J. Appl. Numer. Optim. 6 (2024), No. 1, pp. 71-82
Available online at http://jano.biemdas.com
https://doi.org/10.23952/jano.6.2024.1.04

A NEW HYBRID HALPERN-BASED EXTRAGRADIENT ALGORITHM FOR A PSEUDOMONOTONE EQUILIBRIUM PROBLEM AND A FIXED POINT PROBLEM OF NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we propose a hybrid Halpern-based extragradient algorithm for finding a common solution of a pseudomonotone equilibrium problem and a fixed point problem of a nonexpansive mapping. We prove the strong convergence of the proposed algorithm under some mixed conditions. Some numerical examples are provided to illustrate the effectiveness of the proposed algorithm.

Keywords. Banach space; Halpern-based extragradient algorithm; Nonexpansive mapping; Pseudomonotone Equilibrium problem;.

2020 Mathematics Subject Classification. 47H09, 65K15, 90C30.

1. Introduction

Let C be a nonempty, convex, and closed subset of a Hilbert space H, and let $T : C \to C$ be a nonexpansive mapping, i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Denote the set of fixed points of T by $Fix(T)$. Fixed point problem is an important branch in the nonlinear functional field and is widely studied by numerous scholars due to its variational real applications. For the approximation of fixed points of nonexpansive mappings, Halpern [1] first investigated the following iteration, which is now called *Halpern iteration* in the literature:

$$x_0 \in C,\ x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n,\ n \geq 0,$$

where u is a fixed vector in set C and $\{\alpha_n\} \subset (0, 1)$ is a real sequence. Halpern pointed out that the following conditions (C1) $\lim_{n\to\infty} \alpha_n = 0$ and (C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$ are necessary conditions for the convergence of iterative sequences generated in Halpern iteration. Recently, Halpern iteration has been extensively studied; see, e.g., [2, 3, 4, 5] and the references therein.

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Received 25 April 2023; Accepted 31 August 2023; Published online 5 March 2014.

It is believe that Halpern iteration converges slowly due to (C2). In 2004, Martinez-Yanes and Xu [2] proposed a modification, which is called a hybrid Halpern iteration as follows:

$$\begin{cases} x_0 \in C, \\ y_n = \alpha_n x_0 + (1-\alpha_n) T x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 - \alpha_n(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \ n \ge 0. \end{cases}$$

The authors proved that if $\{\alpha_n\} \subset (0,1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$, then $\{x_n\}$ converges strongly to the element $P_{Fix(T)}x_0$, where P is the projection from H onto C.

It is known that the strong convergence the algorithm above holds only in Hilbert spaces. For approximating the fixed point of a nonexpansive mapping in Banach spaces, Xu [6] introduced the following iterative algorithm: $x_0 = x \in C$ and

$$\begin{cases} C_n = \overline{co}\{z \in C : \|z - Tz\| \le t_n \|x_n - Tx_n\|\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x, \ n \ge 0, \end{cases}$$

where C is a nonempty, closed, and convex subset of a smooth and uniformly convex Banach space, $T : C \to C$ is a nonexpansive mapping, $\overline{co}D$ denotes the convex closure of the set D, $\{t_n\}$ is a sequence in $(0,1)$ with $t_n \to 0$, and $\Pi_{C_n \cap Q_n}$ is the generalized projection from E onto $C_n \cap Q_n$. The author proved that the sequence $\{x_n\}$ generated in algorithm converges strongly to the element $\Pi_{Fix(T)}x_0$.

Later on, Matsushita and Takahashi [7] also proposed a modification by replacing the generalized projection $\Pi_{C_n \cap Q_n}$ with the metric projection $P_{C_n \cap Q_n}$. It is worth noting that the subset C is assumed to be bounded in [7].

On the other hand, let $f : C \times C \to \mathbb{R}$ be a bifunction with $f(x,x) = 0$ for all $x \in C$. The *equilibrium problem* in the sense of [8] is to find $z \in C$ such that

$$f(z,y) \ge 0, \quad \forall y \in C.$$

The set of solutions of the equilibrium problem is denoted by $EP(f,C)$ from now on. The equilibrium problem has a mass of applications in economics, management and others, and many efficient algorithms for solving the problem have been introduced; see, e.g., [9, 10, 11, 12, 13, 14, 15] and the references therein.

An interesting problem is to find a common solution the fixed point problem of a nonexpansive mapping and the equilibrium problem from viewpoint of multi-constraints. Takahashi and Takahashi [16] introduced the following iterative algorithm for finding a common solution problem in Hilbert space:

$$\begin{cases} x_0 \in C, \\ \text{find } u_n \in C \text{ such that } f(u_n,y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T u_n, \ n \ge 0, \end{cases}$$

where $f : C \to C$ is a contraction. The authors proved that, under some mixed conditions, the sequence $\{x_n\}$ generated in their algorithm converges strongly to an element in $Fix(T) \cap EP(f,C)$.

For more iteration algorithms on common solution problems in Hilbert spaces and Banach spaces, we refer the readers to [9, 10, 17, 18, 19].

