2), we obtain

$$\int_0^t \|x_\lambda(s) - x(s)\|ds \leq \frac{e^{\alpha T}-1}{\alpha}\|x_\lambda^0 - x_0\| + \frac{\beta}{\alpha}\left(\frac{e^{\alpha T}-1}{\alpha} - T\right)|\lambda - \overline{\lambda}|.$$

Thus, by using Lemma 3.3 and the previous inequality, we obtain

$$\frac{1}{2}\left|\int_0^T x_\lambda^T(s)Qx_\lambda(s) - x^T(s)Qx(s)ds\right|$$
$$\leq d\|Q\|\left(\frac{e^{\alpha T}-1}{\alpha}(\|x_\lambda^0 - x_0\| + \|x_\lambda(T) - x(T)\|) + \left(\frac{\beta T(e^{\alpha T}-1)}{\alpha} + \frac{\beta}{\alpha}\left(\frac{e^{\alpha T}-1}{\alpha} - T\right)\right)|\lambda - \overline{\lambda}|\right). \tag{3.5}$$

Since u^* and u_λ^* are in U_{ad}, then

$$\left|\int_0^T u_\lambda^*(s)^2 - u(s)^2 ds\right| \leq 2\tilde{M}\int_0^T \left|u_\lambda^*(s) - u^*(s)\right| ds.$$

Hence, from Theorem 3.1, it results that

$$\frac{1}{2}\left|\int_0^T u_\lambda^*(s)^2 - u(s)^2 ds\right| \leq T\tilde{M}\left(A_1\|x_\lambda^0 - x_0\| + A_2|\lambda - \overline{\lambda}| + A_3\|x_\lambda(T) - x(T)\|\right). \tag{3.6}$$

From (3.4) (3.5), and (3.6), we have

$$\left|J_\lambda(u_\lambda^*) - J(u)\right| \leq B_1\|x_\lambda(T) - x(T)\| + B_2\|x_\lambda^0 - x_0\| + B_3|\lambda - \overline{\lambda}|,$$

with

$$\begin{cases} B_1 = d\|G\| + d\|Q\| + rT\tilde{M}A_1 \\ B_2 = d\|Q\|\left(\frac{e^{\alpha T}-1}{\alpha}\right) + rT\tilde{M}A_2 \\ B_3 = d\|Q\| + \left(\frac{\beta T(e^{\alpha T}-1)}{\alpha} + \frac{\beta}{\alpha}\left(\frac{e^{\alpha T}-1}{\alpha} - T\right)\right) + rT\tilde{M}A_3. \end{cases}$$

This completes the proof. □

3.2. **Quantitative stability under global parametric perturbation of the data.** In this section, we suppose that u is fixed but all of the other data (the cost functional as well as the constraints system) depend on a parameter λ, which varies around a nominal value $\overline{\lambda}$. In this way, our parametric cost function is given by

$$J_\lambda(u) = \frac{1}{2}x^T(T)G_\lambda x(T) + \frac{1}{2}\int_0^T x^T(s)Q_\lambda x(s)ds + \frac{1}{2}\int_0^T ru(s)^2 ds.$$

Accordingly, we consider the following parametric minimization problem

$$\begin{cases} \min J_\lambda(u) \\ u \in U_{ad} \end{cases} \tag{3.7}$$

subject to the constraint given by

$$\begin{cases} x'(t) = A_\lambda x(t) + u(t)B_\lambda x(t) \\ x(0) = x_\lambda^0 \in \mathbb{R}^n. \end{cases} \tag{3.8}$$

For $\lambda \in \Lambda(\overline{\lambda})$, we denote by $x_{\lambda,u}$ the unique solution of (3.8) and by $p_{\lambda,u}$ we mean the associated adjoint vector defined as the unique solution of the following system

$$\begin{cases} \dot{p}(t) = -(A_\lambda + u(t)B_\lambda)^T p(t) - Q_\lambda x(t) \\ p(T) = G_\lambda x(T). \end{cases} \tag{3.9}$$

Suppose that:

$$(i)\ \exists d' > 0,\ \forall \lambda \in \Lambda(\overline{\lambda}),\ \forall u \in U_{ad}\ \|x_\lambda\| \leq d' \qquad (ii)\ \exists \rho > 0,\ \forall \lambda \in \Lambda(\overline{\lambda}),\ \|B_\lambda\| \leq \rho. \tag{3.10}$$

We then make the following assumption on parameter λ: There exists a constant $L' > 0$ such that, for all $\lambda \in \Lambda(\overline{\lambda})$,

$$\text{the matrices } A_\lambda, B_\lambda, Q_\lambda, G_\lambda \text{ are } L' - \text{Lipschitz in } \lambda. \tag{3.11}$$

Therefore, we introduce the function $F_\lambda := F_{x_0^\lambda, A_\lambda, B_\lambda, Q_\lambda, G_\lambda} : L^2([0,T],[m,M]) \longrightarrow L^2([0,T],[m,M])$ defined for every admissible control u by

$$F_\lambda(u)(t) = \max(m, \min(M, -\frac{1}{r}p_{u,\lambda}^T(t)B_\lambda x_{u,\lambda}(t))).$$

Lemma 3.5. *Let (3.10) hold. Then, for all $t \in [0,T]$ and all $\lambda \in \Lambda(\overline{\lambda})$, $\|p_\lambda(t)\| \leq d'\rho(1+T)e^{\alpha(T-t)}$.*

The proof is similar with the proof of Lemma 3.2. Hence, we omit the proof here.

Throughout the rest of this paper, we write $\delta' = d'\rho(1+T)e^{\alpha T}$.

