J. Appl. Numer. Optim. 6 (2024), No. 2, pp. 177-202
Available online at http://jano.biemdas.com
https://doi.org/10.23952/jano.6.2024.2.02

GENERALIZED HUKUHARA DINI HADAMARD ε-SUBDIFFERENTIAL AND $\mathbf{H}_\varepsilon$-SUBGRADIENT AND THEIR APPLICATIONS IN INTERVAL OPTIMIZATION

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Abstract. In this paper, we develop and analyze the concepts of gH-Dini Hadamard ε-subdifferential and $\mathbf{H}_\varepsilon$-subgradient for interval-valued functions (IVFs). Some important characteristics of gH-Dini Hadamard ε-subdifferential such as closedness, convexity, and monotonicity are studied. The interrelations between gH-subgradient and gH-Dini Hadamard ε-subgradient, and between gH-Fréchet derivative and gH-Dini Hadamard ε-subdifferential are investigated. To define the concept of $\mathbf{H}_\varepsilon$-subgradient, the notions of the sponge of a set around a point and gH-calm IVF at a point are studied. A variational description of gH-Dini Hadamard ε-subgradient with $\mathbf{H}_\varepsilon$-subgradient is proposed. Various necessary and sufficient conditions for obtaining an ε-efficient solution to an interval optimization problem (IOP) with the help of gH-Dini Hadamard ε-subgradient of an IVF are derived. Lastly, an application of proposed results is discussed in the sparsity regularizer for IOPs.

Keywords. gH-Dini Hadamard ε-subdifferential; $\mathbf{H}_\varepsilon$-subgradient; ε-efficient solution; Interval-valued sparsity regularizer problem.

2020 Mathematics Subject Classification. 49N15, 49J40, 90C31.

1. Introduction

Optimization theory for nonsmooth functions is known to have been hugely influenced by approximate subdifferentials or ε-subgradients. Various numerical methods have been constructed with the help of ε-subgradients to minimize convex functions. The contribution of approximate subdifferentials on the calculus of convex subdifferentials helps to develop several generalized gradients of Clarke [1, 2, 3]. As Dini Hadamard derivative preserves the linearity of derivative with respect to the direction [1], ε-Dini Hadamard plays a prominent role [4] in the advancement of nonsmooth analysis, especially nondifferentiable functions in the absence of convexity. In 1976, Mordukhovich [5] first obtained ε-subdifferentials as a byproduct of certain approximative techniques. After that, various methods and theories were developed for finding an approximate solution to optimization problems [6, 7, 8].

Observe that many real-world situations are not always expressible with conventional mathematics due to uncertainty or inexact data. Thus there is a need to advance optimization tools for

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Received 19 November 2022; Accepted 30 January 2023; Published online 1 April 2024.

uncertain optimization. The optimization problem due to uncertainty in a random variable comes under stochastic optimization, and the optimization problem involving a membership function is expressed by a fuzzy optimization problem. The optimization problem with interval coefficients comes under interval optimization, named interval optimization problem (IOP), is indispensable in dealing with many real-life uncertain problems. Some applications of interval optimization in real life can be found in [8, 9, 10].

1.1. **Literature Survey.** The class of subdifferentials developed by Rockafellar [11] is a crucial factor in the body of optimization theory that perfectly replaces the role of gradients to identify optima for convex functions. However, this is inadequate in developing the optimality conditions for nonconvex optimization problems. Mordukhovich [5] first obtained byproducts of certain finite dimensional approximative optimization techniques of nonconvex functions, which were named as approximate subdifferentials or ε-subdifferentials. After [5], several subsequent attempts were made to further elaborate the definition to a more general setting with Gâteaux [12] or Fréchet [13] differentiable norms. However, these results were restricted to Banach spaces. In [14], a new definition of approximate subdifferentials for arbitrary locally convex spaces was given. With this, numerous analytic results of approximate subdifferentials were discussed. In [15] and [16], a general Banach theory of approximate subdifferentials was discussed. With the influence of these developed results, Azimov and Gasimov [17, 18] proposed the concept of weak subdifferentials, which are the generalization of classical subdifferential in which supporting hyperplanes were replaced by supporting conic surfaces [19, 20]. To further extend this concept of weak subdifferentials to nonsmooth analysis, the notion of sponges was defined by Treiman [21]. Based on this notion, Boţ [22] defined the Dini Hadamard ε-subdifferentiability. In the developed results of Boţ [22], it can be observed that the existence of a neighborhood around a point implies the existence of a sponge at that point. However, if the set is convex, then a sponge around a point also implies the existence of a neighborhood around that point (see Example 2.2 in [22]).

The theory of gH-Dini Hadamard ε-subdifferentials for IVFs is not studied yet. In the analysis of gH-Dini Hadamard ε-subdifferentiability of an IVF, calculus plays a significant role. Initially, Hukuhara [23] defined the notion of differentiability of IVFs by using the Hukuhara difference (known as H-difference). However, this definition suffers certain drawbacks (see [27]). Nevertheless, Wu [24] presented the notions of limit, continuity, and differentiability of IVFs. Thereafter, Markov [25] removed this downside of H-differentiability by introducing the idea of a new subtraction (known as nonstandard subtraction) and proposed the generalized calculus on intervals. Furthermore, Stefanini and Bede [26] refined the concept of Hukuhara difference with the commencement of generalized Hukuhara difference (known as gH-difference). After that, in [27], the notion of generalized differentiability was discussed by using gH-difference of intervals for IVFs. Also, Ghosh et al. [28] presented the notion of gH-directional derivative for IVFs. Motivated by [29], the theory of gH-subdifferential and gH-subgradient of IVFs were discussed in [30]. With this, numerous researchers [28, 31, 32, 33, 34, 35] have contributed to the gH-subdifferentiability of IVFs.

This article involves the concept of gH-Dini Hadamard ε-subdifferential for IVFs. Towards this, we have observed that the notion of gH-Dini Hadamard ε-subdifferential is more general than the concept of gH-subdifferentiability. In the sequel, the relation of gH-subdifferentiability and gH-Fréchet differentiability with gH-Dini Hadamard ε-subdifferentiability is given. Further, an important concept of $\mathbf{H}_\varepsilon$-subgradient is given. To define this concept, the notions of sponge of a set and gH-calm IVF are discussed. The reason for defining these notions is that the existing results are limited for neighbourhoods. It is known that every neighbourhood is a sponge, however, the converse need not be true [21]. Thus the notion of sponge of a set is more general. Based on this, several necessary and sufficient conditions for finding an ε-efficient solution to an IOP are derived.

1.2. **Motivation and work done.** It is known that, in conventional optimization theory, the approximate subdifferentials and the Dini Hadamard ε-subdifferentials are found to be minimal among other conceivable subdifferentials. Also, on analyzing the literature on IOPs, it can be noticed that the theory on gH-subgradients and gH-subdifferentiability has been developed recently (see, e.g., [30, 31, 32, 33, 34, 35]). From the existing results on IOPs, one sees that the gH-subdifferential set may be empty and there is no theory to study the behaviour of such IVFs (see Example 3.1). However, with the help of the defined notion of gH-Dini Hadamard ε-subdifferentials, these IVFs can be studied. Moreover, the concept of gH-Dini Hadamard ε-subdifferential contains the set of gH-subdifferential and set of Fréchet derivatives; however, the converse is not true (see Theorem 3.1 and 3.4). We have proposed the notion of $\mathbf{H}_\varepsilon$-subgradient, which is more general than all the existing subdifferentials on IOPs (see [30, 31, 32, 33, 34, 35]) and also contains the set of gH-Dini Hadamard ε-subdifferential (see Theorem 4.1). A variational description of gH-Dini Hadamard ε-subgradient with $\mathbf{H}_\varepsilon$-subgradient was performed. To observe the application of gH-Dini Hadamard ε-subdifferentiability in IOPs, we define the concept of ε-efficient solution to an IOP. Based on the idea of ε-efficient solutions to an IOP, necessary and sufficient optimality conditions to an IOP are given.

1.3. **Delineation.** The whole work is demonstrated in the following order. Section 2 covers some basic tools of arithmetic on intervals and calculus of IVFs. In Section 3, the gH-Dini Hadamard ε-subdifferentiability for IVFs with its several important characteristics is proposed. In the same section, a few relations between gH-Fréchet differentiability and gH-Dini Hadamard ε -subdifferentiability are given. Next, an important concept of $\mathbf{H}_\varepsilon$-subgradient is given in Section 4, which is based on the criterion of sponge of a set. Further, a variational interpretation of gH-Dini Hadamard ε-subdifferential based on the sponge of a set is discussed. In Section 5, the concept of ε-efficient solution followed by necessary and sufficient efficient conditions for finding an ε-efficient solution to an IOP with the gH-Dini Hadamard ε-subgradient of its objective function are given. An example to demonstrate the application of proposed results in sparsity regularizer for IOPs is given. Finally, Section 6 covers the conclusion and future scopes.

2. Preliminaries and Terminologies

In this section, basic tools on intervals, and convexity and calculus of IVFs are provided. In the whole paper,

- $\mathbb{R}_+$ and $\mathbb{R}$ denote the collection of nonnegative real numbers and set of real numbers, respectively;
- $I(\mathbb{R})$ refers to the collection of all compact intervals;
- the elements of $I(\mathbb{R})$ and $I(\mathbb{R})^n$ are denoted by bold capital letters and bold capital letters with a cap, respectively;
- $-\infty$ and $+\infty$ represent the intervals $[-\infty, -\infty]$ and $[+\infty, +\infty]$;
- $\overline{I(\mathbb{R})} = I(\mathbb{R}) \cup \{-\infty, +\infty\}$; and
- $\mathscr{B}(h, \delta)$ represents a ball with centre at h and radius δ in $\mathbb{R}^n$.

2.1. **Fundamental operations and dominance relations on intervals.** Let $\mathbf{P}, \mathbf{Q} \in I(\mathbb{R})$ and $\beta \in \mathbb{R}$. Moore's [36] interval addition, subtraction, product, division, and scalar multiplication are represented by $\mathbf{P} \oplus \mathbf{Q}$, $\mathbf{P} \ominus \mathbf{Q}$, $\mathbf{P} \odot \mathbf{Q}$, $\mathbf{P} \oslash \mathbf{Q}$, and $\beta \odot \mathbf{P}$, respectively. In defining $\mathbf{P} \oslash \mathbf{Q}$, it is assumed that $0 \notin \mathbf{Q}$.

For any nondegenerate interval $\mathbf{P}$, the relation $\mathbf{P} \ominus \mathbf{P} = \mathbf{0}$ does not hold. The following definition of difference of intervals is used throughout this article.

Definition 2.1. (gH-difference [26]). Let $\mathbf{P}, \mathbf{Q} \in I(\mathbb{R})$. Then, the gH-difference between $\mathbf{P} = [\underline{p}, \overline{p}]$ and $\mathbf{Q} = [\underline{q}, \overline{q}]$ is denoted by $\mathbf{P} \ominus_{gH} \mathbf{Q}$, defined by

$$\mathbf{P} \ominus_{gH} \mathbf{Q} = \left[\min\{\underline{p} - \underline{q}, \overline{p} - \overline{q}\},\ \max\{\underline{p} - \underline{q}, \overline{p} - \overline{q}\}\right] \text{ and } \mathbf{P} \ominus_{gH} \mathbf{P} = \mathbf{0}.$$

If the interval $\mathbf{X}$ is the gH-difference of $\mathbf{P}$ and $\mathbf{Q}$, then

$$\mathbf{P} = \mathbf{Q} \oplus \mathbf{X} \text{ or } \mathbf{Q} = \mathbf{P} \ominus \mathbf{X}.$$

For two elements $\widehat{\mathbf{C}} = (\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_n)^\top$ and $\widehat{\mathbf{D}} = (\mathbf{D}_1, \mathbf{D}_2, \ldots, \mathbf{D}_n)^\top$ in $I(\mathbb{R})^n$, the algebraic operation $\star$ between $\widehat{\mathbf{C}}$ and $\widehat{\mathbf{D}}$ on the product space $I(\mathbb{R})^n = I(\mathbb{R}) \times I(\mathbb{R}) \times \cdots \times I(\mathbb{R})$ (n times) is denoted by $\widehat{\mathbf{C}} \star \widehat{\mathbf{D}}$ and defined as

$$\widehat{\mathbf{C}} \star \widehat{\mathbf{D}} = (\mathbf{C}_1 \star \mathbf{D}_1, \mathbf{C}_2 \star \mathbf{D}_2, \ldots, \mathbf{C}_n \star \mathbf{D}_n)^\top,$$

where $\star \in \{\oplus, \ominus, \ominus_{gH}\}$.

Definition 2.2. (Norms on $I(\mathbb{R})$ and $I(\mathbb{R})^n$ [36]). The norm of an interval $\mathbf{S} = [\underline{s}, \overline{s}] \in I(\mathbb{R})$ and an interval vector $\widehat{\mathbf{S}} = (\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_n)^\top \in I(\mathbb{R})^n$ are defined by

$$\|\mathbf{S}\|_{I(\mathbb{R})} = \max\{|\underline{s}|, |\overline{s}|\} \text{ and } \|\widehat{\mathbf{S}}\|_{I(\mathbb{R})^n} = \sum_{j=1}^{n} \|\mathbf{S}_j\|_{I(\mathbb{R})}, \text{ respectively.}$$

Definition 2.3. (Dominance of intervals [24]). Consider two intervals $\mathbf{C} = [\underline{c}, \overline{c}]$ and $\mathbf{D} = [\underline{d}, \overline{d}]$ in $I(\mathbb{R})$.

(i) $\mathbf{C}$ is called dominated by $\mathbf{D}$ if '$\underline{d} \leq \underline{c}$ and $\overline{d} \leq \overline{c}$', and we write $\mathbf{D} \preceq \mathbf{C}$;
(ii) $\mathbf{C}$ is said to be strictly dominated by $\mathbf{D}$ if either '$\underline{d} \leq \underline{c}$ and $\overline{d} < \overline{c}$' or '$\underline{d} < \underline{c}$ and $\overline{d} \leq \overline{c}$', and we write $\mathbf{D} \prec \mathbf{C}$;
(iii) if $\mathbf{C}$ is not dominated by $\mathbf{D}$, then we write $\mathbf{D} \npreceq \mathbf{C}$. If $\mathbf{C}$ is not strictly dominated by $\mathbf{D}$, then we say that $\mathbf{C}$ and $\mathbf{D}$ are not comparable and we write $\mathbf{D} \nprec \mathbf{C}$.

Remark 2.1. It can be noted that, for any $\widehat{\mathbf{C}}$ and $\widehat{\mathbf{D}}$ in $I(\mathbb{R})^n$,

$$\widehat{\mathbf{C}} \preceq \widehat{\mathbf{D}} \iff \mathbf{C}_j \preceq \mathbf{D}_j \text{ for all } j = 1, 2, \ldots, n.$$

Lemma 2.1. *(See [35]). Let $\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{R}$, and $\boldsymbol{S} \in I(\mathbb{R})$.*

(i) If $\boldsymbol{P} \preceq \boldsymbol{Q}$ and $\boldsymbol{Q} \preceq \boldsymbol{R}$, then $\boldsymbol{P} \preceq \boldsymbol{R}$ and
(ii) If $\boldsymbol{P} \oplus \boldsymbol{Q} \preceq \boldsymbol{R} \oplus \boldsymbol{S}$, then $\boldsymbol{P} \ominus_{gH} \boldsymbol{R} \preceq \boldsymbol{S} \ominus_{gH} \boldsymbol{Q}$.