It needs to point out that the spaces considered in [9, 10, 17, 18, 19] are Banach spaces and the involved mappings are relatively nonexpansive mappings or quasi-ϕ-nonexpansive mappings. To the best of our knowledge, until now the solution algorithm for the common solution problems in Banach spaces where the nonlinear mapping is the nonexpansive mapping has not been introduced yet in the literature. In this paper, inspired by [2, 3, 6, 17], we propose a new hybrid Halpern-baesd extragradient algorithm for finding a common solution of an equilibrium problem and the fixed point problem of a nonexpansive mapping in a Banach space. The strong convergence for the proposed algorithm is proved under some mixed conditions. Our results improves and develops the results [6, 20, 21, 22, 23] and many others. Finally, some numerical examples are given to illustrate the convergence of the proposed algorithm.

2. Preliminaries

Let E be a Banach space, and let $S(E) = \{z \in E : \|z\| = 1\}$ be the unit sphere of space E. Recall that E is said to be smooth provided, for each $x, y \in S(E)$,

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists. Let $\rho_E : [0,\infty) \to [0,\infty)$ be the f smooth modulus of E defined by

$$\rho_E(t) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x \in S(E), \|y\| \le t\right\}.$$

If $\frac{\rho_E(t)}{t} \to 0$ as $t \to 0$, E is said to be uniformly smooth. For $q > 1$, one knows that L^q is a uniformly smooth Banach space. Furthermore, any Hilbert space H is uniformly smooth.

A Banach space E is said to be strictly convex if, for all $x, y \in S(E)$, $\frac{\|x+y\|}{2} < 1$. The convex modulus of E is the function $\delta_E : [0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x-y\| = \varepsilon\right\}, \ \forall \varepsilon \in [0,2].$$

E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $0 < \varepsilon \le 2$ and $\delta_E(0) = 0$. Let $q > 1$. A uniformly convex Banach space E is said to be q-uniformly convex if there exists some constant $c > 0$ such that $\delta_E(\varepsilon) \ge c\varepsilon^q$. It is known that L^q is q-uniformly convex when $q > 2$ and 2-uniformly convex when $1 < q \le 2$. Furthermore, any Hilbert space is 2-uniformly convex.

Let E^* be the duality space of a Banach space E. For all $x \in E$ and $\bar{x} \in E^*$, we denote the value of $\bar{x}$ at x by $\langle x, \bar{x}\rangle$. The normalized duality mapping J on E is defined by

$$J(x) = \{\bar{x} \in E^* : \langle x, \bar{x}\rangle = \|x\|^2 = \|\bar{x}\|^2\}, \ \forall x \in E.$$

It is known that if E is smooth, then J is single-valued and if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E. Also, if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

Let E be a smooth, strictly convex, and reflexive Banach space, and let C be a nonempty, closed, and convex subset of E. The *Lyapunov function* $\phi : E \times E \to [0,\infty)$ is defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \ \forall x, y \in E.$$

The mapping $\Pi_C : E \to C$ is called the generalized projection [24] if it assigns any point $x \in E$ to the minimum point of the functional $\phi(x,y)$; that is, $\Pi_C x = \arg\min_{y\in C} \phi(y,x)$.

Lemma 2.1. [24] *Let E be a smooth, strictly convex, and reflexive Banach space, and let C be a nonempty, closed, and convex subset of E. Then the following conclusions holds:*

(a) $\phi(x,\Pi_C y) + \phi(\Pi_C y, y) \le \phi(x,y)$ *for all $x \in C$, $\forall y \in E$;*

(b) $\phi(x, J^{-1}(\lambda Jy + (1-\lambda)Jz)) \le \lambda\phi(x,y) + (1-\lambda)\phi(x,z)$ *for all $x,y,z \in E$ and $\lambda \in [0,1]$;*

(c) *for $\{x_n\}, \{y_n\} \subset E$, if either $\{x_n\}$ or $\{y_n\}$ is bounded, and $\phi(x_n,y_n) \to 0$, then $\|x_n - y_n\| \to 0$ as $n \to \infty$;*

(d) *for $x \in E$ and $z \in C$, $z = \Pi_C x$ if and only if $\langle y - z, Jx - Jz\rangle \le 0$ for all $y \in C$.*

Let E be a Banach space and define the function $V : E \times E^* \to \mathbb{R}$ [24] by

$$V(x,z) = \|x\|^2 - 2\langle x,z\rangle + \|z\|^2,\ \forall x \in E, \forall z \in E^*.$$

From the definitions of ϕ and V, it follows that

$$V(x,z) = \phi(x, J^{-1}z),\ \forall x \in E, \forall z \in E^*.$$

The following result characters the property of the function V.

Lemma 2.2. [24] *Let E be a reflexive, strictly convex, and smooth Banach space with its dual E^*. Then $V(x,x^*) + 2\langle J^{-1}x - x^*, y^*\rangle \le V(x, x^* + y^*)$ for all $x \in E$ and for all $x^*, y^* \in E^*$.*

Lemma 2.3. [25] *Let C be a nonempty, bounded, closed, and convex subset of a uniformly convex Banach space E. Then there exists a strictly increasing, convex, and continuous function $\gamma : [0,\infty) \to [0,\infty)$with $\gamma(0) = 0$ such that, for a nonexpansive mapping $S : C \to C$, and any finite many element $\{z_j\}_{j=1}^m$ in C, the following inequality holds:*

$$\gamma\left(\left\|S\left(\sum_{j=1}^m \eta_j z_j\right) - \sum_{j=1}^m \eta_j S z_j\right\|\right) \le \gamma^{-1}\left(\max_{1\le j,k\le m}(\|z_j - z_k\| - \|Sz_j - Sz_k\|)\right),$$

where $\{\eta_j\}_{j=1}^m \subset [0,1]$ with $\sum_{j=1}^m \eta_j = 1$.