Lemma 3.6. *Let* (3.10) *hold. Let* $x_{\lambda,u}$ *be the unique solution to (3.8), and let* $x_{\lambda',u}$ *be the unique solution to (3.8) for* $\lambda = \lambda'$. *Then, for all* $t \in [0,T]$,

$$\|x_{\lambda,u}(t) - x_{\lambda',u}(t)\| \leq e^{\rho(1+\tilde{M})t}\left(\|x_\lambda^0 - x_{\lambda'}^0\| + Td'(\|A_\lambda - A_{\lambda'}\| + \tilde{M}\|B_\lambda - B_{\lambda'}\|)\right). \qquad (3.12)$$

If, moreover, (3.11) *is satisfied, then, for all* $t \in [0,T]$,

$$\|x_{\lambda,u}(t) - x_{\lambda',u}(t)\| \leq e^{\rho(1+\tilde{M})T}\left(\|x_\lambda^0 - x_{\lambda'}^0\| + (1+\tilde{M})Td'L'|\lambda - \lambda'|\right).$$

Proof. By definitions of $x_{\lambda,u}$ and $x_{\lambda',u}$ we are able to write

$$\begin{aligned} x_{\lambda,u}(t) - x_{\lambda',u}(t) &= x_\lambda^0 - x_{\lambda'}^0 + \int_0^t (A_\lambda + u(s)B_\lambda)x_{\lambda,u}(s) - (A_{\lambda'} + u(s)B_{\lambda'})x_{\lambda',u}(s)ds \\ &= x_\lambda^0 - x_{\lambda'}^0 + \int_0^t (A_\lambda + u(s)B_\lambda)(x_{\lambda,u}(s) - x_{\lambda',u}(s))ds \\ &\quad + \int_0^t (A_\lambda - A_{\lambda'} + u(s)(B_\lambda - B_{\lambda'}))x_{\lambda',u}(s)ds. \end{aligned}$$

Then

$$\begin{aligned} \|x_{\lambda,u}(t) - x_{\lambda',u}(t)\| &\leq \|x_\lambda^0 - x_{\lambda'}^0\| + \rho(1+\tilde{M})\int_0^t \|x_{\lambda,u}(s) - x_{\lambda',u}(s)\|ds \\ &\quad + d'\int_0^t \left(\|A_\lambda - A_{\lambda'}\| + \tilde{M}\|B_\lambda - B_{\lambda'}\|\right)ds \\ &\leq \|x_\lambda^0 - x_{\lambda'}^0\| + Td'\left(\|A_\lambda - A_{\lambda'}\| + \tilde{M}\|B_\lambda - B_{\lambda'}\|\right) \\ &\quad + \rho(1+\tilde{M})\int_0^t \|x_{\lambda,u}(s) - x_{\lambda',u}(s)\|ds. \end{aligned}$$

Therefor, Gronwall Lemma yields

$$\|x_{\lambda,u}(t) - x_{\lambda',u}(t)\| \leq e^{\rho(1+\tilde{M})t}\left(\|x_\lambda^0 - x_{\lambda'}^0\| + Td'(\|A_\lambda - A_{\lambda'}\| + \tilde{M}\|B_\lambda - B_{\lambda'}\|)\right).$$

This completes the proof. □

Now, for $\lambda,\ \lambda' \in \Lambda(\overline{\lambda})$, let $p_{\lambda,u}$ be the unique solution to (3.9), and let $p_{\lambda',u}$ be the unique solution to (3.9) for $\lambda = \lambda'$. With this notation, we state and prove the following.

Lemma 3.7. *Let* (3.10) *hold. Then, for all* $t \in [0,T]$, $\|p_{\lambda,u}(t) - p_{\lambda',u}(t)\| \leq C_1\|x_\lambda^0 - x_{\lambda'}^0\| + C_2\|Q_\lambda - Q_{\lambda'}\| + C_3\|G_\lambda - G_{\lambda'}\| + C_4\|A_\lambda - A_{\lambda'}\| + C_5\|B_\lambda - B_{\lambda'}\|$, *where*

$$\begin{cases} C_1 = (1+T)\rho e^{T\rho(1+\tilde{M})}e^{(\rho+1)\tilde{M}T}; \\ C_2 = Td'e^{(\rho+1)\tilde{M}T}; \\ C_3 = d'e^{(\rho+1)\tilde{M}T}; \\ C_4 = e^{(\rho+1)\tilde{M}T}(Td'e^{T\rho(1+\tilde{M})} + \delta'T + T^2d'\rho e^{T\rho(1+\tilde{M})}); \\ C_5 = \tilde{M}e^{(\rho+1)\tilde{M}T}(Td'e^{T\rho(1+\tilde{M})} + \delta'T + T^2d'\rho e^{T\rho(1+\tilde{M})}). \end{cases}$$

If, in addition, (3.11) *is satisfied, then, for all* $t \in [0,T]$,

$$\|p_{\lambda,u}(t) - p_{\lambda',u}(t)\| \leq C_1\|x_\lambda^0 - x_{\lambda'}^0\| + L'(C_2 + C_3 + C_4 + C_5)|\lambda - \lambda'|.$$

Proof. By integrating system (3.9) we obtain

$$p_{\lambda,u}(t) = p_{\lambda,u}(T) + \int_t^T (A_\lambda + u(s)B_\lambda)^T p_{\lambda,u}(s) + Q_\lambda x_{\lambda,u}(s)ds.$$

It follows that

$$
\begin{aligned}
&p_{\lambda,u}(t)-p_{\lambda',u}(t)\\
&=\int_t^T (A_\lambda+u(s)B_\lambda)^T p_{\lambda,u}(s)-(A_{\lambda'}+u(s)B_{\overline{\lambda}})^T p_{\lambda',u}(s)+Q_\lambda x_{\lambda,u}(s)-Q_{\lambda'}x_{\lambda',u}(s)ds\\
&\quad+\quad p_{\lambda,u}(T)-p_{\overline{\lambda},u}(T).
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&(A_\lambda+u(s)B_\lambda)^T p_{\lambda,u}(s)-(A_{\lambda'}+u(s)B_{\lambda'})^T p_{\lambda',u}(s)\\
&=(A_\lambda+u(s)B_\lambda)^T(p_{\lambda,u}(s)-p_{\lambda',u}(s))+(A_\lambda-A_{\lambda'}+u(s)(B_\lambda-B_{\lambda'}))^T p_{\lambda,u}(s)
\end{aligned}
$$

