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MODIFIED RELAXED CQ-ALGORITHMS FOR A SPLIT EQUALITY PROBLEM IN HILBERT SPACES

YU CAO[1], YISHUO PENG[1], LUOYI SHI[2], YASONG CHEN[1,*]

[1]*School of Mathematical Sciences, Tiangong University, Tianjin, 300387, China*
[2]*School of Software, Tiangong University, Tianjin 300387, China*

Abstract. In this paper, we propose three self-adaptive relaxed CQ-algorithms with projections onto half-spaces for solving a split equality problem. The stepsize of the algorithms is dynamically calculated without any prior information regarding operator norms. Moreover, we prove the strong convergence to the minimum-norm solution of the split equality problem. Finally, we test the validity of our results by conducting some numerical experiments and consider signal recovery problems as applications.

Keywords. Minimum-norm solution; Polyak's gradient method; Split equality problem; Self-adaptive stepsize; Signal recovery problem.

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1. Introduction

Let $A : H_1 \to H_2$ be a bounded and linear operator, where H_1 and H_2 are real Hilbert spaces, and let C and Q be nonempty, convex, and closed subsets of H_1 and H_2, respectively. The Split Feasibility Problem (SFP), which was firstly introduced by Censor and Elfving [1], is formulated as:

$$\text{finding } x \in C \text{ such that } Ax \in Q. \tag{1.1}$$

The SFP finds wide applications in medical imaging and there are various efficient algorithms for solving it; see, e.g., [2, 3, 4, 5, 6, 7] and the references therein.

Recently, Moudafi [8] proposed a Split Equality Problem (SEP): Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded and linear operators, and let $C \subset H_1$ and $Q \subset H_2$ be two nonempty, convex, and closed sets, where H_1, H_2, and H_3 are real Hilbert spaces. The SEP is formulated as:

$$\text{finding } x \in C \text{ and } y \in Q \text{ such that } Ax = By. \tag{1.2}$$

If $B = I$, then SEP (1.2) reduces to the celebrated SFP (1.1). Observe that the SEP allows for asymmetric and partial relations between x and y. This covers numerous problems such as the intensity-modulated radiation therapy. Assume that SEP (1.2) is consistent, which means that the

*Corresponding author.
E-mail address: yasongchen@126.com (Y. Chen).
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solution set is nonempty. In 2013, Moudafi [8] introduced a popular CQ-algorithm to solve the SEP, which is presented as follows:

$$\begin{cases} x_{n+1} = P_C\ (x_n - \gamma_n\ A^*(Ax_n - By_n)), \\ y_{n+1} = P_Q\ (y_n + \gamma_n\ B^*(Ax_n - By_n)), \end{cases} \tag{1.3}$$

where P_C and P_Q are the metric (nearest point) projections onto C and Q, respectively, $A^* : H_3 \to H_1$ and $B^* : H_3 \to H_2$ are the adjoint of A and B, and the stepsize $\{\gamma_n\} \subset (\varepsilon, \min\{\frac{1}{\lambda_A}, \frac{1}{\lambda_B}\} - \varepsilon)$ for a small enough $\varepsilon > 0$, where λ_A and λ_B stand for the spectral radius of A^*A and B^*B, respectively. Moudafi's CQ-algorithm can be considered as a specific instance of the gradient-projection technique employed in constrained convex minimization problems. The CQ-algorithm could be used to solve the SEP (1.2) when the P_C, P_Q, and operator norms are calculable. Note that the task of determining the metric projection to a convex and closed set is not an easy job due to the absence of an explicit formula. In order to overcome these technical difficulties in Algorithm (1.3), Moudafi [9] further proposed the following relaxed alternating CQ-algorithm

$$\begin{cases} x_{n+1} = P_{C_n}\ (x_n - \gamma_n\ A^*(Ax_n - By_n)), \\ y_{n+1} = P_{Q_n}\ (y_n + \gamma_n\ B^*(Ax_n - By_n)). \end{cases} \tag{1.4}$$

In this situation, $\{C_n\}$ and $\{Q_n\}$ are two sequences of convex and closed sets defined by:

$$C_n = \{x \in H_1 : c(x_n) + \langle \xi_n, x - x_n \rangle \leq 0\},$$

where $\xi_n \in \partial c(x_n)$ and

$$Q_n = \{y \in H_2 : q(y_n) + \langle \eta_n, y - y_n \rangle \leq 0\},$$

where $\eta_n \in \partial q(y_n)$.

Algorithm (1.4) is devised by replacing the C and Q to C_n and Q_n, respectively. Hence, P_{C_n} and P_{Q_n} as the projection onto half-spaces are easy to calculate. This technique expands the range of the algorithm in the practical applications from the viewpoint of numerical computation.

It is noted that the calculation of the stepsize γ_n in Algorithms (1.3) and (1.4) is dependent on the operator matrix norms $\|A\|$ and $\|B\|$ (or the highest eigenvalues of A^*A and B^*B). In order to execute the alternating CQ-algorithm, one must calculate (at the very least estimate) the operator norms of A and B, which is in general not an easy task in practice.

In 2012, López et al. [10] and Zhao and Shi [11] presented an efficient technique for estimating the stepsize, which does not need any prior information of the operator norms for solving SFP (1.1) and multiple-set split problems, respectively. Inspired by this, Dong, He and Zhao [12] introduced a new algorithm with a choice of the stepsize sequence γ_n and a new relaxed algorithm with a choice of the stepsize sequence γ_n in 2014 as follows:

$$(\mathrm{DHZ}-1): \begin{cases} x_{n+1} = P_C\ (x_n - \gamma_n\ A^*(Ax_n - By_n)), \\ y_{n+1} = P_Q\ (y_n + \gamma_n\ B^*(Ax_n - By_n)), \end{cases} \tag{1.5}$$

where γ_n is chosen in such a way that

$$\gamma_n = \rho_n\ \min\{\frac{\|Ax_n - By_n\|^2}{\|A^*(Ax_n - By_n)\|^2}, \frac{\|Ax_n - By_n\|^2}{\|B^*(Ax_n - By_n)\|^2}\},$$

where $0 < \rho_n < 1$,

$$(\mathrm{DHZ}-2): \begin{cases} x_{n+1} = P_{C_n}\ (x_n - \gamma_n\, A^*(Ax_n - By_n)), \\ y_{n+1} = P_{Q_n}\ (y_n + \gamma_n\, B^*(Ax_n - By_n)), \end{cases} \tag{1.6}$$

where $\{C_n\}$ and $\{Q_n\}$ are the same as the ones in Algorithm (1.4), and the stepsize is chosen as follows:

$$\gamma_n = \rho_n\ \min\{\frac{\|Ax_n - By_n\|^2}{\|A^*(Ax_n - By_n)\|^2}, \frac{\|Ax_n - By_n\|^2}{\|B^*(Ax_n - By_n)\|^2}\},$$

where $0 < \rho_n < 1$.

Note that Algorithms (1.3), (1.4), (1.5), and (1.6) only have the weak convergence. Thus Shi, Chen and Wu [13] investigated the strong convergence for SEP (1.2). In this situation, we use Γ to denote the solution set of SEP (1.2), i.e., $\Gamma = \{(x,y) \in C \times Q : Ax = By, x \in C, y \in Q\}$. Let $S = C \times Q$ in $H = H_1 \times H_2$, and define $G : H \to H_3$ by $G = [A, -B]$. Then, $G^*G : H \to H$ has the matrix form:

$$G^*G = \begin{bmatrix} A^*A & -A^*B \\ -B^*A & B^*B \end{bmatrix}.$$

Observe that (1.2) can be rephrased as finding $w = (x,y) \in S$ with $Gw = 0$, or, more generally, minimizing $\|Gw\|$ over $w \in S$. The algorithm is proposed as $w_{n+1} = P_S\ \{(1-\alpha_n)[I - \gamma\, G^*G]w_n\}$, i.e.,

$$\begin{cases} x_{n+1} = P_C\ \{(1-\alpha_n)[x_n - \gamma\, A^*(Ax_n - By_n)]\}, \\ y_{n+1} = P_Q\ \{(1-\alpha_n)[y_n + \gamma\, B^*(Ax_n - By_n)]\}, \end{cases}$$

where $\alpha_n > 0$ and $\gamma > 0$ are under simple and straightforward conditions.

In addition, Shi, Chen and Wu [13] also gave another KM-CQ-like algorithm that converges strongly to a solution of SEP (1.2) as follows

$$w_{n+1} = (1-\beta_n)w_n + \beta_n P_S\ \{(1-\alpha_n)[I - \gamma\, G^*G]w_n\},$$

i.e.,

$$\begin{cases} x_{n+1} = (1-\beta_n)x_n + \beta_n P_C\ \{(1-\alpha_n)[x_n - \gamma\, A^*(Ax_n - By_n)]\}, \\ y_{n+1} = (1-\beta_n)y_n + \beta_n P_Q\ \{(1-\alpha_n)[y_n + \gamma\, B^*(Ax_n - By_n)]\}, \end{cases}$$

where $\alpha_n > 0$ and $\beta_n > 0$ are under simple and straightforward conditions.

