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AN OUTER QUADRATIC APPROXIMATION METHOD FOR SOLVING SPLIT FEASIBILITY PROBLEMS

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Abstract. In this paper, we consider the multiple-sets split feasibility problem in real Hilbert space and propose a self-adaptive method that uses projections onto quadratic (balls) approximations of the problem's associated sets. Our algorithm has several major advantages over existing methods in the literature. The first is its simple implementation as it uses closed-formula projection onto balls, and the second is that strong convergence is obtained under mild conditions. Several numerical experiments illustrate and compare the performances of the proposed scheme.

Keywords. Balls approximation; Outer quadratic approximation method; Projection method; Split feasibility problem.

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1. Introduction

Censor, Gibali and Reich [11] introduced the split inverse problem (SIP). Given are two vector spaces X and Y and a bounded and linear operator $T : X \to Y$, let the two inverse problems IP_1 be formulated in space X and IP_2 be formulated in space Y. Given these data, the Split Inverse Problem (SIP) is formulated as follows:

$$\text{find a point } x^* \in X \text{ that solves } \mathrm{IP}_1$$
$$\text{and such that}$$
$$\text{the point } y^* = Tx^* \in Y \text{ solves } \mathrm{IP}_2.$$

The first instance of the SIP is the split convex feasibility problem (SCFP) [9]. Here the spaces are H_1 and H_2, real Hilbert spaces. Let T be the operator as above with its adjoint $T^* : H_2 \to H_1$.

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The split feasibility problem consists of finding a point

$$x^* \in C \text{ such that } Tx^* \in Q \tag{1.1}$$

where C and Q are nonempty, convex, and closed subsets of H_1 and H_2, respectively. One denotes the solutions set of the SCFP (1.1) by $D = C \cap T^{-1}(Q)$ and always assume its nonempty.

SCFPs reformulations have been successfully employed for many real-world problems, such as intensity-modulated radiation therapy [8, 10], medical image reconstruction [3, 9], gene regulatory network inference [28], just to name a few.

One of the well-known methods for solving SCFP (1.1) is Byrne CQ-algorithm [3, 4]. Given the current iterate x_n, update the next iterate via the rule

$$x_{n+1} = P_C(x_n - \tau_n T^*(I - P_Q)Tx_n)), \tag{1.2}$$

where P_C and P_Q are the nearest point projections onto C and Q, respectively, and $\tau_n \in \left(0, 2/\|T\|^2\right)$ with $\|T\|^2$ being the spectral radius of T^*T.

Examining the CQ-algorithm from the computational point of view, it can be seen that it bears two major drawbacks. The first is the need to compute P_C and P_Q per each iteration. When C and Q, the involved sets, are not "simple" enough, this task might be very costly. Second, τ_n, the step-size, depends on the evaluation of $\|T\|^2$, which could be expansive.

In a way to overcome the first drawback, Yang [30] introduced the relaxed CQ-algorithm that uses projections onto outer linear approximations (half-space) of the sets C and Q. For introducing Yang's algorithm, assume that sets C and Q are given as a sublevel sets of some convex functions, that is,

$$C := \{x \in H_1 : c(x) \leq 0\} \text{ and } Q := \{y \in H_2 : q(y) \leq 0\}, \tag{1.3}$$

where $c : H_1 \to \mathbb{R}$ and $q : H_2 \to \mathbb{R}$ are convex and subdifferentiable functions on H_1 and H_2, respectively, and that subdifferentials $\partial c(x)$ and $\partial q(y)$ of c and q, respectively, are bounded operators (i.e., bounded on bounded sets).

The outer linear (half-space) approximations ($C \subseteq C_n$ and $Q \subseteq Q_n$ for all $n \geq 1$) for the sets C and Q given as in (1.3) are presented next. Let $x_n \in H_1$, $\xi_n \in \partial c(x_n)$, and $\eta_n \in \partial q(Tx_n)$. Define

$$C_n := \begin{cases} \{x \in H_1 : c(x_n) \leq \langle \xi_n, x_n - x\rangle\}, & \text{if } \xi_n \neq 0, \\ H_1, & \text{if } \xi_n = 0, \end{cases} \tag{1.4}$$

and

$$Q_n := \begin{cases} \{y \in H_2 : q(Tx_n) \leq \langle \eta_n,\ Tx_n - y\rangle\}, & \text{if } \eta_n \neq 0, \\ H_2, & \text{if } \eta_n = 0. \end{cases} \tag{1.5}$$

Next, we define the convex and differentiable functions $f_n(\cdot)$ and its associated gradient functions $\nabla f_n(\cdot)$

$$f_n(x_n) := \frac{1}{2}\|(I - P_{Q_n})Tx_n\|^2,\ \nabla f_n(x_n) := T^*(I - P_{Q_n})Tx_n. \tag{1.6}$$

With the above data, for a given iterate x_n, Yang's relaxed CQ iterative procedure is given as

$$x_{n+1} = P_{C_n}(x_n - \tau_n \nabla f_n(x_n)), \tag{1.7}$$

where τ_n is chosen as in Byrne's CQ-algorithm (1.2). While overcoming the first computational obstacle of Byrne's original algorithm, Yang's method still require to evaluate the norm of T. Thus, López et al. [19] introduced a new relaxed CQ method with adaptive step-size rules. The step-size τ_n is then determined as follows

$$\tau_n := \frac{\rho_n f_n(x_n)}{\|\nabla f_n(x_n)\|^2}, \tag{1.8}$$

where $\rho_n \in (0,4)$ such that $\liminf_{n\to\infty} \rho_n(4-\rho_n) > 0$ for all $n \geq 1$. Under suitable conditions, the weak convergence of (1.8) was established.

As strong convergence methods are more desirable in infinite dimensional spaces, researchers proposed CQ extensions that converges strongly to a solution of the SCFP (1.1); see, e.g., [14, 15, 17, 19, 27, 31]. In particular, for a fixed point $u \in H_1$ and arbitrary $x_0 \in H_1$, López et al. [19] introduced the so-called Halpern-CQ method

$$x_{n+1} = \alpha_n u + (1-\alpha_n) P_{C_n}\Big(x_n - \tau_n \nabla f_n(x_n)\Big), \forall n \geq 1. \tag{1.9}$$

Another related result is of [27]:

$$x_{n+1} = P_{C_n}\Big((1-\alpha_n)(x_n - \tau_n \nabla f_n(x_n))\Big), \forall n \geq 1, \tag{1.10}$$

where $\{\alpha_n\} \subset (0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = +\infty$, and C_n, $\nabla f_n(x_n)$, and τ_n are given as in (1.4), (1.6), and (1.8), respectively. Under some standard conditions, it was shown that any sequence $\{x_n\}$ generated by (1.9) converges strongly to the point $x^* = P_D(u)$ whereas the sequence $\{x_n\}$ generated by (1.10) converges strongly to the point $x^* = P_D(0)$.

