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# SELF-ADAPTIVE INERTIAL SHRINKING TSENG'S EXTRAGRADIENT METHOD FOR SOLVING A PSEUDOMONOTONE VARIATIONAL INEQUALITIES IN BANACH SPACES

O.T. MEWOMO*, A.O.E. OWOLABI, T.O. ALAKOYA

*School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa*

**Abstract.** In this paper, we propose an inertial shrinking Tseng's extragradient algorithm with a self-adaptive step size for solving pseudomonotone variational inequality problem with non-Lipschitz operators in the framework of 2-uniformly convex Banach spaces which are also uniformly smooth. Moreover, we prove a strong convergence result for the proposed algorithm under mild conditions on the control parameters. The main advantages of our algorithm are: our proposed algorithm solves the variational inequality problem with a larger class of mappings (pseudomonotone and non-Lipschitz operators); unlike the existing results in the literature, our algorithm does not require any linesearch technique even while the operator is non-Lipschitz; minimized number of projections per iteration compared to related results in the literature; and the inertial technique employed which speeds up the rate of convergence. Finally, we present some numerical examples to illustrate the efficiency of our algorithm in comparison with related methods in the literature.

**Keywords.** Inertial technique; Lyapunov functional; Pseudomonotone operator; Self-adaptive.

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## 1. INTRODUCTION

Let $C$ be a nonempty, closed, and convex subset of a real Banach space $E$ with induced norm $\|\cdot\|$, and let $E^*$ be the dual of $E$. Let $A : C \to E^*$ be a single-valued mapping. The variational inequality problem (VIP) is to find $z \in C$ such that $\langle x-z, Az\rangle \geq 0$ for all $x \in C$. We denote the solution set of the VIP by $VI(C,A)$. If $A$ is monotone, the VIP is known as monotone variational inequality problem, while it is known as pseudomonotone variational inequality problem if $A$ is pseudomonotone. It is known that the variational inequality theory has gradually become a significant tool for solving several problems arising from sciences, engineering, economics, minimization problems, mathematical programming, structural analysis, and optimization theory. Due to the wide applications, several iterative algorithms have been proposed for approximating the solutions of the VIP and related optimization problems; see, e.g., [1, 2, 3, 4, 5, 6, 7] and the references therein.

*Corresponding author.
E-mail address: mewomoo@ukzn.ac.za (O.T. Mewomo).
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An iterative method for solving the VIP in a Hilbert space is the projected gradient method, which is defined as follows:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = P_C(x_n - \lambda A x_n), \forall n \geq 1, \end{cases}$$

where $P_C$ is the projection operator onto the closed convex subset $C$ of $H$ and $\lambda > 0$ is a suitable step size. It is known that the sequence generated by this method converges weakly to a solution of the VIP if $A$ is inverse-strongly monotone, and converges strongly under some appropriate conditions to a solution of the VIP if $A$ is $\alpha$-strongly monotone and $L$-Lipschitz continuous, where $\lambda \in \left(0, \frac{2\alpha}{L^2}\right)$.

In order to relax the strong monotonicity assumption, Korpelevich introduced the following extragradient method (EGM) for solving the VIP in a finite dimensional Euclidean space $\mathbb{R}^m$:

$$\begin{cases} x_0 \in C, \\ y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n), \quad n \geq 1, \end{cases}$$

where $C \subset \mathbb{R}^m$ is a nonempty, closed, and convex set, $A : C \to \mathbb{R}^m$ is monotone and $L$-Lipschitz continuous, and $\lambda \in \left(0, \frac{1}{L}\right)$. Korpelevich proved that the sequence generated by (1) converges weakly to a solution of the VIP in a finite dimensional space. The extragradient method was further extended by Nadezhkina and Takahashi [8] to the framework of real Hilbert spaces. We note that the EGM requires computation of two projections in every iteration. This is really difficult to calculate in numerical simulation when set $C$ is a general closed and convex set and the efficiency of the method is seriously affected. In order to overcome this weakness, Censor et al. [9] (see also [10]) introduced the Subgradient Extragradient Method (SEGM) which involves the modification of one of the projections. The SEGM is given as follows:

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \lambda A x_n), \\ Q_n = \{z \in H : \langle x_n - \lambda A x_n - y_n, z - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{Q_n}(x_n - \lambda A y_n). \end{cases}$$

Censor et al. [10] proved that provided the solution set VI(C, A) is nonempty, the sequence $\{x_n\}$ generated by SEGM converges weakly to an element $p \in VI(C,A)$, where $p = \lim_{n\to\infty} P_{VI(C,A)} x_n$. Tseng [11] also proposed the following iterative scheme known as the Tseng's extragradient method (TEGM) in order to overcome the drawback in EGM :

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = y_n - \lambda (A y_n - A x_n), \end{cases}$$

where $A$ is a monotone and Lipschitz continuous operator and $\lambda \in \left(0, \frac{1}{L}\right)$. Clearly, the TEGM requires one projection to be computed per iteration and then has an advantage in computing projection over the EGM.

In optimization theory, the inertial technique plays a vital role in speeding up the rate of convergence of iterative algorithms. For recent works on this technique, we refer to [12, 13, 14, 15, 16, 17, 18, 19]. This technique originates from an implicit discretization method of the second-order dynamical systems in solving the smooth convex minimization problem. Alvarez and Attouch [20] employed the idea of the heavy ball method in order to construct the following algorithm for treating a maximal monotone operator:

$$\begin{cases} x_0, x_1 \in H, \\ y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = J^A_{\lambda_n} y_n, \end{cases}$$

where $J^A_{\lambda_n}$ is the resolvent operator of $A$, $\lambda_n > 0$, and $\theta_n(x_n - x_{n-1})$ is called the inertial extrapolation with $\theta_n \in [0,1)$. They proved that if $\{\lambda_n\}$ is increasing and $\theta_n \in [0,1)$ are selected so that $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty$, then the sequence $\{x_n\}$ generated by (1) converges weakly to a zero point of $A$. Thong et al. [21] employed the inertial technique for solving a monotone VIP in real Hilbert spaces. They proposed some hybrid projection methods and shrinking projection methods for solving the problem. One of the hybrid projection methods that the authors proposed is presented as follows:

$$\begin{cases} x_0, x_1 \in C, \\ u_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(u_n - \lambda A u_n), \\ z_n = \alpha_n u_n + (1-\alpha_n)(y_n - \lambda(Ay_n - Au_n)), \\ C_n = \{w \in H : \|z_n - w\| \leq \|u_n - w\|\}, \\ Q_n = \{w \in H : \langle w - x_n, x_1 - x_n \rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{1.1}$$

and one of the shrinking projection algorithms that they proposed is presented as follows:

$$\begin{cases} C_1 = C, \\ x_0, x_1 \in C, \\ u_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(u_n - \lambda A u_n), \\ z_n = \alpha_n u_n + (1-\alpha_n)(y_n - \lambda(Ay_n - Au_n)), \\ C_{n+1} = \{w \in C_n : \|z_n - w\| \leq \|u_n - w\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \end{cases} \tag{1.2}$$

where $\{\alpha_n\} \subset [0,1)$ with $0 \leq \alpha_n < \alpha < 1$, and $\{\theta_n\}$ is a bounded real sequence. They proved that the sequences $\{x_n\}$ generated by (1.1) and (1.2) converge strongly to an element in $VI(C,A)$ provided that $\lambda \in \left(0, \frac{1}{L}\right)$. Also, Cholamjiak et al. [22] proposed an algorithm which combines the inertial projection and contraction method with Mann-type technique for solving monotone variational inequality problems in real Hilbert spaces. They proved strong convergence of the proposed method under some appropriate conditions. Their proposed algorithm is presented as follows.

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**Algorithm 1.1.**

**Initialization:** Let $\gamma \in (0,2), \lambda \in \left(0, \frac{1}{L}\right), \theta > 0$, and $x_0, x_1 \in H$ be chosen arbitrarily. Let $\{\tau_n\} \in (a, 1-\beta_n)$ for some $a > 0$, where $\{\beta_n\} \in (0,1)$ is a sequence satisfying $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$.

**Iterative step:** Calculate $x_{n+1}$ as follows:

**Step 1:** Given $x_{n-1}$ and $x_n$ for each $n \geq 1$, choose $\theta_n$ such that $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta,\ \frac{\delta_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases}$$

**Step 2:** Compute

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda F(w_n)), \end{cases}$$

If $y_n = w_n$ or $F(y_n) = 0$, then Stop, $y_n$ is the solution. Otherwise go to **Step 3.**
**Step 3:** Calculate $z_n = w_n - \gamma\eta_n d_n$, where

$$d_n := w_n - y_n - \lambda(F(y_n) - F(w_n)), \quad \eta_n := \frac{\langle w_n - y_n, d_n\rangle}{\|d_n\|^2}.$$

**Step 4:** Calculate $x_{n+1} = (1 - \tau_n - \beta_n)x_n + \tau_n z_n$.
**Step 5:** Set $n := n+1$ and return to **Step 1.**

To extend the subgradient extragradient method to the framework of 2-uniformly convex and uniformly smooth Banach spaces, Chidume and Nnakwe [23] proposed the following iterative method:

$$\begin{cases} x_1 \in C, \text{ and } \sigma > 0, \\ y_n = \Pi_C J^{-1}(Jx_n - \sigma F(x_n)), \\ C_n = \{w \in E : \langle w - y_n, Jx_n - \sigma F(x_n) - Jy_n\rangle \leq 0\}, \\ x_{n+1} = \Pi_{C_n} J^{-1}(Jx_n - \sigma F(y_n)), n \geq 1, \end{cases} \tag{1.3}$$

where $J : E \to 2^{E^*}$ is the normalized duality mapping and $\Pi_C$ is the generalized projection of the Banach space $E$ onto $C$. They proved that the sequence generated by (1.3) converges weakly to the solution of the VIP.