Lemma 2.2. *(See [36]). For $\boldsymbol{A}, \boldsymbol{B} \in I(\mathbb{R})$ and $y \in \mathbb{R}$, we have*

$$y \odot (\boldsymbol{A} \oplus \boldsymbol{B}) = y \odot \boldsymbol{A} \oplus y \odot \boldsymbol{B}.$$

2.2. **Calculus of IVFs.** Throughout this article, we assume that $\mathscr{X}$ is a nonempty subset of $\mathbb{R}^n$, unless mentioned otherwise. A function $\Psi : \mathscr{X} \to I(\mathbb{R})$ is called an IVF on $\mathscr{X}$. For any $z \in \mathscr{X}$, Ψ is represented as

$$\Psi(z) = \left[\underline{\psi}(z), \overline{\psi}(z)\right],$$

where $\underline{\psi}$ and $\overline{\psi}$ are real-valued functions on $\mathscr{X}$ satisfying $\underline{\psi}(z) \leq \overline{\psi}(z)$ for each $z \in \mathscr{X}$.

Definition 2.4. (Proper IVF). Let $\Psi : \mathscr{X} \to \overline{I(\mathbb{R})}$ be an extended IVF. Then, Ψ is said to be proper if there exists $\bar{z} \in \mathscr{X}$ such that

$$\Psi(\bar{z}) \prec [+\infty, +\infty] \text{ and } [-\infty, -\infty] \prec \Psi(z) \text{ for all } z \in \mathscr{X}.$$

Definition 2.5. (Effective domain of IVF). Let $\Psi : \mathscr{X} \to \overline{I(\mathbb{R})}$ be an extended IVF. The effective domain of Ψ is defined as

$$\operatorname{dom}(\Psi) = \left\{z \in \mathscr{X} : \|\Psi(z)\|_{I(\mathbb{R})} < +\infty\right\}.$$

Definition 2.6. (Linear IVF [28]). Let $\mathscr{X}$ be a linear subspace of $\mathbb{R}^n$. The function $\Psi : \mathscr{X} \to I(\mathbb{R})$ is said to be linear if

(i) $\Psi(\lambda z) = \lambda \odot \Psi(z)$ for all $z \in \mathscr{X}$ and for all $\lambda \in \mathbb{R}$ and
(ii) for all $z,\ w \in \mathscr{X}$,

either $\Psi(z) \oplus \Psi(w) = \Psi(z+w)$ or none of $\Psi(z) \oplus \Psi(w)$ and $\Psi(z+w)$ dominates the other.

Definition 2.7. *(Convex IVF* [24]*).* Let $\mathscr{X}$ be a convex subset of $\mathbb{R}^n$. Then, an IVF Ψ is said to be *convex* on $\mathscr{X}$ if, for any $z_1, z_2 \in \mathscr{X}$ and $\delta_1,\ \delta_2 \in [0,1]$ with $\delta_1 + \delta_2 = 1$,

$$\Psi(\delta_1 z_1 + \delta_2 z_2) \preceq \delta_1 \odot \Psi(z_1) \oplus \delta_2 \odot \Psi(z_2).$$

If $\underline{\psi}$ and $\overline{\psi}$ are convex on a convex set $\mathscr{X} \subseteq \mathbb{R}^n$, then the IVF Ψ is convex on $\mathscr{X}$ and vice-versa.

Definition 2.8. (gH-continuity [37]). Let Ψ be an IVF on $\mathscr{X}$. If, for any $\bar{z} \in \mathscr{X}$ and $h \in \mathbb{R}^n$ such that $\bar{z}+h \in \mathscr{X}$, the limit

$$\lim_{\|h\| \to 0} \left(\Psi(\bar{z}+h) \ominus_{gH} \Psi(\bar{z})\right) = \mathbf{0} \text{ exists},$$

then Ψ is gH-continuous at $\bar{z} \in \mathscr{X}$. Moreover, if Ψ is gH-continuous at each $z \in \mathscr{X}$, then Ψ is said to be gH-continuous on $\mathscr{X}$.

Definition 2.9. (gH-derivative [27]). Let $\mathscr{X}$ be a nonempty subset of $\mathbb{R}$. The gH-derivative of an IVF Ψ at a point $\bar{z} \in \mathscr{X}$ and $h \in \mathbb{R}$ such that $\bar{z}+h \in \mathscr{X}$, is defined by

$$\Psi'(\bar{z}) = \lim_{h \to 0} \frac{1}{h} \odot \left(\Psi(\bar{z}+h) \ominus_{gH} \Psi(\bar{z})\right), \text{ provided the limit exists.}$$

Definition 2.10. (gH-directional derivative [28]). The gH-directional derivative of an IVF Ψ at $\bar{z} \in \mathscr{X}$ in the direction $h \in \mathbb{R}^n$ such that $\bar{z} + \beta h \in \mathscr{X}$ for sufficiently small $\beta > 0$, denoted by $\Psi_{\mathscr{D}}(\bar{z})(h)$, is defined by

$$\lim_{\beta \to 0+} \frac{1}{\beta} \odot \left(\Psi(\bar{z}+\beta h) \ominus_{gH} \Psi(\bar{z})\right), \text{ provided that the limit exists.}$$

Definition 2.11. (gH-Gâteaux derivative [28]). Let Ψ be an IVF on a nonempty open subset $\mathscr{X}$ of $\mathbb{R}^n$. If, for each $h \in \mathbb{R}^n$ and at $\bar{z} \in \mathscr{X}$, the limit

$$\Psi_{\mathscr{G}}(\bar{z})(h) = \lim_{\lambda \to 0+} \frac{1}{\lambda} \odot (\Psi(\bar{z}+\lambda h) \ominus_{gH} \Psi(\bar{z}))$$

exists and $\Psi_{\mathscr{G}}(\bar{z})$ is a gH-continuous linear IVF from $\mathbb{R}^n$ to $I(\mathbb{R})$, then $\Psi_{\mathscr{G}}(\bar{z})$ is called gH-Gâteaux derivative of Ψ at $\bar{z}$. If Ψ has a gH-Gâteaux derivative at $\bar{z}$, then Ψ is called gH-Gâteaux differentiable at $\bar{z}$.

Definition 2.12. (gH-Fréchet derivative [28]). Let Ψ be an IVF on be a nonempty open subset $\mathscr{X}$ of $\mathbb{R}^n$. For $\bar{z} \in \mathscr{X}$ and $h \in \mathbb{R}^n$, if there exists a gH-continuous and linear mapping $\Psi_{\mathscr{F}} : \mathscr{X} \to I(\mathbb{R})$ with the following property

$$\lim_{\|h\| \to 0} \frac{\|\Psi(\bar{z}+h) \ominus_{gH} \Psi(\bar{z}) \ominus_{gH} \Psi_{\mathscr{F}}(h)\|_{I(\mathbb{R})}}{\|h\|} = 0,$$

then Ψ is said to have a gH-Fréchet derivative at $\bar{z}$, denoted by $\Psi_{\mathscr{F}}$.

Theorem 2.1. *(See [28]). Let $\mathscr{X}$ be a nonempty open subset of $\mathbb{R}^n$ and the Fréchet derivative of IVF $\Psi : \mathscr{X} \to I(\mathbb{R})$ exists at some $\bar{z} \in \mathscr{X}$. Then, the gH-Gâteaux derivative of Ψ at $\bar{z}$ exists along any $h \in \mathbb{R}^n$ and values of both the derivatives are equal.*

Remark 2.2. (See [28]). Let Ψ be an IVF on a nonempty open subset $\mathscr{X}$ of $\mathbb{R}^n$. Let Ψ has gH-Gâteaux derivative at $\bar{z} \in \mathscr{X}$. Then, Ψ has gH-directional derivative at $\bar{z}$ in every direction $h \in \mathbb{R}^n$ also.

Definition 2.13. (gH-subgradient [30]). For an IVF $\Psi : \mathscr{X} \to I(\mathbb{R})$, an element $\widehat{\mathbf{S}} = (\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_n)^\top \in I(\mathbb{R})^n$ is called a gH-subgradient of Ψ at $\bar{z}$ if

$$(w - \bar{z})^\top \odot \widehat{\mathbf{S}} \preceq \Psi(w) \ominus_{gH} \Psi(\bar{z}) \text{ for all } w \in \mathscr{X}.$$

The set of all gH-subgradients of Ψ at $\bar{z}$ is said to be gH-subdifferential and is denoted by $\partial\Psi(\bar{z})$.

Definition 2.14. (Sequence in $I(\mathbb{R})^n$ [30]). An IVF $\widehat{\Psi} : \mathbb{N} \to I(\mathbb{R})^n$ is called a sequence in $I(\mathbb{R})^n$.

Definition 2.15. (Convergence of a sequence in $I(\mathbb{R})^n$ [30]). Let $\{\widehat{\mathbf{S}}_k\}$ be a sequence in $I(\mathbb{R})^n$. If, for every $\varepsilon > 0$, we can find a $p \in \mathbb{N}$ such that

$$\|\widehat{\mathbf{S}}_k \ominus_{gH} \widehat{\mathbf{S}}\|_{I(\mathbb{R})^n} < \varepsilon \text{ for each } k \geq p,$$

then the sequence $\{\widehat{\mathbf{S}}_k\}$ is said to be convergent to $\widehat{\mathbf{S}}$. Then, $\widehat{\mathbf{S}}$ is called the limit of the sequence $\{\widehat{\mathbf{S}}_k\}$ and is represented as $\lim_{k\to\infty} \widehat{\mathbf{S}}_k = \widehat{\mathbf{S}}$.

Remark 2.3. (See [30]). If a sequence $\{\widehat{\mathbf{S}}_k\}$ in $I(\mathbb{R})^n$ converges to some $\widehat{\mathbf{S}} \in I(\mathbb{R})^n$, where $\widehat{\mathbf{S}}_k = (\mathbf{S}_{1k}, \mathbf{S}_{2k}, \ldots, \mathbf{S}_{nk})^\top$ and $\widehat{\mathbf{S}} = (\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_n)^\top$, then it can be noted from Definition 2.2 that each $\{\mathbf{S}_{jk}\}$ in $I(\mathbb{R})$ converges to $\mathbf{S}_j \in I(\mathbb{R})$ for all $j = 1, 2, \ldots, n$.

3. gH-Dini Hadamard ε-Subdifferentiability

We define the gH-Dini Hadamard ε-subdifferential set of an IVF by using gH-Dini Hadamard derivative of an IVF followed by its several characterizations. Further, we prove that the concept of gH-Dini Hadamard ε-subdifferential is more general than the concept of gH-subdifferentiability. In the sequel, the relation of gH-subdifferentiability and gH-Fréchet differentiability with gH-Dini Hadamard ε-subdifferentiability is discussed.

Definition 3.1. (gH-Dini Hadamard derivative of an IVF). Let Ψ be an IVF on $\mathscr{X}$. If, for $\bar{z} \in \mathscr{X}$ and $h \in \mathbb{R}^n$, the limit inferior

$$\Psi_{\mathscr{D}\mathscr{H}}(\bar{z})(h) = \liminf_{\substack{u \to h \\ \lambda \to 0+}} \frac{1}{\lambda} \odot (\Psi(\bar{z}+\lambda u) \ominus_{gH} \Psi(\bar{z})),$$

exists and $\Psi_{\mathscr{D}\mathscr{H}}(\bar{z})(h)$ is a linear IVF from $\mathscr{X}$ to $I(\mathbb{R})$, then the limit value is called gH-Dini Hadamard derivative of Ψ at $\bar{z}$ in the direction h.

Definition 3.2. (gH-Dini Hadamard ε-subdifferentiability of an IVF). Let $\Psi : \mathscr{X} \to \overline{I(\mathbb{R})}$ be an extended IVF that is finite at $\bar{z} \in \mathscr{X}$. Then, for $\varepsilon > 0$, the gH-Dini Hadamard ε-subdifferential of Ψ at $\bar{z}$, denoted by $\partial_\varepsilon^{\mathscr{D}\mathscr{H}}\Psi(\bar{z})$, is defined as

$$\partial_\varepsilon^{\mathscr{D}\mathscr{H}}\Psi(\bar{z}) = \left\{ \widehat{\mathbf{S}} \in I(\mathbb{R})^n : (w-\bar{z})^\top \odot \widehat{\mathbf{S}} \preceq \Psi_{\mathscr{D}\mathscr{H}}(\bar{z})(w-\bar{z}) \oplus \varepsilon \|w-\bar{z}\| \text{ for all } w \in \mathscr{X} \right\}.$$

Then, $\widehat{\mathbf{S}}$ is called the gH-Dini Hadamard ε-subgradient of Ψ at $\bar{z}$. Further, if $\partial_\varepsilon^{\mathscr{D}\mathscr{H}}\Psi(\bar{z}) \neq \emptyset$, we say that IVF Ψ is gH-Dini Hadamard ε-subdifferentiable at $\bar{z}$.

Example 3.1. Let $\Psi : \mathbb{R} \to I(\mathbb{R})$ be an IVF given by $\Psi(z) = [1,2] \odot |z|$. Let us check the gH-Dini Hadmard ε-subdifferentiability of Ψ at 0. Let us assume $\mathbf{S} = [\underline{s}, \overline{s}] \in \partial_\varepsilon^{\mathscr{D}\mathscr{H}}\Psi(0)$ for $\varepsilon > 0$. Therefore, for $w \in \mathbb{R}$, we have

$$\begin{aligned} & [\underline{s},\overline{s}] \odot (w-0) \preceq \Psi_{\mathscr{D}\mathscr{H}}(0)(w-0) \oplus \varepsilon|w-0| \\ \Longrightarrow \quad & [\underline{s},\overline{s}] \odot w \preceq \liminf_{\substack{u \to w \\ \lambda \to 0+}} \frac{1}{\lambda} \odot ([1,2] \odot |\lambda u|) \oplus \varepsilon|w-0| \\ \Longrightarrow \quad & [\underline{s},\overline{s}] \odot w \preceq [1,2] \odot |w| \oplus \varepsilon|w|. \end{aligned}$$

We have the following two cases:

(i) For $w \geq 0$,

$$[\underline{s},\overline{s}] \preceq [1,2] \oplus \varepsilon \implies \underline{s} \leq 1+\varepsilon \text{ and } \overline{s} \leq 2+\varepsilon.$$

(ii) For $w < 0$,

$$\begin{aligned} [\underline{s},\overline{s}] \odot w \preceq [1,2] \odot (-w) \oplus \varepsilon(-w) \quad & \Longrightarrow \quad [1,2] \odot (-1) \oplus \varepsilon(-1) \preceq [\underline{s},\overline{s}] \\ & \Longrightarrow \quad [-2-\varepsilon, -1-\varepsilon] \preceq [\underline{s},\overline{s}] \\ & \Longrightarrow \quad -2-\varepsilon \leq \underline{s} \text{ and } -1-\varepsilon \leq \overline{s}. \end{aligned}$$

Hence, in view of Case (i) and Case (ii), we have

$$\partial_\varepsilon^{\mathscr{D}\mathscr{H}}\Psi(0) = \{\mathbf{S} \in I(\mathbb{R}) : -2-\varepsilon \leq \underline{s} \leq 1+\varepsilon \text{ and } -1-\varepsilon \leq \overline{s} \leq 2+\varepsilon\}.$$

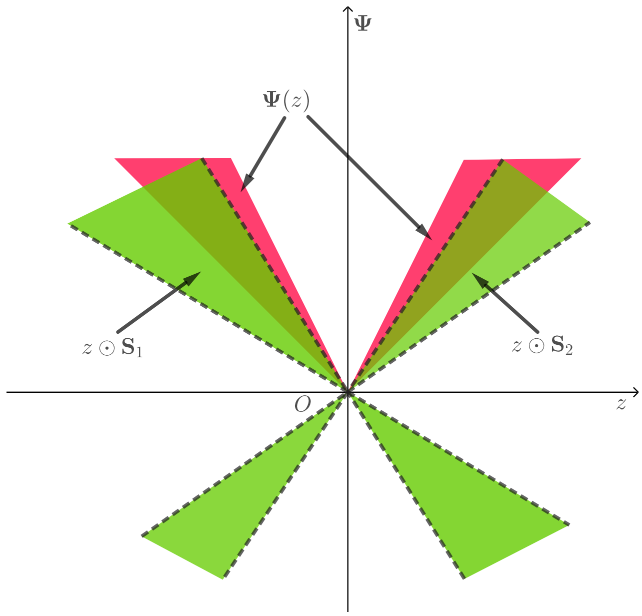


FIGURE 1. The Geometrical representation of two possible gH-Dini Hadamard ε-subgradients of Ψ of Example 3.1

A geometrical view of gH-Dini Hadamard ε-subdifferentiability of Ψ of Example 3.1 is given in Figure 1. The IVF Ψ is depicted by the pink region. For $\varepsilon = 1$, the two possible gH-Dini Hadamard ε-subgradients of Ψ at 0 are denoted by $\mathbf{S}_1$ and $\mathbf{S}_2$ and demonstrated by green region.