Recently, Yu et al. [33] considered the sets representations (1.3) with the functions $c : H_1 \to (-\infty, +\infty]$ and $q : H_2 \to (-\infty, +\infty]$ as λ-strongly and ϖ-strongly convex subdifferentiable functions on H_1 and H_2, respectively such that

$$c(x) \geq c(x_n) + \langle \xi_n, x - x_n \rangle + \frac{\lambda}{2}\|x - x_n\|^2, \text{ where } \xi_n \in \partial c(x_n),$$

and

$$q(y) \geq q(Tx_n) + \langle \eta_n, y - Tx_n \rangle + \frac{\varpi}{2}\|y - Tx_n\|^2, \text{ where } \eta_n \in \partial q(Tx_n).$$

Then, an outer quadratic approximation (ball-relaxed CQ-algorithm) method for solving the SCFP (1.1) was introduced by replacing the sets C_n (1.4) and Q_n (1.5), respectively, by C_n^* and Q_n^*, where

$$C_n^* = \Big\{x \in H_1 : c(x_n) + \langle \xi_n, x - x_n \rangle + \frac{\lambda}{2}\|x - x_n\|^2 \leq 0\Big\}, \tag{1.11}$$

and

$$Q_n^* = \Big\{y \in H_2 : q(Tx_n) + \langle \eta_n, y - Tx_n \rangle + \frac{\varpi}{2}\|y - Tx_n\|^2 \leq 0\Big\}. \tag{1.12}$$

For an arbitrary starting point $x_0 \in H_1$, Yu et al. [33] proposed the following weak convergent ball-relaxed method

$$x_{n+1} = P_{C_n^*}\left(x_n - \frac{\rho_n \|(I - P_{Q_n^*})Tx_n\|^2}{2\|T^*(I - P_{Q_n^*})Tx_n\|^2} T^*(I - P_{Q_n^*})Tx_n\right), \tag{1.13}$$

where $\rho_n \in (0,4)$ with $\liminf_{n\to\infty} \rho_n(4-\rho_n) > 0$.

Now, we wish to extend our scope to Censor et al. [10] multiple-sets split feasibility problem (MSSCFP). Let H_1 and H_2 be two real Hilbert spaces. Let $T : H_1 \to H_2$ be a linear and bounded operator and $T^*: H_2 \to H_1$ its adjoint. The multiple-sets split feasibility problem consists of finding a point $x^* \in H_1$ such that

$$x^* \in \bigcap_{i=1}^{t} C_i \text{ such that } Tx^* \in \bigcap_{j=1}^{r} Q_j, \tag{1.14}$$

where $C_1, \ldots, C_t$ and $Q_1, \ldots, Q_r$ are non-empty, closed, and convex subsets of H_1 and H_2, respectively and $t \geq 1$ and $r \geq 1$ are given integers. The solution set of (1.14) is define as

$$\Pi := \left(\cap_{i=1}^{t} C_i\right) \cap T^{-1}\left(\cap_{j=1}^{r} Q_j\right).$$

For solving the MSSCFP (1.14), Censor et al. [10] proposed the following proximity function $p(x)$ that measures the "distance" of a point to all sets:

$$p(x) := \frac{1}{2}\sum_{i=1}^{t} \alpha_i \|(I - P_{C_i})x\|^2 + \frac{1}{2}\sum_{j=1}^{r} \beta_j \|(I - P_{Q_j})Tx\|^2, \tag{1.15}$$

where α_i $(i = 1,2,\ldots,t) > 0$ and β_j $(j = 1,2,\ldots,r) > 0$ and $\sum_{i=1}^{t} \alpha_i + \sum_{j=1}^{r} \beta_j = 1$. Clearly, if the MSSCFP is feasible ($\Pi \neq \emptyset$) then $p(x^*) = 0$ and otherwise, it yields the best least solution. Following this work, many extensions were proposed; see, e.g., [13, 18, 20, 25, 32]. Moreover, extensions to fixed points, null points, and more were also proposed in [2, 5, 7, 12, 21, 22, 26].

Reich and Tuyen [23] introduced the following generalized split feasibility problem (GSCFP). Let H_j, $j = 1,2,\ldots,M$, be real Hilbert spaces and C_j, $j = 1,2,\ldots,M$, be closed and convex subsets of H_j, respectively. Let $B_j: H_j \to H_{j+1}$, $j = 1,2,\ldots,M-1$, be bounded linear operators such that

$$S := C_1 \cap B_1^{-1}(C_2) \cap \cdots \cap B_1^{-1}\left(B_2^{-1} \ldots \left(B_{M-1}^{-1}(C_M)\right)\right) \neq \emptyset.$$

The generalized split feasibility problem consists of finding a point

$$x^* \in S, \tag{1.16}$$

that is, $x^* \in C_1, B_1 x^* \in C_2, \ldots, B_{M-1}B_{M-2}\ldots B_1 x^* \in C_M$. In [23], Reich and Tuyen proved a strong convergence theorem for a modification of the CQ-algorithm which solves the GSCFP (1.16).

The split feasibility problem with multiple output sets (SCFPMOS) of Reich et al. [22] is another related SCFP generalization. Let H, H_j, $j = 1,2,\ldots,M$, be real Hilbert spaces and let $T_j: H \to H_j$, $j = 1,2,\ldots,M$, be bounded linear operators. It is to find an element x^* such that

$$x^* \in \Gamma := C \cap \left(\cap_{j=1}^{M} T_j^{-1}(Q_j)\right) \neq \emptyset \tag{1.17}$$

where C and Q_j, $j = 1,2,\ldots,M$, are non-empty, closed, and convex subsets of H and H_j, $j = 1,2,\ldots,M$, respectively.

A projection gradient algorithm and a viscosity approximation iterative method for solving the SCFPMOS (1.17) in infinite-dimensional Hilbert spaces were introduced in [22], but both methods still require to compute the metric projections on to the sets C and Q_i and the operator norm. In [24], a self-adaptive step-size algorithm for solving the SCFPMOS (1.17) was introduced.

Motivated by the problems and methods above, we consider the following multiple-sets split feasibility problem with multiple output sets (MSSCFPMOS). Let H, H_j, $j = 1,2,\ldots,M$, be real Hilbert spaces and let $T_j: H \to H_j$, $j = 1,2,\ldots,M$, be bounded linear operators. The multiple-sets split feasibility problem with multiple output sets consists of finding a point x^* such that

$$x^* \in \Omega := \left(\cap_{i=1}^{N} C_i\right) \cap \left(\cap_{j=1}^{M} T_j^{-1}\left(Q_j\right)\right) \neq \emptyset \tag{1.18}$$

where C_i, $i=1,2,\dots,N$, and Q_j, $j=1,2,\dots,M$, are non-empty, closed and convex subsets of H and H_j, $j=1,2,\dots,M$, respectively, $N,M\geq 1$ are given integers. Solutions of (1.18) fulfil $x^*\in C_i$ for each $i=1,2,\dots,N$, and $T_jx^*\in Q_j$ for each $j=1,2,\dots,M$.

It can be easily confirmed that, with $N=1$, MSSCFPMOS (1.18) reduced to SCFPMOS (1.17). Moreover, if $N=1=M$, then MSSCFPMOS (1.18) reduced to SCFP (1.1). Our aim is to establish a simple, strong convergenc,e and self-adaptive step-size method for solving the MSSCFPMOS (1.18) in real Hilbert spaces.

The paper is organized as follows. We start with recalling some basic definitions and results in Section 2. The algorithm and its analysis are presented in Section 3 and then in Section 4, the last section, we demonstrate and compare the performances of our new scheme for several numerical examples.

2. Preliminaries

Throughout this paper, let H, H_1 or H_2 be a real Hilbert space with inner product $\langle .,. \rangle$, and induced norm $\|.\|$. Let I stand for the identity operator on H, H_1 or H_2. Let " $\rightharpoonup$ " and " $\to$ ", denote the weak and strong convergence, respectively. For any sequence $\{x_n\}\subseteq H$, $\omega_w(x_n)=\big\{x\in H:\exists\{x_{n_k}\}\subseteq\{x_n\}$ such that $x_{n_k}\rightharpoonup x\big\}$ denotes the weak ω-limit set of $\{x_n\}$. We denote the set of fixed points of an operator $T:H\to H$ (if T has a fixed point) by $F(T)=\{x\in H: Tx=x\}$.