and $Q_\lambda x_{\lambda,u}(s)-Q_{\lambda'}x_{\lambda',u}(s)=Q_\lambda(x_{\lambda,u}(s)-x_{\lambda',u}(s))+(Q_\lambda-Q_{\lambda'})x_{\lambda',u}(s)$. We have

$$
\begin{aligned}
\|p_{\lambda,u}(t)-p_{\lambda',u}(t)\|\le \|p_{\lambda,u}(T)-p_{\lambda',u}(T)\|&+\int_t^T \|^t(A_\lambda+u(s)B_\lambda)\|\|p_{\lambda,u}(s)-p_{\lambda',u}(s)\|ds\\
&+\int_t^T \|^t(A_\lambda-A_{\lambda'}+u(s)(B_\lambda-B_{\lambda'}))\|\|p_{\lambda,u}(s)\|\\
&+\int_t^T \|Q_\lambda\|\|x_{\lambda,u}(s)-x_{\lambda',u}(s)\|ds+\int_t^T \|Q_\lambda-Q_{\lambda'}\|\|x_{\lambda',u}(s)\|ds.
\end{aligned}
$$

Keeping in mind that

$$
\begin{cases}
\|(A_\lambda+u(s)B_\lambda)^T\|\le(\rho+1)\tilde{M}\\
\|(A_\lambda-A_{\lambda'}+u(s)(B_\lambda-B_{\lambda'}))^T\|\le\|A_\lambda-A_{\lambda'}\|+\tilde{M}\|B_\lambda-B_{\lambda'}\|\\
\|p_{\lambda,u}(s)\|\le\delta' \text{ and } \|p_{\lambda',u}(s)\|\le\delta',
\end{cases}
$$

we have

$$
\begin{aligned}
&\|p_{\lambda,u}(t)-p_{\lambda',u}(t)\|\\
&\le\|p_{\lambda,u}(T)-p_{\lambda',u}(T)\|+(\rho+1)\tilde{M}\int_0^T\|p_{\lambda,u}(s)-p_{\lambda',u}(s)\|ds\\
&\quad+T\delta'(\|A_\lambda-A_{\lambda'}\|+\tilde{M}\|B_\lambda-B_{\lambda'}\|)+\rho\int_0^T\|x_{\lambda,u}(s)-x_{\lambda',u}(s)\|ds+Td'\|Q_\lambda-Q_{\lambda'}\|.
\end{aligned}
$$

In turns, condition $p_{\lambda,u}(T)=G_\lambda x_{\lambda,u}(T)$ leads to

$$
\|p_{\lambda,u}(T)-p_{\lambda',u}(T)\|\le\|G_\lambda\|\|x_{\lambda,u}(T)-x_{\lambda',u}(T)\|+\|x_{\lambda',u}(T)\|\|G_\lambda-G_{\lambda'}\|.
$$

Thanks to inequality (3.12), we conclude that

$$
\begin{aligned}
\|p_{\lambda,u}(t)-p_{\overline{\lambda},u}(t)\|\le\rho e^{T\rho(1+\tilde{M})}(1+T)\|x^0_\lambda-x^0_{\lambda'}\|&+Td'\|Q_\lambda-Q_{\lambda'}\|+d'\|G_\lambda-G_{\lambda'}\|\\
&+(Td'e^{T\rho(1+\tilde{M})}+\delta'T+T^2d'\rho e^{T\rho(1+\tilde{M})})(\|A_\lambda-A_{\lambda'}\|+\tilde{M}\|B_\lambda-B_{\lambda'}\|)\\
&+(\rho+1)\tilde{M}\int_0^T\|x_{\lambda,u}(s)-x_{\overline{\lambda},u}(s)\|ds.
\end{aligned}
$$

Accordingly, the Gronwall inequality implies that

$$
\begin{aligned}
&\|p_{\lambda,u}(t)-p_{\lambda',u}(t)\|\\
&\le C_1\|x^0_\lambda-x^0_{\lambda'}\|+C_2\|Q_\lambda-Q_{\lambda'}\|+C_3\|G_\lambda-G_{\lambda'}\|+C_4\|A_\lambda-A_{\lambda'}\|+C_5\|B_\lambda-B_{\lambda'}\|,
\end{aligned}
$$

where

$$\begin{cases} C_1 = (1+T)\rho e^{T\rho(1+\tilde{M})} e^{(\rho+1)\tilde{M}T}; \\ C_2 = T d' e^{(\rho+1)\tilde{M}T}; \\ C_3 = d' e^{(\rho+1)\tilde{M}T}; \\ C_4 = e^{(\rho+1)\tilde{M}T}(T d' e^{T\rho(1+\tilde{M})} + \delta' T + T^2 d' \rho e^{T\rho(1+\tilde{M})}); \\ C_5 = \tilde{M} e^{(\rho+1)\tilde{M}T}(T d' e^{T\rho(1+\tilde{M})} + \delta' T + T^2 d' \rho e^{T\rho(1+\tilde{M})}). \end{cases}$$

This completes the proof. □

In [11], the non-parametric form of operator F_λ was proved to be Lipschitz continuous over the control space of continuous functions $C(0,T,V)$. In the next result, we show that this property holds true in the largest space $L^2(0,T,V)$ even within the presence of an external parameter.

Theorem 3.3. *Let (3.10) be satisfied. Then, there exists a constant $K \geq 0$ such that, for all $\lambda \in \Lambda(\overline{\lambda})$ and for all $u_1, u_2 \in L^2(0,T,V)$, $\|F_\lambda(u_1) - F_\lambda(u_2)\|_{L^2(0,T,V)} \leq KT\|u_1 - u_2\|_{L^2(0,T,V)}$, where*

$$K = \frac{\|B\| \max(d', \delta')}{r}\left(\rho d' e^{(\rho+1)\tilde{M}T} + e^{\rho T}\left(\rho^2 d' e^{(\rho+1)\tilde{M}T} + d'\rho + d'\rho T e^{(\rho+1)\tilde{M}T}\right)\right).$$

In particular, if $KT \in [0,1)$, then, for all $\lambda \in \Lambda(\overline{\lambda})$, F_λ is a contraction over $L^2(0,T,V)$.