Example 3.2. In this example, we check the gH-Dini Hadamard ε-subdifferentiability of an IVF $\Psi : \mathbb{R} \to I(\mathbb{R})$ which is given by $\Psi(z) = [-2,-1] \odot |z|$ at $z = 0$. Let us assume $\mathbf{S} \in \partial_{\varepsilon}^{\mathscr{D}\mathscr{H}} \Psi(0)$ for $0 < \varepsilon < 2$. Therefore, for $w \in \mathbb{R}$, we have

$$\begin{aligned}
& (w-0) \odot \mathbf{S} \preceq \Psi_{\mathscr{D}\mathscr{H}}(0)(w-0) \oplus \varepsilon |w-0| \\
\implies \quad & w \odot \mathbf{S} \preceq \liminf_{\substack{u \to w \\ \lambda \to 0+}} \frac{1}{\lambda} \odot (\Psi(\lambda u) \ominus_{gH} \Psi(0)) \oplus \varepsilon |w| \\
\implies \quad & w \odot \mathbf{S} \preceq \liminf_{\substack{u \to w \\ \lambda \to 0+}} \frac{1}{\lambda} \odot ([-2,-1] \odot |\lambda u|) \oplus \varepsilon |w| \\
\implies \quad & w \odot \mathbf{S} \preceq [-2,-1] \odot |w| \oplus \varepsilon |w|.
\end{aligned}$$

We have the following two cases:

(i) For $w \geq 0$,

$$w \odot \mathbf{S} \preceq [-2,-1] \odot w \oplus \varepsilon w \implies \mathbf{S} \preceq [-2,-1] \oplus \varepsilon.$$

(ii) For $w < 0$,

$$w \odot \mathbf{S} \preceq [-2,-1] \odot (-w) \oplus \varepsilon(-w) \implies [1,2] \ominus_{gH} \varepsilon \preceq \mathbf{S}.$$

From Case (i) and Case (ii), we can observe that, for any $0 < \varepsilon < 2$, there does not exist any $\mathbf{S} \in I(\mathbb{R})$ which satisfies both the cases simultaneously. Thus $\partial_{\varepsilon}^{\mathscr{D}\mathscr{H}} \Psi(\bar{z})$ is empty.

Theorem 3.1. *Let Ψ be a gH-subdifferentiable IVF at $\bar{z}$ in $\mathscr{X}$. Then, Ψ is gH-Dini Hadamard ε-subdifferentiable at $\bar{z}$ as well.*

Proof. If Ψ is gH-subdifferentiable at $\bar{z}$, then there exists an $\widehat{\mathbf{S}} \in \partial\Psi(\bar{z})$ such that

$$(w-\bar{z})^\top \odot \widehat{\mathbf{S}} \preceq \Psi(w) \ominus_{gH} \Psi(\bar{z}) \text{ for all } w \in \mathscr{Z}. \tag{3.1}$$

Also, for any $\varepsilon > 0$, the following relation holds:

$$\Psi(w) \ominus_{gH} \Psi(\bar{z}) \preceq \Psi(w) \ominus_{gH} \Psi(\bar{z}) \oplus \varepsilon\|w-\bar{z}\| \text{ for all } w \in \mathscr{Z}. \tag{3.2}$$

Therefore, from (3.1) and (3.2), we conclude that, for all $w \in \mathscr{Z}$,

$$\begin{aligned}
& (w-\bar{z})^\top \odot \widehat{\mathbf{S}} \preceq \Psi(w) \ominus_{gH} \Psi(\bar{z}) \oplus \varepsilon\|w-\bar{z}\| \\
\implies\ & (w-\bar{z})^\top \odot \widehat{\mathbf{S}} \preceq \liminf_{\substack{u\to(w-\bar{z})\\ \lambda\to 0+}} \frac{1}{\lambda} \odot (\Psi(\bar{z}+\lambda u) \ominus_{gH} \Psi(\bar{z})) \oplus \varepsilon\|w-\bar{z}\| \\
\implies\ & (w-\bar{z})^\top \odot \widehat{\mathbf{S}} \preceq \Psi_{\mathscr{D}\mathscr{H}}(\bar{z})(w-\bar{z}) \oplus \varepsilon\|w-\bar{z}\| \\
\implies\ & \widehat{\mathbf{S}} \in \partial_\varepsilon^{\mathscr{D}\mathscr{H}}\Psi(\bar{z}).
\end{aligned}$$

Thus Ψ is gH-Dini Hadamard ε-subdifferentiable at $\bar{z}$. □

Remark 3.1. It is to be noted that the converse of the conclusion made in Theorem 3.1 need not be true. For instance, consider the IVF discussed in Example 3.2: $\Psi(z) = [-2,-1] \odot |z|$. Let us check the gH-subdifferentiability and gH-Dini Hadamard ε-subdifferentiability of Ψ at 0 for any $\varepsilon \geq 2$. From Example 3.2, it can be observed that the gH-Dini Hadamard ε-subdifferentiability of Ψ at 0 is given by

$$\partial_\varepsilon^{\mathscr{D}\mathscr{H}}\Psi(0) = \{\mathbf{S} \in I(\mathbb{R}) : [1,2] \ominus_{gH} \varepsilon \preceq \mathbf{S} \preceq [-2,-1] \oplus \varepsilon\}.$$

Now the gH-subdifferential of Ψ at 0 is

$$\begin{aligned}
& w \odot \mathbf{S} \preceq \Psi(w) \ominus_{gH} \Psi(0) \\
\implies\ & w \odot \mathbf{S} \preceq [-2,-1] \odot |w|.
\end{aligned}$$

There arise the following two cases:

(i) For $w \geq 0$,

$$\mathbf{S} \odot w \preceq [-2,-1] \odot w \implies \mathbf{S} \preceq [-2,-1].$$

(ii) For $w < 0$,

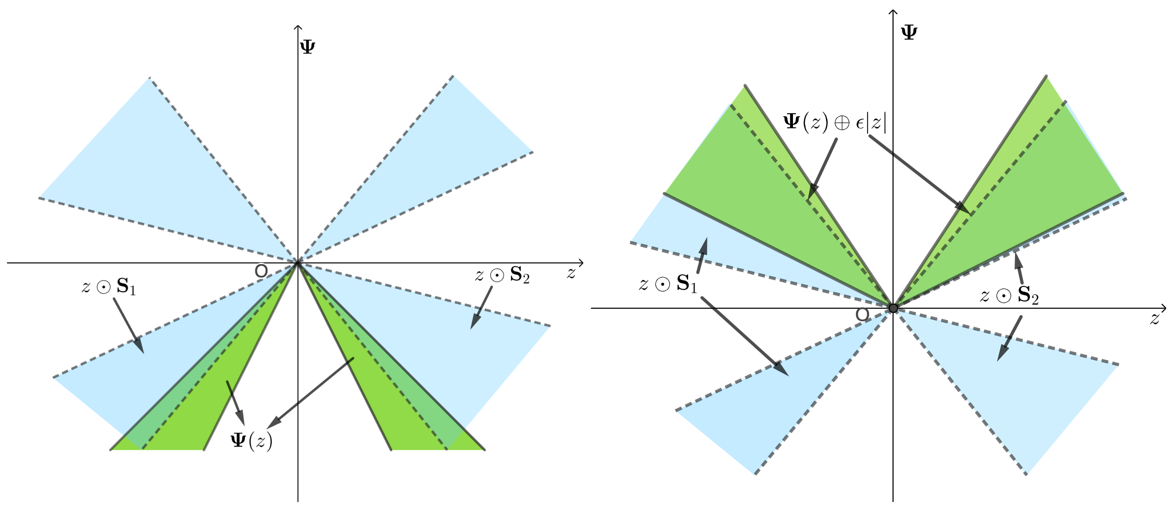
$$\mathbf{S} \odot w \preceq [-2,-1] \odot (-w) \implies [1,2] \preceq \mathbf{S}.$$

From Case (i) and Case (ii), we observe that for any $\varepsilon \geq 2$ there does not exist any $\mathbf{S} \in I(\mathbb{R})$, which satisfies both the cases simultaneously. Hence, $\partial\Psi(0) = \emptyset$.

Note 3.1. The graph of gH-Dini Hadamard ε-subdifferentiability of IVF Ψ at $z = 0$ of Remark 3.1 is depicted in Figure 2. For $\varepsilon = 2.5$, the two possible gH-Dini Hadamard ε-subgradients of Ψ at 0 are denoted by $\mathbf{S}_1$ and $\mathbf{S}_2$. It can be observed that with the help of gH-Dini Hadamard ε-subdifferentiability, the two subgradients $z \odot \mathbf{S}_1$ and $z \odot \mathbf{S}_2$ always supports the IVF $(\Psi(z) \oplus \varepsilon|z|)$ from below as demonstrated in Figure 2(b), which fails for IVF $\Psi(z)$ (see Figure 2(a)).

Lemma 3.1. *(Monotonic property of gH-Dini Hadamard ε-subdifferential). Let Ψ be an IVF on $\mathscr{Z}$. Then, for $0 < \varepsilon_1 \leq \varepsilon_2$, and $\bar{z} \in \mathscr{Z}$,*

$$\partial_{\varepsilon_1}^{\mathscr{D}\mathscr{H}}\Psi(\bar{z}) \subseteq \partial_{\varepsilon_2}^{\mathscr{D}\mathscr{H}}\Psi(\bar{z}).$$



(a) The IVF $\Psi(z)$ and gH-Dini Hadamard ε-subgradients of Ψ

(b) The IVF $(\Psi(z) \oplus \varepsilon|z|)$ and gH-Dini Hadamard ε-subgradients of Ψ

FIGURE 2. The geometrical representations of two possible gH-Dini Hadamard ε-subgradients of Ψ of Remark 3.1

Proof. Let $\widehat{\mathbf{S}} \in \partial_{\varepsilon_1}^{\mathscr{D}\mathscr{H}} \Psi(\bar{z})$. Then, for $\varepsilon_1 > 0$ and for all $w \in \mathscr{Z}$, we have

$$(w - \bar{z})^\top \odot \widehat{\mathbf{S}} \;\preceq\; \Psi_{\mathscr{D}\mathscr{H}}(\bar{z})(w - \bar{z}) \oplus \varepsilon_1 \|w - \bar{z}\|.$$

Since $\varepsilon_1 \leq \varepsilon_2$, then we obtain from (i) of Lemma 2.1 that

$$(w - \bar{z})^\top \odot \widehat{\mathbf{S}} \;\preceq\; \Psi_{\mathscr{D}\mathscr{H}}(\bar{z})(w - \bar{z}) \oplus \varepsilon_2 \|w - \bar{z}\| \text{ for all } w \in \mathscr{Z}.$$

Hence, $\partial_{\varepsilon_1}^{\mathscr{D}\mathscr{H}} \Psi(\bar{z}) \subseteq \partial_{\varepsilon_2}^{\mathscr{D}\mathscr{H}} \Psi(\bar{z})$. □

Theorem 3.2. *Let Ψ be an IVF on $\mathscr{Z}$. Then, for any $\varepsilon > 0$ and $\bar{z} \in \mathscr{Z}$, $\partial_{\varepsilon}^{\mathscr{D}\mathscr{H}} \Psi(\bar{z})$ is a convex set.*

Proof. Let us assume $\partial_{\varepsilon}^{\mathscr{D}\mathscr{H}} \Psi(\bar{z})$ is nonempty and there exist $\widehat{\mathbf{L}}$, $\widehat{\mathbf{K}} \in \partial_{\varepsilon}^{\mathscr{D}\mathscr{H}} \Psi(\bar{z})$ for any $\varepsilon > 0$. Then, for all $w \in \mathscr{Z}$,

$$\begin{aligned} &(w - \bar{z})^\top \odot \widehat{\mathbf{L}} \preceq \Psi_{\mathscr{D}\mathscr{H}}(\bar{z})(w - \bar{z}) \oplus \varepsilon \|w - \bar{z}\| \text{ and} \\ &(w - \bar{z})^\top \odot \widehat{\mathbf{K}} \preceq \Psi_{\mathscr{D}\mathscr{H}}(\bar{z})(w - \bar{z}) \oplus \varepsilon \|w - \bar{z}\| \\ \Longrightarrow\; &\bigoplus_{i=1}^{n} \mathbf{L}_i \odot h_i \preceq \Psi_{\mathscr{D}\mathscr{H}}(\bar{z})(h) \oplus \varepsilon \|h\| \text{ and} \\ &\bigoplus_{i=1}^{n} \mathbf{K}_i \odot h_i \preceq \Psi_{\mathscr{D}\mathscr{H}}(\bar{z})(h) \oplus \varepsilon \|h\| \text{ taking } h = (w - \bar{z}) \in \mathscr{Z}. \end{aligned}$$

Thus, for $h \in \mathscr{X}$ and $\lambda_1, \lambda_2 > 0$ with $\lambda_1 + \lambda_2 = 1$, we have

$$\lambda_1 \odot \left(\bigoplus_{i=1}^{n} \mathbf{L}_i \odot h_i \right) \oplus \lambda_2 \odot \left(\bigoplus_{i=1}^{n} \mathbf{K}_i \odot h_i \right) \preceq \lambda_1 \odot \Psi_{\mathscr{DH}}(\bar{z})(h) \oplus \lambda_2 \odot \Psi_{\mathscr{DH}}(\bar{z})(h) \oplus \varepsilon \|h\|$$

$$\implies \bigoplus_{i=1}^{n} (\lambda_1 \odot \mathbf{L}_i \oplus \lambda_2 \odot \mathbf{K}_i) \odot h_i \preceq \Psi_{\mathscr{DH}}(\bar{z})(h) \oplus \varepsilon \|h\| \text{ from Lemma 2.2}$$

$$\implies h^\top \odot (\lambda_1 \odot \widehat{\mathbf{L}} \oplus \lambda_2 \odot \widehat{\mathbf{K}}) \preceq \Psi_{\mathscr{DH}}(\bar{z})(h) \oplus \varepsilon \|h\|.$$

Therefore, $\lambda_1 \odot \widehat{\mathbf{L}} \oplus \lambda_2 \odot \widehat{\mathbf{K}} \in \partial_\varepsilon^{\mathscr{DH}} \Psi(\bar{z})$. Hence, $\partial_\varepsilon^{\mathscr{DH}} \Psi(\bar{z})$ is a convex set. □

Theorem 3.3. *Let Ψ be an IVF on $\mathscr{X}$. Then, for $\bar{z} \in \mathscr{X}$ and $\varepsilon > 0$, $\partial_\varepsilon^{\mathscr{DH}} \Psi(\bar{z})$ is closed.*