It is widely believed that the gradient-projection method is one of the most popular methods for tackling SFP (1.1) among the various algorithms: $x_{n+1} = x_n - \alpha_n \nabla f(x_n)$, where the stepsize $\alpha_n \geq 0$ can be selected by using diverse ways. Among various gradient-projection methods, Polyak [14] suggested the following way to select the stepsize: $\alpha_n = \lambda_n \frac{f(x_n) - f^*}{\|\nabla f(x_n)\|^2}$ with $\lambda_n \in (0,2)$. In 2018, Wang [15] initially proposed an algorithm, which is a combination of the relaxed CQ method and Polyak's gradient method with weak convergence to solve the SFP (1.1), which is described as follows: $x_{n+1} = x_n - \gamma_n[(x_n - P_{C_n}x_n) + A^*(I - P_{Q_n})Ax_n]$, where

$$\gamma_n = \frac{\rho_n(\|x_n - P_{C_n}x_n\|^2 + \|(I - P_{Q_n})Ax_n\|^2)}{2\|(x_n - P_{C_n}x_n) + A^*(I - P_{Q_n})Ax_n\|^2},$$

with $0 < \varepsilon \leq \rho_n \leq 4 - \varepsilon$ for a small enough $\varepsilon > 0$.

On the other hand, the calculation processes of some algorithms are unexpectedly tedious, which limit its practical applications. Thus scholars also proposed various techniques to accelerate their algorithms, such as inertial techniques or alternated inertial techniques.

In 2022, Yu and Wang [16] suggested a number of relaxed CQ-algorithms for SFP (1.1). The core of their algorithms lies in substituting the projections to the half-spaces C_n and Q_n with the projections to the intersection of C_n and C_{n-1} and the intersection of Q_n and Q_{n-1}. The special projections expedite the rate of convergence. Their algorithm is described as follows:

$$x_{n+1} = P_{C_n^2}(x_n - \gamma_n A^*(I - P_{Q_n^2})Ax_n),$$

where $x_0, x_1 \in H_1, C_n^2 = C_n \cap C_{n-1}, Q_n^2 = Q_n \cap Q_{n-1}$, and $\gamma_n = \frac{\rho_n \|(I-P_{Q_n^2})Ax_n\|^2}{\|A^*(I-P_{Q_n^2})Ax_n\|^2}$ with $\rho_n \in (0,2)$.

They numerically demonstrated that their algorithm is obviously accelerated by using the special half-spaces C_n^2 and Q_n^2. Based on the research of Yu and Wang [16], Ling, Tong and Shi [17] developed a number of algorithms for SFP (1.1) with the projections to the intersection of C_n and C_{n-1} and the intersection of Q_n and Q_{n-1}, respectively. Their proposed algorithms with strong convergence are presented as following:

$$(\text{LTS}-1): x_{n+1} = P_{C_n^2}[(1-\alpha_n)(x_n - \gamma_n A^*(I - P_{Q_n^2})Ax_n)],$$

where $C_n^2 = C_n \cap C_{n-1}, Q_n^2 = Q_n \cap Q_{n-1}$, and $\gamma_n = \frac{\rho_n \|(I-P_{Q_n^2})Ax_n\|^2}{\|A^*(I-P_{Q_n^2})Ax_n\|^2}$, where $\rho_n \in (0,2)$,

$$(\text{LTS}-2): x_{n+1} = (1-\alpha_n)[x_n - \gamma_n ((x_n - P_{C_n^2}x_n) + A^*(I - P_{Q_n^2})Ax_n)],$$

where $C_n^2 = C_n \cap C_{n-1}, Q_n^2 = Q_n \cap Q_{n-1}$, and

$$\gamma_n = \frac{\rho_n(\|x_n - P_{C_n^2}x_n\|^2 + \|(I - P_{Q_n^2})Ax_n\|^2)}{\|(x_n - P_{C_n^2}x_n) + A^*(I - P_{Q_n^2})Ax_n\|^2}$$

with $\rho_n \in (0,2)$.

In this paper, based on the algorithms in [16], we devise three strong convergence schemes for solving SEP (1.2). The organization of this paper is as follows: In Section 2, we provide some definitions and preliminary results for the convergence analysis of our iterative algorithms. In Section 3, we introduce our first iterative algorithm and discuss its strong convergence analysis. In Section 4, we introduce the second algorithm and present its the analysis of the strong convergence. In Section 5, we present the last algorithm and demonstrate its strong convergence. In Section 6, we provide some numerical experiments in signal recovery problems to demonstrate the effectiveness of the suggested iterative algorithms. In Section 7, the last section, we give a concluding conclusion.

2. PRELIMINARIES

In this paper, we use the following notations:

- M stands for a convex, closed, and nonempty subset of a real Hilbert space of H;
- I stands for the identity operator on H;
- $\to$ stands for the strong convergence and $\rightharpoonup$ stands for the weak convergence;
- $\omega_w(w_n) = \{w \in H : \exists \{w_{n_i}\} \subset \{w_n\} \textit{ such that } w_{n_i} \rightharpoonup w\}$ stands for the weak ω-limit set of $\{w_n\}_{n\in N}$.

Recall that an operator $D : M \to H$ is called: (1) nonexpansive on if $\|Dx - Dy\| \le \|x - y\|$ for all $x, y \in M$; firmly nonexpansive if $\|Dx - Dy\|^2 \le \|x - y\|^2 - \|(I - D)x - (I - D)y\|^2$ for all $x, y \in M$, which is equivlent to $\|Dx - Dy\|^2 \le \langle Dx - Dy, x - y\rangle$ for all $x, y \in M$; (3) κ-inverse strongly monotone (κ-ism) if $\langle Dx-Dy, x-y\rangle \geq \kappa\|Dx-Dy\|^2$, where k is some positive constant, for all $x,y \in M$.

The metric projection, which plays an important role in this paper, is defined as follows: for each $x \in H$, there is a unique nearest point $P_M x \in M$ such that $\|x-P_M x\| = \min\{\|x-y\| : y \in M\}$. Notice that both P_M and $I-P_M$ are 1-ism, nonexpansive, and firmly nonexpansive.

Recall that the subdifferential of a convex function $f : H \to R$ at $x \in H$ is defined as:

$$\partial f(x) = \{\phi \in H : f(y) - f(x) \geq \langle \phi, y-x\rangle, \forall y \in H\}.$$

Recall that $f : H \to R$ is said to be weakly lower semi-continuous (w-lsc) at x if $x_n \rightharpoonup x$ implies $f(x) \leq \liminf_{n\to\infty} f(x_n)$.

Finally, we present a crucial lemma for our convergence analysis in the following sections.

Lemma 2.1. [18] *Let $\{h_n\}$ be a nonnegative sequence with*

$$\begin{cases} h_{n+1} \leq h_n - \xi_n + \tau_n, \\ h_{n+1} \leq (1-\delta_n)h_n + \delta_n\theta_n, \end{cases}$$

where $\{\delta_n\} \subset (0,1)$ with $\sum_{n=1}^{\infty}\delta_n = \infty$, $\{\tau_n\}$ is a real sequence with $\lim_{n\to\infty}\tau_n = 0$, $\{\theta_n\}$ is a real sequence, and $\{\xi_n\}$ is nonnegative real sequence such that $\lim_{i\to\infty}\xi_{n_i} = 0$ yields that $\limsup_{i\to\infty}\theta_{n_i} \leq 0$ for every subsequence $\{n_i\}$ of $\{n\}$. Then $\lim_{n\to\infty} h_n = 0$.

3. The First Algorithm

In this section, we still use the notations Γ, G, S, and w, which are the same as in [13]. The sets S_n, C_n, and Q_n at points w_n, x_n, and y_n are defined by:

$$\begin{cases} S_n = \{w \in H : s(w_n) \leq \langle \eta_n, w_n - w\rangle\}, \\ C_n = \{x \in H_1 : c(x_n) \leq \langle \eta_n', x_n - x\rangle\}, \\ Q_n = \{y \in H_2 : q(y_n) \leq \langle \eta_n'', y_n - y\rangle\}, \end{cases}$$

where $\eta_n \in \partial s(w_n), \eta_n' \in \partial c(x_n)$, and $\eta_n'' \in \partial q(y_n)$. It is clear to see that $C \subset C_n$, $Q \subset Q_n$, and $S \subset S_n$ for all $n \geq 1$, and S_n is a half-space and therefore the corresponding projection is easy to calculate.

Next, we present two assumptions.

Assumption 1:

(A1): The solution set of SEP (1.2), $\Gamma = \{\widehat{w} \in S : G\widehat{w} = 0\}$, is convex, closed, and nonempty.

(A2): The function $s : H \to R$ are subdifferentiable, weakly lower semi-continuous, and convex on H.

Assumption 2:

$\{\rho_n\}_{n=1}^{\infty}$ is a positive sequence which satisfies the following condition:

(A3): $\{\rho_n\} \subset (0,2)$ with $\inf_{n\in N}\rho_n(2-\rho_n) > 0$.

