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RAVINES OF QUADRATIC FUNCTIONS

NGUYEN NANG TAM[1], NGUYEN DONG YEN[2,*]

[1] *Institute of Theoretical and Applied Research, Duy Tan University, Hanoi 100000, Vietnam; Faculty of Natural Sciences, Duy Tan University, Da Nang 550000, Vietnam*
[2] *Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet, Hanoi 10307, Vietnam*

Abstract. In this paper, the notion of the ravine of real-valued functions is extended from the finite-dimensional setting to an infinite-dimensional setting. Ravines of quadratic functions are studied in detail. The obtained results solve a problem raised by Professor Joachim Gwinner. In addition, it is proved that a weakly continuous real-valued convex function defined on a reflexive Banach space cannot have any ravine along the null subspace.

Keywords. Normed space; Ravine; Convex function; Proper linear subspace; Quadratic function.

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1. Introduction

The notion of the ravine of a function was first introduced by Belousov and Andronov in [1]. Later, a detailed study on the ravine of a function was given by the same authors in [2, 3]. For more comments and discussions on the importance of the ravine of a function, we refer to [8]. Since quadratic functions were widely used in the literature (see, e.g., [4, 5, 6] and the references therein), it is of interest to investigate what the ravine means for quadratic functions. In a finite-dimensional setting, Tam et al. [8, Theorem 4.1] have proved that quadratic functions cannot have ravines along linear subspaces.

Due to their variety of applications, quadratic functions defined on infinite-dimensional spaces have been studied by many authors, especially in optimization (see, e.g., [4, 9]). Since the notion of the ravine of a function has been considered only in finite-dimensional settings so far, it is desirable to obtain some analogues of the results on the ravines of quadratic functions in [8] in an infinite-dimensional setting. This problem was shown to us by Professor Joachim Gwinner in private communication.

In the present paper, we extend the notion of the ravine of a function in [1] to a normed space setting and show that any real-valued function defined on a normed space has a ravine along any dense proper linear subspace. Since the proof of the result on real-valued convex functions given in [2, Remark 6, p. 37] seems to be incorrect, we give a completely different proof and

*Corresponding author.
E-mail address: nguyennangtam@duytan.edu.vn (N.N. Tam), ndyen@math.ac.vn (N.D. Yen).
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show that the result is true not only in finite dimensions but also in infinite dimensions. Then, we establish a result on the ravines of quadratic functions.

The definition of a ravine of a real-valued function defined on a normed space and a sufficient condition for a function to have a ravine are presented in Section 2. The non-existence of ravines of a weakly continuous real-valued convex function defined on a reflexive Banach space along the null subspace is proved in the same section. A theorem on the ravines of quadratic functions, which is our main result, is obtained in Section 3. Some concluding remarks are given in Section 4, the last section.

2. Ravines of Real-valued Functions

Let $(X,\|\cdot\|)$ be a real normed space. Suppose that $f: X \to \mathbb{R}$ is a function defined on X, and $L \subset X$ is a fixed proper linear subspace. One says that L is *dense* in X if $\overline{L} = X$, where $\overline{L}$ denotes the topological closure of L in X. The distance from $z \in X$ to a subset $A \subset X$ is defined by $d(z,A) := \inf_{x\in A} \|z-x\|$. The closed ball (resp., the open ball) centered at $\bar{x} \in X$ with radius $\rho > 0$ is denoted by $\bar{B}(\bar{x},\rho)$ (resp., $B(\bar{x},\rho)$). Let $S(\bar{x},\rho) := \bar{B}(\bar{x},\rho)\setminus B(\bar{x},\rho)$ be the corresponding sphere. By $\mathbb{N}$, we denote the set of positive natural numbers.

Definition 2.1. (See [2, p. 34] for the definition in the case $X = \mathbb{R}^n$) We say that f has a *ravine along the proper subspace* L or, shorter, *L-ravine*, if there exists a sequence $\{x^k\}$, called an *L-ravine sequence*, such that, for all positive numbers δ and ε, and for all sequences $\{y^k\}$ and $\{z^k\}$ satisfying the conditions

$$\begin{cases} \|x^k - y^k\| < \delta, & y^k \in x^k + L, \\ \|x^k - z^k\| < \delta, & d(z^k, x^k + L) > \varepsilon \end{cases} \tag{2.1}$$

for all $k \in \mathbb{N}$, the equality $\lim_{k\to+\infty} \left[f(z^k) - f(y^k)\right] = +\infty$ is fulfilled.