Proof. Let $\{\widehat{\mathbf{S}}_k\}$ be a sequence in $\partial_\varepsilon^{\mathscr{DH}} \Psi(\bar{z})$ which converges to $\widehat{\mathbf{S}} \in I(\mathbb{R})^n$, where $\widehat{\mathbf{S}}_k = (\mathbf{S}_{k1}, \mathbf{S}_{k2}, \ldots, \mathbf{S}_{kn})^\top$ and $\widehat{\mathbf{S}} = (\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_n)^\top$. Then, for all $w \in \mathscr{X}$ and $\widehat{\mathbf{S}}_k \in \partial_\varepsilon^{\mathscr{DH}} \Psi(\bar{z})$, we have

$$\begin{aligned} & (w - \bar{z})^\top \odot \widehat{\mathbf{S}}_k \preceq \Psi_{\mathscr{DH}}(\bar{z})(w - \bar{z}) \oplus \varepsilon \|w - \bar{z}\| \\ \implies \quad & h^\top \odot \widehat{\mathbf{S}}_k \preceq \Psi_{\mathscr{DH}}(\bar{z})(h) \oplus \varepsilon \|h\| \text{ taking } h = (w - \bar{z}) \in \mathscr{X} \\ \implies \quad & \bigoplus_{i=1}^{n} h_i^\top \odot \mathbf{S}_{ki} \preceq \Psi_{\mathscr{DH}}(\bar{z})(h) \oplus \varepsilon \|h\|. \end{aligned} \tag{3.3}$$

Without loss of generality, let first m number of components of h in (3.3) be nonnegative and rest $(n - m)$ number of components be negative. Also, from Definition 3.1, we assume $\Psi_{\mathscr{DH}}(\bar{z})(h) = \bigoplus_{i=1}^{n} \mathbf{R}_i \odot h_i$ and $\mathbf{R}_i \in I(\mathbb{R})$ for each $i = 1, 2, \ldots, n$. Therefore,

$$\bigoplus_{i'=1}^{m} h_{i'} \odot \widehat{\mathbf{S}}_{ki'} \oplus \bigoplus_{j=m+1}^{n} h_j \odot \widehat{\mathbf{S}}_{kj} \preceq \bigoplus_{i'=1}^{m} \mathbf{R}_{i'} \odot h_{i'} \oplus \bigoplus_{j=m+1}^{n} \mathbf{R}_j \odot h_j \oplus \varepsilon \|h\|$$

$$\text{or,} \quad \bigoplus_{i'=1}^{m} [\underline{s}_{ki'} h_{i'}, \bar{s}_{ki'} h_{i'}] \oplus \bigoplus_{j=m+1}^{n} [\bar{s}_{kj} h_j, \underline{s}_{kj} h_j] \preceq \bigoplus_{i'=1}^{m} [\underline{r}_{i'} h_i, \bar{r}_{i'} h_i] \oplus \bigoplus_{j=m+1}^{n} [\bar{r}_j h_j, \underline{r}_j h_j] \oplus \varepsilon \|h\|.$$

Hence,

$$\sum_{i'=1}^{m} \underline{s}_{ki' h_{i'}} + \sum_{j=m+1}^{n} \bar{s}_{kj} h_j \leq \sum_{i=1'}^{m} \underline{r}_{i'} h_{i'} + \sum_{j=m+1}^{n} \bar{r}_j h_j + \varepsilon \|h\| \text{ and} \tag{3.4}$$

$$\sum_{i'=1}^{m} \bar{s}_{ki'} h_{i'} + \sum_{j=m+1}^{n} \underline{s}_{kj} h_j \leq \sum_{i'=1}^{m} \bar{r}_{i'} h_{i'} + \sum_{j=m+1}^{n} \underline{r}_j h_j + \varepsilon \|h\|. \tag{3.5}$$

Since $\widehat{\mathbf{S}}_k$ converges to $\widehat{\mathbf{S}}$, from Remark 2.3, the sequences $\underline{s}_{ki'}$ and $\bar{s}_{ki'}$ converge to $\underline{s}_{i'}$, and $\bar{s}_{i'}$, respectively, for each i' and similarly for each j. Thus

$$\left(\sum_{i'=1}^{m} \underline{s}_{ki'} h_{i'} + \sum_{j=m+1}^{n} \bar{s}_{kj} h_j \right) \to \left(\sum_{i'=1}^{m} \underline{s}_{i'} h_{i'} + \sum_{j=m+1}^{n} \bar{s}_j h_j \right) \text{ and} \tag{3.6}$$

$$\left(\sum_{i'=1}^{m} \bar{s}_{ki'} h_{i'} + \sum_{j=m+1}^{n} \underline{s}_{kj} h_j \right) \to \left(\sum_{i'=1}^{m} \bar{s}_{i'} h_{i'} + \sum_{j=m+1}^{n} \underline{s}_j h_j \right). \tag{3.7}$$

From (3.4), (3.5), (3.6), and (3.7), we obtain

$$\left(\sum_{i'=1}^{m} \underline{s}_{i'}h_{i'} + \sum_{j=m+1}^{n} \overline{s}_j h_j\right) \leq \sum_{i'=1}^{m} \underline{r}_{i'}h_{i'} + \sum_{j=m+1}^{n} \overline{r}_j h_j + \varepsilon\|h\| \text{ and} \tag{3.8}$$

$$\left(\sum_{i'=1}^{m} \overline{s}_{i'}h_{i'} + \sum_{j=m+1}^{n} \underline{s}_j h_j\right) \leq \sum_{i'=1}^{m} \overline{r}_{i'}h_{i'} + \sum_{j=m+1}^{n} \underline{r}_j h_j + \varepsilon\|h\|. \tag{3.9}$$

In view of (3.8) and (3.9), we obtain

$$\begin{aligned}
&\left[\sum_{i'=1}^{m} \underline{s}_{i'}h_{i'} + \sum_{j=m+1}^{n} \overline{s}_j h_j, \sum_{i'=1}^{m} \overline{s}_{i'}h_{i'} + \sum_{j=m+1}^{n} \underline{s}_j h_j\right] \preceq \Psi_{\mathscr{DH}}(\bar{z})(h) \oplus \varepsilon\|h\| \\
\implies\ & \bigoplus_{i'=1}^{m} [\underline{s}_{i'}h_{i'}, \overline{s}_{i'}h_{i'}] \oplus \bigoplus_{j=m+1}^{n} [\overline{s}_j h_j, \underline{s}_j h_j] \preceq \Psi_{\mathscr{DH}}(\bar{z})(h) \oplus \varepsilon\|h\| \\
\implies\ & \bigoplus_{i'=1}^{m} \mathbf{S}_{i'} \odot h_{i'} \oplus \bigoplus_{j=m+1}^{n} \mathbf{S}_j \odot h_j \preceq \Psi_{\mathscr{DH}}(\bar{z})(h) \oplus \varepsilon\|h\| \\
\implies\ & h^\top \odot \widehat{\mathbf{S}} \preceq \Psi_{\mathscr{DH}}(\bar{z})(h) \oplus \varepsilon\|h\|.
\end{aligned}$$

Thus $\widehat{\mathbf{S}} \in \partial_\varepsilon^{\mathscr{DH}}\Psi(\bar{z})$. Hence, $\partial_\varepsilon^{\mathscr{DH}}\Psi(\bar{z})$ is closed. □

Theorem 3.4. *Let Ψ be a gH-subdifferentiable and gH-Fréchet differentiable IVF at $\bar{z} \in \mathscr{Z}$. Then, for any $\varepsilon > 0$, $\{\Psi_{\mathscr{F}}(\bar{z})\} \subseteq \partial_\varepsilon^{\mathscr{DH}}\Psi(\bar{z})$, where $\Psi_{\mathscr{F}}(\bar{z})$ is the Fréchet derivative of Ψ at $\bar{z}$.*

Proof. If Ψ is gH-subdifferentiable at $\bar{z}$, then there exists $\widehat{\mathbf{S}} \in I(\mathbb{R})^n$ satisfying

$$(w - \bar{z})^\top \odot \widehat{\mathbf{S}} \preceq \Psi(w) \ominus_{gH} \Psi(\bar{z}) \text{ for any } w \in \mathscr{Z}. \tag{3.10}$$

Taking $w = \bar{z} + \lambda e$, $\lambda \geq 0$, and $e \in \mathscr{Z}$ in (3.10) such that $\|e\| = 1$, we have

$$\begin{aligned}
&(\lambda e)^\top \odot \widehat{\mathbf{S}} \preceq \Psi(\bar{z} + \lambda e) \ominus_{gH} \Psi(\bar{z}) \\
\implies\ & e^\top \odot \widehat{\mathbf{S}} \preceq \frac{1}{\lambda} \odot (\Psi(\bar{z} + \lambda e) \ominus_{gH} \Psi(\bar{z})).
\end{aligned}$$

Also, Ψ is gH-Fréchet differentiable at $\bar{z}$. From Theorem 2.1 and Remark 2.2, for $\lambda \to 0+$, we have

$$e^\top \odot \widehat{\mathbf{S}} \preceq \Psi_{\mathscr{F}}(\bar{z})(e). \tag{3.11}$$

Replacing e by $-e$ in the last relation, we see that

$$\begin{aligned}
&(-e)^\top \odot \widehat{\mathbf{S}} \preceq \Psi_{\mathscr{F}}(\bar{z})(-e) \\
\implies\ & \Psi_{\mathscr{F}}(\bar{z})(e) \preceq e^\top \odot \widehat{\mathbf{S}}.
\end{aligned} \tag{3.12}$$

Hence, from (3.11), (3.12), and Theorem 3.1, we obtain

$$\Psi_{\mathscr{F}}(\bar{z}) \in \partial\Psi(\bar{z}) \implies \{\Psi_{\mathscr{F}}(\bar{z})\} \subseteq \partial_\varepsilon^{\mathscr{DH}}\Psi(\bar{z}).$$

□

Remark 3.2. The conclusion in Theorem 3.4 can be strict. For instance, consider an IVF $\Psi : \mathbb{R} \to I(\mathbb{R})$ given by $\Psi(z) = \mathbf{0} \odot z$. Let us calculate the gH-Fréchet derivative and gH-Dini Hadamard ε-subdifferential of Ψ at $\bar{z} = 0$. Let us assume that there exists an $\mathbf{S} \in \partial_\varepsilon^{\mathscr{D}\mathscr{H}} \Psi(0)$ satisfying that for any $w \in \mathbb{R}$,

$$\begin{aligned}
& (w-0) \odot \mathbf{S} \preceq \Psi_{\mathscr{D}\mathscr{H}}(0)(w-0) \oplus \varepsilon |w-0| \\
\Longrightarrow \quad & w \odot \mathbf{S} \preceq \liminf_{\substack{u \to w \\ \lambda \to 0+}} \frac{1}{\lambda} \odot (\Psi(\lambda u) \ominus_{gH} \Psi(0)) \oplus \varepsilon |w| \\
\Longrightarrow \quad & w \odot \mathbf{S} \preceq \mathbf{0} \oplus \varepsilon |w| \\
\Longrightarrow \quad & w \odot \mathbf{S} \preceq \varepsilon |w|.
\end{aligned}$$

There arise the following two cases:

(i) For $w \geq 0$, $w \odot \mathbf{S} \preceq \varepsilon w \implies \mathbf{S} \preceq \varepsilon$.
(ii) For $w < 0$, $\mathbf{S} \odot w \preceq \varepsilon(-w) \implies -\varepsilon \preceq \mathbf{S}$.

In view of Case (i) and Case (ii), we have $\partial_\varepsilon^{\mathscr{D}\mathscr{H}} \Psi(0) = \{\mathbf{S} : -\varepsilon \preceq \mathbf{S} \preceq \varepsilon\}$. Now assume that $\Psi_{\mathscr{F}}(\bar{z})$ is the gH-Fréchet derivative of Ψ at $\bar{z} = 0$. Then,

$$\lim_{|h| \to 0} \frac{\|\Psi(\bar{z}+h) \ominus_{gH} \Psi(\bar{z}) \ominus_{gH} \Psi_{\mathscr{F}}(h)\|_{I(\mathbb{R})}}{|h|} = 0 \implies \lim_{|h| \to 0} \frac{\|\Psi_{\mathscr{F}}(h)\|_{I(\mathbb{R})}}{|h|} = 0 \implies \Psi_{\mathscr{F}}(0) = \mathbf{0}.$$

Thus $\Psi_{\mathscr{F}}(0) \subset \partial_\varepsilon^{\mathscr{D}\mathscr{H}} \Psi(0)$.

Theorem 3.5. *Let Ψ be gH-Dini Hadamard ε-subdifferentiable IVF at $\bar{z}$ in $\mathscr{Z}$. Then, for any $\varepsilon > 0$ and $\delta > 0$,*

$$\partial_{\varepsilon'}^{\mathscr{D}\mathscr{H}} (\delta \odot \Psi)(\bar{z}) = \delta \odot \partial_\varepsilon^{\mathscr{D}\mathscr{H}} (\Psi)(\bar{z}), \text{ where } \varepsilon' = \varepsilon\delta.$$

Proof. Let $\widehat{\mathbf{S}} \in \delta \odot \partial_\varepsilon^{\mathscr{D}\mathscr{H}} (\Psi)(\bar{z})$. Then, we can write $\widehat{\mathbf{S}} = \delta \odot \widehat{\mathbf{S}}'$ such that $\widehat{\mathbf{S}}' \in \partial_\varepsilon^{\mathscr{D}\mathscr{H}} (\Psi)(\bar{z})$ for any $w \in \mathscr{Z}$ and

$$\begin{aligned}
& (w-\bar{z})^\top \odot \widehat{\mathbf{S}}' \preceq \Psi_{\mathscr{D}\mathscr{H}}(\bar{z})(w-\bar{z}) \oplus \varepsilon \|w-\bar{z}\| \\
\Longrightarrow\ & (w-\bar{z})^\top \odot \left(\frac{1}{\delta} \odot \widehat{\mathbf{S}} \right) \preceq \Psi_{\mathscr{D}\mathscr{H}}(\bar{z})(w-\bar{z}) \oplus \varepsilon \|w-\bar{z}\| \\
\Longrightarrow\ & (w-\bar{z})^\top \odot \widehat{\mathbf{S}} \preceq \delta \odot (\Psi_{\mathscr{D}\mathscr{H}}(\bar{z})(w-\bar{z}) \oplus \varepsilon \|w-\bar{z}\|) \\
\Longrightarrow\ & (w-\bar{z})^\top \odot \widehat{\mathbf{S}} \preceq \delta \odot \left(\liminf_{\substack{u \to (w-\bar{z}) \\ \lambda \to 0+}} \frac{1}{\lambda} \odot (\Psi(\bar{z}+\lambda u) \ominus_{gH} \Psi(\bar{z})) \right) \oplus \varepsilon\delta \|w-\bar{z}\| \\
\Longrightarrow\ & (w-\bar{z})^\top \odot \widehat{\mathbf{S}} \preceq \liminf_{\substack{u \to (w-\bar{z}) \\ \lambda \to 0+}} \frac{1}{\lambda} \odot (\delta \odot \Psi(\bar{z}+\lambda u) \ominus_{gH} \delta \odot \Psi(\bar{z})) \oplus \varepsilon\delta \|w-\bar{z}\| \\
\Longrightarrow\ & (w-\bar{z})^\top \odot \widehat{\mathbf{S}} \preceq (\delta \odot \Psi)_{\mathscr{D}\mathscr{H}}(\bar{z})(w-\bar{z}) \oplus \varepsilon' \|w-\bar{z}\|, \text{ where } \varepsilon' = \varepsilon\delta \\
\Longrightarrow\ & \widehat{\mathbf{S}} \in \partial_{\varepsilon'}^{\mathscr{D}\mathscr{H}} (\delta \odot \Psi)(\bar{z}).
\end{aligned}$$