We start with a known and useful norm inequality in real Hilbert space H, $\|\sigma x+(1-\sigma)y\|^2\leq \sigma\|x\|^2+(1-\sigma)\|y\|^2$ for all $x,y\in H$ and for all $\sigma\in\mathbb{R}$.

Definition 2.1. Let C be a nonempty, closed, and convex subset of H. An operator $T:C\to H$ is called:

(1) Lipschitz continuous with constant $\sigma>0$ on C if $\|Tx-Ty\|\leq\sigma\|x-y\|$, $\forall x,y\in C$;
(2) nonexpansive on C if $\|Tx-Ty\|\leq\|x-y\|$, $\forall x,y\in C$;
(3) firmly nonexpansive on C if $\|Tx-Ty\|^2\leq\|x-y\|^2-\|(I-T)x-(I-T)y\|^2$, $\forall x,y\in C$, which is equivalent to $\|Tx-Ty\|^2\leq\langle Tx-Ty,\, x-y\rangle$, $\forall x,y\in C$.

Next, we recall the definition and properties of the metric projection of H onto the set C.

Definition 2.2. Let $C\subseteq H$ be a nonempty, closed, and convex set. For every element $x\in H$, there exists a unique nearest point in C, denoted by $P_C(x)$ such that $\|x-P_C(x)\|=\min\{\|x-y\|: y\in C\}$. The operator $P_C:H\to C$ is called a metric projection of H onto C. It is readily seen that $F(P_C):=C$. Moreover, the metric projection mapping P_C has the following well-known properties.

Lemma 2.1. *Let $C\subseteq H$ be a nonempty, closed, and convex set. Then, the following assertions hold, for any $x,y\in H$ and $z\in C$,*

(1) $\langle x-P_C(x),\, z-P_C(x)\rangle\leq 0$;
(2) $\|P_C(x)-P_C(y)\|\leq\|x-y\|$;
(3) $\|P_C(x)-P_C(y)\|^2\leq\langle P_C(x)-P_C(y),\, x-y\rangle$;
(4) $\|P_C(x)-z\|^2\leq\|x-z\|^2-\|x-P_C(x)\|^2$.

Definition 2.3. Given a function $f:H\to(-\infty,+\infty]$,

(1) f is called proper if $\{x\in H: f(x)<+\infty\}\neq\emptyset$;
(2) f is called convex if, for each $\sigma\in(0,1)$, $f(\sigma x+(1-\sigma)y)\leq\sigma f(x)+(1-\sigma)f(y), \forall x,y\in H$;

(3) f is called σ-strongly convex if $f(x)-(\sigma/2)\|x\|^2$ is convex;
(4) f is called lower semi-continuous (lsc) at x if $x_n \to x$ implies $f(x) \leq \liminf_{n\to\infty} f(x_n)$;
(5) f is called weakly lower semi-continuous (w-lsc) at x if $x_n \rightharpoonup x$ implies $f(x) \leq \liminf_{n\to\infty} f(x_n)$;
(6) f is called lower semi-continuous on H if it is lower semi-continuous at every point $x \in H$ and f is weakly lower semi-continuous on H if it is weakly lower semi-continuous at every point $x \in H$;
(7) A vector $\xi \in H$ is a subgradient of f at a point x if $f(y) \geq f(x) + \langle \xi, y-x\rangle$, $\forall y \in H$;
(8) The set of all subgradients of f at $x \in H$, denoted by $\partial f(x)$, is called the subdifferential of f, and is defined by $\partial f(x) = \{\xi \in H : f(y) \geq f(x) + \langle \xi, y-x\rangle$, for each $y \in H\}$;
(9) If $\partial f(x) \neq \emptyset$, f is said to be subdifferentiable at x. If the function f is continuously differentiable then $\partial f(x) = \{\nabla f(x)\}$.

Lemma 2.2. *([1]) Let $f : H \to (-\infty,+\infty]$ be a proper and convex function. Then f is lower semi-continuous if and only if it is weakly lower semi-continuous.*

Lemma 2.3. *([1]) Let $f : H \to (-\infty,+\infty]$ be a σ-strongly convex function. Then, for all $x,y \in H$, $f(y) \geq f(x) + \langle \xi, y-x\rangle + \frac{\sigma}{2}\|y-x\|^2$, $\xi \in \partial f(x)$.*

Lemma 2.4. *([29]) Let C and Q be closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively, and $f : H_1 \to (-\infty,+\infty]$ be given by $f(x) = \frac{1}{2}\|(I-P_Q)Tx\|^2$, where $T : H_1 \to H_2$ is a bounded and linear operator. Then, for $\sigma > 0$ and $x^* \in H_1$, the following statements are equivalent.*

(1) the point x^* solves the SCFP (1.1);
(2) the point x^* is the fixed point of the mapping $P_C(I-\sigma\nabla f)$.

Lemma 2.5. *([4]) Let H_1 and H_2 be real Hilbert spaces and let $f : H_1 \to (-\infty,+\infty]$ be given by $f(x) = \frac{1}{2}\|(I-P_Q)Tx\|^2$, where Q is closed and convex subset of H_2. Let $T : H_1 \to H_2$ be a bounded and linear operator. Then,*

(1) f is convex and weakly lower semi-continuous on H_1;
(2) $\nabla f(x) = T^*(I-P_Q)Tx$, for $x \in H_1$;
(3) ∇f is $\|T\|^2$-Lipschitz, i.e., $\|\nabla f(x) - \nabla f(y)\| \leq \|T\|^2\|x-y\|$, $\forall x,y \in H_1$.

Lemma 2.6. *([16]) Let $\{\Sigma_n\}$ be a sequence of nonnegative real numbers such that*

$$\Sigma_{n+1} \leq (1-\varsigma_n)\Sigma_n + \varsigma_n\Lambda_n,\ n \geq 1,$$
$$\Sigma_{n+1} \leq \Sigma_n - \Phi_n + \Xi_n,\ n \geq 1,$$

where $\{\varsigma_n\} \subset (0,1)$, $\{\Phi_n\}$ is a nonnegative real sequence, and $\{\Lambda_n\}$ and $\{\Xi_n\}$ are real sequences such that

(1) $\sum_{n=1}^{\infty} \varsigma_n = \infty$;
(2) $\lim_{n\to\infty} \Xi_n = 0$;
(3) $\lim_{k\to\infty} \Phi_{n_k} = 0$ implies $\limsup_{k\to\infty} \Lambda_{n_k} \leq 0$ for any subsequence $\{n_k\}$ of $\{n\}$. Then $\lim_{n\to\infty} \Sigma_n = 0$.

3. Main Result

Focusing on the MSSCFPMOS (1.18) with the sets C_i ($i \in \{1,2,\ldots,N\}$) and Q_j ($j \in \{1,2,\ldots,M\}$) representations

$$C_i = \{x \in H : c_i(x) \leq 0\} \ \text{ and } \ Q_j = \{y \in H_2 : q_j(y) \leq 0\},$$

for $c_i : H \to (-\infty, +\infty]$, $i \in \{1,2,\dots,N\}$ and $q_j : H_j \to (-\infty, +\infty]$, $j \in \{1,2,\dots,M\}$ being λ_i and ϖ_j strongly convex functions, respectively, we give our method. Moreover, we assume the following.