Cai et al. [24] proposed the following algorithm which combines the Halpern's technique and the subgradient extragradient idea for solving the VIP with monotone and Lipschitz continuous mappings in 2-uniformly convex and uniformly smooth Banach spaces:

**Algorithm 1.2.**

**Step 0:** Let $x_1 \in E$ be a given starting point. Set $n = 1$.
**Step 1:** Given the current iterate $x_n$, compute $y_n = \Pi_C(Jx_n - \lambda_n Ax_n)$. If $x_n - y_n = 0$: STOP. Else, construct the set

$$T_n := \{z \in E : \langle Jx_n - \lambda_n Ax_n - Jy_n, z - y_n\rangle \leq 0\}$$

and compute $w_n = \Pi_{T_n}(Jx_n - \lambda_n Ay_n)$ and update the next iterate via

$$x_{n+1} = J^{-1}(\alpha_n Jx_1 + (1 - \alpha_n)Jw_n).$$

**Step 2:** Set $n \leftarrow n+1$ and go to **Step 1**.

where $A : E \to E^*$ is monotone and Lipschitz continuous. Under some certain assumptions, strong convergence was obtained. Furthermore, they modified the algorithm by employing the linesearch approach, and strong convergence was also obtained for this modification.

Shehu [25] introduced the following algorithm for approximating a solution of the VIP in a 2-uniformly convex and uniformly smooth Banach space $E$:

**Algorithm 1.3.**

$$\begin{cases} x_1 \in E, \\ y_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ x_{n+1} = J^{-1}(Jy_n - \lambda_n(Ay_n - Ax_n)), \end{cases}$$

where $A : E \to E^*$ is monotone and $L$-Lipschitz continuous, $\Pi_C$ is the generalized projection from $E$ onto $C$, $J$ is the normalized duality mapping on $E$, and the sequence of step sizes satisfies the following inequality:

$$0 < a \leq \lambda_n \leq b < \frac{1}{\sqrt{2\mu}\kappa L},$$

where $\kappa > 0$ is the 2-uniform smoothness constant of $E^*$, $\mu > 0$ is the 2-uniform convexity constant of $E$, and $L$ is the Lipschitz constant of $A$. It was proved that the sequence $\{x_n\}$ generated by Algorithm 1.3 converges weakly to a point in $VI(C,A)$ when $J$ is assumed to be weakly sequentially continuous. The author further proposed a modification of Algorithm 1.3 by employing the linesearch technique for solving the VIP. The new modification is presented as follows:

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**Algorithm 1.4.**

**Step 0:** Give $\gamma > 0, \ell \in (0,1)$ and $\theta \in \left(0, \frac{1}{\sqrt{2\mu}\kappa}\right)$. Let $x_1 \in E$ be a given starting point. Set $n := 1$.
**Step 1:** Compute $y_n := \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)$, where $\lambda_n$ is chosen to be the largest $\lambda \in \{\gamma, \gamma\ell, \gamma\ell^2, \cdots\}$ satisfying $\lambda \|Ax_n - Ay_n\| \leq \theta \|x_n - y_n\|$. If $x_n - y_n = 0$: STOP.
**Step 2:** Compute $x_{n+1} = J^{-1}(Jy_n - \lambda_n(Ay_n - Ax_n))$.
**Step 3:** Set $n \leftarrow n+1$, and go to **Step 1**.

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In this paper, inspired and motivated by the works above, we propose and study an inertial algorithm which combines the Tseng's extragradient method with shrinking projection technique for approximating a solution of the VIP with pseudomonotone operators in 2-uniformly convex and uniformly smooth Banach spaces. We establish a strong convergence theorem for the proposed method. Finally, we present some numerical examples to illustrate the efficacy of our algorithm as well as compare it with some of the existing works in the literature. We highlight below the advantages of our method over existing results in the literature.

(i) The ease in evaluating minimal number of projections onto the feasible set $C$ per iteration makes our method efficient for computation.
(ii) Our method solves the VIP with a larger class of mappings (pseudomonotone and non-Lipschitz mappings).
(iii) While the cost operator is non-Lipschitz, our method does not require any linesearch technique which slows down convergence rate of algorithms. We employ a more efficient self-adaptive step size technique.
(iv) Our method employs the inertial technique to accelerate the rate of convergence.

Subsequent sections of this paper are organised as follows: In Section 2, we recall some basic definitions and lemmas that are relevant in establishing our main result. In Section 3, we present our proposed method and highlight some of its important features while in Section 4, we establish some lemmas that are useful in proving the strong convergence theorem of our proposed algorithm and then prove the strong convergence theorem. In Section 5, we present some numerical examples to illustrate the performance of our method and compare it with some related methods in the literature. Finally, in Section 6, we give a concluding remark.

## 2. PRELIMINARIES

In this section, we recall some useful lemmas and definitions required to establish our result. Let $C$ be a nonempty, closed, and convex subset of a real Banach space $E$. Let $E^*$ and $\langle \cdot,\cdot\rangle$ denote the dual space of $E$ and the duality pairing between elements of $E$ and $E^*$, respectively. We denote the strong convergence of sequence $\{x_n\}$ to $x$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$. Let $E$ be a real Banach space and $1 < q \leq 2 \leq p < +\infty$, where $\frac{1}{p}+\frac{1}{q}=1$. The modulus of smoothness of $E$ denoted by $\rho_E(\varepsilon)$ is the function $\rho_E : [0,+\infty) \to [0,+\infty)$ defined by

$$\begin{aligned}\rho_E(\tau) &= \sup\left\{\frac{\|x-y\|+\|x+y\|}{2}-1 : \|x\|=1, \|y\|=\tau\right\}\\ &= \sup\left\{\frac{\|x-\tau y\|+\|x+\tau y\|}{2}-1 : \|x\|=1=\|y\|\right\}.\end{aligned}$$

$\lim_{\tau\to 0^+}\frac{\rho_E(\tau)}{\tau}=0$ if and only if $E$ is uniformly smooth, and $E$ is said to be $q$-uniformly smooth if there exists a constant $D_q>0$ such that $\rho_E(\tau)\leq D_q\tau^q$. It is 2-uniformly smooth if there exists a constant $D>0$ such that $\rho_E(\tau)<D\tau^2$. $E$ is said to be smooth if

$$\lim_{\tau\to 0}\frac{\|x+\tau y\|-\|x\|}{\tau}$$

exists for all $x,y\in S_E$. It is known that every 2-uniformly smooth Banach space is uniformly smooth. The modulus of convexity of $E$ denoted by $\delta_E(\varepsilon)$ is the function $\delta_E : (0,2]\to[0,1]$ defined by

$$\delta_E(\varepsilon)=\inf\left\{1-\frac{\|x+y\|}{2} : \|x\|=\|y\|=1; = \|x-y\|\geq\varepsilon\right\}.$$

$\delta_E(\varepsilon)>0$ for all $\varepsilon\in(0,2]$ if and only if $E$ is uniformly convex, and $E$ is $p$-uniformly convex if there exists a constant $C_p>0$ such that $\delta_E(\varepsilon)\geq C_p\varepsilon^p$ for all $\varepsilon\in(0,2]$. Also, $E$ is 2-uniformly convex if there exists a constant $c$ such that $\delta_E(\varepsilon)>c\varepsilon^2$ for any $\varepsilon\in(0,2]$. It is clear that every 2-uniformly convex Banach space is uniformly convex. The Banach space $E$ is said to be strictly convex if $\|x+y\|<2$ for all $x,y\in S_E$ with $x\neq y$, where $S_E=\{x\in E : \|x\|=1\}$ is the unit sphere of $E$. Every uniformly convex Banach space is strictly convex and reflexive. It is known that if $E$ is $p$-uniformly convex and uniformly smooth, then its dual $E^*$ is $q$-uniformly smooth and uniformly convex. For more details on the geometry of Banach spaces, we refer to [26, 27].

Now, we recall the normalized duality mapping $J : E\to 2^{E^*}$ defined by

$$J(x)=\{x^*\in E^* : \langle x,x^*\rangle=\|x\|^2=\|x^*\|^2, \forall x\in E\}.$$

Note that if $E$ is smooth, then $J$ is one-to-one and single-valued. Furthermore, if $E$ is 2-uniformly convex and uniformly smooth, then the duality mapping $J$ is norm-to-norm uniformly continuous on bounded subsets of $E$. Consider the Lyapunov functional $\phi : E\times E\to\mathbb{R}_+$ defined by

$$\phi(x,y)=\|x\|^2-2\langle x,Jy\rangle+\|y\|^2, \forall x,y\in E.$$

It is easy to obtain from the definition above that $\phi(x,x)=0$ for all $x,y\in E$. If $E$ is strictly convex, then $\phi(x,y)=0\Leftrightarrow x=y$. In real Hilbert spaces, $\phi(x,y)=\|x-y\|^2$ and $\Pi_C=P_C$, where $P_C : H\to C$ is the metric projection of $H$ onto $C$ and $\Pi_C$ is the generalized projection operator given by

$$\Pi_C=\inf_{y\in C}\{\phi(x,y), \quad \forall x\in E\}.$$

From the definition of $\phi$, it is clear that

$$0 \leq (\|x\|^2 - \|y\|^2) \leq \phi(x,y) \leq (\|x\|^2 + \|y\|^2).$$

For all $x,y \in E$ and $\alpha \in (0,1)$, the Lyapunov functional $\phi$ satisfies the following properties:

(P1) $\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x-z, Jz-Jy\rangle$;
(P2) $\phi(x,y) + \phi(y,x) = 2\langle x-y, Jx-Jy\rangle$;
(P3) $\phi(x,y) \leq \langle x, Jx-Jy\rangle + \langle y-x, Jy\rangle \leq \|x\|\|Jx-Jy\| + \|y-x\|\|y\|$;
(P4) $\phi(x, J^{-1}(\alpha Jz + (1-\alpha)Jy)) \leq \alpha\phi(x,z) + (1-\alpha)\phi(x,y)$.