Proof. Fix a value of the parameter λ in $\Lambda(\overline{\lambda})$. Let u_1, u_2 in $L^2([0,T],[m,M])$ and $t \in [0,T]$. It follows that

$$\begin{aligned} &|F_\lambda(u_1)(t) - F_\lambda(u_2)(t)| \\ &\leq \frac{1}{r}\left|p_{u_1}^T(t) B_\lambda x_{u_1}(t) - p_{u_2}^T(t) B_\lambda x_{u_2}(t)\right| \\ &\leq \frac{1}{r}\left|p_{u_1}^T(t) B_\lambda (x_{u_1}(t) - x_{u_2}(t)) + (p_{u_1}(t) - p_{u_2}(t))^T B_\lambda x_{u_2}(t)\right| \\ &\leq \frac{1}{r}\left(\|B_\lambda\|\|p_{u_1}(t)\|\|x_{u_1}(t) - x_{u_2}(t)\| + \|B_\lambda\|\|p_{u_1}(t) - p_{u_2}(t)\|\|x_{u_2}(t)\|\right) \\ &\leq \frac{\rho \max(d,\delta)}{r}\left(\|x_{u_1}(t) - x_{u_2}(t)\| + \|p_{u_1}(t) - p_{u_2}(t)\|\right). \end{aligned} \tag{3.13}$$

For $i \in \{1,2\}$, let x_{u_i} be the unique solution to (3.8). It follows that

$$x_{u_1}(t) - x_{u_2}(t) = \int_0^t (A_\lambda + u_1(s)B_\lambda)(x_{u_1}(s) - x_{u_2}(s)) + \int_0^t (u_1(s) - u_2(s)) B_\lambda x_{u_2}(s) ds.$$

Thus $\|x_{u_1}(t) - x_{u_2}(t)\| \leq (\rho+1)\tilde{M}\int_0^t \|x_{u_1}(s) - x_{u_2}(s)\| ds + d\rho \int_0^T \|u_1(s) - u_2(s)\| ds$, which implies with the help of Gronwall Lemma that

$$\|x_{u_1}(t) - x_{u_2}(t)\| \leq \rho d' e^{(\rho+1)\tilde{M}T}\|u_1 - u_2\|_{L^1(0,T,V)}. \tag{3.14}$$

For $i \in \{1,2\}$, by integrating system (3.9), we obtain

$$p_{u_i}(t) = p_{u_i}(T) + \int_t^T (A_\lambda + u_i(s)B_\lambda)^T p_{u_i}(s) + Q_\lambda x_{u_i}(s) ds.$$

For $t \in [0,T]$, it follows that

$$\|p_{u_1}(t) - p_{u_2}(t)\| \leq \|p_{u_1}(T) - p_{u_2}(T)\| + (\rho+1)\tilde{M}\int_t^T \|x_{u_1}(s) - x_{u_2}(s)\|ds + \rho\int_t^T \|p_{u_1}(s) - p_{u_2}(s)\|ds + d\rho\int_t^T \|u_1(s) - u_2(s)\|ds.$$

On the other hand, by integrating inequality (3.12), we see that

$$\int_t^T \|x_{u_1}(s) - x_{u_2}(s)\|ds \leq d\rho T e^{(\rho+1)\tilde{M}T}\|u_1 - u_2\|_{L^1(0,T,V)}.$$

Observe $p_{u_i}(T) = G_\lambda x_{u_i}(T)$ for $i \in \{1,2\}$. In view of (3.12), we have

$$\|p_{u_1}(T) - p_{u_2}(T)\| \leq \rho^2 d e^{(\rho+1)\tilde{M}T}\|u_1 - u_2\|_{L^1(0,T,V)}.$$

Consequently,

$$\|p_{u_1}(t) - p_{u_2}(t)\| \leq \left(\rho^2 d' e^{(\rho+1)\tilde{M}T} + d'\rho + d\rho T e^{(\rho+1)\tilde{M}T}\right)\|u_1 - u_2\|_{L^1(0,T,V)} + \rho\int_t^T \|p_{u_1}(s) - p_{u_2}(s)\|ds.$$

Again, we also have

$$\|p_{u_1}(t) - p_{u_2}(t)\| \leq e^{\rho T}\left(\rho^2 d' e^{(\rho+1)\tilde{M}T} + d'\rho + d\rho T e^{(\rho+1)\tilde{M}T}\right)\|u_1 - u_2\|_{L^1(0,T,V)}. \tag{3.15}$$

In view of (3.12) and (3.15), we see that inequality (3.13) becomes

$$|F(u_1)(t) - F(u_2)(t)| \leq \frac{\|B\|\max(d',\delta')}{r}\left(\rho d' e^{(\rho+1)\tilde{M}T} + e^{\rho T}\left(\rho^2 d' e^{(\rho+1)\tilde{M}T} + d'\rho + d'\rho T e^{(\rho+1)\tilde{M}T}\right)\right)\|u_1 - u_2\|_{L^1(0,T,V)}.$$

This ends the proof. □

In the next result, K is borrowed to stand for the constant defined in Theorem 3.3 and u^*_λ, $u^*_{\lambda'}$ denotes the optimal control solutions of problem (3.7), respectively for $\lambda, \lambda' \in \Lambda(\overline{\lambda})$.