Conversely, we assume that $\widehat{\mathbf{S}} \in \partial^{\mathscr{D}\mathscr{H}}_{\varepsilon'}(\delta \odot \Psi)(\bar{z})$. Then, for any $\varepsilon > 0$ and $w \in \mathscr{X}$, we have

$$\begin{aligned}
& (w-\bar{z})^\top \odot \widehat{\mathbf{S}} \preceq (\delta \odot \Psi)_{\mathscr{D}\mathscr{H}}(\bar{z})(w-\bar{z}) \oplus \varepsilon' \|w-\bar{z}\| \\
\Longrightarrow\ & (w-\bar{z})^\top \odot \widehat{\mathbf{S}} \preceq \liminf_{\substack{u \to (w-\bar{z}) \\ \lambda \to 0+}} \frac{1}{\lambda} \odot (\delta \odot \Psi(\bar{z}+\lambda u) \ominus_{gH} \delta \odot \Psi(\bar{z})) \oplus \varepsilon\delta \|w-\bar{z}\| \\
\Longrightarrow\ & \frac{1}{\delta} \odot ((w-\bar{z})^\top \odot \widehat{\mathbf{S}}) \preceq \Psi_{\mathscr{D}\mathscr{H}}(\bar{z})(w-\bar{z}) \oplus \varepsilon \|w-\bar{z}\| \\
\Longrightarrow\ & (w-\bar{z})^\top \odot \left(\frac{1}{\delta} \odot \widehat{\mathbf{S}}\right) \preceq \Psi_{\mathscr{D}\mathscr{H}}(\bar{z})(w-\bar{z}) \oplus \varepsilon \|w-\bar{z}\| \\
\Longrightarrow\ & \frac{1}{\delta} \odot \widehat{\mathbf{S}} \in \partial^{\mathscr{D}\mathscr{H}}_{\varepsilon}(\Psi)(\bar{z}) \\
\Longrightarrow\ & \widehat{\mathbf{S}} \in \delta \odot \partial^{\mathscr{D}\mathscr{H}}_{\varepsilon}(\Psi)(\bar{z}).
\end{aligned}$$

Thus we conclude that $\partial^{\mathscr{D}\mathscr{H}}_{\varepsilon'}(\delta \odot \Psi)(\bar{z}) = \delta \odot \partial^{\mathscr{D}\mathscr{H}}_{\varepsilon}(\Psi)(\bar{z})$, where $\varepsilon' = \varepsilon\delta$. □

4. $\mathbf{H}_\varepsilon$-SUBGRADIENT

We define the notion of $\mathbf{H}_\varepsilon$-subgradient for IVFs based on the notion of sponge of a set. Further, it was proved that every gH-Dini Hadamard ε-subgradient at a point $\bar{z}$ is also an $\mathbf{H}_\varepsilon$-subgradient of Ψ at $\bar{z}$. However, the converse need not be true. Furthermore, an interpretation of gH-Dini Hadamard ε-subdifferential is discussed with the help of sponges.

Let Ψ be an IVF on $\mathscr{X}$. For all $\varepsilon > 0$, define an IVF $\Psi_\varepsilon : \mathscr{X} \to I(\mathbb{R})$ by

$$\Psi_\varepsilon(w) = \Psi(w) \oplus \varepsilon \|w-\bar{z}\| \text{ for all } w \in \mathscr{X}.$$

Lemma 4.1. *Let Ψ be an IVF on $\mathscr{X}$. Then, for $\bar{z} \in \mathscr{X}$ and $\varepsilon > 0$,*

$$\Psi_{\varepsilon\mathscr{D}\mathscr{H}}(\bar{z})(w-\bar{z}) = \Psi_{\mathscr{D}\mathscr{H}}(\bar{z})(w-\bar{z}) \oplus \varepsilon \|w-\bar{z}\|.$$

Proof. For $\varepsilon > 0$ and $\bar{z} \in \mathscr{X}$, we have

$$\begin{aligned}
\Psi_{\varepsilon\mathscr{D}\mathscr{H}}(\bar{z})(w-\bar{z}) &= \liminf_{\substack{u \to (w-\bar{z}) \\ \lambda \to 0+}} \frac{1}{\lambda} \odot (\Psi_\varepsilon(\bar{z}+\lambda u) \ominus_{gH} \Psi_\varepsilon(\bar{z})) \\
&= \liminf_{\substack{u \to (w-\bar{z}) \\ \lambda \to 0+}} \frac{1}{\lambda} \odot ((\Psi(\bar{z}+\lambda u) \oplus \varepsilon \|\lambda u\|) \ominus_{gH} \Psi(\bar{z})) \\
&= \liminf_{\substack{u \to (w-\bar{z}) \\ \lambda \to 0+}} \frac{1}{\lambda} \odot (\Psi(\bar{z}+\lambda u) \ominus_{gH} (\Psi(\bar{z})) \oplus \lim_{u \to (w-\bar{z})} \varepsilon \|u\| \\
&= \Psi_{\mathscr{D}\mathscr{H}}(\bar{z})(w-\bar{z}) \oplus \varepsilon \|w-\bar{z}\|.
\end{aligned}$$

Thus we have the desired result. □

Remark 4.1. From Lemma 4.1, we can observe that, for all $w \in \mathscr{X}$ and $\varepsilon > 0$, an $\widehat{\mathbf{S}} \in \partial^{\mathscr{D}\mathscr{H}} \Psi_\varepsilon(\bar{z})$ if and only if

$$\begin{aligned}
& (w-\bar{z})^\top \odot \widehat{\mathbf{S}} \preceq \Psi_{\varepsilon\mathscr{D}\mathscr{H}}(\bar{z})(w-\bar{z}) \\
\Longleftrightarrow\ & (w-\bar{z})^\top \odot \widehat{\mathbf{S}} \preceq \Psi_{\mathscr{D}\mathscr{H}}(\bar{z})(w-\bar{z}) \oplus \varepsilon \|w-\bar{z}\|.
\end{aligned}$$

Therefore, for $\bar{z} \in \mathscr{Z}$ and $\varepsilon > 0$, we have $\partial_\varepsilon^{\mathscr{D}\mathscr{H}} \Psi(\bar{z}) = \partial^{\mathscr{D}\mathscr{H}} \Psi_\varepsilon(\bar{z})$.

Definition 4.1. (Sponge of a set [21]). Let $\mathscr{S} \subseteq \mathscr{Z}$. If for any $\bar{z} \in \mathscr{Z}$ and for all $h \in \mathscr{Z} \setminus \{0\}$, we can find a $\lambda > 0$ and $\delta > 0$ satisfying $\bar{z} + [0,\delta] \cdot \mathscr{B}(h,\delta) \subseteq \mathscr{S}$, then $\mathscr{S}$ is said to be a sponge set around $\bar{z}$.

Definition 4.2. ($\mathbf{H}_\varepsilon$-subgradient for IVF). Let $\Psi : \mathscr{Z} \to \overline{I(\mathbb{R})}$ be an extended IVF on $\mathscr{Z}$. Then, an element $\widehat{\mathbf{S}} \in I(\mathbb{R})^n$ is an $\mathbf{H}_\varepsilon$-subgradient of Ψ at $\bar{z}$ if there exists a sponge $\mathscr{S}$ around $\bar{z} \in \mathscr{Z}$ such that

$$(w-\bar{z})^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \varepsilon \|w-\bar{z}\| \preceq \Psi(w) \ominus_{gH} \Psi(\bar{z}) \text{ for all } w \in \mathscr{S}.$$

Theorem 4.1. *Let Ψ be an IVF on $\mathscr{Z}$ and there exists an $\widehat{\mathbf{S}} \in \partial_\varepsilon^{\mathscr{D}\mathscr{H}} \Psi(\bar{z})$. Then, for all $\gamma > \varepsilon > 0$, $\widehat{\mathbf{S}}$ is an $\boldsymbol{H}_\gamma$-subgradient of Ψ at $\bar{z}$.*

Proof. Let $\widehat{\mathbf{S}} \in \partial_\varepsilon^{\mathscr{D}\mathscr{H}} \Psi(\bar{z})$ and $\gamma > \varepsilon > 0$. Consider the set

$$\mathscr{S} = \{w \in \mathscr{Z} : (w-\bar{z})^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \gamma \|w-\bar{z}\| \preceq \Psi(w) \ominus_{gH} \Psi(\bar{z})\} \text{ for all } w \in \mathscr{Z}.$$

We demonstrate that $\mathscr{S}$ is a sponge around $\bar{z}$. Let $h \in \mathscr{Z} \setminus \{0\}$. Since $\widehat{\mathbf{S}} \in \partial_\varepsilon^{\mathscr{D}\mathscr{H}} \Psi(\bar{z})$, then

$$h^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \varepsilon \|h\| \preceq \Psi_{\mathscr{D}\mathscr{H}}(\bar{z})(h)$$

$$\text{or, } h^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \varepsilon \|h\| \preceq \liminf_{\substack{u \to h \\ \lambda \to 0+}} \frac{1}{\lambda} \odot (\Psi(\bar{z}+\lambda u) \ominus_{gH} \Psi(\bar{z}))$$

$$\text{or, } h^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \left(\frac{\gamma+\varepsilon}{2}\right) \|h\| \preceq \liminf_{\substack{u \to h \\ \lambda \to 0+}} \frac{1}{\lambda} \odot (\Psi(\bar{z}+\lambda u) \ominus_{gH} \Psi(\bar{z})) \text{ from (i) of Lemma 2.1}$$

$$\text{or, } h^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \gamma \|h\| \preceq \liminf_{\substack{u \to h \\ \lambda \to 0+}} \frac{1}{\lambda} \odot (\Psi(\bar{z}+\lambda u) \ominus_{gH} \Psi(\bar{z})) \text{ from (i) of Lemma 2.1.}$$

Therefore, we can find a $\delta_1 > 0$ such that for all $u \in \mathscr{B}(h,\delta_1)$

$$u^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \gamma \|u\| \preceq h^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \left(\frac{\gamma+\varepsilon}{2}\right) \|h\|. \tag{4.1}$$

From the definition of the limit inferior, there exists $\delta_2 > 0$ satisfying that, for all $\lambda \in (0,\delta_2)$ and $u \in \mathscr{B}(h,\delta_2)$,

$$h^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \left(\frac{\gamma+\varepsilon}{2}\right) \|h\| \preceq \frac{1}{\lambda} \odot (\Psi(\bar{z}+\lambda u) \ominus_{gH} \Psi(\bar{z})). \tag{4.2}$$

Hence, from (4.1) and (4.2), there exists $\delta = \frac{1}{2}\min\{\delta_1,\delta_2\} > 0$ such that, for all $\lambda \in (0,\delta]$ and $u \in \mathscr{B}(h,\delta)$,

$$(\lambda u)^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \gamma \|\lambda u\| \preceq \Psi(\bar{z}+\lambda u) \ominus_{gH} \Psi(\bar{z}).$$

Thus, for all $w \in \bar{z} + [0,\delta] \cdot \mathscr{B}(h,\delta)$, we have

$$(w-\bar{z})^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \gamma \|w-\bar{z}\| \preceq \Psi(w) \ominus_{gH} \Psi(\bar{z}).$$

Therefore, $\bar{z} + [0,\delta] \cdot \mathscr{B}(h,\delta) \in \mathscr{S}$. Thus, $\mathscr{S}$ is a sponge around $\bar{z}$. $\square$

Remark 4.2. The converse of Theorem 4.1 need not be true. For example, consider an IVF $\Psi : \mathbb{R} \to I(\mathbb{R})$ such that

$$\Psi(z) = \begin{cases} \mathbf{0}, & z \in \mathscr{S} \\ [-2,-1], & \text{otherwise,} \end{cases}$$

where $\mathscr{S}$ is a sponge set around some $\bar{z} \in \mathbb{R}$. We show that $\mathbf{0}$ is an $\mathbf{H}_\varepsilon$-subgradient of Ψ at $\bar{z}$, however $\mathbf{0} \notin \partial_\varepsilon^{\mathscr{D}\mathscr{H}} \Psi(\bar{z})$.

Let $\mathbf{S} \in I(\mathbb{R})$ be an $\mathbf{H}_\varepsilon$-subgradient of Ψ at $\bar{z}$ and $\mathscr{S}$ be a sponge set around $\bar{z}$. Then, for all $w \in \mathscr{S}$,

$$\begin{aligned} & (w-\bar{z}) \odot \mathbf{S} \ominus_{gH} \varepsilon|w-\bar{z}| \preceq \Psi(w) \ominus_{gH} \Psi(\bar{z}) \\ \implies \quad & (w-\bar{z}) \odot \mathbf{S} \ominus_{gH} \varepsilon|w-\bar{z}| \preceq [0,0] \ominus_{gH} \Psi(\bar{z}). \end{aligned}$$

We have the following cases:

(i) If $\bar{z} \in \mathscr{S}$, then

$$(w-\bar{z}) \odot \mathbf{S} \ominus_{gH} \varepsilon|w-\bar{z}| \preceq \mathbf{0}.$$

There arise the following two subcases:

(a) If $(w-\bar{z}) \geq 0$, then

$$\mathbf{S} \ominus_{gH} \varepsilon \preceq \mathbf{0} \implies \mathbf{S} \preceq \varepsilon.$$

(b) If $(w-\bar{z}) < 0$, then

$$\mathbf{0} \preceq \mathbf{S} \ominus_{gH} (-\varepsilon) \implies -\varepsilon \preceq \mathbf{S}.$$

Since $\varepsilon > 0$ is arbitrary, therefore subcases (a) and (b) of Case (i) occur simultaneously when $\mathbf{S} = \mathbf{0}$.

(ii) If $\bar{z} \notin \mathscr{S}$, then

$$(w-\bar{z}) \odot \mathbf{S} \ominus_{gH} \varepsilon|w-\bar{z}| \preceq [1,2].$$

There arise the following two sub cases:

(a) If $(w-\bar{z}) \geq 0$, then

$$\begin{aligned} & (w-\bar{z}) \odot \mathbf{S} \ominus_{gH} \varepsilon(w-\bar{z}) \preceq [1,2] \\ \implies \quad & (w-\bar{z})\underline{s} \leq 1 + \varepsilon(w-\bar{z}) \text{ and } (w-\bar{z})\bar{s} \leq 2 + \varepsilon(w-\bar{z}). \end{aligned}$$

(b) If $(w-\bar{z}) < 0$, then

$$\begin{aligned} & (w-\bar{z}) \odot \mathbf{S} \ominus_{gH} \varepsilon(\bar{z}-w) \preceq [1,2] \\ \implies \quad & (w-\bar{z})\bar{s} \leq 1 - \varepsilon(w-\bar{z}) \text{ and } (w-\bar{z})\underline{s} \leq 2 - \varepsilon(w-\bar{z}). \end{aligned}$$

Since $(w-\bar{z}) \in \mathbb{R}$ and $\varepsilon > 0$ are arbitrary, therefore subcases (a) and (b) of Case (ii) occur simultaneously when $\mathbf{S} = \mathbf{0}$.

Thus, $\mathbf{0}$ is an $\mathbf{H}_\varepsilon$-subgradient of Ψ at $\bar{z}$. Now, we show that $\mathbf{0} \notin \partial_\varepsilon^{\mathscr{D}\mathscr{H}} \Psi(\bar{z})$. Assume contrarily that $\mathbf{0} \in \partial_\varepsilon^{\mathscr{D}\mathscr{H}} \Psi(\bar{z})$. Therefore, for all $w \in \mathbb{R}$, we get

$$\begin{aligned} & (w-\bar{z}) \odot \mathbf{S} \preceq \Psi_{\mathscr{D}\mathscr{H}}(\bar{z})(w-\bar{z}) \oplus \varepsilon|w-\bar{z}| \\ \implies \quad & \mathbf{0} \preceq \liminf_{\substack{u \to (w-\bar{z}) \\ \lambda \to 0+}} \frac{1}{\lambda} \odot (\Psi(\bar{z}+\lambda u) \ominus_{gH} \Psi(\bar{z})) \oplus \varepsilon|w-\bar{z}|. \end{aligned}$$

We have the following cases:

(i) If $\bar{z} \in \mathscr{S}$, then

$$\mathbf{0} \preceq \liminf_{\substack{u \to (w-\bar{z}) \\ \lambda \to 0+}} \frac{1}{\lambda} \odot \Psi(\bar{z}+\lambda u) \oplus \varepsilon |w-\bar{z}|.$$

Therefore, we observe that for $(\bar{z}+\lambda u) \notin \mathscr{S}$, we get

$$\mathbf{0} \preceq \liminf_{\lambda \to 0+} \frac{1}{\lambda} \odot [-2,-1] \oplus \varepsilon |w-\bar{z}|,$$

which does not exist for any $w \in \mathbb{R}$.