We also consider the functional $V : E \times E \to \mathbb{R}$ which is defined by

$$V(x,x^*) = \|x\|^2 - 2\langle x,x^*\rangle + \|x^*\|^2, \quad \forall x \in E, x^* \in E^*.$$

It is clear that $V(x,x^*) = \phi(x,J^{-1}x^*)$. Observe that if $E$ is a reflexive, strictly convex, and smooth Banach space, then (see [28, 29]) $V(x,x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x,x^*+y^*)$ for all $x \in E, x^*, y^* \in E^*$.

**Definition 2.1.** Let $A : C \to E^*$ be an operator. Then $A$ is said to be

(i) monotone if $\langle x-y, Ax-Ay\rangle \geq 0$ for all $x,y \in C$;
(ii) pseudomonotone if $\langle y-x, Ax\rangle \geq 0 \Rightarrow \langle y-x, Ay\rangle \geq 0$ for all $x,y \in C$;
(iii) $L$-Lipschitz continuous if there exists a constant $L > 0$ such that $\|Ax-Ay\| \leq L\|x-y\|$ for all $x,y \in C$.
(iv) uniformly continuous if, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$, such that $\|Ax-Ay\| < \varepsilon$ whenever $\|x-y\| < \delta$ for all $x,y, \in C$.

It is known that monotone mappings are pseudomonotone. However, the converse is not true. For instance, the mapping $f : (0,+\infty) \to (0,+\infty)$ defined by $fx = \frac{1}{1+x}$ is pseudomonotone but not monotone. Also, we note that uniform continuity is a weaker notion than Lipschitz continuity. For more examples on pseudomonotone operators that are not monotone, we refer to [30, 31]. Also, it is known that if $D$ is a convex subset of $E$, then $A : D \to \text{range}(A)$ is uniformly continuous if and only if, for every $\varepsilon > 0$, there exists a constant $K < +\infty$ such that

$$\|Ax-Ay\| \leq K\|x-y\| + \varepsilon, \quad \forall x,y \in D. \tag{2.1}$$

**Lemma 2.1.** [26] *Let $C$ be a nonempty, closed, and convex subset of a reflexive, strictly convex, and smooth Banach space $E$. Given that $x \in E$ and $z \in C$, $z = \Pi_C x \Leftrightarrow \langle y-z, Jx-Jz\rangle \leq 0$, and $\phi(y,z) + \phi(z,x) \leq \phi(y,x)$ for all $y \in C, x \in E$.*

**Lemma 2.2.** [32] *Let $C$ be a nonempty and convex subset of a Banach space $E$, and let $A$ be a hemicontinuous mapping of $C$ into $E$. Let $z \in C$ such that $\langle x-z, Ax\rangle \geq 0$ for all $x \in C$. Then $z$ is a solution to the VIP.*

**Lemma 2.3.** [33] *Let $E$ be a smooth and uniformly convex Banach space. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in $E$ such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\phi(x_n,y_n) \to 0$ as $n \to \infty$, then $\|x_n - y_n\| \to 0$ as $n \to \infty$.*

**Remark 2.1.** It is known that the converse of Lemma 2.3 also holds if the sequences $\{x_n\}$ and $\{y_n\}$ are bounded.

**Lemma 2.4.** [34] *Let $\frac{1}{p} + \frac{1}{q} = 1, p,q > 1$. Space $E$ is $q$-uniformly smooth if and only if its dual $E^*$ is $p$-uniformly convex.*

**Lemma 2.5.** [35] *Let $E$ be a 2-uniformly smooth Banach space with the best smoothness constant $\kappa > 0$. Then, the following inequality holds: $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, Jx\rangle + 2\|\kappa y\|^2$ for all $x, y \in E$.*

**Lemma 2.6.** [31, 36] *Let $E$ be a $p$-uniformly convex Banach space with $p \geq 2$. Then*

$$\langle x-y, j_p(x) - j_p(y)\rangle \geq \frac{1}{2^{p-2}c^p p}\|x-y\|^p, \quad \forall x,y \in E, \forall j_p(x) \in J_p(x), j_p(y) \in J_p(y),$$

*where $c$ is the $p$-uniformly convexity constant.*

**Lemma 2.7.** [37] *Suppose that $E$ is a 2-uniformly convex Banach space. Then there exists a constant $c \geq 1$ such that $\phi(x,y) \geq \frac{1}{c}\|x-y\|^2$ for all $x, y \in E$.*

**Lemma 2.8.** [38] *Suppose that $\{\lambda_n\}$ and $\{\theta_n\}$ are two nonnegative real sequences such that $\lambda_{n+1} \leq \lambda_n + \phi_n$ for all $n \geq 1$. If $\sum_{n=1}^{\infty} \phi_n < +\infty$, then $\lim_{n\to\infty} \lambda_n$ exists.*

## 3. Proposed Method

In this section, we present our proposed algorithm. Let $C$ be a nonempty, closed, and convex subset of a 2-uniformly convex Banach space $E$ which is also uniformly smooth with dual $E^*$. Let $c$ and $\kappa$ be 2-uniformly convexity constant and 2-uniformly smoothness constant of $E$ and $E^*$, respectively. We establish the convergence of our proposed algorithm under the following conditions:

**Assumption 3.1**

(A1) The feasible set $C$ is nonempty, closed, and convex.
(A2) Mapping $A : C \to E^*$ is pseudomonotone and uniformly continuous on $E$.
(A3) Solution set of the VIP is nonempty, that is, $VI(C,A) \neq \emptyset$.
(A4) $\{\rho_n\}$ is a nonnegative sequence such that $\sum_{n=1}^{\infty} \rho_n < +\infty$.

Now, we present our proposed algorithm as follows:

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**Algorithm 3.1.**

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**Step 0:** Select $\mu \in \left(0, \frac{1}{\kappa\sqrt{2c}}\right), \lambda_1 > 0$, and $\theta_n \in [-\theta, \theta]$ for some $\theta > 0$. Let $x_0, x_1 \in E$ be arbitrary. Set $C_1 = C$ and $n := 1$.
**Step 1:** Compute $w_n = J^{-1}(Jx_n + \theta_n(Jx_n - Jx_{n-1}))$.
**Step 2:** Compute $y_n = \Pi_C J^{-1}(Jw_n - \lambda_n Aw_n)$.
**Step 3:** Compute $z_n = J^{-1}(Jy_n - \lambda_n(Ay_n - Aw_n))$.
**Step 4:** Construct

$$C_{n+1} = \left\{p \in C_n : \phi(p, z_n) \leq \phi(p, w_n) - \left(1 - \frac{2c\mu^2\kappa^2\lambda_n^2}{\lambda_{n+1}^2}\right)\phi(y_n, w_n)\right\},$$

and compute $x_{n+1} = \Pi_{C_{n+1}} x_0$, where the adaptive step-size is given by

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|}, \ \lambda_n + \rho_n\}, & Aw_n - Ay_n \neq 0, \\ \lambda_n + \rho_n, & \text{otherwise.} \end{cases} \tag{3.1}$$

Set $n := n+1$ and go to **Step 1**.

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**Remark 3.1.** (i) We note that the cost operator in our proposed algorithm is pseudomonotone (a broader set of mappings than monotone mappings) and uniformly continuous (a much weaker assumption than the Lipschitz continuity assumption used in several of the existing results in the literature).

(ii) Unlike in several of the existing results in the literature on the VIP with non-Lipschitz operators, our proposed method does not require any linesearch technique. Indeed, it uses a simple step size rule which generates a non-monotonic sequence of step sizes. The step size is constructed such that it reduces the dependence of the algorithm on the initial step size $\lambda_1$.

(iii) Our work is an extension and improvement on the work of [39] from the framework of Hilbert spaces to Banach spaces.

(iv) We employ the inertial technique and self-adaptive step size to speed up the convergence rate.

## 4. Convergence Analysis

In this section, we prove some lemmas required to establish our strong convergence theorem.