(SA1) all functions $c_i (i = 1,2,\dots,N)$ and $q_j (j = 1,2,\dots,M)$ are subdifferentiable on H and H_j, respectively;

(SA2) for any $x \in H$ and for each $i \in \{1,2,\dots,N\}$, subgradient $\xi_i \in \partial c_i(x)$ can be calculated;

(SA3) for any $y \in H_j$ and for each $j \in \{1,2,\dots,M\}$, subgradient $\eta_j \in \partial q_j(y)$ can be calculated;

(SA4) all operators $\partial c_i (i = 1,2,\dots,N)$ and $\partial q_j (j = 1,2,\dots,M)$ are bounded on bounded sets.

Following (SA2)-(SA3), it is clear that all functions c_i and q_j are lower semi-continuous (also weakly from Lemma 2.2) and convex. In our algorithm, given the n-th current iterative x_n, we construct for $i \in \{1,2,\dots,N\}$ the super-sets $C^*_{i,n}$ and for $j \in \{1,2,\dots,M\}$ the super-sets $Q^*_{j,n}$ as follows

$$C^*_{i,n} = \Big\{x \in H : c_i(x_n) + \langle \xi_{i,n},\ x - x_n\rangle + \frac{\lambda_i}{2}\|x - x_n\|^2 \le 0\Big\}, \tag{3.1}$$

where $\xi_{i,n} \in \partial c_i(x_n)$. If $\lambda_i = 0$, then $C^*_{i,n}$ above is reduced to the following half-space

$$C_{i,n} = \Big\{x \in H : c_i(x_n) + \langle \xi_{i,n},\ x - x_n\rangle \le 0\Big\}.$$

If $\lambda_i > 0$, then, for $i \in \{1,2,\dots,N\}$, $C^*_{i,n}$ can be defined by (see [33])

$$C^*_{i,n} = \Big\{x \in H : \Big\|x - \Big(x_n - \frac{1}{\lambda_i}\xi_{i,n}\Big)\Big\|^2 \le \frac{1}{\lambda_i^2}\|\xi_{i,n}\|^2 - \frac{2}{\lambda_i}c_i(x_n)\Big\}$$

and it follows from the fact that $C^*_{i,n} \supseteq C_i \neq \emptyset$ ($i \in \{1,2,\dots,N\}$) the set $C^*_{i,n}$ is nonempty. Furthermore, let $x^* \in C_i$ ($i \in \{1,2,\dots,N\}$). Since each c_i ($i \in \{1,2,\dots,N\}$) is λ_i-strongly convex, it then follows from Lemma 2.3 that

$$c_i(x_n) + \langle \xi_{i,n},\ x^* - x_n\rangle + \frac{\lambda_i}{2}\|x^* - x_n\|^2 \le c_i(x^*) \le 0,$$

which implies that, for each $i \in \{1,2,\dots,N\}$,

$$\frac{2}{\lambda_i}c_i(x_n) \le \frac{2}{\lambda_i}\|\xi_{i,n}\|\|x_n - x^*\| - \|x_n - x^*\|^2 \le \frac{1}{\lambda_i^2}\|\xi_{i,n}\|^2$$

which also yields $\frac{1}{\lambda_i^2}\|\xi_{i,n}\|^2 - \frac{2}{\lambda_i}c_i(x_n) \ge 0$. Therefore, each $C^*_{i,n}$ ($i \in \{1,2,\dots,N\}$) is a nonempty ball of radius $\sqrt{\frac{1}{\lambda_i^2}\|\xi_{i,n}\|^2 - \frac{2}{\lambda_i}c_i(x_n)}$ centred at $x_n - \frac{1}{\lambda_i}\xi_{i,n}$. The set $Q^*_{j,n}$ ($j \in \{1,2,\dots,M\}$) is defined as

$$Q^*_{j,n} = \Big\{y \in H_j : q_j(T_jx_n) + \langle \eta_{j,n},\ y - T_jx_n\rangle + \frac{\varpi_j}{2}\|y - T_jx_n\|^2 \le 0\Big\}, \tag{3.2}$$

where $\eta_{j,n} \in \partial q_j(T_jx_n)$. If $\varpi_j = 0$, then $Q^*_{j,n}$ above is reduced to the following half-space

$$Q_{j,n} = \Big\{y \in H_j : q_j(T_jx_n) + \langle \eta_{j,n},\ y - T_jx_n\rangle \le 0\Big\}.$$

If $\varpi_j > 0$, then $Q^*_{j,n}$ above is noting but a nonempty closed ball. Indeed, $Q^*_{j,n}$ is nonempty because $Q^*_{j,n} \supseteq Q_j \neq \emptyset$ ($j \in \{1,2,\dots,M\}$). Similarly, for all $n \ge 0$ and for each $j \in \{1,2,\dots,M\}$, observe that

$$Q^*_{j,n} = \Big\{y \in H_j : \Big\|y - \Big(T_jx_n - \frac{1}{\varpi_j}\eta_{j,n}\Big)\Big\|^2 \le \frac{1}{\varpi_j^2}\|\eta_{j,n}\|^2 - \frac{2}{\varpi_j}q_j(T_jx_n)\Big\}.$$

That is, each $Q^*_{j,n}$ ($j \in \{1,2,\dots,M\}$) is also a nonempty closed ball of radius

$$\sqrt{\frac{1}{\varpi_j^2}\|\eta_{j,n}\|^2 - \frac{2}{\varpi_j} q_j(T_j x_n)}$$

centred at $T_j x_n - \frac{1}{\varpi_j}\eta_{j,n}$. Therefore, both $C^*_{i,n}$ and $Q^*_{j,n}$ are nothing but nonempty closed balls and it is easy to verify that [33] $C^*_{i,n} \supseteq C_i$ ($i \in \{1,2,\dots,N\}$) and $Q^*_{j,n} \supseteq Q_j$ ($j \in \{1,2,\dots,M\}$) hold for every $n \geq 0$.

With the above, we are now ready to present our new and simple method for solving the MSSCFPMOS (1.18).

Algorithm 1

Step 0. Choose two real sequences $\{\alpha_n\} \subset (0,1)$ and $\{\rho_n\} \subset (0,2)$ satisfying the assumptions:

$$(\mathbf{A1})\ \liminf_{n\to\infty} \rho_n(2-\rho_n) > 0 \qquad (\mathbf{A2})\ \lim_{n\to\infty} \alpha_n = 0 \text{ and } \sum_{n=0}^{\infty} \alpha_n = \infty.$$

Choose arbitrary starting point $x_0 \in H$ and set $n := 0$. Choose weights δ_i^n $(i = 1,2,\dots,N) > 0$ and parameters β_j $(j = 1,2,\dots,M) > 0$ such that

$$\sum_{i=1}^{N} \delta_i^n = 1 \text{ and } \inf_{i\in I_n} \delta_i^n > \delta > 0, \text{ where } I_n = \{i \in \{1,2,\dots,N\} : \delta_i^n > 0\}, \text{ and } \sum_{j=1}^{M} \beta_j = 1.$$