We are now in a position to present first modified relaxed algorithm.

Algorithm 1. *Initialization:* Choose $\{\alpha_n\}_{n=1}^{\infty}$ satisfying the conditions below and $\{\rho_n\}_{n=1}^{\infty}$ satisfying Assumption 2 (A3), respectively. Select initials $w_0, w_1 \in H$ and set $n := 1$.

Iterative Step: Given the iterate w_n, construct w_{n+1} by $w_{n+1} = P_{S_n^2}\{(1-\alpha_n)[I-\gamma_n\, G^*G]w_n\}$, where S_n^2, γ_n, and $\{\alpha_n\}_{n=1}^{\infty}$ are defined as follows respectively $S_n^2 = S_n \cap S_{n-1}$, $\gamma_n = \frac{\rho_n\|Gw_n\|^2}{\|G^*Gw_n\|^2}$, and (i) $\lim_{n\to\infty}\alpha_n = 0$; (ii) $\sum_{n=0}^{\infty}\alpha_n = \infty$; (iii) $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$ or $\lim_{n\to\infty}\frac{|\alpha_{n+1}-\alpha_n|}{\alpha_n} = 0$.

Stopping Criterion: If $w_{n+1} = w_n$ and $\|G^*Gw_n\| = 0$, then terminate. Otherwise, set $n := n+1$ and proceed to Iterative Step.

That is,

$$\begin{cases} x_{n+1} = P_{C_n^2}\{(1-\alpha_n)[x_n - \gamma_n\, A^*(Ax_n - By_n)]\}, \\ y_{n+1} = P_{Q_n^2}\{(1-\alpha_n)[y_n + \gamma_n\, B^*(Ax_n - By_n)]\}, \end{cases}$$

where C_n^2, Q_n^2, and γ_n are respectively defined as $C_n^2 = C_n \cap C_{n-1}$ and $Q_n^2 = Q_n \cap Q_{n-1}$, and

$$\gamma_n = \min\{\frac{\rho_n\|Ax_n - By_n\|^2}{\|A^*(Ax_n - By_n)\|^2}, \frac{\rho_n\|Ax_n - By_n\|^2}{\|B^*(Ax_n - By_n)\|^2}\}.$$

Lemma 3.1. *If $w_{n+1} = w_n$ and $\|G^*Gw_n\| = 0$ for some $n \geq 0$, then $w_n \in \Gamma$.*

Proof. Let $w_{n+1} = w_n$, then it follows that $w_n = P_{S_n^2}\{(1-\alpha_n)[I-\gamma_n\, G^*G]w_n\}$, which means that $w_n \in S_n^2$. According to $\|G^*Gw_n\| = 0$ and the fact that G is bounded, we conclude that $\|Gw_n\| = 0$. Thus, $w_n \in \Gamma$. □

Theorem 3.1. *The sequence $\{w_n\}$ generated by Algorithm 1 converges strongly to $w^* \in \Gamma$, where $w^* = P_\Gamma(0)$.*

Proof. Letting $d_n = (1-\alpha_n)[I-\gamma_n G^*G]w_n$, one has $w_{n+1} = P_{S_n^2}d_n$. Next, we divide the proof into four steps.

Step 1. Let $\hat{w} \in \Gamma$. Since $S \subset S_n^2$, then $\hat{w} = P_S\hat{w} = P_{S_n^2}\hat{w}$. In view of the fact that $P_{S_n^2}$ is firmly nonexpansive, we have

$$\begin{aligned} &\|w_{n+1} - \hat{w}\|^2 \\ &\leq \|d_n - \hat{w}\|^2 - \|(I - P_{S_n^2})d_n\|^2 \\ &= \|(1-\alpha_n)(w_n - \gamma_n G^*Gw_n - \hat{w}) + \alpha_n(-\hat{w})\|^2 - \|(I-P_{S_n^2})d_n\|^2 \\ &= \alpha_n\|\hat{w}\|^2 + (1-\alpha_n)\|w_n - \gamma_n G^*Gw_n - \hat{w}\|^2 - \alpha_n(1-\alpha_n)\|w_n - \gamma_n G^*Gw_n\|^2 - \|(I-P_{S_n^2})d_n\|^2 \\ &\leq \alpha_n\|\hat{w}\|^2 + (1-\alpha_n)\|w_n - \gamma_n G^*Gw_n - \hat{w}\|^2 - \|(I-P_{S_n^2})d_n\|^2. \end{aligned} \tag{3.1}$$

Moreover, we have

$$\begin{aligned} \|w_n - \gamma_n G^*Gw_n - \hat{w}\|^2 &= \|w_n - \hat{w}\|^2 + \gamma_n^2\|G^*Gw_n\|^2 - 2\gamma_n\langle w_n - \hat{w}, G^*Gw_n\rangle \\ &= \|w_n - \hat{w}\|^2 + \gamma_n^2\|G^*Gw_n\|^2 - 2\gamma_n\|Gw_n\|^2 \\ &= \|w_n - \hat{w}\|^2 - \rho_n(2-\rho_n)\frac{\|Gw_n\|^4}{\|G^*Gw_n\|^2}. \end{aligned} \tag{3.2}$$

Combining (3.1) and (3.2), we see that

$$\begin{aligned} &\|w_{n+1} - \hat{w}\|^2 \\ &\leq \alpha_n\|\hat{w}\|^2 + (1-\alpha_n)\|w_n - \hat{w}\|^2 - \rho_n(2-\rho_n)(1-\alpha_n)\frac{\|Gw_n\|^4}{\|G^*Gw_n\|^2} - \|(I-P_{S_n^2})d_n\|^2. \end{aligned} \tag{3.3}$$

Step 2. According to (3.3) and Assumption 2 (A3), we have

$$\begin{aligned}&\|w_{n+1}-\hat{w}\|^2\\&\leq \alpha_n\|\hat{w}\|^2+(1-\alpha_n)\|w_n-\hat{w}\|^2-\rho_n(2-\rho_n)(1-\alpha_n)\frac{\|Gw_n\|^4}{\|G^*Gw_n\|^2}-\|(I-P_{S_n^2})d_n\|^2\\&\leq \alpha_n\|\hat{w}\|^2+(1-\alpha_n)\|w_n-\hat{w}\|^2.\end{aligned}$$

Thus $\|w_{n+1}-\hat{w}\|^2\leq\max\{\|\hat{w}\|^2,\|w_0-\hat{w}\|^2\}$. Hence, $\{\|w_n-\hat{w}\|\}$ is bounded. As a result, $\{w_n\}$ is bounded too.

Step 3. From Assumption 2 (A3) and (3.2), we see that

$$\|w_n-\gamma_nG^*Gw_n-\hat{w}\|^2=\|w_n-\hat{w}\|^2-\rho_n(2-\rho_n)\frac{\|Gw_n\|^4}{\|G^*Gw_n\|^2}\leq\|w_n-\hat{w}\|^2.$$

Since $P_{S_n^2}$ is firmly nonexpansive, we have

$$\begin{aligned}\|w_{n+1}-\hat{w}\|^2&\leq\|(1-\alpha_n)(w_n-\gamma_nG^*Gw_n-\hat{w})+\alpha_n(-\hat{w})\|^2\\&=\alpha_n^2\|\hat{w}\|^2+(1-\alpha_n)^2\|w_n-\gamma_nG^*Gw_n-\hat{w}\|^2\\&\quad+2\alpha_n(1-\alpha_n)\langle w_n-\hat{w},-\hat{w}\rangle+2\alpha_n(1-\alpha_n)\gamma_n\langle G^*Gw_n,\hat{w}\rangle\\&\leq\alpha_n^2\|\hat{w}\|^2+(1-\alpha_n)\|w_n-\gamma_nG^*Gw_n-\hat{w}\|^2\\&\quad+2\alpha_n(1-\alpha_n)\langle w_n-\hat{w},-\hat{w}\rangle+2\alpha_n(1-\alpha_n)\gamma_n\|G\|\|\hat{w}\|\|Gw_n\|\\&\leq(1-\alpha_n)\|w_n-\hat{w}\|^2+\alpha_n[\alpha_n\|\hat{w}\|^2+2(1-\alpha_n)\langle w_n-\hat{w},-\hat{w}\rangle\\&\quad+2(1-\alpha_n)\gamma_n\|G\|\|\hat{w}\|\|Gw_n\|].\end{aligned}\tag{3.4}$$