**Lemma 4.1.** *Let $\{\lambda_n\}$ be the sequence of step sizes generated by Algorithm 3.1. Then, $\{\lambda_n\}$ is well defined and $\lim_{n\to\infty}\lambda_n = \lambda \in [\min\{\frac{\mu}{N},\lambda_1\},\lambda_1+\Psi]$, where $\Psi = \sum_{n=1}^{\infty}\rho_n$ and for some $N>0$.*

*Proof.*
Since $A$ is uniformly continuous, then by (2.1) it follows that for any given $\varepsilon>0$, there exists $K<+\infty$ such that $\|Aw_n - Ay_n\| \le K\|w_n - y_n\| + \varepsilon$. Thus, for the case $Aw_n - Ay_n \neq 0$ for all $n \ge 1$ we have

$$\frac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|} \ge \frac{\mu\|w_n - y_n\|}{K\|w_n - y_n\| + \varepsilon} = \frac{\mu\|w_n - y_n\|}{(K+\varepsilon_1)\|w_n - y_n\|} = \frac{\mu}{N},$$

where $\varepsilon = \varepsilon_1\|w_n - y_n\|$ for some $\varepsilon_1 \in (0,1)$ and $N = K+\varepsilon_1$. Therefore, by the definition of $\lambda_{n+1}$, the sequence $\{\lambda_n\}$ has lower bound $\min\{\frac{\mu}{N},\lambda_1\}$ and has upper bound $\lambda_1+\Psi$. By Lemma 2.8, the limit $\lim_{n\to\infty}\lambda_n$ exists and denoted by $\lambda = \lim_{n\to\infty}\lambda_n$. Clearly, $\lambda \in \left[\min\{\frac{\mu}{N},\lambda_1\},\lambda_1+\Psi\right]$. □

Observe that by (3.1), we have

$$\lambda_{n+1} = \min\left\{\frac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|},\lambda_n+\rho_n\right\} \le \frac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|},$$

which implies that

$$\|Aw_n - Ay_n\| \le \frac{\mu}{\lambda_{n+1}}\|w_n - y_n\|, \quad \forall n \ge 1. \tag{4.1}$$

**Lemma 4.2.** *Let $\{w_n\}$ and $\{y_n\}$ be sequences generated by Algorithm 3.1, and suppose $\{x_n\}$ is bounded. Let $\{w_{n_k}\}$ be a subsequence of $\{w_n\}$ which converges weakly to some $\bar{x} \in E$ as $k \to \infty$ and $\lim_{k\to\infty}\|w_{n_k} - y_{n_k}\| = 0$, then $\bar{x} \in VI(C,A)$.*

*Proof.* By using the definition of $\{y_n\}$ and Lemma 2.1, we obtain $\langle Jw_{n_k} - \lambda_{n_k} Aw_{n_k} - Jy_{n_k}, y - y_{n_k}\rangle \leq 0$ for all $y \in C$. Equivalently, $\frac{1}{\lambda_{n_k}}\langle Jw_{n_k} - Jy_{n_k}, y - y_{n_k}\rangle \leq \langle Aw_{n_k}, y - y_{n_k}\rangle$ for all $y \in C$. It follows that

$$\frac{1}{\lambda_{n_k}}\langle Jw_{n_k} - Jy_{n_k}, y - y_{n_k}\rangle + \langle Aw_{n_k}, y_{n_k} - w_{n_k}\rangle \leq \langle Aw_{n_k}, y - w_{n_k}\rangle, \forall y \in C. \tag{4.2}$$

Since $\|w_{n_k} - y_{n_k}\| \to 0$ as $k \to \infty$ and $J$ is norm-to-norm uniformly continuous on subsets of $E$, we have $\|Jw_{n_k} - Jy_{n_k}\| \to 0$. By taking limit as $k \to \infty$ in (4.2), we arrive at

$$\liminf_{k\to\infty}\langle Aw_{n_k}, y - w_{n_k}\rangle \geq 0, \forall y \in C. \tag{4.3}$$

Furthermore,

$$\langle Ay_{n_k}, y - y_{n_k}\rangle = \langle Ay_{n_k} - Aw_{n_k}, y - w_{n_k}\rangle + \langle Aw_{n_k}, y - w_{n_k}\rangle + \langle Ay_{n_k}, w_{n_k} - y_{n_k}\rangle. \tag{4.4}$$

Since $\|w_{n_k} - y_{n_k}\| \to 0$, then $\lim_{k\to\infty}\|Aw_{n_k} - Ay_{n_k}\| = 0$ due to the uniform continuity of $A$, which together with (4.3) and (4.4) gives

$$\liminf_{n\to\infty}\langle Ay_{n_k}, y - y_{n_k}\rangle \geq 0, \quad \forall y \in C. \tag{4.5}$$

Now, choose a sequence $\{\varepsilon_k\}$ of positive numbers such that $\{\varepsilon_k\}$ is decreasing and $\varepsilon_k \to 0$ as $k \to \infty$. Let $N_k$ represent the smallest positive integer for any $k$ such that

$$\langle Ay_{n_j}, y - y_{n_j}\rangle + \varepsilon_k \geq 0, \forall j \geq N_k, \tag{4.6}$$

where the existence of $N_k$ follows from (4.5). Observe that $\{N_k\}$ is increasing since $\{\varepsilon_k\}$ is decreasing. Moreover, since $\{y_{n_k}\} \subset C$ for each $k$, we can suppose $Ay_{N_k} \neq 0$ (otherwise, $y_{N_k}$ is a solution). For some bounded sequence $\{u_{N_k}\} \subset E$ with $\langle Ay_{N_k}, u_{N_k}\rangle = 1$ for each $k \geq 1$, we deduce from (4.6) that $\langle Ay_{N_k}, y + \varepsilon_k u_{N_k} - y_{N_k}\rangle \geq 0$ for all $y \in C$. Since $A$ is pseudomonotone, we have $\langle A(y + \varepsilon_k u_{N_k}), y + \varepsilon_k u_{N_k} - y_{N_k}\rangle \geq 0$ for all $y \in C$. Therefore, it follows from the last inequality that

$$\langle Ay, y - y_{N_k}\rangle \geq \langle Ay - A(y + \varepsilon_k u_{N_k}), y + \varepsilon_k u_{N_k} - y_{N_k}\rangle - \varepsilon_k\langle Ay, u_{N_k}\rangle, \quad \forall y \in C. \tag{4.7}$$

Since $\{u_{N_k}\}$ is bounded and $\lim\limits_{k\to\infty}\varepsilon_k = 0$, it follows that $\lim\limits_{k\to\infty}\varepsilon_k u_{N_k} = 0$. Since $A$ is uniformly continuous, $\{y_{N_k}\}$ and $\{u_{N_k}\}$ are bounded, together with the fact that $\lim_{k\to\infty}\varepsilon_k u_{N_k} = 0$, it follows from (4.7) that $\liminf_{k\to\infty}\langle Ay, y - y_{N_k}\rangle \geq 0$ for all $y \in C$. Hence, we obtain

$$\langle Ay, y - \bar{x}\rangle = \lim_{k\to\infty}\langle Ay, y - y_{N_k}\rangle = \liminf_{k\to\infty}\langle Ay, y - y_{N_k}\rangle \geq 0, \quad \forall y \in C.$$

Therefore, by Lemma 2.2, we obtain $\bar{x} \in VI(C,A)$. □

**Lemma 4.3.** *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1. Then*

$$\phi(p, z_n) \leq \phi(p, w_n) - \left(1 - \frac{2c\mu^2\kappa^2\lambda_n^2}{\lambda_{n+1}^2}\right)\phi(y_n, w_n). \tag{4.8}$$

*Proof.* Fix $p \in VI(C,A)$. From the definitions of $z_n$ and $\phi$, we have

$$\begin{aligned}\phi(p, z_n) &= \|p\|^2 - 2\langle p, Jy_n - \lambda_n(Ay_n - Aw_n)\rangle + \|Jy_n - \lambda_n(Ay_n - Aw_n)\|^2 \\ &= \|p\|^2 - 2\langle p, Jy_n\rangle + 2\lambda_n\langle p, Ay_n - Aw_n\rangle + \|Jy_n - \lambda_n(Ay_n - Aw_n)\|^2. \end{aligned} \tag{4.9}$$

By using Lemma 2.4, we see that $E^*$ is 2-uniformly smooth. It follows from Lemma 2.5 that

$$\|Jy_n - \lambda_n(Ay_n - Aw_n)\|^2 \leq \|Jy_n\|^2 - 2\lambda_n\langle y_n, Ay_n - Aw_n\rangle + 2\kappa^2\lambda_n^2\|Ay_n - Aw_n\|^2. \tag{4.10}$$

Substituting (4.10) into (4.9), and applying (P1), we obtain

$$\begin{aligned}\phi(p,z_n) &\le \|p\|^2 - 2\langle p, Jy_n\rangle + 2\lambda_n\langle p, Ay_n - Aw_n\rangle + \|Jy_n\|^2 - 2\lambda_n\langle y_n, Ay_n - Aw_n\rangle \\ &\quad + 2\kappa^2\lambda_n^2\|Ay_n - Aw_n\|^2 \\ &= \phi(p,w_n) + \phi(w_n,y_n) + 2\langle p - w_n, Jw_n - Jy_n\rangle + 2\lambda_n\langle p - y_n, Ay_n - Aw_n\rangle \\ &\quad + 2\kappa^2\lambda_n^2\|Ay_n - Aw_n\|^2. \end{aligned} \tag{4.11}$$

By (P2), we have

$$\phi(w_n,y_n) = -\phi(y_n,w_n) + 2\langle y_n - w_n, Jy_n - Jw_n\rangle. \tag{4.12}$$

From the definition of $y_n = \Pi_C J^{-1}(Jw_n - \lambda_n Aw_n)$, we have $\langle p - y_n, Jw_n - \lambda_n Aw_n - Jy_n\rangle \le 0$, which implies that

$$\langle p - y_n, Jw_n - Jy_n\rangle \le \lambda_n\langle p - y_n, Aw_n\rangle. \tag{4.13}$$