(ii) If $\bar{z} \notin \mathscr{S}$, then

$$\mathbf{0} \preceq \liminf_{\substack{u \to (w-\bar{z}) \\ \lambda \to 0+}} \frac{1}{\lambda} \odot (\Psi(\bar{z}+\lambda u) \ominus_{gH} [-2,-1]) \oplus \varepsilon |w-\bar{z}|.$$

Therefore, we observe that, for $(\bar{z}+\lambda u) \in \mathscr{S}$,

$$\mathbf{0} \preceq \liminf_{\lambda \to 0+} \frac{1}{\lambda} \odot [1,2] \oplus \varepsilon |w-\bar{z}|,$$

which does not exist for any $w \in \mathbb{R}$. In view of Case (i) and Case (ii), we can conclude that $\mathbf{0} \notin \partial_\varepsilon^{\mathscr{D}\mathscr{H}} \Psi$ at $\bar{z}$.

Definition 4.3. (gH-calm IVF). Let Ψ be an IVF on $\mathscr{X}$. Then, Ψ is said to be a gH-calm IVF at $\bar{z} \in \mathscr{X}$ if there exist $c \geq 0$ and $\delta > 0$ such that

$$-c\|w-\bar{z}\| \preceq \Psi(w) \ominus_{gH} \Psi(\bar{z}) \text{ for all } w \in \mathscr{B}(\bar{z}, \delta).$$

Remark 4.3. (i) In view of Definition 4.3, it can be observed that there exist some constant $c \geq 0$ such that, for every h in $\mathscr{X}$, we have

$$-c\|h\| \preceq \Psi_{\mathscr{D}\mathscr{H}}(\bar{z})(h).$$

(ii) From Definition 4.3, it can be noted that if Ψ is gH-calm IVF at $\bar{z} \in \mathscr{X}$. Then, $\underline{\psi}, \overline{\psi}$ are calm at $\bar{z} \in \mathscr{X}$ and vice-versa.

Lemma 4.2. *Let Ψ be a gH-calm IVF at $\bar{z} \in \mathscr{X}$, and $\widehat{\mathbf{S}}$ be an $\mathbf{H}_\varepsilon$-subgradient of Ψ at $\bar{z} \in \mathscr{X}$. Then, $\widehat{\mathbf{S}} \in \partial_\varepsilon^{\mathscr{D}\mathscr{H}} \Psi(\bar{z})$.*

Proof. If $\widehat{\mathbf{S}}$ is an $\mathbf{H}_\varepsilon$-subgradient of Ψ at $\bar{z} \in \mathscr{X}$, then there exists a sponge $\mathscr{S}$ around $\bar{z}$ satisfying

$$(w-\bar{z})^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \varepsilon \|w-\bar{z}\| \preceq \Psi(w) \ominus_{gH} \Psi(\bar{z}) \text{ for all } w \in \mathscr{S}.$$

Letting $h \in \mathscr{X} \setminus \{0\}$, we can find a $\beta > 0,\ \delta > 0$ such that $\lambda \in (0,\beta]$ and $u \in \mathscr{B}(h,\delta)$, $\bar{z}+\lambda u \in \mathscr{S}$ and

$$\begin{aligned} & u^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \varepsilon \|u\| \preceq \frac{1}{\lambda} \odot (\Psi(\bar{z}+\lambda u) \ominus_{gH} \Psi(\bar{z})) \\ \Longrightarrow \quad & \liminf_{\substack{u \to h \\ \lambda \to 0+}} u^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \varepsilon \|u\| \preceq \liminf_{\substack{u \to h \\ \lambda \to 0+}} \frac{1}{\lambda} \odot (\Psi(\bar{z}+\lambda u) \ominus_{gH} \Psi(\bar{z})) \\ \Longrightarrow \quad & h^\top \odot \widehat{\mathbf{S}} \ominus_{gH} \varepsilon \|h\| \preceq \Psi_{\mathscr{D}\mathscr{H}}(\bar{z})(h). \end{aligned}$$

Therefore, $\widehat{\mathbf{S}} \in \partial_\varepsilon^{\mathscr{D}\mathscr{H}} \Psi(\bar{z})$. ∎

We next provide an interpretation of gH-Dini Hadamard ε-subdifferential by replacing neighborhood with sponges.

Theorem 4.2. *Let Ψ be an extended IVF on $\mathscr{Z}$. For $\varepsilon > 0$ and $\bar{z} \in \mathscr{Z}$, $\widehat{\mathbf{S}} \in \partial_{\varepsilon}^{\mathscr{D}\mathscr{H}}\Psi(\bar{z})$ if and only if Ψ is gH-calm IVF at $\bar{z}$ and for every $\alpha > 0$, there exists a sponge $\mathscr{S}$ around $\bar{z}$ such that*

$$(w-\bar{z})^{\top} \odot \widehat{\mathbf{S}} \ominus_{gH} (\alpha+\varepsilon)\|w-\bar{z}\| \preceq \Psi(w) \ominus_{gH} \Psi(\bar{z}). \tag{4.3}$$

Proof. Let $\widehat{\mathbf{S}} \in \partial_{\varepsilon}^{\mathscr{D}\mathscr{H}}\Psi(\bar{z})$. Then, from Theorem 4.1, there exists a sponge $\mathscr{S}$ around $\bar{z}$ satisfying

$$(w-\bar{z})^{\top} \odot \widehat{\mathbf{S}} \ominus_{gH} (\alpha+\varepsilon)\|w-\bar{z}\| \preceq \Psi(w) \ominus_{gH} \Psi(\bar{z}) \text{ for all } w \in \mathscr{S}.$$

To prove the converse, we assume that Ψ is gH-calm IVF at $\bar{z}$ and there exists an $\widehat{\mathbf{S}} \in I(\mathbb{R})^n$ such that (4.3) holds. We show that $\widehat{\mathbf{S}} \in \partial_{\varepsilon}^{\mathscr{D}\mathscr{H}}\Psi(\bar{z})$, i.e.,

$$h^{\top} \odot \widehat{\mathbf{S}} \ominus_{gH} \varepsilon\|h\| \preceq \Psi_{\mathscr{D}\mathscr{H}}(\bar{z})(h) \text{ for all } h \in \mathscr{Z}.$$

Now, for all $p \in \mathbb{N}$, take $\alpha_p = \frac{1}{p}$. By the hypothesis, there exists a sponge $\mathscr{S}_p$ around $\bar{z}$ such that

$$(w-\bar{z})^{\top} \odot \widehat{\mathbf{S}} \ominus_{gH} \left(\frac{1}{p}+\varepsilon\right)\|w-\bar{z}\| \preceq \Psi(w) \ominus_{gH} \Psi(\bar{z}) \text{ for all } w \in \mathscr{S}_p.$$

Thus, for $h \in \mathscr{Z}$ and for every $p \in \mathbb{N}$, there exist $t_p > 0$ and $\delta_p > 0$ such that, for all $\lambda \in (0, t_p)$ and $u \in \mathscr{B}(h, \delta_p)$, we have $\bar{z} + \lambda u \in \mathscr{S}_p$ and

$$\begin{aligned} & (\lambda u)^{\top} \odot \widehat{\mathbf{S}} \ominus_{gH} \left(\frac{1}{p}+\varepsilon\right)\|\lambda u\| \preceq \Psi(\bar{z}+\lambda u) \ominus_{gH} \Psi(\bar{z}) \\ \text{or,} \quad & \liminf_{\substack{u \to h \\ \lambda \to 0+}} u^{\top} \odot \widehat{\mathbf{S}} \ominus_{gH} \left(\frac{1}{p}+\varepsilon\right)\|u\| \preceq \liminf_{\substack{u \to h \\ \lambda \to 0+}} \frac{1}{\lambda} \odot (\Psi(\bar{z}+\lambda u) \ominus_{gH} \Psi(\bar{z})) \\ \text{or,} \quad & h^{\top} \odot \widehat{\mathbf{S}} \ominus_{gH} \left(\frac{1}{p}+\varepsilon\right)\|h\| \preceq \Psi_{\mathscr{D}\mathscr{H}}(\bar{z})(h). \end{aligned}$$

Therefore, as $p \to \infty$, $h^{\top} \odot \widehat{\mathbf{S}} \ominus_{gH} \varepsilon\|h\| \preceq \Psi_{\mathscr{D}\mathscr{H}}(\bar{z})(h)$. Thus $\widehat{\mathbf{S}} \in \partial_{\varepsilon}^{\mathscr{D}\mathscr{H}}\Psi(\bar{z})$. □

5. Optimality Conditions on Nonsmooth Interval Optimization

In this section, we define a concept of an ε-efficient solution for IOPs and characterize ε-efficient solutions with the help of the derived results on gH-Dini Hadamard ε-subdifferentiability of IVFs.

Definition 5.1. (ε-efficient solution of an IOP). Let Ψ be an IVF on $\mathscr{Z}$. Then, for $\varepsilon > 0$, a point $\bar{z} \in \mathscr{Z}$ is called an ε-efficient solution to the IOP

$$\min_{z \in \mathscr{Z}} \Psi(z) \tag{5.1}$$

if, for all $w \in \mathscr{Z}$, $\Psi(\bar{z}) \preceq \Psi(w) \oplus \varepsilon\|w-\bar{z}\|$.

Theorem 5.1. *Let Ψ be an IVF on $\mathscr{Z}$. If $\widehat{\mathbf{0}} \in \partial_{\varepsilon}^{\mathscr{D}\mathscr{H}}\Psi(\bar{z})$ for some $\bar{z} \in \mathscr{Z}$, then $\bar{z}$ is an ε-efficient solution of the IOP (5.1).*

Proof. Let $\widehat{\mathbf{0}} \in \partial_\varepsilon^{\mathscr{D}\mathscr{H}} \Psi(\bar{z})$ for some $\bar{z} \in \mathscr{Z}$. Then, for $\varepsilon > 0$ and for all $w \in \mathscr{Z}$, we have

$$(w-\bar{z})^\top \odot \widehat{\mathbf{0}} \preceq \Psi_{\mathscr{D}\mathscr{H}}(\bar{z})(w-\bar{z}) \oplus \varepsilon\|w-\bar{z}\|$$

$$\text{or, } \mathbf{0} \preceq \liminf_{\substack{u\to(w-\bar{z})\\ \lambda\to 0+}} \frac{1}{\lambda} \odot (\Psi(\bar{z}+\lambda u) \ominus_{gH} \Psi(\bar{z})) \oplus \varepsilon\|w-\bar{z}\|$$

$$\text{or, } \mathbf{0} \preceq \left[\min\left\{\liminf_{\substack{u\to(w-\bar{z})\\ \lambda\to 0+}} \frac{\underline{\psi}(\bar{z}+\lambda u)-\underline{\psi}(\bar{z})}{\lambda}, \liminf_{\substack{u\to(w-\bar{z})\\ \lambda\to 0+}} \frac{\overline{\psi}(\bar{z}+\lambda u)-\overline{\psi}(\bar{z})}{\lambda}\right\},\right.$$

$$\left.\max\left\{\liminf_{\substack{u\to(w-\bar{z})\\ \lambda\to 0+}} \frac{\underline{\psi}(\bar{z}+\lambda u)-\underline{\psi}(\bar{z})}{\lambda}, \liminf_{\substack{u\to(w-\bar{z})\\ \lambda\to 0+}} \frac{\overline{\psi}(\bar{z}+\lambda u)-\overline{\psi}(\bar{z})}{\lambda}\right\}\right] \oplus \varepsilon\|w-\bar{z}\|. \tag{5.2}$$

Now there arise following two cases.

(i) Let

$$\left[\min\left\{\liminf_{\substack{u\to(w-\bar{z})\\ \lambda\to 0+}} \frac{\underline{\psi}(\bar{z}+\lambda u)-\underline{\psi}(\bar{z})}{\lambda}, \liminf_{\substack{u\to(w-\bar{z})\\ \lambda\to 0+}} \frac{\overline{\psi}(\bar{z}+\lambda u)-\overline{\psi}(\bar{z})}{\lambda}\right\}\right]$$

$$= \liminf_{\substack{u\to(w-\bar{z})\\ \lambda\to 0+}} \frac{\underline{\psi}(\bar{z}+\lambda u)-\underline{\psi}(\bar{z})}{\lambda}.$$

In this case, from (5.2), we have

$$\mathbf{0} \preceq \left[\liminf_{\substack{u\to(w-\bar{z})\\ \lambda\to 0+}} \frac{\underline{\psi}(\bar{z}+\lambda u)-\underline{\psi}(\bar{z})}{\lambda}, \liminf_{\substack{u\to(w-\bar{z})\\ \lambda\to 0+}} \frac{\overline{\psi}(\bar{z}+\lambda u)-\overline{\psi}(\bar{z})}{\lambda}\right] \oplus \varepsilon\|w-\bar{z}\|.$$

Thus, there exists a $\delta_1 > 0$ such that for all $\lambda \in (0, \delta_1)$, we obtain

$$0 \leq \frac{\underline{\psi}(\bar{z}+\lambda(w-\bar{z}))-\underline{\psi}(\bar{z})}{\lambda} \text{ and } 0 \leq \frac{\overline{\psi}(\bar{z}+\lambda(w-\bar{z}))-\overline{\psi}(\bar{z})}{\lambda}$$

$$\text{or, } 0 \leq \underline{\psi}(\bar{z}+\lambda(w-\bar{z}))-\underline{\psi}(\bar{z}) \;\text{ and }\; 0 \leq \overline{\psi}(\bar{z}+\lambda(w-\bar{z}))-\overline{\psi}(\bar{z}).$$

Therefore, for all $w \in \mathscr{Z}$, we have

$$\begin{aligned} & \mathbf{0} \preceq \Psi(\bar{z}+\lambda(w-\bar{z})) \ominus_{gH} \Psi(\bar{z}) \oplus \varepsilon\|w-\bar{z}\| \\ \Longrightarrow\quad & \mathbf{0} \preceq \Psi(w) \ominus_{gH} \Psi(\bar{z}) \oplus \varepsilon\|w-\bar{z}\| \\ \Longrightarrow\quad & \Psi(\bar{z}) \preceq \Psi(w) \oplus \varepsilon\|w-\bar{z}\|. \end{aligned}$$

(ii) Let

$$\left[\min\left\{\liminf_{\substack{u\to(w-\bar{z})\\ \lambda\to 0+}} \frac{\underline{\psi}(\bar{z}+\lambda u)-\underline{\psi}(\bar{z})}{\lambda}, \liminf_{\substack{u\to(w-\bar{z})\\ \lambda\to 0+}} \frac{\overline{\psi}(\bar{z}+\lambda u)-\overline{\psi}(\bar{z})}{\lambda}\right\}\right]$$

$$= \liminf_{\substack{u\to(w-\bar{z})\\ \lambda\to 0+}} \frac{\overline{\psi}(\bar{z}+\lambda u)-\overline{\psi}(\bar{z})}{\lambda}.$$

Proceeding in a similar manner as in Case (i), we get the desired result. Thus,

$$\Psi(\bar{z}) \preceq \Psi(w) \oplus \varepsilon\|w-\bar{z}\| \text{ for all } w \in \mathscr{Z}.$$

Thus, in view of Case (i) and Case (ii), $\bar{z}$ is an ε-efficient solution to IOP (5.1). □