Theorem 3.4. *Let (3.10) hold. If $KT \in [0,1)$, then $\|u^*_\lambda - u^*_{\lambda'}\|_{L^2(0,T,V)} \leq \frac{\sqrt{T}}{1-KT}(\alpha_1\|x^0_\lambda - x^0_{\lambda'}\| + \alpha_2\|Q_\lambda - Q_{\lambda'}\| + \alpha_3\|G_\lambda - G_{\lambda'}\| + \alpha_4\|A_\lambda - A_{\lambda'}\| + \alpha_5\|B_\lambda - B_{\lambda'}\|)$, where*

$$\begin{cases} \alpha_1 = \frac{\rho\delta' e^{T\rho(1+\tilde{M})} + C_1\rho d'}{r}; \\ \alpha_2 = \frac{C_2\rho d'}{r}; \\ \alpha_3 = \frac{C_3\rho d'}{r}; \\ \alpha_4 = \frac{Td'\rho\delta' e^{T\rho(1+\tilde{M})} + C_4\rho d'}{r}; \\ \alpha_5 = \frac{d'\delta' + \rho T d'\tilde{M}\delta' e^{T\rho(1+\tilde{M})} + \rho d' C_5}{r}. \end{cases}$$

If, in addition, (3.11) holds, then

$$\|u^*_\lambda - u^*_{\lambda'}\|_{L^2(0,T,V)} \leq \frac{\sqrt{T}}{1-KT}\left(\alpha_1\|x^0_\lambda - x^0_{\lambda'}\| + L'(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)|\lambda - \lambda'|\right). \tag{3.16}$$

Proof. Fix $\lambda, \lambda' \in \Lambda(\overline{\lambda}),\ u \in U_{ad},\ t \in [0,T]$. Observe

$$\begin{aligned}
&|F_\lambda(u)(t) - F_{\lambda'}(u)(t)| \\
&\leq \frac{1}{r}\left|p^T_{\lambda,u}(t)B_\lambda x_{\lambda,u}(t) - p^T_{\lambda',u}(t)B_{\lambda'}x_{\lambda',u}(t)\right| \\
&\leq \frac{1}{r}|p^T_{\lambda,u}(t)(B_\lambda x_{\lambda,u}(t) - B_{\lambda'}x_{\lambda',u}(t)) + (p_{\lambda,u}(t) - p_{\lambda',u}(t))^T B_{\lambda'}x_{\lambda',u}(t)| \\
&\leq \frac{1}{r}\left(\|p_{\lambda,u}(t)\|\|B_\lambda x_{\lambda,u}(t) - B_{\lambda'}x_{\lambda',u}(t)\| + \|p_{\lambda,u}(t) - p_{\lambda',u}(t)\|\|B_{\lambda'}\|\|x_{\lambda',u}(t)\|\right) \\
&\leq \frac{1}{r}\left(\delta'\|B_\lambda x_{\lambda,u}(t) - B_{\lambda'}x_{\lambda',u}(t)\| + \rho d'\|p_{\lambda,u}(t) - p_{\lambda',u}(t)\|\right)
\end{aligned}$$

and

$$\begin{aligned}
\|B_\lambda x_{\lambda,u}(t) - B_{\lambda'}x_{\lambda',u}(t)\| &\leq \|x_{\lambda,u}(t)\|\|B_\lambda - B_{\lambda'}\| + \|B_{\lambda'}\|\|x_{\lambda,u}(t) - x_{\lambda',u}(t)\| \\
&\leq d'\|B_\lambda - B_{\lambda'}\| + \rho\|x_{\lambda,u}(t) - x_{\lambda',u}(t)\|.
\end{aligned}$$

Thus

$$r|F_\lambda(u)(t) - F_{\lambda'}(u)(t)| \leq d'\delta'\|B_\lambda - B_{\lambda'}\| + \rho\delta'\|x_{\lambda,u}(t) - x_{\lambda',u}(t)\| + \rho d'\|p_{\lambda,u}(t) - p_{\lambda',u}(t)\|.$$

This, combined with Lemmas 3.6 and 3.7, implies that

$$\begin{aligned}
|F_\lambda(u)(t) - F_{\lambda'}(u)(t)| \leq \frac{\rho\delta' e^{T\rho(1+\tilde{M})} + C_1\rho d'}{r}\|x^0_\lambda - x^0_{\lambda'}\| + \frac{C_2\rho d'}{r}\|Q_\lambda - Q_{\lambda'}\| \\
+ \frac{C_3\rho d'}{r}\|G_\lambda - G_{\lambda'}\| + \frac{Td'\rho\delta' e^{T\rho(1+\tilde{M})} + C_4\rho d'}{r}\|A_\lambda - A_{\lambda'}\| \\
+ \frac{d'\delta' + \rho T d'\tilde{M}\delta' e^{T\rho(1+\tilde{M})} + \rho d' C_5}{r}\|B_\lambda - B_{\lambda'}\|.
\end{aligned}$$

Now, for $\lambda,\ \lambda' \in \Lambda(\overline{\lambda})$, we have

$$\begin{aligned}
\|F_\lambda(u) - F_{\lambda'}(u)\|_{L^2(0,T,V)} \leq \sqrt{T}(\alpha_1\|x^0_\lambda - x^0_{\lambda'}\| + \alpha_2\|Q_\lambda - Q_{\lambda'}\| + \alpha_3\|G_\lambda - G_{\lambda'}\| \\
+ \alpha_4\|A_\lambda - A_{\lambda'}\| + \alpha_5\|B_\lambda - B_{\lambda'}\|),
\end{aligned} \tag{3.17}$$

where

$$\begin{cases}
\alpha_1 = \frac{\rho\delta' e^{T\rho(1+\tilde{M})} + C_1\rho d'}{r}; \\
\alpha_2 = \frac{C_2\rho d'}{r}; \\
\alpha_3 = \frac{C_3\rho d'}{r}; \\
\alpha_4 = \frac{Td'\rho\delta' e^{T\rho(1+\tilde{M})} + C_4\rho d'}{r}; \\
\alpha_5 = \frac{d'\delta' + \rho T d'\tilde{M}\delta' e^{T\rho(1+\tilde{M})} + \rho d' C_5}{r}.
\end{cases}$$