Applying (4.12) and (4.13) in (4.11), we obtain

$$\begin{aligned}\phi(p,z_n) &\le \phi(p,w_n) - \phi(y_n,w_n) + 2\langle y_n - w_n, Jy_n - Jw_n\rangle + 2\langle p - w_n, Jw_n - Jy_n\rangle \\ &\quad + 2\lambda_n\langle p - y_n, Ay_n - Aw_n\rangle + 2\kappa^2\lambda_n^2\|Ay_n - Aw_n\|^2 \\ &\le \phi(p,w_n) - \phi(y_n,w_n) - 2\lambda_n\langle y_n - p, Aw_n\rangle + 2\lambda_n\langle p - y_n, Ay_n - Aw_n\rangle + 2\kappa^2\lambda_n^2\|Ay_n - Aw_n\|^2 \\ &= \phi(p,w_n) - \phi(y_n,w_n) - 2\lambda_n\langle y_n - p, Ay_n\rangle + 2\kappa^2\lambda_n^2\|Ay_n - Aw_n\|^2. \end{aligned} \tag{4.14}$$

Since $p \in VI(C,A)$, we have $\langle Ap, y_n - p\rangle \ge 0, \forall p \in C$. By the pseudomonotonicity of $A$, it follows that $\langle Ay_n, y_n - p\rangle \ge 0$. Hence, we obtain from (4.14) that

$$\phi(p,z_n) \le \phi(p,w_n) - \phi(y_n,w_n) + 2\kappa^2\lambda_n^2\|Ay_n - Aw_n\|^2.$$

By (4.1) and Lemma 2.7, we have

$$\begin{aligned}\phi(p,z_n) &\le \phi(p,w_n) - \phi(y_n,w_n) + \frac{2c\mu^2\kappa^2\lambda_n^2}{\lambda_{n+1}^2}\phi(y_n,w_n) \\ &= \phi(p,w_n) - \left(1 - \frac{2c\mu^2\kappa^2\lambda_n^2}{\lambda_{n+1}^2}\right)\phi(y_n,w_n).\end{aligned}$$

This completes the proof. □

At this point, we state and prove our strong convergence theorem for the proposed algorithm.

**Theorem 4.1.** *Assume that Assumption 3.1 holds. Then, the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to $\bar{x} \in VI(C,A)$, where $\bar{x} = \Pi_{VI(C,A)}x_0$.*

*Proof.* We divide the proof into several steps as follows.
**Step 1.** The set $C_n$ is closed and convex.

Clearly, $C_1 = C$ is closed and convex. Suppose that $C_k$ and $D_k$ are closed and convex for some $k \ge 1$, where

$$D_k = \{p \in C_k : \phi(p,z_k) \le \phi(p,w_k) - \left(1 - \frac{2c\mu^2\kappa^2\lambda_k^2}{\lambda_{k+1}^2}\right)\phi(y_k,w_k)\}.$$

By the construction of $C_{k+1}$, we have $C_{k+1} = C_k \cap D_k$. Since $D_k$ and $C_k$ are closed and convex, then $C_{k+1}$ is closed and convex. By induction, we can conclude that $C_n$ is closed and convex.
**Step 2.** The sequence $\{x_n\}$ is well defined.

It is clear that $VI(C,A) \subset C = C_1$. Let $VI(C,A) \subset C_k$ for some $k \geq 1$, and let $p \in VI(C,A)$. From (4.8), it follows that

$$\phi(p,z_k) \leq \phi(p,w_k) - \left(1 - \frac{2c\mu^2\kappa^2\lambda_k^2}{\lambda_{k+1}^2}\right)\phi(y_k,w_k),$$

which implies that $VI(C,A) \subset D_k$. Hence $VI(C,A) \subset C_{k+1}$. By induction, we have $VI(C,A) \subset C_n$ for all $n \geq 1$. Thus the sequence $\{x_n\}$ is well defined.

**Step 3.** The sequence $\{x_n\}$ is bounded.

Since $VI(C,A)$ is a nonempty, closed, and convex subset of $E$, then there exists an element $y \in VI(C,A)$. It is clear that $x_{n+1} \in C_{n+1} \subset C_n$ for all $n \geq 1$. Also, we have $\langle y - x_{n+1}, Jx_0 - Jx_{n+1}\rangle \leq 0$ for all $y \in C_n$. From Lemma 2.1, we have $x_{n+1} = \Pi_{C_{n+1}}x_0$. Then, it follows that $\phi(x_n,x_0) \leq \phi(x_{n+1},x_0)$ for all $n \in \mathbb{N}$. Similarly, since $VI(C,A) \subset C_n$, we have $\phi(x_n,x_0) \leq \phi(y,x_0)$ for all $y \in VI(C,A)$, $n \in \mathbb{N}$, which implies that $\{\phi(x_n,x_0)\}$ is bounded. Therefore $\{x_n\}$ is bounded. Consequently, $\{y_n\}, \{w_n\}$, and $\{z_n\}$ are bounded.

**Step 4.** We show that the sequence $\{x_n\}$ converges strongly to $\bar{x} \in C$.

It is already known that $x_n = \Pi_{C_n}x_0$ and $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$. Then, it follows from Lemma 2.1 that

$$\phi(x_n,x_0) \leq \phi(x_{n+1},x_0) - \phi(x_{n+1},x_n) \leq \phi(x_{n+1},x_0).$$

Hence, sequence $\{\phi(x_n,x_0)\}$ is increasing. Therefore $\lim_{n\to\infty}\phi(x_n,x_0)$ exists. Clearly, from the construction of $C_n$, we have that $x_m = \Pi_{C_m}x_0 \in C_m \subset C_n$ for $m > n \geq 1$. Recall that $x_n = \Pi_{C_n}x_0$. From Lemma 2.1, it follows that $\phi(x_m,x_n) \leq \phi(x_m,x_0) - \phi(x_n,x_0) \to 0$ as $m,n \to \infty$, which implies by Lemma 2.3 that

$$\lim_{n\to\infty}\|x_m - x_n\| = 0. \tag{4.15}$$

This implies that $\{x_n\}$ is a Cauchy sequence. Hence, there exists an element $\bar{x} \in C$ such that $\lim_{n\to\infty}x_n = \bar{x}$.

**Step 5.** We show that $\bar{x} \in VI(C,A)$.

Letting Let $m = n+1$ in (4.15), we have $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Since $J$ is norm-to-norm uniformly continuous on each bounded subset of $E$, we have

$$\lim_{n\to\infty}\|Jx_{n+1} - Jx_n\| = 0. \tag{4.16}$$

From the definition of $w_n$, we see $Jw_n - Jx_n = \theta_n(Jx_n - Jx_{n-1})$, which together with (4.16) yields

$$\|Jw_n - Jx_n\| = |\theta_n|\|Jx_n - Jx_{n-1}\| \to 0 \ \text{ as } \ n \to \infty. \tag{4.17}$$

By Lemma 2.6, we obtain

$$\|w_n - x_n\|^2 \leq 2c^2\langle w_n - x_n, Jw_n - Jx_n\rangle \leq 2c^2\|w_n - x_n\|\|Jw_n - Jx_n\|,$$

which implies that

$$\|w_n - x_n\| \leq 2c^2\|Jw_n - Jx_n\| \to 0 \ \text{ as } \ n \to \infty. \tag{4.18}$$

Since $\{x_n\}$ and $\{w_n\}$ are bounded, it follows from Remark 2.1 that $\phi(x_n,w_n) \to 0$ as $n \to \infty$. By (4.16) and (4.17), we have $\|Jx_{n+1} - Jw_n\| \leq \|Jx_{n+1} - Jx_n\| + \|Jx_n - Jw_n\| \to 0$ as $n \to \infty$. By (P2), we obtain

$$\phi(x_{n+1},w_n) \leq \langle x_{n+1} - w_n, Jx_{n+1} - Jw_n\rangle \leq M\|Jx_{n+1} - Jw_n\| \to 0. \tag{4.19}$$

By Lemma 2.3, we have $\lim_{n\to\infty}\|x_{n+1}-w_n\|=0$. Recall that $x_{n+1}\in C_{n+1}$. Hence, we have

$$\phi(x_{n+1},z_n)\leq\phi(x_{n+1},w_n)-\left(1-\frac{2c\mu^2\kappa^2\lambda_n^2}{\lambda_{n+1}^2}\right)\phi(y_n,w_n). \tag{4.20}$$

Moreover, since $\lim_{n\to\infty}\lambda_n=\lambda>0$ and $\mu\in\left(0,\frac{1}{\kappa\sqrt{2c}}\right)$, we obtain

$$\lim_{n\to\infty}\left(1-\frac{2c\mu^2\kappa^2\lambda_n^2}{\lambda_{n+1}^2}\right)=(1-2c\mu^2\kappa^2)>0.$$