Step 1. Given the current iterate $x_n \in H$, compute the next iterate x_{n+1} by

$$x_{n+1} = \sum_{i=1}^{N} \delta_i^n P_{C^*_{i,n}}\left((1-\alpha_n)\left(x_n - \tau_n \sum_{j=1}^{M} \beta_j T_j^*\left(I - P_{Q^*_{j,n}}\right)T_j x_n\right)\right),$$

where $C^*_{i,n}$ and $Q^*_{j,n}$ are the sets defined in (3.1) and (3.2), respectively and the step-size τ_n is updated via

$$\tau_n := \frac{\rho_n \sum_{j=1}^{M} \beta_j \left\|\left(I - P_{Q^*_{j,n}}\right)T_j x_n\right\|^2}{\Theta_n^2},$$

where

$$\Theta_n := \max\left\{1,\ \left\|\sum_{j=1}^{M} \beta_j T_j^*\left(I - P_{Q^*_{j,n}}\right)T_j x_n\right\|\right\}.$$

Step 2. If $x_{n+1} = x_n$, then stop; otherwise, set $n := n+1$ and return to **Step 1.**

Remark 3.1. If $\lambda_i = \varpi_j = 0$, then all functions c_i and q_j for $i \in \{1,2,\dots,N\}$ and $j \in \{1,2,\dots,M\}$ are convex, then Algorithm 1 reduced to a outer linear (half-spaces) approximation method projections. Moreover, only one family of sets is convex and the other is strongly convex, and we obtain another new algorithm for solving the MSSCFPMOS (1.18).

3.1. **Convergence Analysis.**

Lemma 3.1. *Assume that (SA1)-(SA4) hold and let $\{x_n\}$ be any sequence generated by Algorithm 1. Then*

$$\sum_{j=1}^{M}\beta_j\left\|\left(I-P_{Q^*_{j,n}}\right)T_jx_n\right\|^2=0 \quad\Leftrightarrow\quad \left\|\sum_{j=1}^{M}\beta_jT_j^*\left(I-P_{Q^*_{j,n}}\right)T_jx_n\right\|=0.$$

Proof. Suppose that $\sum_{j=1}^{M}\beta_j\left\|\left(I-P_{Q^*_{j,n}}\right)T_jx_n\right\|^2=0$. Thus

$$\left\|\sum_{j=1}^{M}\beta_jT_j^*\left(I-P_{Q^*_{j,n}}\right)T_jx_n\right\|^2 \le M\Big(\max_{1\le j\le M}\beta_j\Big)\Big(\max_{1\le j\le M}\|T_j\|^2\Big)\sum_{j=1}^{M}\beta_j\left\|(I-P_{Q^*_{j,n}})T_jx_n\right\|^2,$$

which yields $\left\|\sum_{j=1}^{M}\beta_jT_j^*\left(I-P_{Q^*_{j,n}}\right)T_jx_n\right\|=0$.

On the other hand, let $\left\|\sum_{j=1}^{M}\beta_jT_j^*\left(I-P_{Q^*_{j,n}}\right)T_jx_n\right\|=0$ and fix $x^*\in\Omega$. By Lemma 2.1, we have

$$\begin{aligned}\sum_{j=1}^{M}\beta_j\left\|(I-P_{Q^*_{j,n}})T_jx_n\right\|^2 &\le \left\langle\sum_{j=1}^{M}\beta_j(I-P_{Q^*_{j,n}})T_jx_n,\ T_jx_n-T_jx^*\right\rangle\\ &= \left\langle\sum_{j=1}^{M}\beta_jT_j^*(I-P_{Q^*_{j,n}})T_jx_n,\ x_n-x^*\right\rangle\\ &\le \left\|\sum_{j=1}^{M}\beta_jT_j^*(I-P_{Q^*_{j,n}})T_jx_n\right\|\|x_n-x^*\|.\end{aligned}$$

So, it is clear that $\sum_{j=1}^{M}\beta_j\left\|(I-P_{Q^*_{j,n}})T_jx_n\right\|^2=0$ and the desired result is obtained. □

Lemma 3.2. *Assume that the solution set of the MSSCFPMOS (1.18) $\Omega\neq\emptyset$ and let $\{\rho_n\}$ and $\{\alpha_n\}$ be the sequences defined in Algorithm 1. Let $\{x_n\}$ be any sequence generated by Algorithm 1. Then,*

(1): *for all $x^*\in\Omega$ and $n\in\mathbb{N}$, it holds*

$$\|x_{n+1}-x^*\|^2\le\alpha_n\|x^*\|^2+(1-\alpha_n)\|x_n-x^*\|^2 -\rho_n(2-\rho_n)(1-\alpha_n)\frac{\left(\sum_{j=1}^{M}\beta_j\left\|\left(I-P_{Q^*_{j,n}}\right)T_jx_n\right\|^2\right)^2}{\Theta_n^2},$$

(2): *sequence $\{x_n\}$ is bounded,*
(3): *for all $x^*\in\Omega$ and $n\in\mathbb{N}$, it holds*

$$\|x_{n+1}-x^*\|^2\le(1-\alpha_n)\|x_n-x^*\|^2+\alpha_n\Big[\alpha_n\|x^*\|^2+2(1-\alpha_n)\langle x_n-x^*,\ -x^*\rangle +2\tau_n(1-\alpha_n)\|x^*\|\left\|\sum_{j=1}^{M}\beta_jT_j^*\left(I-P_{Q^*_{j,n}}\right)T_jx_n\right\|\Big].$$

Proof. **(1)** Let $x^* \in \Omega$. Note that, for each $j = 1,2,\ldots,M$, $I - P_{Q^*_{j,n}}$ is firmly nonexpansive and $\sum_{j=1}^{M} \beta_j T_j^* \left(I - P_{Q^*_{j,n}}\right) T_j x^* = 0$. Hence, it follows from Lemma 2.1 that

$$\begin{aligned}\left\langle \tau_n \sum_{j=1}^{M} \beta_j T_j^* \left(I - P_{Q^*_{j,n}}\right) T_j x_n, x_n - x^* \right\rangle &= \tau_n \sum_{j=1}^{M} \beta_j \left\langle \left(I - P_{Q^*_{j,n}}\right) T_j x_n, T_j x_n - T_j x^* \right\rangle \\ &\geq \tau_n \sum_{j=1}^{M} \beta_j \left\| \left(I - P_{Q^*_{j,n}}\right) T_j x_n \right\|^2,\end{aligned}$$

which together with the definition of τ_n and $\left\| \sum_{j=1}^{M} \beta_j T_j^* \left(I - P_{Q^*_{j,n}}\right) T_j x_n \right\| \leq \Theta_n$ implies that

$$\begin{aligned}&\left\| x_n - \tau_n \sum_{j=1}^{M} \beta_j T_j^* \left(I - P_{Q^*_{j,n}}\right) T_j x_n - x^* \right\|^2 \\ =\ & \|x_n - x^*\|^2 + \tau_n^2 \left\| \sum_{j=1}^{M} \beta_j T_j^* \left(I - P_{Q^*_{j,n}}\right) T_j x_n \right\|^2 - 2\tau_n \left\langle \sum_{j=1}^{M} \beta_j T_j^* \left(I - P_{Q^*_{j,n}}\right) T_j x_n, x_n - x^* \right\rangle \\ \leq\ & \|x_n - x^*\|^2 + \tau_n^2 \Theta_n^2 - 2\tau_n \sum_{j=1}^{M} \beta_j \left\| \left(I - P_{Q^*_{j,n}}\right) T_j x_n \right\|^2 \\ =\ & \|x_n - x^*\|^2 - \rho_n(2-\rho_n) \frac{\left(\sum_{j=1}^{M} \beta_j \left\| \left(I - P_{Q^*_{j,n}}\right) T_j x_n \right\|^2 \right)^2}{\Theta_n^2}. \qquad (3.3)\end{aligned}$$