Now, let us consider an example of a function of the cubic polynomial type on an infinite-dimensional normed space, which has a ravine along a proper subspace L. By ℓ_2 we denote the Hilbert space of real sequences $x = (x_1, x_2, \ldots)$ with $\sum_{i=1}^{+\infty} (x_i)^2 < +\infty$. The inner product and the norm are defined respectively by $\langle x,y\rangle = \sum_{i=1}^{+\infty} x_i y_i$ and $\|x\| = \langle x,x\rangle^{1/2} = \left(\sum_{i=1}^{+\infty} (x_i)^2\right)^{1/2}$ for all $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots)$ from ℓ_2.

Example 2.1. (cf. [2, Example 1, p. 48]) Consider the function $f(x) = (x_1)^2 x_2$ of the argument $x = (x_1, x_2, \ldots) \in \ell_2$ and let $L = \{x = (x_1, x_2, \ldots) \in \ell_2 \mid x_1 = 0\}$. Choose $x^k = (0, t_k, 0, 0, \ldots)$, where $\lim_{k\to+\infty} t_k = +\infty$. Suppose that the constants $\delta > 0$ and $\varepsilon > 0$ are given arbitrarily. If $\{y^k\}$ and $\{z^k\}$ are two sequences of vectors in ℓ_2 satisfying the conditions in (2.1), then one must have $y^k = (0, t_k + \alpha_k, y_3^k, y_4^k, \ldots)$ and $z^k = (z_1^k, t_k + \beta_k, z_3^k, z_4^k, \ldots)$ with

$$\left(|\alpha_k|^2 + (y_3^k)^2 + (y_4^k)^2 + \ldots\right)^{1/2} < \delta,$$

$$\left((z_1^k)^2 + \beta_k^2 + (z_3^k)^2 + (z_4^k)^2 + \ldots\right)^{1/2} < \delta,$$

and $|z_1^k| > \varepsilon$ for all $k \in \mathbb{N}$. It follows that $|\beta_k| < \delta$ for every $k \in \mathbb{N}$. Therefore,

$$f(z^k) - f(y^k) = f(z^k) = (z_1^k)^2(t_k + \beta_k) \to +\infty \text{ as } k \to +\infty.$$

This shows that $\{x^k\}$ is an L-ravine sequence of f.

The following theorem reveals a pathological situation related to Definition 2.1.

Theorem 2.1. *If the proper subspace L is dense in X, then any function $f : X \to \mathbb{R}$ has a ravine along L. Moreover, any sequence $\{x^k\} \subset X$ is an L-ravine sequence.*

Proof. To prove the theorem, fix any sequence $\{x^k\} \subset X$ and let $\delta > 0$ and $\varepsilon > 0$ be given arbitrarily. Suppose that $\{y^k\}$ is a sequence satisfying the conditions $\|x^k - y^k\| < \delta$ and $y^k \in x^k + L$ for all k. Then, *there is no sequence $\{z^k\}$ in X such that $\|x^k - z^k\| < \delta$ and*

$$d(z^k, x^k + L) > \varepsilon.$$

Indeed, since $d(z^k, x^k + L) = d(z^k - x^k, L)$, the last inequality implies that $d(z^k - x^k, L) > \varepsilon$. This is impossible because $z^k - x^k \in \overline{L}$ by the density of L. Thus, one cannot find any sequences $\{y^k\}$ and $\{z^k\}$ satisfying (2.1). Therefore, in accordance with Definition 2.1, $\{x^k\}$ is an L-ravine sequence. □

Theorem 2.1 tells us that the above definition of ravine is meaningful only if $\overline{L} \neq X$.

The next theorem is on the non-existence of any ravine of a convex function along the null subspace. This result was given in [2, Remark 6, p. 37] for real-valued convex functions defined on finite-dimensional Euclidean spaces. But, as far as we know, the proof there is incorrect. By a completely different proof, we now show that the result is true not only in finite dimensions but also in infinite dimensions. The exact formulation of our result is as follows.