Step 4. We prove that $\{w_n\}$ converges strongly to $w^*=P_\Gamma(0)$. Without loss of generality, we assume that there exists $\varepsilon>0$ such that $\rho_n(2-\rho_n)(1-\alpha_n)\geq\varepsilon$. It follows from (3.3) that

$$\begin{aligned}&\|w_{n+1}-w^*\|^2\\&\leq\alpha_n\|w^*\|^2+(1-\alpha_n)\|w_n-w^*\|^2-\rho_n(2-\rho_n)(1-\alpha_n)\frac{\|Gw_n\|^4}{\|G^*Gw_n\|^2}-\|(I-P_{S_n^2})d_n\|^2\\&\leq\alpha_n\|w^*\|^2+(1-\alpha_n)\|w_n-w^*\|^2-\frac{\varepsilon\|Gw_n\|^4}{\|G^*Gw_n\|^2}-\|(I-P_{S_n^2})d_n\|^2\\&\leq\alpha_n\|w^*\|^2+\|w_n-w^*\|^2-\frac{\varepsilon\|Gw_n\|^4}{\|G^*Gw_n\|^2}-\|(I-P_{S_n^2})d_n\|^2.\end{aligned}\tag{3.5}$$

In view of (3.4) and (3.5), we have

$$\begin{cases}\|w_{n+1}-w^*\|^2\leq(1-\alpha_n)\|w_n-w^*\|^2+\alpha_n\theta_n,\\\|w_{n+1}-w^*\|^2\leq\|w_n-w^*\|^2-\xi_n+\alpha_n\|w^*\|^2,\end{cases}$$

where $\theta_n=\alpha_n\|w^*\|^2+2(1-\alpha_n)\langle w_n-w^*,-w^*\rangle+2(1-\alpha_n)\gamma_n\|G\|\|w^*\|\|Gw_n\|$ and

$$\xi_n=\frac{\varepsilon\|Gw_n\|^4}{\|G^*Gw_n\|^2}+\|(I-P_{S_n^2})d_n\|^2$$

with $\{\alpha_n\}\subset(0,1),\lim_{n\to\infty}\alpha_n=0$, and $\sum_{n=0}^{\infty}\alpha_n=\infty$. Let $h_n=\|w_n-w^*\|^2$. To utilize Lemma (2.1), it suffices to confirm that, for every subsequence $\{n_i\}\subset\{n\}$, $\lim_{i\to\infty}\xi_{n_i}=0\Rightarrow\limsup_{i\to\infty}\theta_{n_i}\leq0$.

Suppose that $\lim_{i\to\infty}\xi_{n_i}=0$. Thus

$$\lim_{i\to\infty}\left(\frac{\varepsilon\|Gw_{n_i}\|^4}{\|G^*Gw_{n_i}\|^2}+\|(I-P_{S^2_{n_i}})d_{n_i}\|^2\right)=0,$$

which implies that $\lim_{i\to\infty}\frac{\varepsilon\|Gw_{n_i}\|^4}{\|G^*Gw_{n_i}\|^2}=0$ and $\lim_{i\to\infty}\|(I-P_{S^2_{n_i}})d_{n_i}\|=0$. Since G is a bounded linear operator and $\{w_n\}$ is a bounded sequence, we obtain that $\lim_{i\to\infty}\|Gw_{n_i}\|=\lim_{i\to\infty}\|w_{n_i}\|=0$.

Next, we show that $\omega_w(w_{n_i})\in\Gamma$. Since $\{w_{n_i}\}$ is bounded, then $\omega_w(w_{n_i})\neq\emptyset$. Let $\bar{w}\in\omega_w(w_{n_i})$. Then there exists a subsequence $\{w_{n_{i_j}}\}$ of $\{w_{n_i}\}$ such that $w_{n_{i_j}}\rightharpoonup\bar{w}$. Without loss of generality, we may assume that $w_{n_i}\rightharpoonup\bar{w}$. Since $P_{S^2_{n_i}}(w_{n_i})\in S^2_{n_i}\subset S_{n_i}$, we have $s(w_{n_i})\leq\langle\eta_{n_i},w_{n_i}-P_{S^2_{n_i}}w_{n_i}\rangle$, where $\eta_{n_i}\in\partial s(w_{n_i})$. From the hypothesis that η_{n_i} is bounded and the fact that $I-P_{S^2_{n_i}}$ is nonexpansive, we see that

$$s(w_{n_i})\leq\langle\eta_{n_i},w_{n_i}-P_{S^2_{n_i}}w_{n_i}\rangle\leq\|\eta_{n_i}\|\|I-P_{S^2_{n_i}}\|\|w_{n_i}\|\to 0$$

as $i\to\infty$. Since s is weakly lower semi-continuous, it follows that $\bar{w}\in S$ and

$$0\leq\|G\bar{w}\|^2=\langle G\bar{w},G\bar{w}\rangle=\lim_{i\to\infty}\langle Gw_{n_i},G\bar{w}\rangle\leq\lim_{i\to\infty}\|Gw_{n_i}\|\|G\bar{w}\|\to 0$$

as $i\to\infty$. This implies $G\bar{w}=0$, that is, $\bar{w}\in\Gamma$, i.e., $\omega_w(w_{n_i})\in\Gamma$. Note that

$$\begin{aligned}&\limsup_{i\to\infty}\theta_{n_i}\\&=\limsup_{i\to\infty}[\alpha_{n_i}\|w^*\|^2+2(1-\alpha_{n_i})\langle w_{n_i}-w^*,-w^*\rangle+2(1-\alpha_{n_i})\gamma_{n_i}\|G\|\|w^*\|\|Gw_{n_i}\|]\\&=2\limsup_{i\to\infty}\langle w_{n_i}-w^*,-w^*\rangle=2\max_{\bar{w}\in\omega_w(w_{n_i})}\langle\bar{w}-w^*,-w^*\rangle\leq 0.\end{aligned}$$

From Lemma 2.1, we can infer that $\{w_n\}$ converges strongly to $w^*=P_\Gamma(0)$. The proof is complete. □

4. The Second Algorithm

In this section, we propose our second modified relaxed algorithm for solving SEP (1.2).

Algorithm 2. *Initialization:* Choose $\{\rho_n\}_{n=1}^\infty$ satisfying Assumption 2 (A3) and $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ satisfying the conditions: (i) $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=0}^\infty\alpha_n=\infty$; (ii) $\sum_{n=0}^\infty|\alpha_{n+1}-\alpha_n|=0$; and (iii) $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$. Select initials $w_0,w_1\in H$ and set $n:=1$.

Iterative Step: Given the iterate w_n, construct w_{n+1} by $w_{n+1}=(1-\beta_n)w_n+\beta_nP_{S^2_n}\{(1-\alpha_n)[I-\gamma_n G^*G]w_n\}$, where S^2_n and γ_n are the same as Algorithm 1, respectively.

Stopping Criterion: If $w_{n+1}=w_n$ and $\|G^*Gw_n\|=0$, then terminate. Otherwise, set $n:=n+1$ and proceed to Iterative Step.

That is,

$$\begin{cases}x_{n+1}=(1-\beta_n)x_n+\beta_nP_{C^2_n}\{(1-\alpha_n)[x_n-\gamma_n A^*(Ax_n-By_n)]\},\\y_{n+1}=(1-\beta_n)y_n+\beta_nP_{Q^2_n}\{(1-\alpha_n)[y_n+\gamma_n B^*(Ax_n-By_n)]\},\end{cases}$$

where C^2_n, Q^2_n, and γ_n are respectively defined by $C^2_n=C_n\cap C_{n-1}$, $Q^2_n=Q_n\cap Q_{n-1}$, and

$$\gamma_n=\min\left\{\frac{\rho_n\|Ax_n-By_n\|^2}{\|A^*(Ax_n-By_n)\|^2},\frac{\rho_n\|Ax_n-By_n\|^2}{\|B^*(Ax_n-By_n)\|^2}\right\}.$$

Lemma 4.1. *If $w_{n+1}=w_n$ and $\|G^*Gw_n\|=0$ for some $n\geq 0$, then $w_n\in\Gamma$.*

Since the proof of this lemma is similar to Lemma 3.1, we omit the proof here.

Theorem 4.1. *The sequence* $\{w_n\}$ *generated by Algorithm 2 converges strongly to* $w^* \in \Gamma$, *where* $w^* = P_\Gamma(0)$.