Lemma 2.4. [26] *Let C be a nonempty, closed, and convex subset of a uniformly convex Banach space E, and let $T : C \to C$ be a nonexpansive mapping. Then the mapping $I - T$ is demi-closed at zero, i.e., for any sequence $\{x_n\} \subset C$, if $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

Lemma 2.5. [27] *Let $\{a_n\}$ be a nonnegative real sequence such that $a_{n+1} \le (1-b_n)a_n + b_n c_n$ for all $n \ge 1$, where $\{b_n\}$ is a real sequence in $(0,1)$ and $\{c_n\}$ is a real sequence satisfy that* $\lim_{n\to\infty} b_n = 0$, $\sum_{n=1}^\infty b_n = \infty$, *and* $\limsup_{n\to\infty} c_n \le 0$. *Then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6. [28] *Let $\{k_n\}$ be a nonnegative real sequence. Suppose that, for any integer m, there exists an integer j such that $j \ge m$ and $k_j \le k_{j+1}$. Let n_0 be an integer such that $k_{n_0} \le k_{n_0+1}$ and for all integer $n \ge n_0$, and define $\tau(n) = \max\{l \in \mathbb{N} : n_0 \le l \le n, k_l \le k_{l+1}\}$. Then $0 \le k_n \le k_{\tau(n)+1}$ for all $n \ge n_0$. Furthermore, the sequence $\{\tau(n)\}_{n\ge n_0}$ is non-decreasing and tends to $+\infty$ as $n \to \infty$.*

3. MAIN RESULTS

In this section, let $\mathbb{N}$ denote the positive integer set, E a uniformly convex and uniformly smooth Banach space, and C a nonempty, convex, and closed subset of E. Let $T : C \to C$ be a nonexpansive mapping, and let $f : C \times C \to \mathbb{R}$ be a bifunction with $\Omega \neq \emptyset$, where $\Omega = EP(f,C) \cap Fix(T)$. For the desired result, we assume that f satisfies the following conditions:

(A0) Either $\mathrm{int}(C) \neq \emptyset$, or, for each $x \in C$, $f(x,\cdot)$ is continuous at a point in C.
(A1) f is pseudomonotone on C, i.e., $f(x,y) \geq 0 \Longrightarrow f(y,x) \leq 0$ for all $x,y \in C$.
(A2) f is Lipschitz-type continuous with the positive constants c_1 and c_2, i.e.,

$$f(x,y) + f(y,z) \geq f(x,z) - c_1\|x-y\|^2 - c_2\|y-z\|^2, \ \forall x,y,z \in C.$$

(A3) $\limsup_{n\to\infty} f(u_n,y) \leq f(x,y)$ for each sequence u_n weakly converging to $q \in C$ and $y \in C$.
(A4) $f(x,\cdot)$ is convex and subdifferentiable on C for each $x \in C$.

For finding a point $\bar{x} \in \Omega$, we present the hybrid Halpern-based extragradient algorithm as follows.

Algorithm 1. Hybrid Halpern-based Extragradient Algorithm

Initialization: Choose the initial points $x_1 = x \in C$, the sequences $\{\alpha_n\} \subset (0,1)$ with $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{t_n\} \subset (0,\infty)$ with $t_n \to 0$, and $\{\lambda_n\} \subset [\lambda', \lambda'']$ with $0 < \lambda' < \lambda_n < \lambda'' < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$.

Iteration Step Compute

$$\begin{cases} y_n = \arg\min\left\{y \in C : \lambda_n f(x_n,y) + \frac{1}{2}\phi(y,x_n)\right\}, \\ z_n = \arg\min\left\{y \in C : \lambda_n f(y_n,y) + \frac{1}{2}\phi(y,x_n)\right\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(J^{-1}(\alpha_n Jx + (1-\alpha_n)Jz_n)), \end{cases}$$

where

$$C_n = \overline{co}\{z \in C : \|z - Tz\| \leq t_n\|x_n - Tx_n\|\} \text{ and } Q_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geq 0\}.$$

Stop Criterion If $x_n = y_n = Tx_n$, then stop and $x_n \in \Omega$.

Remark 3.1. Conditions (A1) and (A2) imply that $f(x,x) = 0$ for all $x \in C$; see [23].

Remark 3.2. Although f is required to satisfy condition (A3), Lipschitz constants c_1 and c_2 need not to be known because they are not used as the input parameters in the proposed algorithm.

Remark 3.3. Condition (A0) guarantees that sequences $\{y_n\}$ and $\{z_n\}$ are well defined; see [17].

Remark 3.4. It is known that $Fix(T)$ is closed and convex. Set $EP(f,C)$ is also closed convex under the conditions (A1), (A3), and (A4). Thus Ω is closed and convex.

The following remark demonstrates that the stop criterion can well work.

Remark 3.5. From [17, Lemma 3.1 (ii)], it follows that

$$\lambda_n(f(x_n,y)-f(x_n,y_n)) \geq \langle y_n-y, Jy_n-Jx_n\rangle,\ \forall y\in C.$$

If $x_n=y_n$ for some $n\in\mathbb{N}$, we have $f(x_n,y)\geq 0$ for all $y\in C$. It follows that $x_n\in EP(f,C)$. Furthermore, if $x_n=Tx_n$, one has $x_n\in Fix(T)$. Therefore, the stop criterion can well work.