Theorem 2.2. *A weakly continuous real-valued convex function on a real reflexive Banach space cannot have any ravine along the null subspace.*

Proof. Let $f : X \to \mathbb{R}$ be a weakly continuous convex function, where X is a real reflexive Banach space. To prove by contradiction, suppose that f has a ravine along the subspace $L = \{0\}$. Then, by Definition 2.1, there exists a sequence $\{x^k\}$ such that for any numbers $\delta > 0$ and $\varepsilon > 0$, and for any sequences $\{y^k\}$ and $\{z^k\}$ satisfying the conditions in (2.1) for all $k \in \mathbb{N}$, the equality $\lim\limits_{k\to+\infty} \left[f(z^k) - f(y^k)\right] = +\infty$ holds. Since $L = \{0\}$, this means that $y^k = x^k$ for all k,

$$\varepsilon < \|x^k - z^k\| < \delta \quad (\forall k \in \mathbb{N}) \tag{2.2}$$

and

$$\lim_{k\to+\infty} \left[f(z^k) - f(x^k)\right] = +\infty. \tag{2.3}$$

Fix any positive numbers ε and δ with $\varepsilon < \delta$. Take a number $\delta_1 \in (\varepsilon, \delta)$.

CLAIM 1. *Sequence $\{x^k\}$ is unbounded.*

Indeed, if $\{x^k\}$ is bounded, then there is $\rho > 0$ such that $x^k \in \bar{B}(0,\rho)$ for all $k \in \mathbb{N}$. Select a sequence $\{z^k\}$ satisfying (2.2). Then (2.3) holds. On one hand, since $z^k \in \bar{B}(0, \rho + \delta)$ for all $k \in \mathbb{N}$ and X is a reflexive Banach space, there exists a subsequence $\{z^{k'}\}$ of $\{z^k\}$ that weakly converges to a vector $\hat{z} \in \bar{B}(0, \rho + \delta)$ as $k' \to +\infty$. On the other hand, by the lower semicontinuity of f in the weak topology of X and the Weierstrass theorem, one can find $\hat{x} \in \bar{B}(0, \rho + \delta)$

such that $f(\hat{x}) \le f(x)$ for every $x \in \bar{B}(0,\rho+\delta)$. In particular,

$$f(z^k) - f(\hat{x}) \ge f(z^k) - f(x^k) \quad (\forall k \in \mathbb{N}).$$

Then, it follows from (2.3) that

$$\liminf_{k\to+\infty} \left[f(z^k) - f(\hat{x})\right] \ge \liminf_{k\to+\infty} \left[f(z^k) - f(x^k)\right] = \lim_{k\to+\infty} \left[f(z^k) - f(x^k)\right] = +\infty.$$

Obviously, this yields $\lim_{k'\to+\infty} f(z^{k'}) = +\infty$. Meanwhile, by the weak continuity of f and the weak convergence of $\{z^{k'}\}$ to $\hat{z}$, we have $\lim_{k'\to+\infty} f(z^{k'}) = f(\hat{z})$. This reaches a contradiction. Thus sequence $\{x^k\}$ must be unbounded.

Thanks to Claim 1, by considering a subsequence of $\{x^k\}$ (if necessary), we can assume that $\lim_{k\to+\infty} \|x^k\| = +\infty$.

CLAIM 2. *There exists an index $k_1 \in \mathbb{N}$ such that, for every $k \ge k_1$,*

$$f(z) > f(x^k) \quad \forall z \in S(x^k, \delta_1). \tag{2.4}$$

Indeed, if the claim was false, we would find a subsequence $\{k'\}$ of $\{k\}$ and a sequence $\{z^{k'}\}$ with $z^{k'} \in S(x^{k'}, \delta_1)$ such that $f(z^{k'}) \le f(x^{k'})$ for all k'. This implies that

$$\limsup_{k'\to+\infty} \left[f(z^{k'}) - f(x^{k'})\right] \le 0,$$

which contradicts (2.3). So, the claim is valid.

For every $k \ge k_1$, by the lower semicontinuity of f in the weak topology and the Weierstrass theorem, there exists $u^k \in \bar{B}(x^k, \delta_1)$ with $f(u^k) \le f(x)$ for all $x \in \bar{B}(x^k, \delta_1)$. Property (2.4) guarantees that $u^k \in B(x^k, \delta_1)$. Hence, u^k is a local minimizer of f. As f is a convex function, we have $u^k \in \Sigma_f$ for every $k \ge k_1$, where Σ_f denotes the solution set of optimization problem $\min\{f(x) \mid x \in X\}$. In particular, we have $u^{k_1} \in \Sigma_f$.