By Lemma 2.1, we also obtain the following estimation

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ =\ & \left\| \sum_{i=1}^{N} \delta_i^n P_{C^*_{i,n}} \left((1-\alpha_n) \left(x_n - \tau_n \sum_{j=1}^{M} \beta_j T_j^* \left(I - P_{Q^*_{j,n}}\right) T_j x_n \right) \right) - \sum_{i=1}^{N} \delta_i^n P_{C^*_{i,n}} x^* \right\|^2 \\ \leq\ & \left\| (1-\alpha_n) \left(x_n - \tau_n \sum_{j=1}^{M} \beta_j T_j^* \left(I - P_{Q^*_{j,n}}\right) T_j x_n \right) - x^* \right\|^2 \\ \leq\ & (1-\alpha_n) \left\| x_n - \tau_n \sum_{j=1}^{M} \beta_j T_j^* \left(I - P_{Q^*_{j,n}}\right) T_j x_n - x^* \right\|^2 + \alpha_n \|x^*\|^2. \qquad (3.4)\end{aligned}$$

Substituting (3.3) into (3.4), we have that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \leq \alpha_n \|x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 \\ - \rho_n(2-\rho_n)(1-\alpha_n) \frac{\left(\sum_{j=1}^{M} \beta_j \left\| \left(I - P_{Q^*_{j,n}}\right) T_j x_n \right\|^2 \right)^2}{\Theta_n^2}. \qquad (3.5)\end{aligned}$$

(2) Since $\liminf_{n\to\infty} \rho_n(2-\rho_n) > 0$, we obtain from (3.5) that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\leq \alpha_n\|x^*\|^2 + (1-\alpha_n)\|x_n-x^*\|^2 \\
&\leq \max\{\|x^*\|^2,\ \|x_n-x^*\|^2\} \\
&\leq \max\{\|x^*\|^2,\ \|x_{n-1}-x^*\|^2\} \\
&\vdots \\
&\leq \max\{\|x^*\|^2,\ \|x_0-x^*\|^2\}.
\end{aligned}$$

Hence, sequence $\{x_n\}$ is bounded. Consequently, sequence $\{T_jx_n\}$ for each $j = 1,2,\ldots,M$ is also bounded.

(3) Furthermore, since $\liminf_{n\to\infty} \rho_n(2-\rho_n) > 0$, it follows from (3.3) that

$$\left\| x_n - \tau_n \sum_{j=1}^{M} \beta_j T_j^* \left(I - P_{Q_{j,n}^*}\right) T_j x_n - x^* \right\|^2 \leq \|x_n - x^*\|^2. \tag{3.6}$$

From (3.4), we also have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\leq \alpha_n^2\|x^*\|^2 + (1-\alpha_n)^2\left\| x_n - \tau_n \sum_{j=1}^{M} \beta_j T_j^* \left(I - P_{Q_{j,n}^*}\right) T_j x_n - x^* \right\|^2 \\
&+ 2\alpha_n(1-\alpha_n)\left\langle x_n - \tau_n \sum_{j=1}^{M} \beta_j T_j^* \left(I - P_{Q_{j,n}^*}\right) T_j x_n - x^*,\ -x^* \right\rangle,
\end{aligned}$$

which together with (3.6) gives that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\leq \alpha_n^2\|x^*\|^2 + (1-\alpha_n)^2\|x_n-x^*\|^2 + 2\alpha_n(1-\alpha_n)\langle x_n-x^*,\ -x^*\rangle \\
&+ 2\alpha_n\tau_n(1-\alpha_n)\left\langle \sum_{j=1}^{M} \beta_j T_j^* \left(I - P_{Q_{j,n}^*}\right) T_j x_n,\ x^* \right\rangle \\
&\leq (1-\alpha_n)\|x_n-x^*\|^2 + \alpha_n\Big[\alpha_n\|x^*\|^2 + 2(1-\alpha_n)\langle x_n-x^*,\ -x^*\rangle \\
&+ 2\tau_n(1-\alpha_n)\left\langle \sum_{j=1}^{M} \beta_j T_j^* \left(I - P_{Q_{j,n}^*}\right) T_j x_n,\ x^* \right\rangle\Big] \\
&\leq (1-\alpha_n)\|x_n-x^*\|^2 + \alpha_n\Big[\alpha_n\|x^*\|^2 + 2(1-\alpha_n)\langle x_n-x^*,\ -x^*\rangle \\
&+ 2\tau_n(1-\alpha_n)\left\| \sum_{j=1}^{M} \beta_j T_j^* \left(I - P_{Q_{j,n}^*}\right) T_j x_n \right\| \|x^*\|\Big].
\end{aligned} \tag{3.7}$$

This completes the proof. □

Theorem 3.1. *Assume that the solution set of MSSCFPMOS (1.18) is nonempty and the sequences $\{\rho_n\}$ and $\{\alpha_n\}$ satisfy the assumptions* **(A1)** *and* **(A2)** *in Algorithm 1. Then any sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to the point $x^* = P_\Omega 0$.*

Proof. Let $x^* = P_\Omega 0$. From the assumptions imposed on sequences $\{\rho_n\}$ and $\{\alpha_n\}$, there is a constant $\rho > 0$ such that $\rho \leq \rho_n(2-\rho_n)(1-\alpha_n)$ for all $n \in \mathbb{N}$. Thus, it follows from (3.5) that

$$\|x_{n+1}-x^*\|^2 \leq \alpha_n\|x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 - \rho \frac{\left(\sum_{j=1}^M \beta_j \left\| \left(I - P_{Q^*_{j,n}}\right) T_j x_n \right\|^2\right)^2}{\Theta_n^2},$$

which further implies that

$$\|x_{n+1}-x^*\|^2 \leq \alpha_n\|x^*\|^2 + \|x_n - x^*\|^2 - \rho \frac{\left(\sum_{j=1}^M \beta_j \left\| \left(I - P_{Q^*_{j,n}}\right) T_j x_n \right\|^2\right)^2}{\Theta_n^2}. \tag{3.8}$$

By (3.7) and (3.8), we have

$$\begin{aligned} &\|x_{n+1}-x^*\|^2 \leq (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n \Lambda_n, \; n \geq 1, \\ &\|x_{n+1}-x^*\|^2 \leq \|x_n - x^*\|^2 - \Phi_n + \alpha_n\|x^*\|^2, \; n \geq 1, \end{aligned} \tag{3.9}$$

Relating (3.9) to Lemma 2.6, we define for all $n \geq 1$:

$$\begin{aligned} &\Sigma_n = \|x_n - x^*\|^2, \\ &\Lambda_n = \alpha_n\|x^*\|^2 + 2(1-\alpha_n)\langle x_n - x^*, \; -x^*\rangle + 2\tau_n(1-\alpha_n)\left\| \sum_{j=1}^M \beta_j T_j^* \left(I - P_{Q^*_{j,n}}\right) T_j x_n \right\| \|x^*\|, \\ &\Phi_n := \rho \frac{\left(\sum_{j=1}^M \beta_j \left\| \left(I - P_{Q^*_{j,n}}\right) T_j x_n \right\|^2\right)^2}{\Theta_n^2}. \end{aligned}$$

Moreover, setting $\varsigma_n := \alpha_n$, one has $\{\varsigma_n\} \subset (0,1), \lim_{n\to\infty} \varsigma_n = 0$, and $\sum_{n=0}^\infty \varsigma_n = \infty$. One also defines $\Xi_n := \alpha_n\|x^*\|^2$ and obtains that $\lim_{n\to\infty} \Xi_n = 0$