Proof. Letting $d_n = (1-\alpha_n)[I-\gamma_n G^*G]w_n$, one sees that $w_{n+1} = (1-\beta_n)w_n + \beta_n P_{S_n^2} d_n$. Let $\hat{w} \in \Gamma$. From $S \subset S_n^2$, one has $\hat{w} = P_S \hat{w} = P_{S_n^2}\hat{w}$. Since $P_{S_n^2}$ is firmly nonexpansive, one has

$$\begin{aligned}\|w_{n+1}-\hat{w}\|^2 &= \|(1-\beta_n)(w_n-\hat{w}) + \beta_n(P_{S_n^2}d_n - P_{S_n^2}\hat{w})\|^2 \\ &= (1-\beta_n)\|w_n-\hat{w}\|^2 + \beta_n\|P_{S_n^2}d_n - P_{S_n^2}\hat{w}\|^2 - \beta_n(1-\beta_n)\|w_n - P_{S_n^2}d_n\|^2 \\ &\le (1-\beta_n)\|w_n-\hat{w}\|^2 + \beta_n\|P_{S_n^2}d_n - P_{S_n^2}\hat{w}\|^2.\end{aligned} \tag{4.1}$$

Moreover,

$$\begin{aligned}\|P_{S_n^2}d_n - P_{S_n^2}\hat{w}\|^2 &\le \|d_n-\hat{w}\|^2 - \|(I-P_{S_n^2})d_n\|^2 \\ &= \alpha_n\|\hat{w}\|^2 + (1-\alpha_n)\|w_n - \gamma_n G^*Gw_n - \hat{w}\|^2 - \alpha_n(1-\alpha_n)\|w_n - \gamma_n G^*Gw_n\|^2 \\ &\quad - \|(I-P_{S_n^2})d_n\|^2 \\ &\le \alpha_n\|\hat{w}\|^2 + (1-\alpha_n)\|w_n - \gamma_n G^*Gw_n - \hat{w}\|^2 - \|(I-P_{S_n^2})d_n\|^2.\end{aligned} \tag{4.2}$$

It follows from (3.2), (4.1), and (4.2) that

$$\begin{aligned}&\|w_{n+1}-\hat{w}\|^2 \\ &\le (1-\beta_n)\|w_n-\hat{w}\|^2 + \beta_n\|P_{S_n^2}d_n - P_{S_n^2}\hat{w}\|^2 \\ &\le (1-\beta_n)\|w_n-\hat{w}\|^2 + \beta_n[\alpha_n\|\hat{w}\|^2 + (1-\alpha_n)\|w_n - \gamma_n G^*Gw_n - \hat{w}\|^2 - \|(I-P_{S_n^2})d_n\|^2] \\ &\le \alpha_n\beta_n\|\hat{w}\|^2 + (1-\alpha_n\beta_n)\|w_n-\hat{w}\|^2 - (1-\alpha_n)\beta_n\rho_n(2-\rho_n)\frac{\|Gw_n\|^4}{\|G^*Gw_n\|^2} - \beta_n\|(I-P_{S_n^2})d_n\|^2.\end{aligned} \tag{4.3}$$

From Assumption 2 (A3) and (4.3), we have

$$\begin{aligned}\|w_{n+1}-\hat{w}\|^2 &\le \alpha_n\beta_n\|\hat{w}\|^2 + (1-\alpha_n\beta_n)\|w_n-\hat{w}\|^2 - (1-\alpha_n)\beta_n\rho_n(2-\rho_n)\frac{\|Gw_n\|^4}{\|G^*Gw_n\|^2} \\ &\quad - \beta_n\|(I-P_{S_n^2})d_n\|^2 \\ &\le \alpha_n\beta_n\|\hat{w}\|^2 + (1-\alpha_n\beta_n)\|w_n-\hat{w}\|^2.\end{aligned}$$

Thus $\|w_{n+1}-\hat{w}\|^2 \le \max\{\|\hat{w}\|^2, \|w_0-\hat{w}\|^2\}$. This proves that $\{\|w_n-\hat{w}\|\}$ is bounded. As a result, $\{w_n\}$ is bounded too. In view of (3.4) and (4.1), we conclude that

$$\begin{aligned}\|w_{n+1}-\hat{w}\|^2 &\le (1-\beta_n)\|w_n-\hat{w}\|^2 + \beta_n\|P_{S_n^2}d_n - P_{S_n^2}\hat{w}\|^2 \\ &\le (1-\beta_n)\|w_n-\hat{w}\|^2 + \beta_n\|d_n-\hat{w}\|^2 \\ &\le (1-\alpha_n\beta_n)\|w_n-\hat{w}\|^2 + \alpha_n\beta_n[\alpha_n\|\hat{w}\|^2 + 2(1-\alpha_n)\langle w_n-\hat{w}, -\hat{w}\rangle \\ &\quad + 2(1-\alpha_n)\gamma_n\|G\|\|\hat{w}\|\|Gw_n\|].\end{aligned} \tag{4.4}$$

Finally, we prove that $\{w_n\}$ converges strongly to $w^* = P_\Gamma(0)$ (i.e., the minimum norm element of Γ). Without loss of generality, we assume that there exists $\varepsilon' > 0$ such that $(1-\alpha_n)\beta_n\rho_n(2-$

$\rho_n) \geq \varepsilon'$. Thus we find from (4.3) that

$$\begin{aligned}
&\|w_{n+1}-w^*\|^2\\
&\leq \alpha_n\beta_n\|w^*\|^2+(1-\alpha_n\beta_n)\|w_n-w^*\|^2-(1-\alpha_n)\beta_n\rho_n(2-\rho_n)\frac{\|Gw_n\|^4}{\|G^*Gw_n\|^2}\\
&\quad -\beta_n\|(I-P_{S_n^2})d_n\|^2\\
&\leq \alpha_n\beta_n\|w^*\|^2+(1-\alpha_n\beta_n)\|w_n-w^*\|^2-\frac{\varepsilon'\|Gw_n\|^4}{\|G^*Gw_n\|^2}-\beta_n\|(I-P_{S_n^2})d_n\|^2\\
&\leq \alpha_n\beta_n\|w^*\|^2+\|w_n-w^*\|^2-\frac{\varepsilon'\|Gw_n\|^4}{\|G^*Gw_n\|^2}-\beta_n\|(I-P_{S_n^2})d_n\|^2.
\end{aligned} \tag{4.5}$$

From (4.4) and (4.5), we have the two following inequalities:

$$\begin{cases}\|w_{n+1}-w^*\|^2\leq(1-\delta_n)\|w_n-w^*\|^2+\delta_n\theta_n,\\ \|w_{n+1}-w^*\|^2\leq\|w_n-w^*\|^2-\xi_n+\delta_n\|w^*\|^2,\end{cases}$$

where $\theta_n=\alpha_n\|w^*\|^2+2(1-\alpha_n)\langle w_n-w^*,-w^*\rangle+2(1-\alpha_n)\gamma_n\|G\|\|w^*\|\|Gw_n\|$, $\delta_n=\alpha_n\beta_n$, and

$$\xi_n=\frac{\varepsilon'\|Gw_n\|^4}{\|G^*Gw_n\|^2}+\beta_n\|(I-P_{S_n^2})d_n\|^2,$$

with $\{\delta_n\}\subset(0,1),\lim_{n\to\infty}\delta_n=0$, and $\sum_{n=0}^{\infty}\delta_n=\infty$. Let $h_n=\|w_n-w^*\|^2$. To utilize Lemma 2.1, it suffices to confirm that, for every subsequence $\{n_i\}\subset\{n\}$, $\lim_{i\to\infty}\xi_{n_i}=0\Rightarrow\limsup_{i\to\infty}\theta_{n_i}\leq 0$. If $\lim_{i\to\infty}\xi_{n_i}=0$, then $\lim_{i\to\infty}(\frac{\varepsilon'\|Gw_{n_i}\|^4}{\|G^*Gw_{n_i}\|^2}+\beta_n\|(I-P_{S_{n_i}^2})d_{n_i}\|^2)=0$, which implies that

$$\lim_{i\to\infty}\frac{\varepsilon'\|Gw_{n_i}\|^4}{\|G^*Gw_{n_i}\|^2}=\lim_{i\to\infty}\|(I-P_{S_{n_i}^2})d_{n_i}\|=0.$$

Since G is a bounded linear operator and $\{w_n\}$ is a bounded sequence, we obtain that $\lim_{i\to\infty}\|Gw_{n_i}\|$ $=\lim_{i\to\infty}\|w_{n_i}\|=0$.

Next, we show that $\omega_w(w_{n_i})\in\Gamma$. Since $\{w_{n_i}\}$ is bounded, then $\omega_w(w_{n_i})\neq\emptyset$. Let $\bar{w}\in\omega_w(w_{n_i})$. Then there exists a subsequence $\{w_{n_{i_j}}\}$ of $\{w_{n_i}\}$ such that $w_{n_{i_j}}\rightharpoonup\bar{w}$. Without loss of generality, we can assume that $w_{n_i}\rightharpoonup\bar{w}$. Since $P_{S_{n_i}^2}(w_{n_i})\in S_{n_i}^2\subset S_{n_i}$, we have $s(w_{n_i})\leq\langle\eta_{n_i},w_{n_i}-P_{S_{n_i}^2}w_{n_i}\rangle$, where $\eta_{n_i}\in\partial s(w_{n_i})$. Since η_{n_i} is bounded, we obtain from the property of $I-P_{S_{n_i}^2}$ that

$$s(w_{n_i})\leq\langle\eta_{n_i},(I-P_{S_{n_i}^2})w_{n_i}\rangle\leq\|\eta_{n_i}\|\|I-P_{S_{n_i}^2}\|\|w_{n_i}\|\to 0$$

as $i\to\infty$. Since s is w-lsc, it follows that $\bar{w}\in S$. Hence,

$$0\leq\|G\bar{w}\|^2=\langle\bar{w},G^*G\bar{w}\rangle=\lim_{i\to\infty}\langle w_{n_i},G^*G\bar{w}\rangle=\lim_{i\to\infty}\langle Gw_{n_i},G\bar{w}\rangle\leq\lim_{i\to\infty}\|Gw_{n_i}\|\|G\bar{w}\|\to 0$$

as $i\to\infty$. Thus $G\bar{w}=0$, which implies that $\bar{w}\in\Gamma$, i.e., $\omega_w(w_{n_i})\in\Gamma$. Observe that

$$\begin{aligned}
\limsup_{i\to\infty}\theta_{n_i}&=\limsup_{i\to\infty}[\alpha_{n_i}\|w^*\|^2+2(1-\alpha_{n_i})\langle w_{n_i}-w^*,-w^*\rangle\\
&\quad+2(1-\alpha_{n_i})\gamma_{n_i}\|G\|\|w^*\|\|Gw_{n_i}\|]\\
&=2\limsup_{i\to\infty}\langle w_{n_i}-w^*,-w^*\rangle\\
&=2\max_{\bar{w}\in\omega_w(w_{n_i})}\langle\bar{w}-w^*,-w^*\rangle\leq 0.
\end{aligned}$$

From Lemma 2.1, we can infer that $\{w_n\}$ converges strongly to $w^* = P_\Gamma(0)$. The proof is complete. □

5. The Third Algorithm

In this section, we propose the last modified relaxed algorithm for solving SEP (1.2) .