Since $\lim_{k\to+\infty} \|x^k\| = +\infty$, we can find an integer $k_2 > k_1$ such that $\|x^{k_1} - x^{k_2}\| > 2\delta$. As $\delta_1 \in (\varepsilon, \delta)$, this yields $\bar{B}(x^{k_1}, \delta_1) \cap \bar{B}(x^{k_2}, \delta_1) = \emptyset$. Then, $u^{k_2} \notin \bar{B}(x^{k_1}, \delta_1)$. So, the line segment $\left[u^{k_1}, u^{k_2}\right]$ must intersect the sphere $S(x^{k_1}, \delta_1)$ at a point denoted by z^{k_1}. Since $u^{k_2} \in \Sigma_f$ and $u^{k_1} \in \Sigma_f$, by the convexity of Σ_f we have $z^{k_1} \in \Sigma_f$. Hence,

$$f(z^{k_1}) = f(u^{k_1}) \le f(x^{k_1}).$$

But this comes in conflict with (2.4).

The proof of the theorem is complete. □

Remark 2.1. For a real-valued convex function on an infinite-dimensional real reflexive Banach space X, *the weak continuity in Theorem 2.2 is sufficient, but not necessary, for the non-existence of ravines of the function along the null subspace*. To justify this claim, choose $f(x) = \|x\|$ and observe that if (2.2) holds, then (2.3) cannot hold, because

$$\limsup_{k\to+\infty} \left[f(z^k) - f(x^k)\right] = \limsup_{k\to+\infty} \left[\|z^k\| - \|x^k\|\right] \le \limsup_{k\to+\infty} \|z^k - x^k\| \le \delta.$$

Therefore, f cannot have any ravine along the null subspace. As the convex function $f(x) = \|x\|$ is continuous on X, it is weakly lower semicontinuous on X. However, since the reflexive

Banach space X is infinite-dimensional, one can easily show that this function f is not weakly upper semicontinuous on X. Thus, f is not weakly continuous on X.

Corollary 2.1. *A real-valued convex function on $\mathbb{R}^n$ cannot have any ravine along the null subspace.*

Proof. Observe by [7, Corollary 10.1.1] that any convex function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous on $\mathbb{R}^n$. So, since the weak topology of $\mathbb{R}^n$ coincides with the norm topology, the desired result follows from Theorem 2.2. $\square$

3. Ravines of Quadratic Functions

Following [4, p. 193], we say that a function $Q : X \to \mathbb{R}$, where X is a real normed space, is a *quadratic form* on X if there exists a bilinear symmetric function $\psi : X \times X \to \mathbb{R}$ such that $Q(x) = \psi(x,x)$ for all $x \in X$. The symmetry of ψ means that $\psi(x,y) = \psi(y,x)$ for all $x, y \in X$. If ψ is continuous at $(0,0) \in X \times X$, then Q is Fréchet differentiable at any point $u \in X$ and one has $\nabla Q(u) = 2\psi(u,.)$ (see [9, Proposition 2.1] for a proof of this fact). Conversely, if Q is continuous, then using the formula $\psi(x,y) = \frac{1}{4}(Q(x+y) - Q(x-y))$, which holds for all $x, y \in X$, we can infer that ψ is continuous on $X \times X$. We have thus seen that the continuity of Q on X is equivalent to the continuity of ψ on $X \times X$. Examples of discontinuous quadratic forms can be found, e.g., in [9, p. 40].

We will deal with quadratic functions of the type

$$f(x) = Q(x) + \langle b, x \rangle + \gamma, \tag{3.1}$$

where Q is a quadratic form, $\langle b, . \rangle$ is a linear functional on X which is not required to be continuous, and γ is a real number. In the next theorem, there is no condition on the continuity of Q. This means that the result is valid even for discontinuous quadratic functions.

Theorem 3.1. *Quadratic functions on a real normed space cannot have a ravine along any non-dense linear subspace.*

Proof. Let $f(x)$ be a quadratic function of type (3.1). Let $L \subset X$ be a linear subspace with $\overline{L} \neq X$. We will prove that f cannot have any L-ravine sequence.

Suppose that $\{x^k\}$ is an arbitrarily given sequence in X. Fix any vector $c \in X \setminus \overline{L}$. Note that $c \neq 0$ and $d(c, \overline{L}) > 0$. Suppose that $\psi : X \times X \to \mathbb{R}$ is a bilinear symmetric function such that $Q(x) = \psi(x,x)$. Select any $\mu \in (0, \lambda)$, where $\lambda := d(c, \overline{L})$. Put $\delta = \lambda + \|c\|$, $\varepsilon = \lambda - \mu$, and $y^k = x^k$. For each $k \in \mathbb{N}$, define $\alpha(c, x^k) = 2\psi(x^k, c)$.