Next, we focus on the convergence analysis of $\{\Sigma_n\}$. Let $\{n_k\}$ be a subsequence of $\{n\}$ and suppose $\limsup_{k\to\infty} \Phi_{n_k} \leq 0$, which further yields

$$\lim_{k\to\infty} \left[\rho \frac{\left(\sum_{j=1}^M \beta_j \left\| \left(I - P_{Q^*_{j,n_k}}\right) T_j x_{n_k} \right\|^2\right)^2}{\Theta_{n_k}^2} \right] = 0. \tag{3.10}$$

Since $\rho > 0$, (3.10) implies that

$$\lim_{k\to\infty} \left[\frac{\sum_{j=1}^M \beta_j \left\| \left(I - P_{Q^*_{j,n_k}}\right) T_j x_{n_k} \right\|^2}{\Theta_{n_k}} \right] = 0. \tag{3.11}$$

Since $\{x_{n_k}\}$ is bounded and by the Lipschitz continuity of the $\left(I - P_{Q^*_{j,n_k}}\right) T_j x_{n_k}$ for each $j = 1,2,\ldots,M$ and for all $k \in \mathbb{N}$, $\left\{\left\| \sum_{j=1}^M \beta_j T_j^* \left(I - P_{Q^*_{j,n_k}}\right) T_j x_{n_k} \right\|\right\}$ is bound. Hence, $\{\Theta_{n_k}\}$ is bounded as well. Therefore, we obtain from (3.11) that $\lim_{k\to\infty} \sum_{j=1}^M \beta_j \|\left(I - P_{Q^*_{j,n_k}}\right) T_j x_{n_k}\|^2 = 0$, which implies for each $j = 1,2,\ldots,M$ that

$$\lim_{k\to\infty} \left\| \left(I - P_{Q^*_{j,n_k}}\right) T_j x_{n_k} \right\| = 0. \tag{3.12}$$

Furthermore,

$$\lim_{k\to\infty} \tau_{n_k} \left\| \sum_{j=1}^M \beta_j T_j^* \left(I - P_{Q^*_{j,n_k}}\right) T_j x_{n_k} \right\| = 0. \tag{3.13}$$

Next, we prove that each weak cluster point of $\{x_{n_k}\}$ belongs to Ω, that is, $\omega_w(x_{n_k}) \subset \Omega$. Let $p^* \in H$ be a weak cluster point of $\{x_{n_k}\}$. Since $\{x_{n_k}\}$ is a bounded vector sequence, we may assume that there exists a subsequence $\{x_{n_{k_m}}\}$ of $\{x_{n_k}\}$ that convergent to p^* weakly. Furthermore, since each T_j for $j=1,2,\ldots,M$ is bounded and linear, this yields that $\{T_jx_{n_{k_m}}\}$ weakly converges to T_jp^*. We claim here that p^* is a solution to MSSCFPMOS (1.18), that is, $p^*\in\Omega$. To demonstrate this, it suffices to demonstrate that $p^*\in C_i$ for all $i\in\{1,2,\ldots,N\}$ and $T_jp^*\in Q_j$ for all $j\in\{1,2,\ldots,M\}$.

We first demonstrate that $T_jp^*\in Q_j$ for all $j\in\{1,2,\ldots,M\}$. Since ∂q_j for each $j\in\{1,2,\ldots,M\}$ is bounded on bounded sets, we may assume that there is a constant $\eta_0>0$ such that $\|\eta_{j,n_{k_m}}\|\le\eta_0$, where $\eta_{j,n_{k_m}}\in\partial q_j(T_jx_{n_{k_m}})$ for each $j\in\{1,2,\ldots,M\}$. That is, sequence $\{\eta_{j,n_{k_m}}\}$ is bounded. Note that $P_{Q^*_{j,n_{k_m}}}(T_jx_{n_{k_m}})\in Q^*_{j,n_{k_m}}$ for each $j\in\{1,2,\ldots,M\}$. Now, it follows from (3.2) and (3.12) for all $j\in\{1,2,\ldots,M\}$ and as $m\to\infty$ that

$$\begin{aligned} q_j\left(T_jx_{n_{k_m}}\right) &\le \left\langle \eta_{j,n_{k_m}},\, T_jx_{n_{k_m}}-P_{Q^*_{j,n_{k_m}}}(T_jx_{n_{k_m}})\right\rangle-\frac{\varpi_j}{2}\left\|T_jx_{n_{k_m}}-P_{Q^*_{j,n_{k_m}}}(T_jx_{n_{k_m}})\right\|^2\\ &\le \left\langle \eta_{j,n_{k_m}},\, T_jx_{n_{k_m}}-P_{Q^*_{j,n_{k_m}}}(T_jx_{n_{k_m}})\right\rangle\\ &\le \left\|\eta_{j,n_{k_m}}\right\|\left\|\left(I-P_{Q^*_{j,n_{k_m}}}\right)T_jx_{n_{k_m}}\right\|\\ &\le \eta_0\left\|\left(I-P_{Q^*_{j,n_{k_m}}}\right)T_jx_{n_{k_m}}\right\|\to 0. \end{aligned}\tag{3.14}$$

The weakly lower semi-continuity of q_j together with (3.14) implies for all $j\in\{1,2,\ldots,M\}$ that

$$q_j(T_jp^*)\le\liminf_{m\to\infty}q_j\left(T_jx_{n_{k_m}}\right)\le\lim_{k\to\infty}\eta_0\left\|\left(I-P_{Q^*_{j,n_{k_m}}}\right)T_jx_{n_{k_m}}\right\|=0.$$

It turns out that, $T_jp^*\in Q_j$, $\forall j\in\{1,2,\ldots,M\}$.

We next prove that $p^*\in C_i$ for all $i\in\{1,2,\ldots,N\}$. Indeed, it follows from the definition of x_{n+1} that

$$\begin{aligned} &\|x_{n_{k_m}+1}-x_{n_{k_m}}\|\\ \le\ &\left\|(1-\alpha_{n_{k_m}})\left(x_{n_{k_m}}-\tau_{n_{k_m}}\sum_{j=1}^M\beta_jT_j^*\left(I-P_{Q^*_{j,n_{k_m}}}\right)T_jx_{n_{k_m}}\right)-x_{n_{k_m}}\right\|\\ \le\ &\alpha_{n_{k_m}}\left\|x_{n_{k_m}}-\tau_{n_{k_m}}\sum_{j=1}^M\beta_jT_j^*\left(I-P_{Q^*_{j,n_{k_m}}}\right)T_jx_{n_{k_m}}\right\|+\tau_{n_{k_m}}\left\|\sum_{j=1}^M\beta_jT_j^*\left(I-P_{Q^*_{j,n_{k_m}}}\right)T_jx_{n_{k_m}}\right\|\to 0, \end{aligned}$$

as $m\to\infty$. That is,

$$\lim_{m\to\infty}\left\|x_{n_{k_m}}-x_{n_{k_m}+1}\right\|=0.\tag{3.15}$$

Since ∂c_i for each $i\in\{1,2,\ldots,N\}$ is bounded on bounded sets, we may again assume that, for all $n_{k_m}\ge0$, there is a constant $\xi_0>0$ such that $\|\xi_{i,n_{k_m}}\|\le\xi_0$, where $\xi_{i,n_{k_m}}\in\partial c_i(x_{n_{k_m}})$ for each $i\in\{1,2,\ldots,N\}$. That is, $\{\xi_{i,n_{k_m}}\}$ is bounded. Using the fact that $x_{n_{k_m}+1}\in C^*_{i,n_{k_m}}$ for all $i\in\{1,2,\ldots,N\}$ and employing (3.1) and (3.15), we obtain for all $i\in\{1,2,\ldots,N\}$ as $m\to\infty$ that

$$\begin{aligned} c_i(x_{n_{k_m}}) &\le \left\langle \xi_{i,n_{k_m}},\, x_{n_{k_m}}-x_{n_{k_m}+1}\right\rangle-\frac{\lambda_i}{2}\left\|x_{n_{k_m}}-x_{n_{k_m}+1}\right\|^2\\ &\le \left\|\xi_{i,n_{k_m}}\right\|\left\|x_{n_{k_m}}-x_{n_{k_m}+1}\right\|\\ &\le \xi_0\left\|x_{n_{k_m}}-x_{n_{k_m}+1}\right\|\to 0. \end{aligned}\tag{3.16}$$