To prove the convergence of Algorithm 1, we assume that the stop criterion is not satisfied for all $n\in\mathbb{N}$ and hence $\{x_n\}$ is an infinite sequence.

Lemma 3.1. *The sequence $\{x_n\}$ generated by Algorithm 1 is bounded.*

Proof. For any $x^*\in\Omega$, by [17, Lemma 3.1 (ii)] we have

$$\phi(x^*,z_n)\leq\phi(x^*,x_n)-(1-2\lambda_n c_1)\phi(y_n,x_n)-(1-2\lambda_n c_2)\phi(z_n,y_n). \tag{3.1}$$

On account of $\lambda_n<\min\left\{\frac{1}{2c_1},\frac{1}{2c_2}\right\}$, (3.1), and Lemma 2.1 (a) and (b), we have

$$\begin{aligned}\phi(x^*,x_{n+1}) &\leq \phi(x^*,J^{-1}(\alpha_nJx+(1-\alpha_n)Jz_n))\\ &\leq \alpha_n\phi(x^*,x)+(1-\alpha_n)\phi(x^*,z_n)\\ &\leq \alpha_n\phi(x^*,x)+(1-\alpha_n)[\phi(x^*,x_n)-(1-2\lambda_nc_1)\phi(y_n,x_n)\\ &\quad -(1-2\lambda_nc_2)\phi(z_n,y_n)]\\ &\leq \alpha_n\phi(x^*,x)+(1-\alpha_n)\phi(x^*,x_n)\\ &\leq \max\{\phi(x^*,x),\phi(x^*,x_n)\}\\ &\leq\cdots\leq\max\{\phi(x^*,x),\phi(x^*,x_1)\}.\end{aligned} \tag{3.2}$$

It follows that $\{\phi(x^*,x_n)\}$ is bounded. Furthermore, (3.1) implies that $\{\phi(x^*,z_n)\}$ is also bounded. Since $\|x_n\|\leq\sqrt{\phi(x^*,x_n)}+\|x^*\|$, $\{x_n\}$ is bounded. Similarly, $\{z_n\}$ is also bounded from the boundedness of $\{\phi(x^*,z_n)\}$. This completes the proof. □

Now we give the convergence result for Algorithm 1 as follows.

Theorem 3.1. *The sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to $\bar{x}=\Pi_\Omega x$.*

Proof. Set $u_n=J^{-1}(\alpha_nJx+(1-\alpha_n)Jz_n)$. By Lemma 2.1 (a) and (b), Lemma 2.2, and (3.2), we obtain

$$\begin{aligned}\phi(\bar{x},x_{n+1}) &\leq \phi(\bar{x},J^{-1}(\alpha_nJx+(1-\alpha_n)Jz_n)\\ &\leq V(\bar{x},\alpha_nJx+(1-\alpha_n)Jz_n-\alpha_n(Jx-J\bar{x}))\\ &\quad -2\langle J^{-1}(\alpha_nJx+(1-\alpha_n)Jz_n)-\bar{x},-\alpha_n(Jx-J\bar{x})\rangle\\ &= V(\bar{x},\alpha_nJ\bar{x}+(1-\alpha_n)Jz_n)+2\alpha_n\langle x_{n+1}-\bar{x},Jx-J\bar{x}\rangle\\ &= \phi(\bar{x},J^{-1}(\alpha_nJ\bar{x}+(1-\alpha_n)Jz_n))+2\alpha_n\langle u_n-\bar{x},Jx-J\bar{x}\rangle\\ &\leq \phi(\bar{x},\bar{x})+(1-\alpha_n)\phi(\bar{x},z_n)+2\alpha_n\langle u_n-\bar{x},Jx-J\bar{x}\rangle\\ &\leq (1-\alpha_n)\phi(\bar{x},x_n)+2\alpha_n\langle u_n-\bar{x},Jx-J\bar{x}\rangle.\end{aligned} \tag{3.3}$$

Now we give the further proof by the following two cases:

Case 1. Assume that there exists $n_0>1$ such that $\{\phi(\bar{x},x_n)\}$ is deceasing. It follows that $\{\phi(\bar{x},x_n)\}$ is convergent and

$$\phi(\bar{x},x_n)-\phi(\bar{x},x_{n+1})\to 0 \quad \text{as } n\to\infty. \tag{3.4}$$

Since $\alpha_n \to 0$ and $\{\phi(\bar{x},x_n)\}$ is bounded, we obtain by (3.2) and (3.4) that

$$\begin{aligned}&(1-\alpha_n)[(1-2\lambda_n c_1)\phi(y_n,x_n)+(1-2\lambda_n c_2)\phi(z_n,y_n)]\\ &\leq \alpha_n(\phi(\bar{x},x)-\phi(\bar{x},x_n))+\phi(\bar{x},x_n)-\phi(\bar{x},x_{n+1})\\ &\to 0, \quad \text{as } n\to\infty.\end{aligned} \tag{3.5}$$