Theorem 5.2. *Let $\Psi : \mathscr{Z} \to I(\mathbb{R})$ be an IVF on $\mathscr{Z}$. If $\bar{z}$ is an ε-efficient solution to the IOP (5.1), then, for each $\varepsilon > 0$,*

$$\widehat{\mathbf{0}} \in \partial_{\varepsilon}^{\mathscr{D}\mathscr{H}} \Psi(\bar{z}).$$

Proof. Let $\bar{z}$ be an ε-efficient solution to IOP (5.1). Then, for all $w \in \mathscr{Z}$,

$$\begin{aligned}
& \Psi(\bar{z}) \preceq \Psi(w) \oplus \varepsilon\|w-\bar{z}\| \\
\Longrightarrow \quad & \mathbf{0} \preceq (\Psi(w) \ominus_{gH} \Psi(\bar{z})) \oplus \varepsilon\|w-\bar{z}\| \\
\Longrightarrow \quad & \mathbf{0} \preceq \liminf_{\substack{u \to (w-\bar{z}) \\ \lambda \to 0+}} \frac{1}{\lambda} \odot (\Psi(\bar{z}+\lambda u) \ominus_{gH} \Psi(\bar{z})) \oplus \varepsilon\|w-\bar{z}\| \\
\Longrightarrow \quad & (w-\bar{z})^{\top} \odot \widehat{\mathbf{0}} \preceq \Psi_{\mathscr{D}\mathscr{H}}(\bar{z})(w-\bar{z}) \oplus \varepsilon\|w-\bar{z}\|.
\end{aligned}$$

Therefore, $\widehat{\mathbf{0}} \in \partial_{\varepsilon}^{\mathscr{D}\mathscr{H}} \Psi(\bar{z})$. □

Theorem 5.3. *(Necessary condition for efficient points to an IOP). Let $\Psi : \mathscr{Z} \to I(\mathbb{R})$ be an IVF and $\bar{z}$ be an ε-efficient solution to the IOP (5.1). If there exists an $\widehat{\mathbf{S}} \in \partial_{\varepsilon}^{\mathscr{D}\mathscr{H}} \Psi(\bar{z})$, then*

$$\mathbf{0} \preceq (w-\bar{z})^{\top} \odot \widehat{\mathbf{S}} \text{ for all } w \in \mathscr{Z}.$$

Proof. Let $\bar{z}$ be an ε-efficient solution to IOP (5.1) and there does not exist any $w \in \mathscr{Z}$ such that

$$\mathbf{0} \preceq (w-\bar{z})^{\top} \odot \widehat{\mathbf{S}}.$$

Since $\widehat{\mathbf{S}} \in \partial_{\varepsilon}^{\mathscr{D}\mathscr{H}} \Psi(\bar{z})$, there does not exist any $w \in \mathscr{Z}$ such that

$$\begin{aligned}
& \mathbf{0} \preceq (w-\bar{z})^{\top} \odot \widehat{\mathbf{S}} \preceq \Psi_{\mathscr{D}\mathscr{H}}(\bar{z})(w-\bar{z}) \oplus \varepsilon\|w-\bar{z}\| \\
\text{or,} \quad & \mathbf{0} \preceq \liminf_{\substack{u \to (w-\bar{z}) \\ \lambda \to 0+}} \frac{1}{\lambda} \odot (\Psi(\bar{z}+\lambda u) \ominus_{gH} \Psi(\bar{z})) \oplus \varepsilon\|w-\bar{z}\|.
\end{aligned}$$

Proceeding in a similar manner as in Theorem 5.1, we can conclude that there does not exist any $w \in \mathscr{Z}$ such that

$$\Psi(\bar{z}) \preceq \Psi(w) \oplus \varepsilon\|w-\bar{z}\|,$$

which is a contradiction. Therefore, we obtain

$$\mathbf{0} \preceq (w-\bar{z})^{\top} \odot \widehat{\mathbf{S}} \text{ for all } w \in \mathscr{Z}.$$

□

Theorem 5.4. *(Sufficient condition for efficient points to an IOP). Let Ψ be a convex IVF on $\mathscr{Z}$. If there exists an $\widehat{\mathbf{S}} \in \partial_{\varepsilon}^{\mathscr{D}\mathscr{H}} \Psi(\bar{z})$ for some $\bar{z} \in \mathscr{Z}$ and $\varepsilon > 0$ such that*

$$\mathbf{0} \preceq (w-\bar{z})^{\top} \odot \widehat{\mathbf{S}} \text{ for all } w \in \mathscr{Z}, \tag{5.3}$$

then $\bar{z}$ is an ε-efficient solution to IOP (5.1).

Proof. Let $\widehat{\mathbf{S}} \in \partial_\varepsilon^{\mathscr{D}\mathscr{H}} \Psi(\bar{z})$ be such that the relation (5.3) is true. Therefore, for all $w \in \mathscr{Z}$, we have

$$
\begin{aligned}
&(w-\bar{z})^\top \odot \widehat{\mathbf{S}} \preceq \Psi_{\mathscr{D}\mathscr{H}}(\bar{z}) \oplus \varepsilon\|w-\bar{z}\| \\
&\text{or, } \mathbf{0} \preceq \Psi_{\mathscr{D}\mathscr{H}}(\bar{z}) \oplus \varepsilon\|w-\bar{z}\| \\
&\text{or, } \mathbf{0} \preceq \liminf_{\substack{u\to(w-\bar{z})\\ \lambda\to 0+}} \frac{1}{\lambda} \odot (\Psi(\bar{z}+\lambda u) \ominus_{gH} \Psi(\bar{z})) \oplus \varepsilon\|w-\bar{z}\| \\
&\text{or, } \mathbf{0} \preceq \left[\min\left\{\liminf_{\substack{u\to(w-\bar{z})\\ \lambda\to 0+}} \frac{\underline{\psi}(\bar{z}+\lambda u)-\underline{\psi}(\bar{z}))}{\lambda}, \liminf_{\substack{u\to(w-\bar{z})\\ \lambda\to 0+}} \frac{\overline{\psi}(\bar{z}+\lambda u)-\overline{\psi}(\bar{z}))}{\lambda}\right\},\right. \\
&\qquad \left.\max\left\{\liminf_{\substack{u\to(w-\bar{z})\\ \lambda\to 0+}} \frac{\underline{\psi}(\bar{z}+\lambda u)-\underline{\psi}(\bar{z}))}{\lambda}, \liminf_{\substack{u\to(w-\bar{z})\\ \lambda\to 0+}} \frac{\overline{\psi}(\bar{z}+\lambda u)-\overline{\psi}(\bar{z}))}{\lambda}\right\}\right] \oplus \varepsilon\|w-\bar{z}\|. \qquad (5.4)
\end{aligned}
$$

Proceeding in a similar manner as in Theorem 5.1, we can conclude that $\bar{z}$ is an ε-efficient solution to IOP (5.1). □

Theorem 5.5. *Let $(-\Psi_1)$ be gH-subdifferentiable and gH-Fréchet differentiable IVF at $\bar{z}$ with Fréchet derivative $(-\Psi_{1\mathscr{F}})$ on $\mathscr{Z}$. Let Ψ_2 be an IVF on $\mathscr{Z}$. If, for any $\varepsilon > 0$, $\bar{z}$ is an ε-efficient solution of $\Psi_1 \oplus \Psi_2$, then*

$$(-\Psi_{1\mathscr{F}})(\bar{z}) \in \partial_{\varepsilon'}^{\mathscr{D}\mathscr{H}} \Psi_2(\bar{z}), \text{ where } \varepsilon' = 2\varepsilon. \qquad (5.5)$$

Proof. Let $(-\Psi_1)$ is a gH-subdifferentiable and gH-Fréchet differentiable IVF at $\bar{z}$ with Fréchet derivative $(-\Psi_{1\mathscr{F}})$ on $\mathscr{Z}$, then we have from Theorem 3.4 that

$$\{(-\Psi_{1\mathscr{F}})(\bar{z})\} \subseteq \partial_\varepsilon^{\mathscr{D}\mathscr{H}}(-\Psi_1)(\bar{z}) \text{ for } \varepsilon > 0. \qquad (5.6)$$

Also, as $\bar{z}$ is an ε-efficient solution of $\Psi_1 \oplus \Psi_2$, we have

$$
\begin{aligned}
&(\Psi_1 \oplus \Psi_2)(\bar{z}) \preceq (\Psi_1 \oplus \Psi_2)(w) \oplus \varepsilon\|w-\bar{z}\| \\
&\text{or, } \Psi_1(\bar{z}) \ominus_{gH} \Psi_1(w) \preceq \Psi_2(w) \ominus_{gH} \Psi_2(\bar{z}) \oplus \varepsilon\|w-\bar{z}\| \text{ from (ii) of Lemma 2.1} \\
&\text{or, } \liminf_{\substack{u\to(w-\bar{z})\\ \lambda\to 0+}} \frac{1}{\lambda} \odot (\Psi_1(\bar{z}) \ominus_{gH} \Psi_1(\bar{z}+\lambda u)) \preceq \liminf_{\substack{u\to(w-\bar{z})\\ \lambda\to 0+}} \frac{1}{\lambda} \odot (\Psi_2(\bar{z}+\lambda u) \ominus_{gH} \Psi_2(\bar{z})) \oplus \varepsilon\|w-\bar{z}\| \\
&\text{or, } (-\Psi_{1\mathscr{D}\mathscr{H}}(\bar{z})) \preceq \Psi_{2\mathscr{D}\mathscr{H}}(\bar{z}) \oplus \varepsilon\|w-\bar{z}\| \\
&\text{or, } (-\Psi_{1\mathscr{D}\mathscr{H}}(\bar{z})) \oplus \varepsilon\|w-\bar{z}\| \preceq \Psi_{2\mathscr{D}\mathscr{H}}(\bar{z}) \oplus (\varepsilon+\varepsilon)\|w-\bar{z}\| \\
&\text{or, } (-\Psi_{1\mathscr{D}\mathscr{H}}(\bar{z})) \oplus \varepsilon\|w-\bar{z}\| \preceq \Psi_{2\mathscr{D}\mathscr{H}}(\bar{z}) \oplus \varepsilon'\|w-\bar{z}\|,\ 2\varepsilon = \varepsilon'. \qquad (5.7)
\end{aligned}
$$

In view of (5.6) and (5.7), we have

$$
\begin{aligned}
&(w-\bar{z})^\top \odot (-\Psi_{1\mathscr{F}})(\bar{z}) \preceq (-\Psi_{1\mathscr{D}\mathscr{H}}(\bar{z})) \oplus \varepsilon\|w-\bar{z}\| \preceq \Psi_{2\mathscr{D}\mathscr{H}}(\bar{z}) \oplus \varepsilon'\|w-\bar{z}\| \\
&(w-\bar{z})^\top \odot (-\Psi_{1\mathscr{F}})(\bar{z}) \preceq \Psi_{2\mathscr{D}\mathscr{H}}(\bar{z}) \oplus \varepsilon'\|w-\bar{z}\|.
\end{aligned}
$$

Thus $(-\Psi_{1\mathscr{F}})(\bar{z}) \in \partial_{\varepsilon'}^{\mathscr{D}\mathscr{H}} \Psi_2(\bar{z})$. □

Theorem 5.6. *Let Ψ be an IVF on $\mathscr{Z}$ and $\bar{z}$ be an ε-efficient solution of the IOP (5.1). Then, for any $\varepsilon > 0$,*

$$\partial_\varepsilon^{\mathscr{D}\mathscr{H}} \boldsymbol{0}(\bar{z}) \subseteq \partial_{\varepsilon'}^{\mathscr{D}\mathscr{H}} \Psi(\bar{z}), \text{ where } \varepsilon' = 2\varepsilon.$$

Proof. Let $\widehat{\mathbf{S}} \in \partial_{\varepsilon}^{\mathscr{D}\mathscr{H}} \mathbf{0}(\bar{z})$. Then, for all $w \in \mathscr{X}$ and $\varepsilon > 0$,

$$(w-\bar{z})^{\top} \odot \widehat{\mathbf{S}} \preceq \mathbf{0}_{\mathscr{D}\mathscr{H}}(\bar{z}) \oplus \varepsilon \|w-\bar{z}\|. \tag{5.8}$$

Also, as $\bar{z}$ is an ε-efficient solution of (5.1), for each $w \in \mathscr{X}$ and $\varepsilon > 0$, we have

$$\begin{aligned}
& \Psi(\bar{z}) \preceq \Psi(w) \oplus \varepsilon \|w-\bar{z}\| \\
\implies \quad & \mathbf{0} \preceq \Psi(w) \ominus_{gH} \Psi(\bar{z}) \oplus \varepsilon \|w-\bar{z}\| \\
\implies \quad & \varepsilon \|w-\bar{z}\| \preceq \Psi(w) \ominus_{gH} \Psi(\bar{z}) \oplus (\varepsilon+\varepsilon) \|w-\bar{z}\| \\
\implies \quad & \varepsilon \|w-\bar{z}\| \preceq \liminf_{\substack{u \to (w-\bar{z}) \\ \lambda \to 0+}} \frac{1}{\lambda} \odot (\Psi(\bar{z}+\lambda u) \ominus_{gH} \Psi(\bar{z})) \oplus 2\varepsilon \|w-\bar{z}\| \\
\implies \quad & \varepsilon \|w-\bar{z}\| \preceq \Psi_{\mathscr{D}\mathscr{H}}(\bar{z}) \oplus \varepsilon' \|w-\bar{z}\|, \ \varepsilon' = 2\varepsilon.
\end{aligned} \tag{5.9}$$

From (5.8), (5.9), and (i) of Lemma 2.1, we have

$$(w-\bar{z})^{\top} \odot \widehat{\mathbf{S}} \preceq \Psi_{\mathscr{D}\mathscr{H}}(\bar{z}) \oplus \varepsilon' \|w-\bar{z}\|.$$

□

Therefore, $\widehat{\mathbf{S}} \in \partial_{\varepsilon'}^{\mathscr{D}\mathscr{H}} \Psi(\bar{z})$. Thus $\partial_{\varepsilon}^{\mathscr{D}\mathscr{H}} \mathbf{0}(\bar{z}) \subseteq \partial_{\varepsilon'}^{\mathscr{D}\mathscr{H}} \Psi(\bar{z})$.

Remark 5.1. In Theorem 5.6, the condition on $\bar{z}$ to be an ε-efficient solution is necessary. For instance, consider the IVF that was discussed in Remark 3.1: $\Psi(\bar{z}) = [-2,-1] \odot |z|$. It can be observed that $\mathbf{0}$ is not an ε-efficient point of Ψ and the gH-Dini Hadamard ε-subdifferentials of IVFs Ψ and $\mathbf{0}$ at $\bar{z}=0$, are given by

$$\begin{aligned}
& \partial_{\varepsilon'}^{\mathscr{D}\mathscr{H}} \Psi(0) = \{\mathbf{S} : [1,2] \ominus_{gH} \varepsilon' \preceq \mathbf{S} \preceq [-2,-1] \oplus \varepsilon'\} \text{ and} \\
& \partial_{\varepsilon}^{\mathscr{D}\mathscr{H}} \mathbf{0}(0) = \{\mathbf{S} : -\varepsilon \preceq \mathbf{S} \preceq \varepsilon\}, \text{ respectively.}
\end{aligned}$$

Therefore, for $\varepsilon = 1$, we have

$$\partial_{\varepsilon'}^{\mathscr{D}\mathscr{H}} \Psi(0) = \{\mathbf{S} : [-1,0] \preceq \mathbf{S} \preceq [0,1]\} \text{ and } \partial_{\varepsilon}^{\mathscr{D}\mathscr{H}} \mathbf{0}(0) = \{\mathbf{S} : [-1,-1] \preceq \mathbf{S} \preceq [1,1]\}.$$

It can be observed that $[-1,-1] \in \partial_{\varepsilon}^{\mathscr{D}\mathscr{H}} \mathbf{0}(\bar{z})$ but $[-1,-1] \notin \partial_{\varepsilon'}^{\mathscr{D}\mathscr{H}} \Psi(\bar{z})$.
Thus, $\partial_{\varepsilon}^{\mathscr{D}\mathscr{H}} \mathbf{0}(\bar{z}) \not\subseteq \partial_{\varepsilon'}^{\mathscr{D}\mathscr{H}} \Psi(\bar{z})$.