Algorithm 3. *Initialization:* Choose sequence $\{\alpha_n\}_{n=1}^\infty$ satisfying (i) $\lim_{n\to\infty}\alpha_n = 0$; (ii) $\sum_{n=0}^\infty \alpha_n = \infty$, and sequence $\{\rho_n\}_{n=1}^\infty$ satisfying Assumption 2 (A3), respectively. Select initial $w_0, w_1 \in H$ and set $n := 1$.

Iterative Step: Given the iterate w_n, construct w_{n+1} by

$$w_{n+1} = (1-\alpha_n)[w_n - \gamma_n(w_n - P_{S_n^2}w_n + G^*Gw_n)],$$

where S_n^2 and γ_n are defined as follows respectively $S_n^2 = S_n \cap S_{n-1}$ and

$$\gamma_n = \frac{\rho_n[\|w_n - P_{S_n^2}w_n\|^2 + \|Gw_n\|^2]}{\|w_n - P_{S_n^2}w_n + G^*Gw_n\|^2}.$$

Stopping Criterion: If $\|w_n - P_{S_n^2}w_n + G^*Gw_n\| = 0$, then terminate. Otherwise, set $n := n+1$ and proceed to Iterative Step.

That is,

$$\begin{cases} x_{n+1} = (1-\alpha_n)\{x_n - \gamma_n[x_n - P_{C_n^2}x_n + A^*(Ax_n - By_n)]\}, \\ y_{n+1} = (1-\alpha_n)\{y_n - \gamma_n[y_n - P_{Q_n^2}y_n - B^*(Ax_n - By_n)]\}, \end{cases}$$

where C_n^2, Q_n^2, and γ_n are respectively defined by $C_n^2 = C_n \cap C_{n-1}$, $Q_n^2 = Q_n \cap Q_{n-1}$, and

$$\gamma_n = \min\{\frac{\rho_n[\|x_n - P_{C_n^2}x_n\|^2 + \|A^*(Ax_n - By_n)\|^2]}{\|x_n - P_{C_n^2}x_n + A^*(Ax_n - By_n)\|^2}, \frac{\rho_n[\|y_n - P_{Q_n^2}y_n\|^2 + \|B^*(Ax_n - By_n)\|^2]}{\|y_n - P_{Q_n^2}y_n - B^*(Ax_n - By_n)\|^2}\}.$$

Lemma 5.1. *If $\|w_n - P_{S_n^2}w_n + G^*Gw_n\| = 0$ for some $n \geq 0$, then $w_n \in \Gamma$.*

Proof. Let $\check{w} \in \Gamma$. Since $\|w_n - P_{S_n^2}w_n + G^*Gw_n\| = 0$, then

$$\begin{aligned} 0 &= \langle w_n - P_{S_n^2}w_n + G^*Gw_n, w_n - \check{w}\rangle \\ &= \langle w_n - P_{S_n^2}w_n, w_n - \check{w}\rangle + \langle Gw_n, G(w_n - \check{w})\rangle \\ &\geq \|w_n - P_{S_n^2}w_n\|^2 + \|Gw_n\|^2. \end{aligned} \tag{5.1}$$

Thus $w_n = P_{S_n^2}w_n$ and $Gw_n = 0$. That is, $w_n \in \Gamma$. Hence, Algorithm 3 is well defined. □

Theorem 5.1. *The sequence $\{w_n\}$ generated by Algorithm 3 converges strongly to $w^* \in \Gamma$, where $w^* = P_\Gamma(0)$.*

Proof. Let $e_n = w_n - \gamma_n(w_n - P_{S_n^2}w_n + G^*Gw_n)$. Thus $w_{n+1} = (1-\alpha_n)e_n$. Fixing $\hat{w} \in \Gamma$, one has

$$\|w_{n+1} - \hat{w}\|^2 = \|(1-\alpha_n)(e_n - \hat{w}) + \alpha_n(-\hat{w})\|^2 \leq (1-\alpha_n)\|e_n - \hat{w}\|^2 + \alpha_n\|\hat{w}\|^2. \tag{5.2}$$

From the definition of e_n, we see that

$$\|e_n - \hat{w}\|^2 = \|w_n - \hat{w}\|^2 + \gamma_n^2\|w_n - P_{S_n^2}w_n + G^*Gw_n\|^2 - 2\gamma_n\langle w_n - \hat{w}, w_n - P_{S_n^2}w_n + G^*Gw_n\rangle.$$

According to (5.1), we have

$$\begin{aligned}\|e_n-\hat{w}\|^2 &\le \|w_n-\hat{w}\|^2+\gamma_n^2\|w_n-P_{S_n^2}w_n+G^*Gw_n\|^2-2\gamma_n(\|w_n-P_{S_n^2}w_n\|^2+\|Gw_n\|^2)\\ &\le \|w_n-\hat{w}\|^2+\frac{\rho_n^2(\|w_n-P_{S_n^2}w_n\|^2+\|Gw_n\|^2)^2}{\|w_n-P_{S_n^2}w_n+G^*Gw_n\|^2}-\frac{2\rho_n(\|w_n-P_{S_n^2}w_n\|^2+\|Gw_n\|^2)^2}{\|w_n-P_{S_n^2}w_n+G^*Gw_n\|^2}\\ &\le \|w_n-\hat{w}\|^2-\rho_n(2-\rho_n)\frac{(\|w_n-P_{S_n^2}w_n\|^2+\|Gw_n\|^2)^2}{\|w_n-P_{S_n^2}w_n+G^*Gw_n\|^2}.\end{aligned} \tag{5.3}$$

It then follows from (5.2) that

$$\begin{aligned}\|w_{n+1}-\hat{w}\|^2 &\le (1-\alpha_n)\|e_n-\hat{w}\|^2+\alpha_n\|\hat{w}\|^2\\ &\le (1-\alpha_n)\|w_n-\hat{w}\|^2+\alpha_n\|\hat{w}\|^2\\ &\quad -(1-\alpha_n)\rho_n(2-\rho_n)\frac{(\|w_n-P_{S_n^2}w_n\|^2+\|Gw_n\|^2)^2}{\|w_n-P_{S_n^2}w_n+G^*Gw_n\|^2}.\end{aligned} \tag{5.4}$$

From Assumption 2 (A3) and (5.4), we have

$$\|w_{n+1}-\hat{w}\|^2 \le (1-\alpha_n)\|w_n-\hat{w}\|^2+\alpha_n\|\hat{w}\|^2.$$

Thus $\|w_{n+1}-\hat{w}\|^2 \le \max\{\|\hat{w}\|^2,\|w_0-\hat{w}\|^2\}$. Hence, $\{w_n\}$ is bounded. According to (5.3), we have

$$\|e_n-\hat{w}\|^2 \le \|w_n-\hat{w}\|^2-\rho_n(2-\rho_n)\frac{(\|w_n-P_{S_n^2}w_n\|^2+\|Gw_n\|^2)^2}{\|w_n-P_{S_n^2}w_n+G^*Gw_n\|^2} \le \|w_n-\hat{w}\|^2.$$

Thus

$$\begin{aligned}\|w_{n+1}-\hat{w}\|^2 &= \|(1-\alpha_n)(e_n-\hat{w})+\alpha_n(-\hat{w})\|^2\\ &= (1-\alpha_n)^2\|e_n-\hat{w}\|^2+\alpha_n^2\|\hat{w}\|^2+2(1-\alpha_n)\alpha_n\langle w_n-\hat{w},-\hat{w}\rangle\\ &\quad +2(1-\alpha_n)\alpha_n\gamma_n\langle w_n-P_{S_n^2}w_n+G^*Gw_n,\hat{w}\rangle\\ &\le (1-\alpha_n)\|w_n-\hat{w}\|^2+\alpha_n[\alpha_n\|\hat{w}\|^2+2(1-\alpha_n)\langle w_n-\hat{w},-\hat{w}\rangle\\ &\quad +2(1-\alpha_n)\gamma_n\|\hat{w}\|\|w_n-P_{S_n^2}w_n+G^*Gw_n\|].\end{aligned} \tag{5.5}$$