From $\lambda_n < \lambda'' \leq \min\left\{\frac{1}{2c_1},\frac{1}{2c_2}\right\}$, $\alpha_n \to 0$, (3.4), and (3.5), we assert that $\lim_{n\to\infty}\phi(y_n,x_n) = \lim_{n\to\infty}\phi(z_n,y_n) = 0$, which together with Lemma 2.1 (c) implies that

$$\lim_{n\to\infty}\|y_n-x_n\| = \lim_{n\to\infty}\|z_n-y_n\| = 0. \tag{3.6}$$

From the definition of u_n, it follows that $\|Ju_n - Jz_n\| = \alpha_n\|Jx-Jz_n\| \to 0$ as $n\to\infty$. Since J^{-1} is uniformly continuous on each bounded subset of E^*, we deduce that $\|u_n-z_n\|\to 0$ as $n\to\infty$, which together with (3.6) yields that

$$\|x_n-u_n\| \leq \|x_n-y_n\|+\|y_n-z_n\|+\|z_n-u_n\| \to 0 \quad \text{as } n\to\infty. \tag{3.9}$$

On the other hand, from the definition of Q_n, we have $x_n = \Pi_{Q_n}x$. Hence, Lemma 2.1 (a) implies $\phi(y,x_n)+\phi(x_n,x)\leq\phi(y,x)$ for all $y\in Q_n$. Since $x_{n+1}\in Q_n$, we have $\phi(x_{n+1},x_n)\leq\phi(x_{n+1},x)-\phi(x_n,x)$, which implies that $\{\phi(x_n,x)\}$ is increasing and thus $\lim_{n\to\infty}\phi(x_n,x)$ exists. It follows that $\phi(x_{n+1},x_n)\to 0$. It follows from Lemma 2.1 (c) that $\|x_{n+1}-x_n\|\to 0$ as $n\to\infty$. Since $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that u_{n_j} weakly converges to $q\in C$ and

$$\limsup_{n\to\infty}\langle u_n-\bar{x}, Jx-J\bar{x}\rangle = \lim_{j\to\infty}\langle u_{n_j}-\bar{x}, Jx-J\bar{x}\rangle.$$

It follows from (A3) that $0\leq\limsup_{n\to\infty}f(u_n,y)\leq f(q,y)$, $\forall y\in C$. Thus $q\in EP(f,C)$.

Next, we prove that $q\in Fix(T)$. From the boundedness of $\{x_n\}$, it follows that there exists a bounded closed convex subset D of C with $r=\text{diam}(D)$ such that $\{x_n\}\subset D$ and $\{Tx_n\}\subset D$. Since $x_{n+1}\in C_n$, we see that there exists $\{v_i\}_{i=1}^m\subset C$ such that

$$\left\|x_{n+1}-\sum_{i=1}^m\eta_i v_i\right\| < t_n \tag{3.7}$$

and

$$\|v_i-Tv_i\|\leq t_n\|x_n-Tx_n\|\leq rt_n,\ \forall i\in\{1,\cdots,m\}, \tag{3.8}$$

where $\{\eta_i\}_{i=1}^m\subset[0,1]$ with $\sum_{i=1}^m\eta_i=1$. By Lemma 2.3, (3.7), and (3.8), we have

$$\begin{aligned}&\|x_{n+1}-Tx_{n+1}\|\\ &\leq\left\|x_{n+1}-\sum_{i=1}^m\eta_i v_i\right\|+\left\|\sum_{i=1}^m\eta_i(v_i-Tv_i)\right\|+\left\|\sum_{i=1}^m\eta_i Tv_i-T\Big(\sum_{i=1}^m\eta_i v_i\Big)\right\|\\ &\quad+\left\|T\Big(\sum_{i=1}^m\eta_i v_i\Big)-Tx_{n+1}\right\|\\ &\leq(2+r)t_n+\gamma^{-1}\big(\max(\|v_i-v_j\|-\|Tv_i-Tv_j\|:1\leq i,j\leq m)\big)\\ &\leq(2+r)t_n+\gamma^{-1}\big(\max(\|v_i-Tv_i\|+\|v_j-Tv_j\|:1\leq i,j\leq m)\big)\\ &\leq(2+r)t_n+\gamma^{-1}(2rt_n)\to 0\quad\text{as } n\to\infty.\end{aligned}$$

Note that $\{x_{n_j}\}$ is also converges weakly to q. By using Lemma 2.4, we see that $q \in Fix(T)$. It follows that $q \in \Omega$. Therefore, from Lemma 2.1 (d), it follows that

$$\limsup_{n\to\infty}\langle u_n - \bar{x}, Jx - J\bar{x}\rangle \le \lim_{j\to\infty}\langle q - \bar{x}, Jx - J\bar{x}\rangle \le 0.$$

Applying Lemma 2.5 to (3.3), we have $\phi(\bar{x}, x_n) \to 0$, which together with Lemma 3.1 and Lemma 2.1 (c) implies that $\{x_n\}$ strongly converges to q.