Then, there exists $n_0\in\mathbb{N}$ such that $1-\frac{2c\mu^2\kappa^2\lambda_n^2}{\lambda_{n+1}^2}>\varepsilon>0$ for all $n\geq n_0$. It follows from (4.20) that, for all $n\geq n_0$, $\phi(x_{n+1},z_n)\leq\phi(x_{n+1},w_n)-\varepsilon\phi(y_n,w_n)$. By (4.19), we have that $\lim_{n\to\infty}\phi(x_{n+1},z_n)=0$, which implies that $\lim_{n\to\infty}\|x_{n+1}-z_n\|=0$. Since $J$ is norm-to-norm uniformly continuous on bounded subset of $E$, we have $\|Jx_{n+1}-Jz_n\|\to0$ as $n\to\infty$. Hence, we obtain

$$\|Jz_n-Jw_n\|\leq\|Jz_n-Jx_{n+1}\|+\|Jx_{n+1}-Jw_n\|\to0 \text{ as } n\to\infty. \tag{4.21}$$

Since $J^{-1}$ is norm-to-norm uniformly continuous on bounded subset of $E^*$, we have $\lim_{n\to\infty}\|w_n-z_n\|=0$. $\phi(z_n,w_n)\to0$ as $n\to\infty$. From (4.8), we have

$$\left(1-\frac{2c\mu^2\kappa^2\lambda_n^2}{\lambda_{n+1}^2}\right)\phi(y_n,w_n)\leq\phi(p,w_n)-\phi(p,z_n). \tag{4.22}$$

From (P1), we have

$$\phi(p,w_n)=\phi(p,z_n)+\phi(z_n,w_n)+2\langle p-z_n,Jz_n-Jw_n\rangle,$$

which implies that

$$\begin{aligned}\phi(p,w_n)-\phi(p,z_n)&=\phi(z_n,w_n)+2\langle p-z_n,Jz_n-Jw_n\rangle\\&\leq\phi(z_n,w_n)+2\|p-z_n\|\|Jz_n-Jw_n\|.\end{aligned} \tag{4.23}$$

By taking the limit of (4.23) and applying (4.21), we see that $\lim_{n\to\infty}(\phi(p,w_n)-\phi(p,z_n))=0$. Since $\lim_{n\to\infty}\left(1-\frac{2c\mu^2\kappa^2\lambda_n^2}{\lambda_{n+1}^2}\right)>0$, it follows from (4.22) that $\lim_{n\to\infty}\phi(y_n,w_n)=0$. It follows from Lemma 2.3 that $\lim_{n\to\infty}\|y_n-w_n\|=0$. Since $\{x_n\}$ is a bounded sequence, then there exists a subsequence $\{x_{n_k}\}\subset\{x_n\}$ such that $x_{n_k}\rightharpoonup\bar{x}$. By (4.18) we have $w_{n_k}\rightharpoonup\bar{x}$. Moreover, we have $\lim_{n\to\infty}\|w_{n_k}-y_{n_k}\|=0$, and it follows from Lemma 4.2 that $\bar{x}\in VI(C,A)$.

**Step 6.** We show that $\bar{x}=\Pi_{VI(C,A)}x_0$.

Clearly, $x_n=\Pi_{C_n}x_0$ and $VI(C,A)\subset C_n$. Hence, from Lemma 2.1, it follows that

$$\langle y-x_n,Jx_0-Jx_n\rangle\leq0,\ \ \forall y\in VI(C,A). \tag{4.24}$$

Observe that $J$ is norm-to-norm uniformly continuous on bounded sets. By taking the limit of (4.24), we obtain $\langle y-\bar{x},Jx_0-J\bar{x}\rangle\leq0$ for all $y\in VI(C,A)$, which implies that $\bar{x}=\Pi_{VI(C,A)}x_0$. This completes the proof. □

Next, we have the following consequent result in the framework of Hilbert spaces.

**Corollary 4.1.** *Let $C$ be a nonempty, closed, and convex subset of a real Hilbert space $H$, and let $A:H\to H$ be a pseudomonotone and uniformly continuous mapping. Let $\{x_n\}$ be a sequence generated by the Algorithm presented as follows:*

**Algorithm 4.2.**

**Step 0:** Select $\mu \in \big(0,1\big), \lambda_1 > 0$, and $\theta_n \in [-\theta, \theta]$ for some $\theta > 0$. Let $x_0, x_1 \in E$ be arbitrary. Set $C_1 = C$ and $n := 1$.
**Step 1:** Compute $w_n = (x_n + \theta_n(x_n - x_{n-1}))$.
**Step 2:** Compute $y_n = P_C(w_n - \lambda_n A w_n)$.
**Step 3:** Compute $z_n = y_n - \lambda_n(Ay_n - Aw_n)$.
**Step 4:** Construct

$$C_{n+1} = \left\{ p \in C_n : \|z_n - p\|^2 \leq \|w_n - p\|^2 - \left(1 - \frac{\mu^2 \lambda_n^2}{\lambda_{n+1}^2}\right) \|w_n - y_n\|^2 \right\},$$

and compute $x_{n+1} = P_{C_{n+1}} x_0$, where the adaptive step-size is given by

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|}, \ \lambda_n + \rho_n\}, & Aw_n - Ay_n \neq 0, \\ \lambda_n + \rho_n, & \text{otherwise.} \end{cases}$$

Set $n := n+1$ and go to **Step 1**.

*If $VI(C,A) \neq \emptyset$ and all other conditions of Theorem 4.1 hold, then $\{x_n\}$ converges strongly to $\bar{x} = P_{VI(C,A)} x_0$.*

## 5. Numerical Examples

In this section, we present some numerical examples and compare our proposed method, (Proposed Alg.) Algorithm 3.1 with Algorithm 1.2 proposed by Cai *et al.* (Cai *et al.* Alg.), Algorithm 1.3 proposed by Shehu (Shehu Alg.), Algorithm 1.4 by Shehu (Shehu Alg.), Appendix 6.1 by Liu (Liu Alg.), and Appendix 6.2 by Tan & Cho (Tan & Cho Alg.). All numerical computations were carried out using Matlab version R2021(b). We plot the graphs of errors against the number of iterations in each case. In all the experiments, we use $\|x_{n+1} - x_n\| < 10^{-4}$ as the stopping criterion. We choose $\mu = 0.5, \theta_n = 0.75$, and $\lambda_1 = 0.25$ for our proposed Algorithm 3.1. $\lambda_n = 0.025, \alpha_n = \frac{1}{n+1}$ in Algorithm 1.2 and Algorithm 1.3. $\gamma = 0.65, \ell = 0.05, \theta = 0.1$ in Algorithm 1.4. For Appendix 6.1, $\lambda_n = 0.75, \alpha_n = \frac{2}{3n+1}$, and $\beta_n = \frac{2n+1}{5n+2}$ and $Sx = \frac{x}{3}$. $\mu = 0.5, \theta_n = 0.75$, and $\lambda_1 = 0.25$ in Appendix 6.2.

**Example 5.1.** Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $Ax := (ax_1 + bx_2 + a\sin(x_1), -bx_1 + cx_2 + c\sin(x_2))$ for all $x = (x_1, x_2) \in \mathbb{R}^2$, where $a, b$, and $c$ are real numbers. Then, $A$ is monotone and $L$-Lipschitz continuous with $L = \sqrt{2\max\{4a^2 + b^2, 4c^2 + b^2\}}$. The set of feasible solutions is given by $C = \{(x_1, x_2) \in \mathbb{R}^2 : -1 \leq x_i \leq 1\}$. It is clear that $VI(C,A) = \{(0,0)\}$. Let $z = (0,0)$ and use $\text{TOL}_n = \|x_n - z\| \leq 10^{-5}$ as the stopping criterion.

We consider the following cases for the numerical experiments:

Case 1: Take $x_0 = (2,3), x_1 = (1,2)$.

Case 2: Take $x_0 = (1,1), x_1 = (1,-2)$.

Case 3: Take $x_0 = (3,-2), x_1 = (-2,2)$.

Case 4: Take $x_0 = (0,1), x_1 = (1,2)$.

Te numerical results are reported in Figures 1-4 and Table 1.

TABLE 1. Numerical Results for Example 5.1

| | Case 1 | | Case 2 | | Case 3 | | Case 4 | |
|---|---|---|---|---|---|---|---|---|
| | Iter. | CPU Time | Iter. | CPU Time | Iter. | CPU Time | Iter. | CPU Time |
| Cai *et al. Alg.* | 83 | 0.0089 | 92 | 0.0091 | 73 | 0.0084 | 83 | 0.0086 |
| Shehu Alg. | 80 | 0.0045 | 88 | 0.0050 | 71 | 0.0045 | 80 | 0.0047 |
| Shehu Alg. | 64 | 0.0065 | 71 | 0.0065 | 59 | 0.0066 | 64 | 0.0064 |
| Liu Alg. | 70 | 0.0055 | 54 | 0.0053 | 54 | 0.0052 | 70 | 0.0053 |
| Tan & Cho Alg. | 16 | 0.0070 | 16 | 0.0064 | 16 | 0.0068 | 16 | 0.0066 |
| Proposed Alg. 3.1 | 16 | 0.0131 | 16 | 0.0123 | 16 | 0.0124 | 16 | 0.0124 |