Without loss of generality, we may assume that there exists $\varepsilon>0$ such that $\rho_n(2-\rho_n)(1-\alpha_n)\ge\varepsilon$. It follows from (5.4) that

$$\begin{aligned}\|w_{n+1}-w^*\|^2 &\le (1-\alpha_n)\|w_n-w^*\|^2+\alpha_n\|w^*\|^2\\ &\quad -(1-\alpha_n)\rho_n(2-\rho_n)\frac{(\|w_n-P_{S_n^2}w_n\|^2+\|Gw_n\|^2)^2}{\|w_n-P_{S_n^2}w_n+G^*Gw_n\|^2}\\ &\le \|w_n-w^*\|^2+\alpha_n\|w^*\|^2-\frac{\varepsilon(\|w_n-P_{S_n^2}w_n\|^2+\|Gw_n\|^2)^2}{\|w_n-P_{S_n^2}w_n+G^*Gw_n\|^2}.\end{aligned} \tag{5.6}$$

In view of (5.5) and (5.6), we have

$$\begin{cases}\|w_{n+1}-w^*\|^2 \le (1-\alpha_n)\|w_n-w^*\|^2+\alpha_n\theta_n,\\ \|w_{n+1}-w^*\|^2 \le \|w_n-w^*\|^2-\xi_n+\alpha_n\|w^*\|^2,\end{cases}$$

where $\theta_n = \alpha_n\|\hat{w}\|^2 + 2(1-\alpha_n)\langle w_n - \hat{w}, -\hat{w}\rangle + 2(1-\alpha_n)\gamma_n\|\hat{w}\|\|w_n - P_{S_n^2}w_n + G^*Gw_n\|$ and

$$\xi_n = \frac{\varepsilon(\|w_n - P_{S_n^2}w_n\|^2 + \|Gw_n\|^2)^2}{\|w_n - P_{S_n^2}w_n + G^*Gw_n\|^2},$$

with $\{\alpha_n\} \subset (0,1), \lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$. Let $h_n = \|w_n - w^*\|^2$. To utilize Lemma 2.1, it suffices to confirm that, for every subsequence $\{n_i\} \subset \{n\}$, $\lim_{i\to\infty}\xi_{n_i} = 0 \Rightarrow \limsup_{i\to\infty}\theta_{n_i} \leq 0$. Note that G is a bounded linear operator and $\{w_n\}$ is a bounded vector sequence. If $\lim_{i\to\infty}\xi_{n_i} = 0$, then $\lim_{i\to\infty}\|w_{n_i} - P_{S_{n_i}^2}w_{n_i}\| = \lim_{i\to\infty}\|Gw_{n_i}\| = 0$. Thus $\lim_{i\to\infty}\|w_{n_i}\| = 0$.

Next, we show that $\omega_w(w_{n_i}) \in \Gamma$. Since $\{w_{n_i}\}$ is bounded, then $\omega_w(w_{n_i}) \neq \emptyset$. Let $\bar{w} \in \omega_w(w_{n_i})$. Thus there exists a subsequence $\{w_{n_{i_j}}\}$ of $\{w_{n_i}\}$ such that $w_{n_{i_j}} \rightharpoonup \bar{w}$. Without loss of generality, we can assume that $w_{n_i} \rightharpoonup \bar{w}$. Since $P_{S_{n_i}^2}(w_{n_i}) \in S_{n_i}^2 \subset S_{n_i}$, we have $s(w_{n_i}) \leq \langle \eta_{n_i}, w_{n_i} - P_{S_{n_i}^2}w_{n_i}\rangle$, where $\eta_{n_i} \in \partial s(w_{n_i})$. Since η_{n_i} is bounded and $I - P_{S_{n_i}^2}$ is firmly nonexpansive, we see that

$$s(w_{n_i}) \leq \langle \eta_{n_i}, w_{n_i} - P_{S_{n_i}^2}w_{n_i}\rangle \leq \|\eta_{n_i}\|\|I - P_{S_{n_i}^2}\|\|w_{n_i}\| \to 0 \ (i\to\infty).$$

Since s is w-lsc, it follows that $\bar{w} \in S$. Thus

$$0 \leq \|G\bar{w}\|^2 = \lim_{i\to\infty}\langle Gw_{n_i}, G\bar{w}\rangle \leq \lim_{i\to\infty}\|Gw_{n_i}\|\|G\bar{w}\| \to 0 \ (i\to\infty),$$

which implies that $\bar{w} \in \Gamma$, i.e., $\omega_w(w_{n_i}) \in \Gamma$. Observe that

$$\begin{aligned}\limsup_{i\to\infty}\theta_{n_i} &\leq 2\limsup_{i\to\infty}[\langle w_{n_i} - w^*, -w^*\rangle + \gamma_{n_i}\|w^*\|(\|w_{n_i} - P_{S_{n_i}^2}w_{n_i}\| + \|G^*Gw_{n_i}\|)] \\ &= 2\max_{\bar{w}\in\omega_w(w_{n_i})}\langle \bar{w} - w^*, -w^*\rangle \leq 0.\end{aligned}$$

From Lemma 2.1, we conclude that $\{w_n\}$ converges strongly to $w^* = P_\Gamma(0)$. The proof is complete. □

6. Numerical Experiment

In this section, we present a series of numerical experiments pertaining to signal recovery. We designate Algorithms 1, 2, and 3, López's algorithm in [10] and Yang's algorithm in [19] as Alg1, Alg2, Alg3, López Alg, and Yang Alg, respectively, for the sake of convenience. The code is implemented in MATLAB R2022a and is running on a personal computer with Inter(R) Core(TM) i5-8250U CPU @1.60GHz.

The following lemma is crucial for our numerical experiments.

Lemma 6.1. [20] *Let u_1 and u_2 be two vectors in H. Let η_1 and η_2 be two real numbers. Let $\|u_1\|^2\|u_2\|^2 > |\langle u_1, u_2\rangle|^2$. Let $C = \{x \in H | \langle x, u_1\rangle \leq \eta_1\} \cap \{x \in H | \langle x, u_2\rangle \leq \eta_2\}$. Then $C \neq \Phi$ and $P_C x = x - v_1u_1 - v_2u_2$, where exactly one of the following holds:*

(i) $\langle x, u_1\rangle \leq \eta_1$ and $\langle x, u_2\rangle \leq \eta_2$. Then $v_1 = v_2 = 0$.

(ii) $\|u_2\|^2(\langle x, u_1\rangle - \eta_1) > \langle u_1, u_2\rangle(\langle x, u_2\rangle - \eta_2)$ and $\|u_1\|^2(\langle x, u_2\rangle - \eta_2) > \langle u_1, u_2\rangle(\langle x, u_1\rangle - \eta_1)$. Then

$$v_1 = \frac{\|u_2\|^2(\langle x, u_1\rangle - \eta_1) - \langle u_1, u_2\rangle(\langle x, u_2\rangle - \eta_2)}{\|u_1\|^2\|u_2\|^2 - |\langle u_1, u_2\rangle|^2} > 0 \tag{6.1}$$

and

$$v_2 = \frac{\|u_1\|^2(\langle x,u_2\rangle - \eta_2) - \langle u_1,u_2\rangle(\langle x,u_1\rangle - \eta_1)}{\|u_1\|^2\|u_2\|^2 - |\langle u_1,u_2\rangle|^2} > 0. \tag{6.2}$$

(iii) $\langle x,u_2\rangle > \eta_2$ *and* $\|u_2\|^2(\langle x,u_1\rangle - \eta_1) \leq \langle u_1,u_2\rangle(\langle x,u_2\rangle - \eta_2)$. *Then*

$$v_1 = 0 \text{ and } v_2 = \frac{\langle x,u_2\rangle - \eta_2}{\|u_2\|^2} > 0. \tag{6.3}$$

(iv) $\langle x,u_1\rangle > \eta_1$ *and* $\|u_1\|^2(\langle x,u_2\rangle - \eta_2) \leq \langle u_1,u_2\rangle(\langle x,u_1\rangle - \eta_1)$. *Then*

$$v_1 = \frac{\langle x,u_1\rangle - \eta_1}{\|u_1\|^2} > 0 \text{ and } v_2 = 0. \tag{6.4}$$

The signal recovery problem is established as $\min_{x\in R^N} \frac{1}{2}\|Ax-y\|_2^2$ subject to $\|x\|_1 \leq s$, where $A \in R^{M\times N}, M < N, y \in R^M, s > 0$, $\|\cdot\|_1$ is l_1-norm defined by $\|x\|_1 = \sum_{n=1}^N |x_n|$, and A is a matrix of perceptions, which is constructed from a standard normal distribution. The genuine sparse signal x^* is formed by uniformly distribution within the interval $[-1,1]$ by using random K nonzero elements. The sample data $y = Ax^*$ without any assumption of noise.