Case 2 Assume that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\phi(\bar{x}, x_{n_j}) \le \phi(\bar{x}, x_{n_j+1})$ for all $j \ge 1$. In that case, it follows from Lemma 2.6 that

$$\phi(\bar{x}, x_{\tau(n)}) \le \phi(\bar{x}, x_{\tau(n)+1}), \phi(\bar{x}, x_n) \le \phi(\bar{x}, x_{\tau(n)+1}),\ \forall n \ge n_0. \tag{3.9}$$

where $\tau(n) = \{\max k \in N : n_0 \le k \le n, \phi(\bar{x}, x_k) \le \phi(\bar{x}, x_{k+1})\}$. Furthermore, sequence $\{\tau(n)\}_{n\ge n_0}$ is non-decreasing and $\tau(n) \to +\infty$ as $n \to \infty$. Since $\alpha_{\tau(n)} \to 0$ and $\{\phi(\bar{x}, x_n)\}$ is bounded, by (3.2) and (3.9), we have

$$\begin{aligned}&(1-\alpha_{\tau(n)})[(1-2\lambda_{\tau(n)}c_1)\phi(y_{\tau(n)}, x_{\tau(n)}) + (1-2\lambda_{\tau(n)}c_2)\phi(z_{\tau(n)}, y_{\tau(n)})]\\ &\le \alpha_{\tau(n)}(\phi(\bar{x}, x) - \phi(\bar{x}, x_{\tau(n)})) + \phi(\bar{x}, x_{\tau(n)}) - \phi(\bar{x}, x_{\tau(n)+1})\\ &\le \alpha_{\tau(n)}(\phi(\bar{x}, x) - \phi(\bar{x}, x_{\tau(n)})) \to 0\ as\ n \to \infty.\end{aligned}$$

Note that $\lambda_{\tau(n)} < \lambda'' \le \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$. It follows that

$$\lim_{n\to\infty}\phi(y_{\tau(n)}, x_{\tau(n)}) = \lim_{n\to\infty}\phi(z_{\tau(n)}, y_{\tau(n)}) = 0,$$

which in turn implies that $\|y_{\tau(n)} - x_{\tau(n)}\| \to 0$ and $\|z_{\tau(n)} - y_{\tau(n)}\| \to 0$ as $n \to \infty$, so

$$\|x_{\tau(n)} - z_{\tau(n)}\| \le \|x_{\tau(n)} - y_{\tau(n)}\| + \|y_{\tau(n)} - z_{\tau(n)}\| \to 0, \text{ as } n \to \infty.$$

From the definition of u_n, it follows that $\|Ju_{\tau(n)} - Jz_{\tau(n)}\| = \alpha_{\tau(n)}\|Jx - Jz_{\tau(n)}\| \to 0$ as $n \to \infty$. Since J and J^{-1} is uniformly continuous on each bounded subset of E and E^*(resp.), we deduce that $\|Jy_{\tau(n)} - Jx_{\tau(n)}\| \to 0$, $\|Jx_{\tau(n)} - Jz_{\tau(n)}\| \to 0$, and $\|u_{\tau(n)} - z_{\tau(n)}\| \to 0$ as $n \to \infty$. Then $\|x_{\tau(n)} - u_{\tau(n)}\| \to 0$ as $n \to \infty$. Since $\{u_{\tau(n)}\} \subset C$ is bounded, there exists a subsequence $\{u_{\tau(n_i)}\}$ of $\{u_{\tau(n)}\}$ converging weakly to $q \in C$ such that

$$\limsup_{n\to\infty}\langle u_{\tau(n)} - \bar{x}, Jx - J\bar{x}\rangle = \lim_{i\to\infty}\langle u_{\tau(n_i)} - \bar{x}, Jx - J\bar{x}\rangle.$$

By a similar arguing as in Case 1, we can obtain $q \in \Omega$ and

$$\limsup_{n\to\infty}\langle u_{\tau(n)} - \bar{x}, Jx - J\bar{x}\rangle \le 0. \tag{3.10}$$

It follows from (3.3) that

$$\phi(\bar{x}, x_{\tau(n)+1}) \le (1-\alpha_{\tau(n)})\phi(\bar{x}, x_{\tau(n)}) + 2\alpha_{\tau(n)}\langle u_{\tau(n)} - \bar{x}, Jx - J\bar{x}\rangle. \tag{3.11}$$

On account of (3.9) and (3.11), we have

$$\begin{aligned}\alpha_{\tau(n)}\phi(\bar{x}, x_{\tau(n)}) &\le \phi(\bar{x}, x_{\tau(n)}) - \phi(\bar{x}, x_{\tau(n)+1}) + 2\alpha_{\tau(n)}\langle u_{\tau(n)} - \bar{x}, Jx - J\bar{x}\rangle\\ &\le 2\alpha_{\tau(n)}\langle u_{\tau(n)} - \bar{x}, Jx - J\bar{x}\rangle,\end{aligned}$$

which together with $\alpha_{\tau(n)} > 0$ leads to $\phi(\bar{x}, x_{\tau(n)}) \le 2\langle u_{\tau(n)} - \bar{x}, Jx - J\bar{x}\rangle$. This fact with (3.10) implies that $\limsup_{n\to\infty}\phi(\bar{x}, x_{\tau(n)}) \le 0$ and hence $\lim_{n\to\infty}\phi(\bar{x}, x_{\tau(n)}) = 0$. Thus it follows from (3.11) and $\alpha_{\tau(n)} \to 0$ that $\phi(\bar{x}, x_{\tau(n)+1}) \to 0$ as $n \to \infty$. Since $\phi(\bar{x}, x_n) \le \phi(\bar{x}, x_{\tau(n)+1})$ for all $n \ge n_0$ by (3.9), we have $\phi(\bar{x}, x_n) \to 0$ as $n \to \infty$. It follows from Lemma 3.1 that $x_n \to \bar{x}$ as $n \to \infty$. This completes the proof. □

Remark 3.6. Let $\{T_i\}_{i=1}^l : C \to C$ be a family of nonexpansive mappings with $\cap_{i=1}^l Fix(T_i) \neq \emptyset$, and let $T = \sum_{i=1}^l \beta_i T_i$, where $\{\beta_i\}_{i=1}^l \subset [0,1]$ with $\sum_{i=1}^l \beta_i = 1$. Obviously, $T : C \to C$ is a nonexpansive mapping from C into itself. From [29, Lemma 3], it follows that $Fix(T) = \cap_{i=1}^l Fix(T_i)$. Thus one can extend the results above from a nonexpansive mapping to a family of nonexpansive mappings.