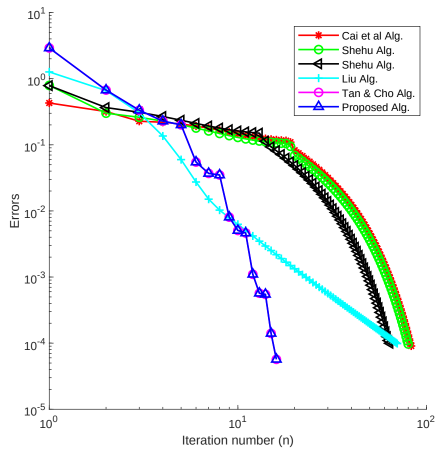


FIGURE 1. Example 5.1: Case 1

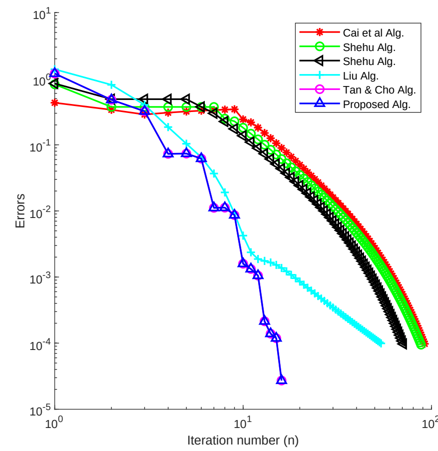


FIGURE 2. Example 5.1: Case 2

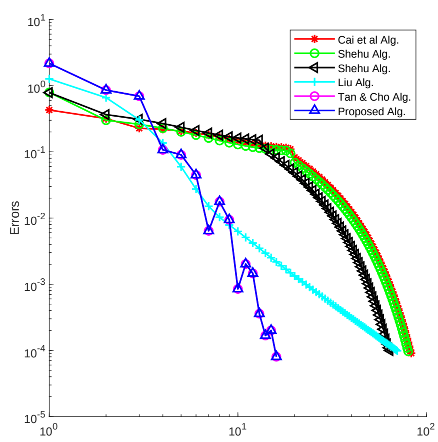


FIGURE 3. Example 5.1: Case 3

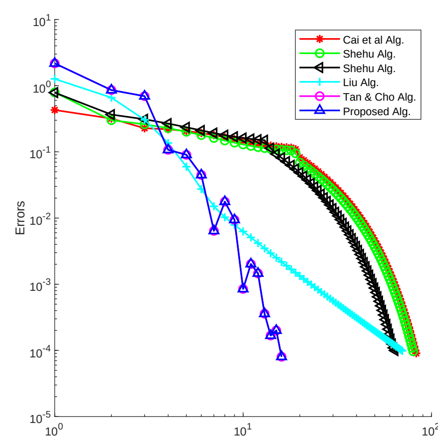


FIGURE 4. Example 5.1: Case 4

**Example 5.2.** Let $E = (l_2(\mathbb{R}),\ \|.\|_{l_2})$, where $l_2(\mathbb{R}) := \{x = (x_1, x_2, x_3, \dots),\ x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ and $\|x\|_{l_2} := \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{\frac{1}{2}}$ for all $x \in l_2(\mathbb{R})$. Let $\mathscr{C} = \{x \in l_2(\mathbb{R}) : ||x - a||_{l_2} \leq r\}$, where $a = (1, \frac{1}{3}, \frac{1}{9}, \cdots)$, $r = 2$. Then $\mathscr{C}$ is a nonempty, closed, and convex subset of $l_2(\mathbb{R})$. Define the operator $A, : l_2(\mathbb{R}) \to l_2(\mathbb{R})$ by $Ax = \frac{x}{2} + (1, 1, 0, 0, \dots,)$. Then, $A$ is pseudomonotone and uniformly continuous.

Consider the following cases for the numerical experiments:
Case 1: Take $x_0 = (3, 1, \frac{1}{3}, \cdots)$ and $x_1 = (2, 1, \frac{1}{2}, \cdots)$.
Case 2: Take $x_0 = (2, 1, \frac{1}{2}, \cdots)$ and $x_1 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots)$.
Case 3: Take $x_0 = (3, 1, \frac{1}{3}, \cdots)$ and $x_1 = (\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \cdots)$.
Case 4: Take $x_0 = (2, 1, \frac{1}{2}, \cdots)$ and $x_1 = (0.1, 0.01, 0.001, \cdots)$.
The numerical results are reported in Figures 5-8 and Table 2.

TABLE 2. Numerical Results for Example 5.2

| | Case 1 | | Case 2 | | Case 3 | | Case 4 | |
|---|---|---|---|---|---|---|---|---|
| | Iter. | CPU Time | Iter. | CPU Time | Iter. | CPU Time | Iter. | CPU Time |
| Cai *et al.* Alg. | 481 | 0.0192 | 491 | 0.0195 | 487 | 0.0233 | 481 | 0.0233 |
| Shehu Alg. | 475 | 0.0041 | 484 | 0.0045 | 482 | 0.0044 | 475 | 0.0044 |
| Shehu Alg. | 383 | 0.0134 | 390 | 0.0143 | 388 | 0.0133 | 383 | 0.0133 |
| Liu Alg. | 44 | 0.0046 | 44 | 0.0044 | 44 | 0.0045 | 44 | 0.0045 |
| Tan & Cho Alg. | 53 | 0.0055 | 53 | 0.0057 | 54 | 0.0057 | 53 | 0.0060 |
| Proposed Alg. 3.1 | 37 | 0.0116 | 37 | 0.0113 | 37 | 0.0110 | 37 | 0.0123 |

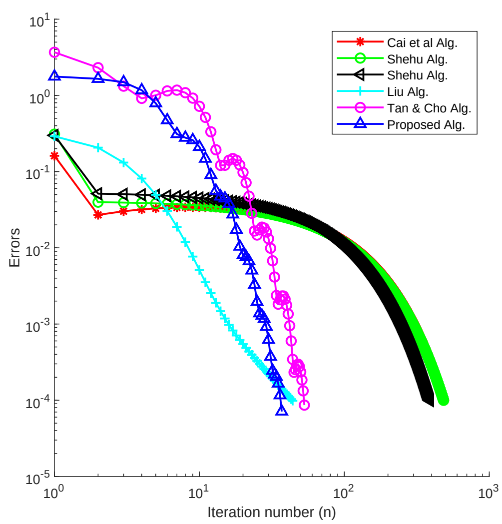


FIGURE 5. Example 5.2: Case 1

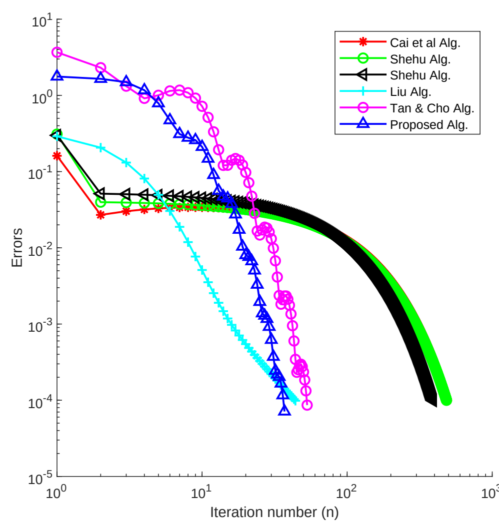


FIGURE 6. Example 5.2: Case 2

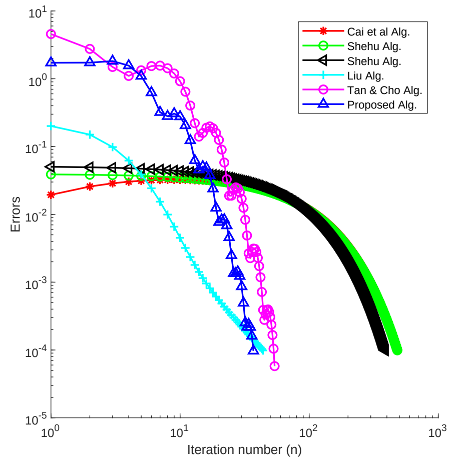


FIGURE 7. Example 5.2: Case 3

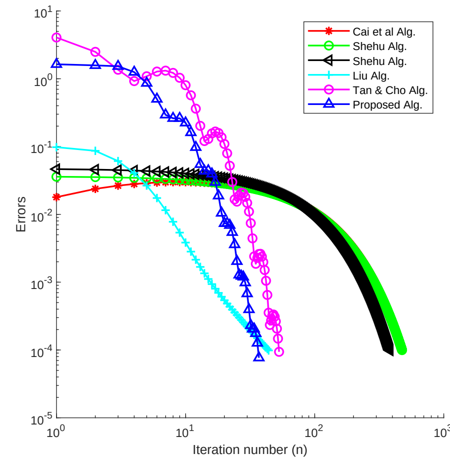


FIGURE 8. Example 5.2: Case 4

## 6. CONCLUSION

In this paper, we studied the class of pseudomonotone variational inequalities with non-Lipschitz operators and proposed an inertial shrinking Tseng's extragradient algorithm with self-adaptive step sizes for approximating the solution of the problem in the framework of 2-uniformly convex Banach spaces which are also uniformly smooth. Moreover, we proved a strong convergence result for the proposed algorithm under mild conditions on the control parameters. We also provided some numerical experiments in order to illustrate the efficiency of our algorithm as well as compare it with related methods in the literature.

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## REFERENCES

[1] M.U. Wickramasinghe, O.T. Mewomo, T.O. Alakoya, O.S. Iyiola, Mann-type approximation scheme for solving a new class of split inverse problems in Hilbert spaces, Appl. Anal. (2023). DOI: 10.1080/00036811.2023.2233977.
[2] L. Liu, B. Tan, S.Y. Cho, On the resolution of variational inequality problems with a double-hierarchical structure, J. Nonlinear Convex Anal. 21 (2020), 377-386.
[3] L.V. Nguyen, Q.H. Ansari, X. Qin, Weak sharpness and finite convergence for solutions of nonsmooth variational inequalities in Hilbert spaces, Appl. Math. Optim. 84 (2021), 807-828.
[4] J.S. Jung, A general iterative algorithm for split variational inclusion problems and fixed point problems of a pseudocontractive mapping, J. Nonlinear Funct. Anal. 2022 (2022), 13.