For convenience, we define $B = I$ in Algorithm 1, Algorithm 2, and Algorithm 3. Then SEP (1.2) reduces to SFP (1.1). In this situation, we define $C = \{x \in R^N : \|x\|_1 \leq s\}, s = K$, and $Q = \{y\}$. By using the relaxed CQ algorithms, we define the convex function $c(x) := \|x\|_1 - s$ and designate the half space C_n as: $C_n = \{x \in R^N : c(t_n) \leq \langle \xi_n, t_n - x\rangle\}$, where $\xi_n \in \partial c(t_n)$. The subdifferential ∂c at $t_n \in R^N$ is defined by $[\partial c(t_n)]_i = sign((t_n)_i)$, where $sign(\cdot)$ presents the sign function.

The metric projection of a point $x \in R^N$ onto C_n is as follows:

$$P_{C_n}(x) = \begin{cases} x, \ if \ c(t_n) + \langle \xi_n, x - t_n\rangle \leq 0, \\ x - \frac{c(t_n)+\langle \xi_n, x-t_n\rangle}{\|\xi_n\|^2}\xi_n, \ otherwise. \end{cases}$$

The initials are as $x_0 = x_1 = (0,0,...,0)^T \in R^N$. The parameters for Alg1, Alg2, Alg3, López's Alg, and Yang Alg are adjusted as in Table 1. To gauge the precision of the recovery, we employ the following mean square error technique: $MSE = \frac{1}{N}\|x_n - x^*\|$, where x_n is the recovered signal at nth iteration.

On the other hand, we use the stop criterion of the iteration as $MSE < 10^{-6}$ and $MSE < 10^{-8}$. It is demonstrated that our relaxed CQ-algorithms exhibit lower iterations and CPU time compared with the López algorithm and Yang algorithm across various K-sparse scenarios in Table 2. Figure 2 illustrates the relationship between MSE values and the number of iterations for $M = 256$, $N = 512$, and $K = 40$ when $MSE < 10^{-8}$. As illustrated in Figure 1 and Figure 2, our proposed algorithms have the capability to accurately estimate the signal x^*. Simultaneously, the CPU time of our proposed methods is less and the MSE is smaller with the same number of iterations.

In this numerical example, it is evident that our algorithms outperform the López's algorithm and Yang's algorithm when the parameters are suitable.

7. Conclusion

In this paper, we proposed three novel relaxed CQ-algorithms for solving SEP (1.2) and obtained the strong convergence of the three algorithms under mild conditions. For our algorithms, there are two features. One is the projection onto the intersection of two half-spaces, and the other one

FIGURE 1. Comparison of Different Algorithms of Signal processing

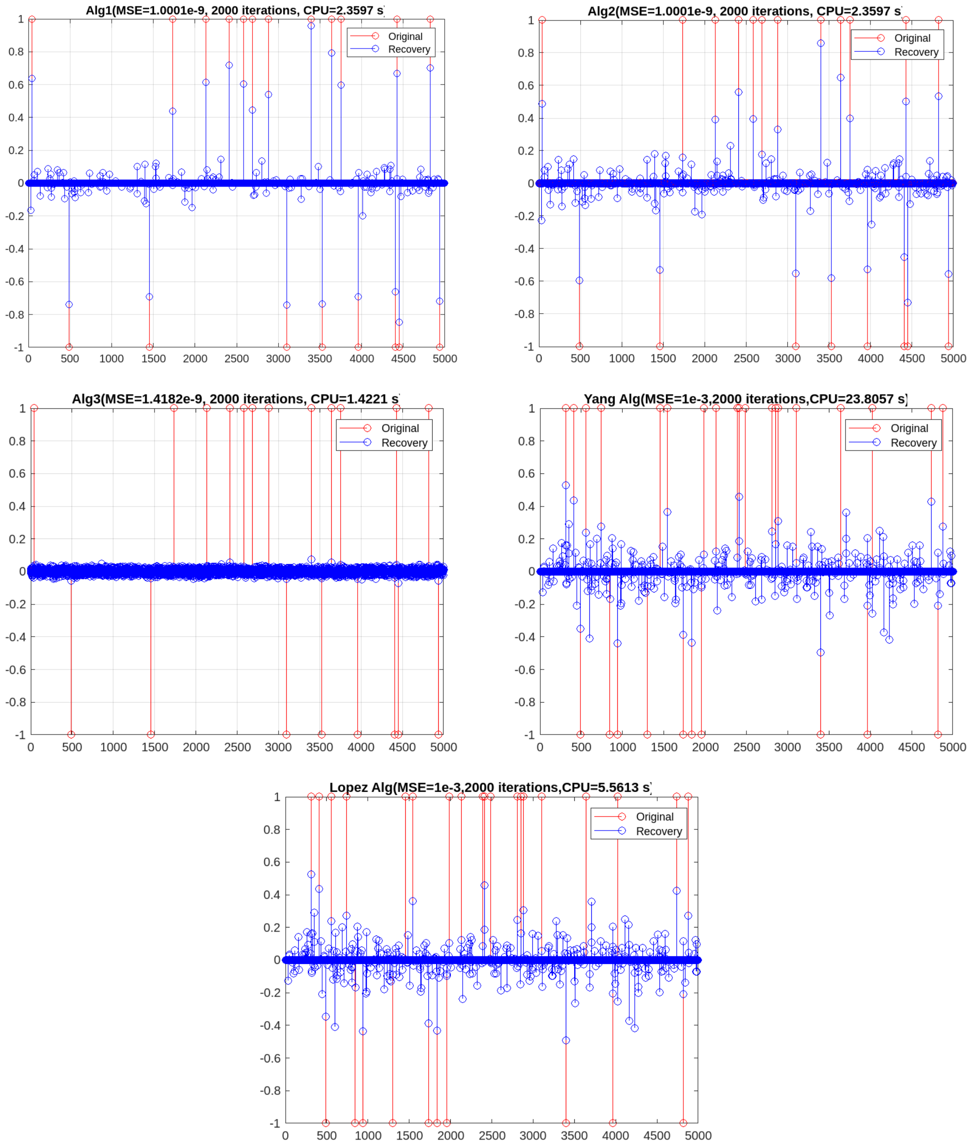

FIGURE 2. Graph of MSE values and number of iterations for $M = 256$, $N = 512$, $K = 40$ when $MSE < 10^{-8}$

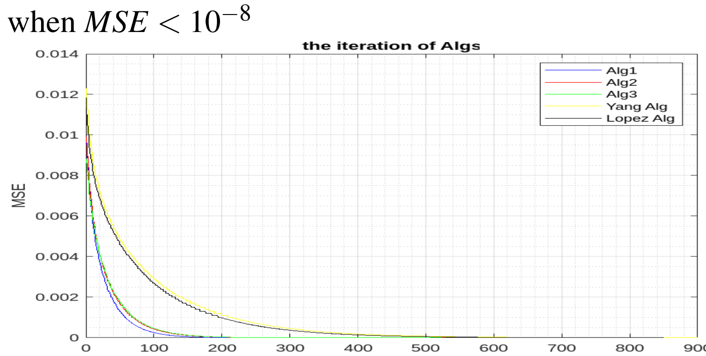

TABLE 1. Algorithms and Their Setting of Parameters

Algorithm	Setting of Parameters
Alg1	$\alpha_n = \frac{1}{10^5\ n},\ \rho_n = 1$
Alg2	$\alpha_n = \frac{1}{10^5\ n},\ \rho_n = 1\ ,\ \beta_n = \frac{1}{2}$
Alg3	$\alpha_n = \frac{1}{10^5\ n},\ \rho_n = 1$
Yang Alg	$\gamma = \frac{1}{\|A\|^2}$
López Alg	$\rho_n = 1$

TABLE 2. Calculative Results of the Five Algorithms with $M = 256,\ N = 512$ and different K-sparse signal, $MSE < 10^{-6}$ and $MSE < 10^{-8}$

K-sparse signal	Algorithms	$MSE < 10^{-6}$		$MSE < 10^{-8}$	
		Iter	CPU time	Iter	CPU time
K=20	Alg1	169	0.0783	260	0.1103
	Alg2	213	0.0840	335	0.1240
	Alg3	201	0.1349	303	0.1830
	Yang Alg	619	5.1785	852	7.0907
	López Alg	571	0.4551	773	0.6279
K=30	Alg1	270	0.1181	416	0.1823
	Alg2	341	0.1262	524	0.2301
	Alg3	302	0.1616	496	0.1802
	Yang Alg	923	17.1308	1239	16.0637
	López Alg	1477	1.0487	1496	1.6438
K=40	Alg1	279	0.1251	433	0.1685
	Alg2	338	0.1297	529	0.2041
	Alg3	335	0.2156	479	0.1776
	Yang Alg	1023	16.7836	1345	18.2422
	López Alg	1687	1.4419	1544	1.5986

is the step-sizes, which does not need the prior operator norm. The efficacy of our algorithms is demonstrated through numerical experiments conducted on signal recovery problems.

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