4. Numerical Examples

In this section, we present two numerical examples to illustrate the convergence of Algorithm 1. The codes were written by Matlab 2016b and conducted on a PC Intel(R) Core (TM) i5-4260U CPU, 2.00 GHz, Ram 4.00 GB.

Example 4.1. Let $E = \mathbb{R}^m$ and $C = \{(x_1, \cdots, x_m) : x_1 \geq -1, x_i \geq 1, i = 2, \cdots, m\}$. Define a bifunction $f : E \times E \to \mathbb{R}$ by

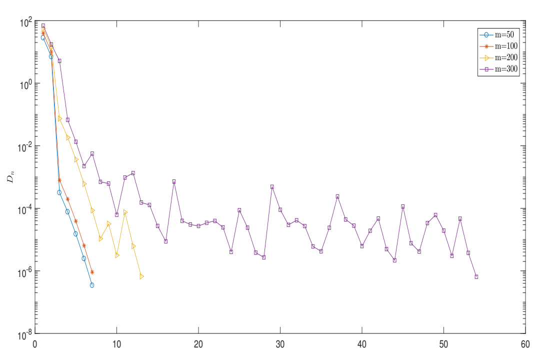
$$f(x,y) = \sum_{i=2}^{m} (y_i - x_i)\|x\|, \ \forall x = (x_1, \cdots, x_m), y = (y_1, \cdots, y_m) \in C,$$

where $\|\cdot\|$ is the standard norm in Euclidean space. It is known that f satisfies the conditions (A1)-(A4) and the Lipschitz-type constants in (A2) are $c_1 = c_2 = 2$; see [12] for details. Define a nonexpansive mapping $T : C \to C$ by

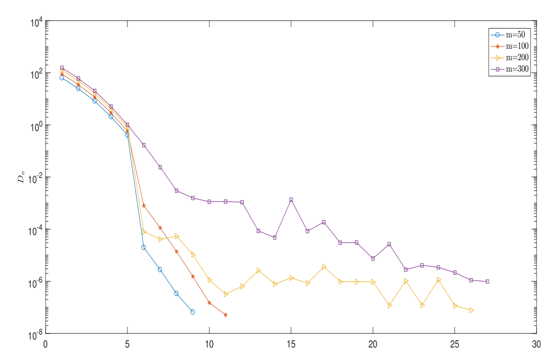
$$Tx = \left(x_1, \frac{1+x_2}{2}, \cdots, \frac{1+x_m}{2}\right), \ \forall x = (x_1, \cdots, x_m) \in C.$$

It is easy to see that $\Omega = \{(x_1, 1, \cdots, 1) : x_1 \geq -1\}$.

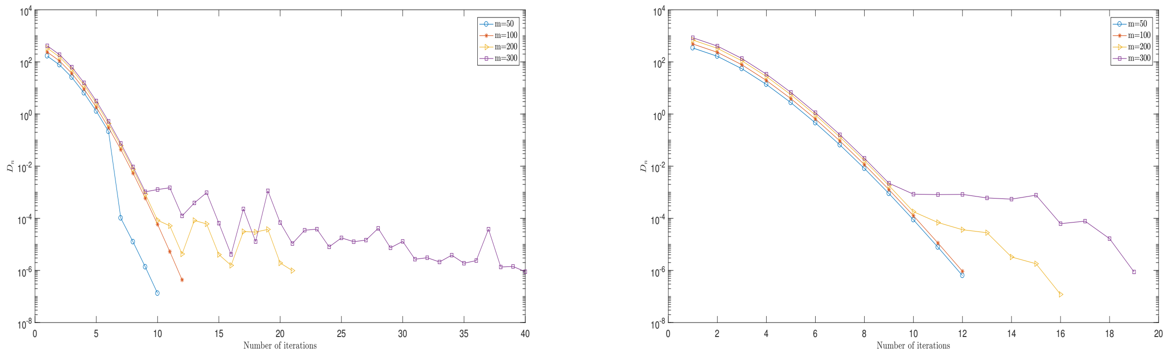
In this example, we choose the parameter sequences $\alpha_n = t_n = \frac{1}{n}$, and $\lambda_n = \frac{n+1}{12n}$. Set $u = (1, \cdots, 1) \in \mathbb{R}^{m-1}$ and $u_n = (x_2, \cdots, x_m) \in \mathbb{R}^{m-1}$ with $x_n = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m$, where x_n is the nth iteration point generated by Algorithm 1. If $x_1 \geq -1$ and $\|u_n - u\| = 0$, then $x_n \in \Omega$. Hence we illustrate the convergence of Algorithm 1 by the convergence of $\{\|u_n - u\|\}$ for this example. We choose the initial point x and use $D_n = \|u_n - u\| < 10^{-6}$ as the stopping criterion. The convergence of $\{D_n\}$ with the different dimension m and initial point x is demonstrated in Figures 1. In the figure, the x-axis represents for the number of iterations while the y-axis represents for the value of D_n. From the figure we can see that sequence $\{D_n\}$ has the better convergence for each different dimension m and initial point x.