First, consider the case where $\alpha(c, x^k)$ does not tend to $+\infty$ as $k \to +\infty$. In this situation, setting $z^k = x^k + c$, we have

$$\begin{cases} \|x^k - y^k\| = 0 < \delta, \quad y^k \in x^k + L, \\ \|x^k - z^k\| = \|c\| < \delta, \\ d(z^k, x^k + L) = d(z^k, x^k + \overline{L}) = d(c, \overline{L}) = \lambda > \varepsilon. \end{cases} \tag{3.2}$$

Besides,

$$\begin{aligned} f(z^k) &= Q(z^k) + \langle b, z^k \rangle + \gamma \\ &= Q(x^k + c) + \langle b, x^k + c) + \gamma \\ &= \psi(x^k + c, x^k + c) + \langle b, x^k + c \rangle + \gamma \\ &= Q(x^k) + Q(c) + \psi(x^k, c) + \psi(c, x^k) + \langle b, x^k \rangle + \langle b, c \rangle + \gamma \\ &= f(x^k) + f(c) + \alpha(c, x^k) - \gamma. \end{aligned}$$

By (3.2), sequences $\{y^k\}$ and $\{z^k\}$ satisfy the conditions in (2.1). Since

$$f(z^k) - f(y^k) = f(z^k) - f(x^k) = \alpha(c, x^k) + f(c) - \gamma,$$

one sees that sequence $\{f(z^k) - f(y^k)\}$ does not tend to $+\infty$ as $k \to +\infty$.

Now, let us consider the case where $\lim\limits_{k\to+\infty} \alpha(c, x^k) = +\infty$. Set $z^k = x^k - c$ for every $k \in \mathbb{N}$. Since $\overline{L}$ is a linear subspace of X, we have

$$\begin{cases} \|x^k - y^k\| = 0 < \delta, \quad y^k \in x^k + L, \\ \|x^k - z^k\| = \|c\| < \delta, \\ d(z^k, x^k + L) = d(z^k, x^k + \overline{L}) = d(-c, \overline{L}) = d(c, \overline{L}) = \lambda > \varepsilon. \end{cases} \tag{3.3}$$

In addition, it holds that

$$\begin{aligned} f(z^k) &= Q(z^k) + \langle b, z^k \rangle + \gamma \\ &= Q(x^k - c) + \langle b, x^k - c) + \gamma \\ &= \psi(x^k - c, x^k - c) + \langle b, x^k - c \rangle + \gamma \\ &= Q(x^k) + Q(c) - \Big(\psi(x^k, c) + \psi(c, x^k)\Big) + \langle b, x^k \rangle - \langle b, c \rangle + \gamma \\ &= f(x^k) + f(c) - \alpha(c, x^k) - 2\langle b, c \rangle - \gamma. \end{aligned}$$

Thanks to (3.3), sequences $\{y^k\}$ and $\{z^k\}$ satisfy the conditions in (2.1). Since

$$f(z^k) - f(y^k) = f(z^k) - f(x^k) = -\alpha(c, x^k) + f(c) - 2\langle b, c \rangle - \gamma,$$

one obtains $\lim\limits_{k\to+\infty} \left[f(z^k) - f(y^k)\right] = -\infty$.

We have thus shown that there exist constants $\delta > 0$ and $\varepsilon > 0$ such that one can find some sequences $\{y^k\}$ and $\{z^k\}$ satisfying the conditions in (2.1) for which the property

$$\lim_{k\to+\infty} \left[f(z^k) - f(y^k)\right] = +\infty$$

is invalid. Therefore, $\{x^k\}$ is not an L-ravine sequence. Since $\{x^k\}$ was given arbitrarily, this completes the proof. □

4. CONCLUSIONS

We extended some results on ravines of quadratic functions to an infinite dimensional setting. Theorem 3.1 shows that quadratic functions cannot have a ravine along any non-dense subspace. Combining this with Theorem 2.1, we can infer that quadratic functions on a real normed space X have a ravine along a linear subspace L if and only if L is dense in X.

Another result of the paper is Theorem 2.2 on the non-existence of ravines of real-valued convex functions along the null subspace, which shows that the statement given in [2, Remark 6, p. 37] is correct and it is valid for weakly continuous real-valued convex functions defined on reflexive Banach spaces.

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