Therefore, we conclude that

$$\begin{aligned}
\sup_{u \in L^2(0,T,V)} \|F_\lambda(u) - F_{\lambda'}(u)\|_{L^2(0,T,V)} \leq \sqrt{T}(\alpha_1\|x^0_\lambda - x^0_{\lambda'}\| + \alpha_2\|Q_\lambda - Q_{\lambda'}\| \\
+ \alpha_3\|G_\lambda - G_{\lambda'}\| + \alpha_4\|A_\lambda - A_{\lambda'}\| + \alpha_5\|B_\lambda - B_{\lambda'}\|).
\end{aligned}$$

Now, if $KT \in [0,1)$, then $\text{Fix}(F_\lambda) = \{u^*_\lambda\}$ and $\text{Fix}(F_{\lambda'}) = \{u^*_{\lambda'}\}$. From Theorem 2.2, we have

$$h(\text{Fix}(F_\lambda), \text{Fix}(F_{\lambda'})) \leq \frac{1}{1-KT} \sup_{u \in L^2(0,T,V)} \|F_\lambda(u) - F_{\lambda'}(u)\|_{L^2(0,T,V)},$$

which can also be written as

$$\|u^*_\lambda - u^*_{\lambda'}\|_{L^2(0,T,V)} \leq \frac{\sqrt{T}}{1-KT}(\alpha_1\|x^0_\lambda - x^0_{\lambda'}\| + \alpha_2\|Q_\lambda - Q_{\lambda'}\| + \alpha_3\|G_\lambda - G_{\lambda'}\| \\ + \alpha_4\|A_\lambda - A_{\lambda'}\| + \alpha_5\|B_\lambda - B_{\lambda'}\|).$$

The proof of (3.16) is immediate from the previous inequality and condition (3.11). This achieves the proof. □

Theorem 3.5. *Under the same notation and assumption of Theorem 3.4, if $KT \in [0,1)$, then there exists positive constants $(\beta_i)_{1\leq i\leq 6}$ such that $|J_\lambda(u^*_\lambda) - J_{\lambda'}(u^*_{\lambda'})| \leq \beta_1\|x^0_\lambda - x^0_{\lambda'}\| + \beta_2\|Q_\lambda - Q_{\lambda'}\| + \beta_3\|G_\lambda - G_{\lambda'}\| + \beta_4\|A_\lambda - A_{\lambda'}\| + \beta_5\|B_\lambda - B_{\lambda'}\| + \beta_6\|x_\lambda(T) - x_{\lambda'}(T)\|$, where*

$$\begin{cases} \beta_1 = Td'\rho e^{\rho(1+\tilde{M})T} + \frac{\alpha_1 rT\tilde{M}}{1-KT}; \\ \beta_2 = \frac{Td'^2}{2} + \frac{\alpha_2 rT\tilde{M}}{1-KT}; \\ \beta_3 = \frac{d'^2}{2} + \frac{\alpha_3 rT\tilde{M}}{1-KT}; \\ \beta_4 = \rho T^2 d'^2 e^{\rho(1+\tilde{M})T} + \frac{\alpha_4 rT\tilde{M}}{1-KT}; \\ \beta_5 = \tilde{M}\rho T^2 d'^2 e^{\rho(1+\tilde{M})T} + \frac{\alpha_5 rT\tilde{M}}{1-KT}; \\ \beta_6 = d'\rho. \end{cases}$$

If, in addition, hypothesis (3.11) is satisfied, then

$$|J_\lambda(u^*_\lambda) - J_{\lambda'}(u^*_{\lambda'})| \\ \leq (\beta_1 + \beta_6 e^{\rho(1+\tilde{M})T})\|x^0_\lambda - x^0_{\lambda'}\| + (L'(\beta_2+\beta_3+\beta_4+\beta_5) + (1+\tilde{M})Td'L'e^{\rho(1+\tilde{M})T})|\lambda - \lambda'|.$$

Proof. Fix $\lambda, \lambda' \in \Lambda(\overline{\lambda})$, and let u^*_λ, $u^*_{\lambda'}$ the optimal control solutions to problem (3.8) respectively for λ, $\lambda' \in \Lambda(\overline{\lambda})$. It can be easily seen that

$$J_\lambda(u^*_\lambda) - J_{\lambda'}(u^*) = \frac{1}{2}\left(x^T_\lambda(T)Gx_\lambda(T) - x^T_{\lambda'}(T)G_{\lambda'}x_{\lambda'}(T)\right) \\ + \frac{1}{2}\int_0^T x^T_\lambda(s)Qx_\lambda(s) - x^T_{\lambda'}(s)Q_{\lambda'}x_{\lambda'}(s)ds + \frac{r}{2}\int_0^T u^*_\lambda(s)^2 - u_{\lambda'}(s)^2 ds.$$

Observe

$$x^T_\lambda(T).G_\lambda x_\lambda(T) - x^T_{\lambda'}(T).G_{\lambda'}x_{\lambda'}(T) \\ = x^T_\lambda(T).G_\lambda(x_\lambda(T) - x_{\lambda'}(T)) + x^T_\lambda(T).(G_\lambda - G_{\lambda'})x_{\lambda'}(T) + (x_\lambda(T) - x_{\lambda'}(T))^T.G_{\lambda'}x_{\lambda'}(T).$$

Since $\|x_\lambda(T)\| \leq d'$ and $\|x_{\lambda'}(T)\| \leq d'$, then the Cauchy-Schwarz inequality implies that

$$|x^T_\lambda(T).G_\lambda x_\lambda(T) - x^T_{\lambda'}(T).G_{\lambda'}x_{\lambda'}(T)| \leq 2\rho d'\|x_\lambda(T) - x_{\lambda'}(T)\| + d'^2\|G_\lambda - G_{\lambda'}\|.$$

Similarly, for all $s \in [0,T]$,

$$|x^T_\lambda(s).Q_\lambda x_\lambda(s) - x^T_{\lambda'}(s).Q_{\lambda'}x_{\lambda'}(s)| \leq 2\rho d'\|x_\lambda(s) - x_{\lambda'}(s)\| + d'^2\|Q_\lambda - Q_{\lambda'}\|.$$