(A) $x = (5, \cdots, 5)$



(B) $x = (10, \cdots, 10)$



(C) $x = (25, \cdots, 25)$ (D) $x = (50, \cdots, 50)$

FIGURE 1. Numerical results for Example 4.1 with different dimension m and initial point x

Example 4.2. Let $E = l_3(\mathbb{R})$, which is defined by $l_3(\mathbb{R}) = \{x = (x_1, x_2, \cdots) : x_i \in \mathbb{R}, \sum_{i=1}^{\infty} |x_i|^3 < \infty\}$ with norm $\|x\| = (\sum_{i=1}^{\infty} |x_i|^3)^{\frac{1}{3}}$ for each $x \in E$. E is a uniformly convex and uniformly smooth Banach space. For each $x = (x_1, x_2, \cdots) \in E$ with $x \neq 0$, the normalized duality mapping

$$Jx = \frac{1}{\|x\|}(x_1^2 \mathrm{sgn}(x_1), x_2^2 \mathrm{sgn}(x_2), \cdots).$$

For each $z = (z_1, z_2, \cdots) \in E^*$, let $x = (x_1, x_2, \cdots) \in E$ such that $z = Jx$. That is,

$$(z_1, z_2, \cdots) = \frac{1}{\|x\|}(x_1^2 \mathrm{sgn}(x_1), x_2^2 \mathrm{sgn}(x_2), \cdots).$$

Note that $\|x\| = \|z\|$. It follows that

$$x_i = \|x\| \sqrt{|z_i|} \mathrm{sgn}(z_i) = \|z\| \sqrt{|z_i|} \mathrm{sgn}(z_i), \ i = 1, 2, \cdots .$$

Hence, for each $z = (z_1, z_2, \cdots) \in E^*$,

$$J^{-1} z = J^{-1} Jx = x = \|z\| (\sqrt{|z_1|} \mathrm{sgn}(z_1), \sqrt{|z_2|} \mathrm{sgn}(z_2), \cdots).$$

Let $u = (u_1, u_2, \cdots, u_m, 0, 0, \cdots)$, where $u_1 = u_2 = \cdots = u_m = 1$ with $m \in \mathbb{N}$, and

$$C = \{x \in E : \|x\| \leq 1, x_i \geq 0, i = 1, 2, \cdots\}.$$

Define the bifunction $f : C \times C \to \mathbb{R}$ by

$$f(x, y) = \langle y - x, Ju \rangle, \ \forall x, y \in C.$$

It is obvious that conditions (A0)-(A4) are satisfied. Define the mapping $T : C \to C$ by

$$Tx = \left(\frac{x_1}{2}, \frac{x_2}{2}, 0, 0, \cdots \right), \ \forall x = (x_1, x_2, \cdots) \in C.$$

It is easy to see that T is a nonexpansive mapping on C. Moreover, $\Omega = Fix(T) \cap EP(f, C) = \{x^*\}$ with $x^* = (0, 0, \cdots)$.

In this example, we choose the parameter sequences

$$\alpha_n = t_n = \frac{1}{n}, \lambda_n = \frac{n+1}{4n},$$

and the initial point $x_1 = (0.1, 0.1, \cdots, 0.1, 0, 0, \cdots)$, where the number of 0.1 is m. We perform Algorithm 1 for this example and compute the values of $\{\|x_n\|\}_{n=1}^{100}$ with the different m. These values are drawn in Figures 2. The curves of these values in Figure 2 demonstrate the convergence of the sequence $\{x_n\}$ generated by Algorithm 1.

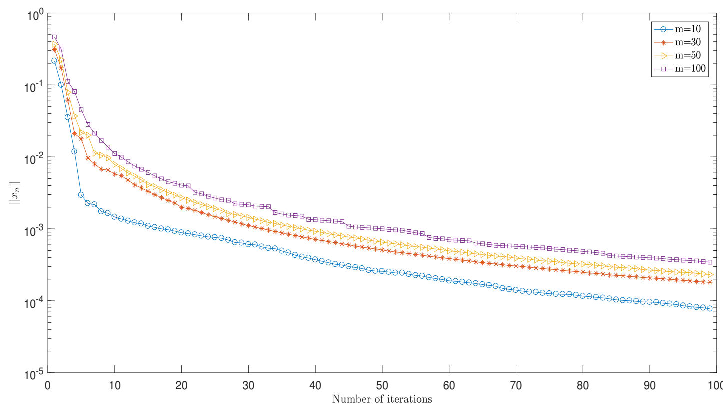


FIGURE 2. Numerical results for Example 4.2 with different m

5. CONCLUSION

In this paper, we presented a hybrid Halpern-based extragradient algorithm for finding a common solution of a pseudomonotone equilibrium problem and a fixed point problem of a nonexpansive mapping in a uniformly convex and uniformly smooth Banach space. The strong convergence of the proposed algorithm is proved. Some numerical examples are given to support the convergence of the proposed algorithm.

Acknowledgments
The authors are grateful to the reviewers for useful suggestions which improved the contents of this paper.

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