Example 5.1. (An application example: Sparsity regularizer for IOPs). In many classification problems, the data set may not be precise and thus involves uncertainty. This may be due to errors in measurement, implementation, etc. We know that overfitting in a model is a common problem which one faces; to remove this, we induce sparsity in our model. Let us consider the following interval-valued regression problem:

$$\min_{w \in \mathbb{R}^n} \tfrac{1}{2} \odot \|y-w\|_2^2 \odot \mathbf{P}, \tag{5.10}$$

where $y \in \mathbb{R}^n$ and $\mathbf{0} \prec \mathbf{P}$. Let us assume that $w^* = (w_1, w_2, \ldots, w_n)^{\top}$ be an efficient solution to the IOP (5.10), and our aim is to constrain the efficient solution w^* to be zero for some range of y. To achieve our aim, we consider the following approximated IOP:

$$\min_{w \in \mathbb{R}^n} (\Psi_1(w,y) \oplus \Psi_2(w,y)) = \min_{w \in \mathbb{R}^n} \left(\tfrac{1}{2} \odot \|y-w\|_2^2 \odot \mathbf{P} \oplus \lambda \odot \|w\|_1 \odot \mathbf{Q}\right),$$

where $\Psi_1(y,w) = \frac{1}{2} \odot \|y-w\|_2^2 \odot \mathbf{P}$, $\Psi_2(y,w) = \lambda \odot \|w\|_1 \odot \mathbf{Q}$, $\lambda > 0$, and $\mathbf{0} \preceq \mathbf{Q}$. From Theorem 5.5, we can see that if w^* is an ε-efficient solution of $(\Psi_1(w,b) \oplus \Psi_2(w,b))$, then relation (5.5) holds. In the below, we characterize w^* with the help of (5.5).

From Definition 2.12, we observe that Ψ_1 is gH-Fréchet differentiable at $w^* \in \mathbb{R}^n$, and we have

$$\begin{aligned}\Psi_{1\mathscr{F}}(w^*) = \nabla\Psi(w^*) =& (D_1\Psi(w^*), D_2\Psi(w^*), \ldots, D_n\Psi(w^*))^\top \\ =& ((w_1^* - y_1) \odot \mathbf{P}, (w_2^* - y_2) \odot \mathbf{P}, \ldots, (w_n^* - y_n) \odot \mathbf{P})^\top.\end{aligned}$$

Also, by using Example 3.1, the gH-Dini Hadamard ε-subgradient of Ψ_2 at w^* is given by

$$\partial_{\varepsilon}^{\mathscr{DH}} \Psi_2(w^*) \in \begin{cases} \lambda \odot \mathbf{Q}, & \text{if } w_i^* > 0 \\ (-1) \odot \lambda \odot \mathbf{Q}, & \text{if } w_i^* < 0 \\ \mathbf{G}_i \in I(\mathbb{R}) : (-1) \odot \lambda \odot \mathbf{Q} \oplus (-\varepsilon) \preceq \mathbf{G}_i \preceq \lambda \odot \mathbf{Q} \oplus \varepsilon, & \text{if } w_i^* = 0. \end{cases}$$

Therefore, in view of Theorem 5.5, w^* is an ε-efficient solution of $(\Psi_1 \oplus \Psi_2)$ if

$$(w_i^* - y_i) \odot \mathbf{P} \in \begin{cases} \lambda \odot \mathbf{Q}, & \text{if } w_i^* > 0 \\ (-1) \odot \lambda \odot \mathbf{Q}, & \text{if } w_i^* < 0 \\ \mathbf{G}_i \in I(\mathbb{R}) : (-1) \odot \lambda \odot \mathbf{Q} \oplus (-\varepsilon) \preceq \mathbf{G}_i \preceq \lambda \odot \mathbf{Q} \oplus \varepsilon, & \text{if } w_i^* = 0. \end{cases}$$

In view of the above relation, we can observe that, for $w^* = 0$,

$$(w_i^* - y_i) \odot \mathbf{P} \in \mathbf{G}_i \implies y_i^* \in [w_i^*, w_i^*] \ominus_{gH} (\mathbf{G}_i \oslash \mathbf{P}).$$

With the help of the above relation, we can obtain a range of y_i for which w^* is zero, and this will help in achieving w^* to be an optimal solution to the problem.

6. Conclusion

In this paper, the concept of gH-Dini Hadamard ε-subdifferentiabilty of IVF (Definition 3.1) with its several characterizations were studied. It was observed that the gH-subdifferentiability implies the gH-Dini Hadamard ε-subdifferentiability (Theorem 3.1). However, the converse need not be true (Remark 3.1). Further, a relation of gH-Dini Hadamard ε-subdifferentiability with the Fréchet derivative of an IVF (Theorem 3.4) was discussed. We proposed the notion of $\mathbf{H}_{\varepsilon}$-subgradient (Definition 4.2) of IVF with the help of sponge of a set. A variational interpretation of gH-Dini Hadamard ε-subdifferentiabilty of an IVF with sponges and gH-calm IVF (Theorem 4.2) was given. To develop this relation, we derived two important results (Theorem 4.1 and Lemma 4.2) based on sponges and gH-calm IVF. We further defined the notion of ε-efficient solution of an IOP. Thereafter, we discussed several necessary and sufficient conditions for an ε-efficient solution of an IOP (Theorems 5.1, 5.2, 5.3, and 5.4). Finally, an example to demonstrate the application of the proposed results in sparsity regularizer was given (Example 5.1).

In the future, we try to apply the developed theory on the stability and duality of non-convex IVFs via augmented Lagrangian IVF. The related Lagrangian IVF can be constructed with the help of supporting cones to the epigraph of a usual perturbed IVF. Basically, we attempt to solve the following IOP:

$$\left.\begin{array}{ll} \inf & \Psi(z) \\ \text{subject to} & g(z) \preceq 0, z \in Z, \end{array}\right\}, \tag{6.1}$$

where $\Psi : Z \to I(\mathbb{R})$ is an IVF and g is a mapping from Z to Y. Here X is a real linear space and Z is a nonempty subset of X. Let Y be a real normed space and Y^* be the dual of Y. Let C be a closed and convex pointed cone (i.e., $C \cap (-C) = \{0\}$) with its vertex at the origin of Y and C^* is the dual cone of the cone C defined as

$$C^* = \{u^* \in Y : \langle u, u^* \rangle \geq 0 \text{ for all } u \in C\}.$$

The set $D = \{x \in X : x \in Z,\ g(x) \leq 0\}$ is the feasible set of IOP (6.1). Let $B^* = \{b^* \in Y^* : \|b^*\| \leq 1\}$ be the unit ball in Y^*. The augmented Lagrangian IVF for the IOP (6.1) can be defined as

$$\mathbf{L}(z, u^*, \varepsilon) = \Psi(z) \ominus_{gH} \langle u^*, g(z) \rangle \oplus \lambda(g(z), u^*, \varepsilon)),\ z \in Z, (u^*, \varepsilon) \in V,$$

where $\lambda : Y \times Y^* \times \mathbb{R}_+ \to \mathbb{R}$ is a real-valued function such that

$$\lambda(u, u^*, \varepsilon) = \sup\{\langle u, \varepsilon b^* \rangle : b^* \in B^* \text{ and } \varepsilon b^* - y^* \in C^*\}$$

and V is the following subset of $Y^* \times \mathbb{R}_+$:

$$V = \{(y^*, \varepsilon) \in \mathscr{Y}^*\} \times \mathbb{R}_+ : \varepsilon b^* - y^* \in C^*\} \text{ for some } b^* \in B^*.$$

Then, a related dual IOP is given by

$$\sup_{(y^*, \varepsilon) \in V} \mathbf{H}(y^*, \varepsilon),$$

where $\mathbf{H} : Y^* \times \mathbb{R}_+ \to I(\mathbb{R})$ is the dual IVF and $\mathbf{H}(y^*, \varepsilon) = \inf_{z \in Z} \mathbf{L}(z, y^*, \varepsilon)$.
The supporting cones to the epigraph of a usual perturbation function can be defined as

$$C(y^*, \varepsilon) = \{(u, a) \in Y \times \mathbb{R} : -\varepsilon\|u\| + \langle u, u^* \rangle \geq a\} \text{ for all } \varepsilon > 0 \text{ and } u^* \in Y^*.$$

We shall attempt to prove that the use of supporting cones instead of supporting hyperplanes may lead to the notion of a gH-Dini Hadamard ε-subdifferential for IVFs.

Acknowledgments
The authors are truly thankful to the editor and reviewers for their constructive comments that substantially improved the quality of the paper. Debdas Ghosh acknowledges the financial support of the research grants MATRICS (MTR/2021/000696) and Core Research Grant (CRG/2022/001347) by the Science and Engineering Research Board, India.

REFERENCES

[1] F. H. Clarke, Generalized gradients and applications, Trans. Amer. Math. Soc/ 205 (1975), 247–262.
[2] A. Ioffe, Approximate subdifferentials and applications. I. the finite-dimensional theory, Trans. Amer. Math. Soc. 281 (1984), 389–416.
[3] J. Chen, Z. Liu, F.E. Lomovtsev, V. Obukhovskii, Optimal feedback control for a class of second-order evolution differential inclusions with Clarke's subdifferential, J. Nonlinear Var. Anal. 6 (2022), 551-565.
[4] J. Dutta, Generalized derivatives and nonsmooth optimization, a finite dimensional tour, Top 13 (2005), 185–279.
[5] B. S. Mordukhovich, Maximum principle in the optimal time control problem with nonsmooth constraints, J. Appl. Math. Mech. 40 (1976), 960-968.
[6] X. Zhao, Q. Sun, L. Liu, S. Y. Cho. Convergence analysis of a projected gradient method for multiobjective optimization problems. J. Nonlinear Var. Anal. 5 (2021), 929–938.
[7] M. Laghdir, I. Dali, M.B. Moustaid, A sequential formula for the ε-subdifferential of multi-composed functions via a perturbation approach with an application to location problems, Appl. Set-Valued Anal. Optim. 4 (2022), 145-166.

[8] T. Westerlund, I. Harjunkoski, J. Isaksson, Solving a production optimization problem in a paper-converting mill with milp, Comput. Chem. Eng. 22 (1998), 563–570.

[9] Z. Hao, L. Haowen, W. Pengcheng, C. Guobiao, H. Feng. Uncertainty analysis and design optimization of solid rocket motors with finocyl grain, Struct. Multidiscip. Optim. 62 (2020), 3521–3537.

[10] X. Wan, T. Hao, X. Rong, Q. Meng, The robust analysis of supply chain based on uncertainty computation: insight from open innovation, Clust. Comput. 22 (2019), 10009–10018.

[11] R.T. Rockafellar, The theory of subgradients and its applications to problems of optimization. Convex and non-convex functions, vol. 1. RJ Res Math Educ. Heldermann Verlag, Berlin, 1970.

[12] A. Ioffe, Nonsmooth analysis: differential calculus of nondifferentiable mappings, Trans. Amer. Math. Soc. 266 (1981), 1–56.

[13] A. Y. Kruger, Properties of generalized differentials, Sib. Math. J. 26 (1985), 822–832.

[14] A. D. Ioffe, Sous-différentielles approchées de fonctions numériques, C. R. Acad. Sci. Ser. 292 (1981), 675–678.

[15] A. Ioffe, Approximate subdifferentials and applications II. Mathematika 33 (1986), 111–128.

[16] A. Ioffe, Approximate subdifferentials and applications 3: the metric theory, Mathematika 36 (1989), 1–38.

[17] A. Azimov, R. Gasimov, Stability and duality of nonconvex problems via augmented Lagrangian, Cybern. Syst. Anal. 38 (2002), 412–421.

[18] A. Azimov, R. Kasimov, On weak conjugacy, weak subdifferentials and duality with zero gap in nonconvex optimization, Int. J. Appl. Math. 1 (1999), 171–192.

[19] R. Kasimbeyli, G. İnceoğlu, The properties of the weak subdifferentials, Gazi Univ. J. Sci. 23 (2010), 49–52.

[20] R. Kasimbeyli, M. Mammadov, On weak subdifferentials, directional derivatives, and radial epiderivatives for nonconvex functions, SIAM J. Optim. 20 (2009), 841–855.

[21] J. S. Treiman, Clarke's gradients and epsilon-subgradients in Banach spaces, Trans. Amer. Math. Soc. 294 (1986), 65–78.

[22] R. I. Boţ, D.-M. Nechita, On the Dini-Hadamard subdifferential of the difference of two functions, J Glob. Optim. 50 (2011), 485–502.

[23] M. Hukuhara, Integration des applications mesurables dont la valeur est un compact convexe, Funkc. Ekvacioj 10 (1967), 205–223.

[24] H.-C. Wu, The Karush–Kuhn–Tucker optimality conditions in an optimization problem with interval-valued objective function, Eur. J. Oper. Res. 176 (2007), 46–59.

[25] S. Markov, Calculus for interval functions of a real variable, Computing 22 (1979), 325–337.

[26] L. Stefanini, B. Bede, Generalized Hukuhara differentiability of interval-valued functions and interval differential equations, Nonlinear Anal. 71 (2009), 1311–1328.

[27] Y. Chalco-Cano, A. Rufián-Lizana, H. Román-Flores, and M.-D. Jiménez-Gamero, Calculus for interval-valued functions using generalized Hukuhara derivative and applications, Fuzzy Sets Syst. 219 (2013), 49–67.

[28] D. Ghosh, R. S. Chauhan, R. Mesiar, A. K. Debnath, Generalized Hukuhara Gˆateaux and Frechet derivatives of interval-valued functions and their application in optimization with interval-valued functions, Info. Sci. 510 (2020), 317–340.

[29] S. Hai, Z. Gong, The differential and subdifferential for fuzzy mappings based on the generalized difference of n-cell fuzzy-numbers, J. Comput. Anal. Appl. 24 (2018), 184–195.

[30] D. Ghosh, A. K. Debnath, R. S. Chauhan, R. Mesiar, Generalized-Hukuhara subgradient and its application in optimization problem with interval-valued functions, Sādhanā, 47, 2022.

[31] D. Ghosh, A. K. Debnath, R. S. Chauhan, O. Castillo, Generalized-Hukuhara-gradient efficient-direction method to solve optimization problems with interval-valued functions and its application in least squares problems, Int. J. Fuzzy Syst. 24 (2022), 1275–1300.

[32] G. Kumar, D. Ghosh, Ekeland's variational principle for interval-valued functions, Comput. Appl. Math. 42 (2023), 28.

[33] K. Kumar, D. Ghosh, G. Kumar, Weak sharp minima for interval-valued functions and its primal-dual characterizations using generalized Hukuhara subdifferentiability, Soft Comput. 26 (2022), 10253–10273.

[34] Anshika, D. Ghosh, Interval-valued value function and its application in interval optimization problems, Comput. Appl. Math. 41 (2022), 1–26.

[35] Anshika, D. Ghosh, R. Mesiar, H. R. Yao, R. S. Chauhan, Generalized-Hukuhara subdifferential analysis and its application in nonconvex composite optimization problems with interval-valued functions, Inf. Sci. 622 (2023), 771-793.

[36] R. E. Moore, Methods and Applications of Interval Analysis, SIAM, Philadelphia, 1979.

[37] D. Ghosh, Newton method to obtain efficient solutions of the optimization problems with interval-valued objective functions, J. Appl. Math. Comput. 53 (2017), 709–731.