Thus

$$\frac{1}{2}\int_0^T |x_\lambda^T(s).Q_\lambda x_\lambda(s) - x_{\lambda'}^T(s).Q_{\lambda'}x_{\lambda'}(s)|ds \leq \rho d' \int_0^T \|x_\lambda(s) - x_{\lambda'}(s)\|ds + \frac{Td'^2}{2}\|Q_\lambda - Q_{\lambda'}\|.$$

It follows from inequality (3.12) that

$$\begin{aligned}&\frac{1}{2}\int_0^T |x_\lambda^T(s).Q_\lambda x_\lambda(s) - x_{\lambda'}^T(s).Q_{\lambda'}x_{\lambda'}(s)|ds \\ &\leq \rho T d' e^{\rho(1+\tilde{M})T}\left(\|x_\lambda^0 - x_{\lambda'}^0\| + Td'(\|A_\lambda - A_{\lambda'}\| + \tilde{M}\|B_\lambda - B_{\lambda'}\|)\right).\end{aligned}$$

Therefore, since u_λ^* and $u_{\lambda'}^*$ are in U_{ad}, we have

$$\int_0^T |u_\lambda^*(s)^2 - u_{\lambda'}^*(s)^2|ds \leq 2\tilde{M}\int_0^T |u_\lambda^*(s) - u_{\lambda'}^*(s)|ds \leq 2\tilde{M}\sqrt{T}\|u_\lambda^* - u_{\lambda'}^*\|_{L^2(0,T,V)}.$$

Observe that

$$\begin{aligned}\frac{r}{2}\int_0^T |u_\lambda^*(s)^2 - u_{\lambda'}^*(s)^2|ds \leq \frac{r\tilde{M}T}{1-KT}(&\alpha_1\|x_\lambda^0 - x_{\lambda'}^0\| + \alpha_2\|Q_\lambda - Q_{\lambda'}\| + \alpha_3\|G_\lambda - G_{\lambda'}\| \\ &+ \alpha_4\|A_\lambda - A_{\lambda'}\| + \alpha_5\|B_\lambda - B_{\lambda'}\|).\end{aligned}$$

Thus

$$\begin{aligned}|J_\lambda(u_\lambda^*) - J_{\lambda'}(u_{\lambda'}^*)| \leq \beta_1\|x_\lambda^0 - x_{\lambda'}^0\| + \beta_2\|Q_\lambda - Q_{\lambda'}\| + \beta_3\|G_\lambda - G_{\lambda'}\| + \beta_4\|A_\lambda - A_{\lambda'}\| \\ + \beta_5\|B_\lambda - B_{\lambda'}\| + \beta_6\|x_\lambda(T) - x_{\lambda'}(T)\|,\end{aligned}$$

where

$$\begin{cases}\beta_1 = Td'\rho e^{\rho(1+\tilde{M})T} + \frac{\alpha_1 rT\tilde{M}}{1-KT};\\ \beta_2 = \frac{Td'^2}{2} + \frac{\alpha_2 rT\tilde{M}}{1-KT};\\ \beta_3 = \frac{d'^2}{2} + \frac{\alpha_3 rT\tilde{M}}{1-KT};\\ \beta_4 = \rho T^2 d'^2 e^{\rho(1+\tilde{M})T} + \frac{\alpha_4 rT\tilde{M}}{1-KT};\\ \beta_5 = \tilde{M}\rho T^2 d'^2 e^{\rho(1+\tilde{M})T} + \frac{\alpha_5 rT\tilde{M}}{1-KT};\\ \beta_6 = d'\rho.\end{cases}$$

The second part of the conclusion of the Theorem follows immediately from the first one. This completes de proof. □

Corollary 3.2. *Let (3.10) hold. Assume that the initial and the terminal state are the same for the two values λ and λ' of the parameter. Then $|J_\lambda(u_\lambda^*) - J_{\lambda'}(u_{\lambda'}^*)| \leq \beta_2\|Q_\lambda - Q_{\lambda'}\| + \beta_3\|G_\lambda - G_{\lambda'}\| + \beta_4\|A_\lambda - A_{\lambda'}\| + \beta_5\|B_\lambda - B_{\lambda'}\|$, where*

$$\begin{cases}\beta_2 = \frac{Td'^2}{2} + \frac{\alpha_2 rT\tilde{M}}{1-KT};\\ \beta_3 = \frac{d'^2}{2} + \frac{\alpha_3 rT\tilde{M}}{1-KT};\\ \beta_4 = \rho T^2 d'^2 e^{\rho(1+\tilde{M})T} + \frac{\alpha_4 rT\tilde{M}}{1-KT};\\ \beta_5 = \tilde{M}\rho T^2 d'^2 e^{\rho(1+\tilde{M})T} + \frac{\alpha_5 rT\tilde{M}}{1-KT};\end{cases}$$

If, in addition, (3.11) hold, then $|J_\lambda(u_\lambda^*) - J_{\lambda'}(u_{\lambda'}^*)| \leq (\beta_2 + \beta_3 + \beta_4 + \beta_5)|\lambda - \lambda'|$.

3.3. **Extension of stability to ε-optimal control.** Let us now introduce the concept of ε-optimal control for problem (3.7)-(3.8).

We say that the function $u \in U_{ad}$ is a ε-optimal control of the problem if and only if $u \in \varepsilon$-fixed point of $F_{x_0,A,B,Q,G}$ i.e, $u \in \varepsilon - \mathrm{Fix}(F_{x_0,A,B,Q,G})$.

Some examples of approximate fixed points and related discussions can be found in [12, 2]. Now, we denote the set $\varepsilon - \mathrm{Fix}(F_{x_0,A,B,Q,G})$ by $S^{\varepsilon}(x_0,A,B,Q,G)$, which actually expresses the set of approximate solutions to our optimal control problem (3.7)-(3.8).

Notation: In what follows, for all $\varepsilon > 0$ and all $\lambda \in \Lambda(\overline{\lambda})$, we write for simplicity $S^{\varepsilon}(\lambda) := S^{\varepsilon}(x_0^{\lambda}, A_\lambda, B_\lambda, Q_\lambda, G_\lambda)$.