[5] L.C. Ceng, A subgradient-extragradient method for bilevel equilibrium problems with the constraints of variational inclusion systems and fixed point problems, Commun. Optim. Theory 2021 (2021), 4.
[6] T.O. Alakoya, V.A. Uzor, O.T. Mewomo, J.-C. Yao, On a system of monotone variational inclusion problems with fixed-point constraint, J. Inequ. Appl. 2022 (2022), 47.
[7] L.C. Ceng, A. Petruse, X. Qin, J.C. Yao, A modified inertial subgradient extragradient method for solving pseudomonotone variational inequalities and common fixed point problems, Fixed Point Theory 21 (2020), 93-108.
[8] N. Nadezhkina, W. Takahashi, Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings, J. Optim. Theory. Appl. 128 (2006), 191–201.
[9] Y. Censor, A. Gibali, S. Reich, Strong convergence of subgradient extragradient methods for the variational inequality problem in Hilbert space, Optim. Meth. Softw. 26 (2011), 827–845.
[10] Y. Censor, A. Gibali, S. Reich, The subgradient extragradient method for solving variational inequalities in Hilbert spaces, J. Optim. Theory Appl. 148 (2011), 318-335.
[11] P. Tseng, A modified forward-backward splitting method for maximal method for maximal monotone mappings, SIAM J.Control Optim. 38 (2000), 431-446.
[12] L. Liu, S.Y. Cho, J.C. Yao, Convergence analysis of an inertial Tseng's extragradient algorithm for solving pseudomonotone variational inequalities and applications, J. Nonlinear Var. Anal. 5 (2021), 627-644.
[13] L. Liu, X. Qin, Strong convergence theorems for solving pseudo-monotone variational inequality problems and applications, Optimization, 71 (2022), 3603-3626.
[14] X. Qin, L. Wang, J.C. Yao, Inertial splitting method for maximal monotone mappings, J. Nonlinear Convex Anal. 21 (2020), 2325-2333.
[15] G.C. Ugwunnadi, C. Izuchukwu, L.O. Jolaoso, C.C Okeke, K.O. Aremu, A modified inertial projection and contraction method for solving bilevel split variational inequality problems, Appl. Set-Valued Anal. Optim. 4 (2022), 55-71.
[16] B. Tan, X. Qin, S.Y. Cho, Revisiting subgradient extragradient methods for solving variational inequalities, Numer. Algor. 90 (2022), 1593-1615.
[17] E.C. Godwin, T.O. Alakoya, O.T. Mewomo, J.-C. Yao, Relaxed inertial Tseng extragradient method for variational inequality and fixed point problems, Appl. Anal. (2022), DOI:10.1080/00036811.2022.2107913.
[18] E.C. Godwin, O.T. Mewomo, T.O. Alakoya, A strongly convergent algorithm for solving multiple set split equality equilibrium and fixed point problems in Banach spaces, Proc. Edinb. Math. Soc. 66 (2023), 475-515.
[19] J. Fan, L. Liu, X. Qin, A subgradient extragradient algorithm with inertial effects for solving strongly pseudomonotone variational inequalities, Optimization 69 (2020), 2199-2215.
[20] F. Alvarez, H. Attouch, An inertial proximal method for monotone operators via discretization of a nonlinear oscillator with damping, Set-Valued Var. Anal.9 (2001), 3–11.
[21] D.V. Thong, N.T. Vinh, D.V. Hieu, Accelerated hybrid and shrinking projection methods for variational inequality problems, Optimization, 68 (2019), 981–998.
[22] P. Cholamjiak, D.V. Thong, Y.J. Cho, A novel inertial projection and contraction method for solving pseudomonotone variational inequality problems, Acta Appl. Math. 169 (2020), 217–245.
[23] C.E. Chidume, M.O. Nnakwe, Convergence theorems of subgradient extragradient algorithm for solving variational inequalities and a convex feasibility problem, Fixed Point Theory Appl. 2018 (2018), 16.
[24] G. Cai, A. Gibali, O. S. Iyiola, Y. Shehu, A new double-projection method for solving variational inequalities in Banach spaces, J. Optim. Theory Appl. 178 (2018), 219–239.
[25] Y. Shehu, Single projection algorithm for variational inequalities in Banach spaces with application to contact problem, Acta Math. Sci. 40 (2020), 1045–1063.
[26] I. Cioranescu, Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems, Kluwer, Dordrecht, 1990.
[27] K. Goebel, S. Reich, Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings, Marcel Dekker, New York and Basel, 1984.
[28] Y. Alber, I. Ryazantseva, Nonlinear Ill-Posed Problems of Monotone Type, Springer, London, 2006.
[29] R.T. Rockfellar, Monotone operators and the proximal point algorithm, SIAM J. Control Optim. 14 (1977), 877–808.

[30] R. I. Bot, E. R. Csetnek, P. T. Vuong, The forward-backward-forward method from continuous and discrete perspective for pseudomonotone variational inequalities in Hilbert spaces, European J. Oper. Res. 287 (2020), 49-60.
[31] P. D. Khanh, P. T. Vuong, Modified projection method for strongly pseudomonotone variational inequalities, J. Global Optim. 58 (2014), 341-350.
[32] E. Resmerita, On total convexity, Bregman projections and stability in Banach spaces, J. Convex Anal. 11 (2004), 1–16.
[33] S. Kamimura, W. Takahashi, Strong convergence of a proximal-type algorithm in a Banach space, SIAM J. Optim. 13 (2003), 938-945.
[34] K. Avetisyan, O. Djordjevi'ĉ, M. Pavlovi'ĉ, Littlewood-palew inequalities in uniformly convex and uniformly smooth Banach spaces, J. Math. Anal. Appl. 336 (2007), 31–43.
[35] H.K. Xu, Inequalities in Banach spaces with applications, Nonlinear Anal. 16 (1991), 1127–1138.
[36] H. Iiduka, W. Takahashi, Weak convergence of a projection algorithm for variational inequalities in a Banach space, J. Math. Anal. Appl. 339 (2008), 668-679.
[37] K. Aoyama, F. Kohsaka, Strongly relatively nonexpansive sequences generated by firmly nonexpansive-like mappings, Fixed Point Theory Appl. 2014 (2014), 95.
[38] K. K. Tan, H. K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, J. Math. Anal. Appl. 178 (1993), 301-308.
[39] B. Tan, S. Y. Cho, Self adaptive inertial shrinking projection algorithms for solving pseudomonotone variational inequalities, J. Nonlinear Convex Anal. 22 (2021), 613-627.
[40] Y. Liu, Strong convergence of the Halpern subgradient extragradient method for solving variational inequalities in Banach spaces, J Nonlinear Sci. Appl. 10 (2017), 395-409.

**Appendix 6.1.** [40, Algorithm 4.1]

$$\begin{cases} x_0 \in E, \\ y_n = \Pi_C J^{-1}(Jx_n - \lambda_n A(x_n)), \\ T_n = \{w_n \in E : \langle w - y_n, Jx_n - \lambda_n Ax_n - Jy_n\rangle \leq 0\}, \\ w_n = \Pi_{T_n}(Jx_n - \lambda_n A(y_n)), \\ z_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jw_n), \\ x_{n+1} = J^{-1}(\beta_n Jx_n + (1-\beta_n)JSz_n), \end{cases}$$

where $E$ is a 2-uniformly convex and uniformly smooth Banach space with 2-uniformly constant $c_1$, $S : E \to E$ is a relatively nonexpansive mapping and $A : E \to E^*$ is a monotone and $L$-Lipschitz mapping with $L > 0, \{\lambda_n\}$ is a real number sequence satisfying $0 < \inf_{n\geq 1} \lambda_n \leq \sup_{n\geq 1} \lambda_n < \frac{c}{L}, \{\beta_n\} \subset [a,b] \subset [0,1]$ for some $a,b \in (0,1), \{\alpha_n\} \subset (0,1)$ with $\lim_{n\to\infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} = \infty$.

**Appendix 6.2.** [39, Algorithm 3.2]

**Algorithm 6.3. Initialization:** Set $\theta_n \in [-\theta, \theta]$ for some $\theta > 0, \lambda_1 > 0, \mu \in (0,1)$, and $C_1 = H$. Let $x_0, x_1 \in H$ be arbitrary.
**Iterative Steps:** Calculate next iteration point $x_{n+1}$ as follows:

$$\begin{cases} w_n = (x_n + \theta_n(x_n - x_{n-1})), \\ y_n = P_C(w_n - \lambda_n Aw_n), \\ z_n = y_n - \lambda_n(Ay_n - Aw_n), \\ x_{n+1} = P_{C_{n+1}} x_0, \end{cases}$$

where

$$C_{n+1} = \left\{ p \in C_n : \|z_n - p\|^2 \le \|w_n - p\|^2 - \left(1 - \frac{\mu^2 \lambda_n^2}{\lambda_{n+1}^2}\right) \|w_n - y_n\|^2 \right\},$$

and the adaptive step-size is given by

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|}, \ \lambda_n\}, & Aw_n - Ay_n \neq 0, \\ \lambda_n, & \text{otherwise.} \end{cases}$$