In the following, we present a quantitative stability result for the sets of approximate solutions to optimal control problem (3.7)-(3.8).

Theorem 3.6. *Let (3.10) hold, and let K be the constant defined in Theorem 3.3. If $KT \in [0,1)$, then, for all $\varepsilon > 0$ and for all $\lambda, \lambda' \in \Lambda(\overline{\lambda})$, the both sets $S^{\varepsilon}(\lambda)$, and $S^{\varepsilon}(\lambda')$ are non-empty. Moreover, $h(S^{\varepsilon}(\lambda), S^{\varepsilon}(\lambda') \leq \frac{\varepsilon}{1-KT} + \frac{\sqrt{T}}{1-KT}(\alpha_1\|x_\lambda^0 - x_{\lambda'}^0\| + \alpha_2\|Q_\lambda - Q_{\lambda'}\| + \alpha_3\|G_\lambda - G_{\lambda'}\| + \alpha_4\|A_\lambda - A_{\lambda'}\| + \alpha_5\|B_\lambda - B_{\lambda'}\|)$. If, in addition, (3.11) is satisfied, then*

$$h(S^{\varepsilon}(\lambda), S^{\varepsilon}(\lambda') \leq \frac{\varepsilon}{1-KT} + \frac{\sqrt{T}}{1-KT}\Big(\alpha_1\|x_\lambda^0 - x_{\lambda'}^0\| + L'(\alpha_2+\alpha_3+\alpha_4+\alpha_5)|\lambda - \lambda'|\Big).$$

Proof. Let $\lambda,\ \lambda' \in \Lambda(\overline{\lambda})$. If $KT \in [0,1)$, then F_λ and $F_{\lambda'}$ are contractions with the same constant KT. By applying Theorem 2.1, we obtain

$$h(S^{\varepsilon}(\lambda), S^{\varepsilon}(\lambda') \leq \frac{\varepsilon}{1-KT} + \frac{1}{1-KT} \sup_{u \in L^2(0,T,V)} h(F_\lambda(u), F_{\lambda'}(u)).$$

Let $u \in U_{ad}$. Using (3.17) we obtain

$$\begin{aligned} &h(F_\lambda(u), F_{\lambda'}(u)) \\ &\leq \sqrt{T}(\alpha_1\|x_\lambda^0 - x_{\lambda'}^0\| + \alpha_2\|Q_\lambda - Q_{\lambda'}\| + \alpha_3\|G_\lambda - G_{\lambda'}\| + \alpha_4\|A_\lambda - A_{\lambda'}\| + \alpha_5\|B_\lambda - B_{\lambda'}\|). \end{aligned}$$

This completes the proof. ☐

Remark 3.1. Fix $\varepsilon > 0$. Let $u^{\star}_{\varepsilon,\overline{\lambda}} = u^{\star}_{\varepsilon} \in S^{\varepsilon}(x_0^{\overline{\lambda}}, A_{\overline{\lambda}}, B_{\overline{\lambda}}, Q_{\overline{\lambda}}, G_{\overline{\lambda}}) = S^{\varepsilon}(x_0,A,B,Q,G)$. Then there exists $u^{\star}_{\varepsilon,\lambda} \in S^{\varepsilon}(x_0^{\lambda}, A_\lambda, B_\lambda, Q_\lambda, G_\lambda)$ such that

$$\|u^{\star}_{\varepsilon,\lambda} - u^{\star}_{\varepsilon}\|_{L^2(0,T,V)} \leq \varepsilon + h\left(S^{\varepsilon}(x_0^{\lambda}, A_\lambda, B_\lambda, Q_\lambda, G_\lambda), S^{\varepsilon}(x_0,A,B,Q,G)\right).$$

Corollary 3.3. *Assume that (3.10) is satisfied. Let K be the constant defined in Theorem 3.3, and let $u^{\star}$ be the optimal control of problem (3.7)-(3.8). If $KT \in [0,1)$, then, for all $\varepsilon > 0$ and for all $\lambda \in \Lambda(\overline{\lambda})$, there exists $u^{\star}_{\varepsilon,\lambda} \in S^{\varepsilon}(x_0^{\lambda}, A_\lambda, B_\lambda, Q_\lambda, G_\lambda)$ such that*

$$\begin{aligned} &\|u^{\star}_{\varepsilon,\lambda} - u^{\star}\|_{L^2(0,T,V)} \\ &\leq \varepsilon(1 + \frac{1}{1-KT}) + \frac{\sqrt{T}}{1-KT}(\alpha_1\|x_\lambda^0 - x_0\| + \alpha_2\|Q_\lambda - Q\| + \alpha_3\|G_\lambda - G\| \\ &\quad + \alpha_4\|A_\lambda - A\| + \alpha_5\|B_\lambda - B\|). \end{aligned}$$

The proof is immediate from Remark 3.1 and Theorem 3.6.

Corollary 3.4. *Under the same assumptions of Corollary 3.3, if $KT \in [0,1)$, then, for all $\varepsilon > 0$, there exists $u^\star_\varepsilon \in S^\varepsilon(x_0, A, B, Q, G)$ such that*

$$\|u^\star_\varepsilon - u^\star\|_{L^2(0,T,V)} \leq \varepsilon(1 + \frac{1}{1-KT}). \tag{3.18}$$

In particular, $(u^\star_\varepsilon)_{\varepsilon>0}$ converges uniformly to the unique solution $u^\star$ when ε goes to 0.

It suffices to take $\lambda = \overline{\lambda}$ in Theorem 3.6. The required convergence in the last part of the conclusion is direct from (3.18).

Remark 3.2. The convergence of the sequence of approximate solutions $(u^\star_\varepsilon)_{\varepsilon>0}$ to the unique solution $u^\star$ is also a straight consequence of the qualitative result on fixed points of general set-valued maps established in [2, Proposition 4, Assertion g)].

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