The weakly lower semi-continuity of c_i together with (3.16) implies for all $i \in \{1,2,\dots,N\}$ that

$$c_i(p^*) \leq \liminf_{m\to\infty} c_i(x_{n_{k_m}}) \leq \lim_{m\to\infty} \xi_0 \left\| x_{n_{k_m}} - x_{n_{k_m}+1} \right\| = 0,$$

Consequently, $p^* \in C_i$ for all $i \in \{1,2,\dots,N\}$. Altogether, we conclude that $p^* \in \Omega$. Since p^* is arbitrary, we conclude that each weak cluster point of $\{x_{n_k}\}$ belongs to Ω. That is, $w_\omega(x_{n_k}) \subset \Omega$, which implies there exists a subsequence $\{x_{n_{k_m}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_m}} \rightharpoonup p^*$.

Furthermore, by Lemma 2.1, assumption **(A2)**, and (3.13), we obtain that

$$\begin{aligned}
\limsup_{m\to\infty} \Lambda_{n_{k_m}} &= \limsup_{m\to\infty} \Big[\alpha_{n_{k_m}} \|x^*\|^2 + 2(1-\alpha_{n_{k_m}}) \langle x_{n_{k_m}} - x^*, \, -x^* \rangle \\
&+ 2\tau_{n_{k_m}} (1-\alpha_{n_{k_m}}) \Big\| \sum_{j=1}^{M} \beta_j T_j^* \Big(I - P_{Q^*_{j,n_{k_m}}} \Big) T_j x_{n_{k_m}} \Big\| \|x^*\| \Big] \\
&= 2 \max_{p^* \in \omega_w(x_{n_{k_m}})} \langle p^* - x^*, \, -x^* \rangle \\
&\leq 0.
\end{aligned}$$

Therefore, applying Lemma 2.6, we conclude that any sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to the minimum-norm element $x^* = P_\Omega 0$ and the proof is complete. □

By setting $N = M = 1$, MSSCFPMOS (1.18) reduces to SCFP (1.1). As a direct consequence of Theorem 3.1, we obtain the following result for solving SCFP (1.1).

Corollary 3.1. *Let H_1 and H_2 be two real Hilbert spaces, and let $T : H_1 \to H_2$ be bounded and linear operator. Let C and Q be nonempty, convex, and closed subsets of H_1 and H_2, respectively. Assume that $D = C \cap T^{-1}(Q) \neq \emptyset$. For any starting point $x_0 \in H_1$, let $\{x_n\}$ be any sequence generated by*

$$x_{n+1} = P_{C_n^*} \Big((1-\alpha_n) \big(x_n - \tau_n T^* (I - P_{Q_n^*}) T x_n \big) \Big)$$

where $\{\alpha_n\} \subset (0,1)$, the step-size τ_n is self-adaptively updated via

$$\tau_n := \frac{\rho_n \|(I - P_{Q_n^*}) T x_n\|^2}{\Big(max\{1, \, \|T^*(I - P_{Q_n^*}) T x_n\|\} \Big)^2}, \ \{\rho_n\} \subset (0,2),$$

and C_n^* and Q_n^* are the balls given by (1.11) and (1.12), respectively. Suppose that the sequences $\{\rho_n\}$ and $\{\alpha_n\}$ satisfy **(A1)** *and* **(A2)** *in Algorithm 1. Then, $\{x_n\}$ converges strongly to the minimum-norm element $x^* = P_D(0)$ of the SCFP (1.1).*

Now, for the special case that $N = 1$, Theorem 3.1 yields the following result for solving the GSCFP (1.16).

Theorem 3.2. *Let $H = H_1, C = C_1, Q_j = C_{j+1}, 1 \leq j \leq M-1, T_1 = B_1, T_2 = B_2 B_1, \dots,$ and $T_{M-1} = B_{M-1} B_{M-2} B_{M-3} \dots B_2 B_1$. Assume that the GSCFP (1.16) is consistent (i.e., $S \neq \emptyset$). Let $x_0 \in C_1$ be an arbitrary initial point, and set $n = 0$. Take the constant parameters β_j $(j = 1,2,\dots,M) > 0$ as in Algorithm 1. Let $\{x_n\}$ be the sequence generated by*

$$x_{n+1} = P_{C^*_{1,n}} \Big((1-\alpha_n) \Big(x_n - \tau_n \sum_{j=1}^{M-1} \beta_j T_j^* \Big(I - P_{C^*_{j+1,n}} \Big) T_j x_n \Big) \Big)$$

where $C^*_{1,n}$ and $C^*_{j+1,n}$ are balls containing C_1 and C_{j+1}, respectively, the step-size τ_n is self-adaptively updated via

$$\tau_n := \frac{\rho_n \sum_{j=1}^{M-1} \beta_j \left\| \left(I - P_{C^*_{j+1,n}}\right) T_j x_n \right\|^2}{\Theta_n^2}$$

where

$$\Theta_n := max\left\{1,\ \left\| \sum_{j=1}^{M-1} \beta_j T_j^* \left(I - P_{C^*_{j+1,n}}\right) T_j x_n \right\| \right\},$$

$\{\alpha_n\} \subset (0,1)$, $\{\rho_n\} \subset (0,2)$ *satisfying the assumptions:* $\liminf_{n\to\infty} \rho_n(2-\rho_n) > 0$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0} \alpha_n = \infty$. *Then,* $\{x_n\}$ *converges strongly to the minimum-norm element* $x^* \in S$, *where* $x^* = P_S(0)$.

Remark 3.2. For the particular case, $M = 1$, MSSCFPMOS (1.18) reduced to the following problem.

Let H_1 and H_2 be two real Hilbert spaces, and let $T : H_1 \to H_2$ be bounded and linear operator with its adjoint $T^* : H_2 \to H_1$. Find an element x^* such that

$$x^* \in E := \left(\cap_{i=1}^N C_i\right) \cap T^{-1}(Q) \neq \emptyset \tag{3.17}$$

where C_i, $i = 1,2,\ldots,N$, and Q are nonempty, closed, and convex subsets of H_1 and H_2, respectively, and N is a given positive integer. That is, $x^* \in C_i$ for each $i = 1,2,\ldots,N$, and $Tx^* \in Q$.

It can be easily seen that (3.17) is a special case of the MSSCFP (1.14) with $r = 1$. Moreover, we present the following result for solving (3.17).

Theorem 3.3. *Assume that the solution set of (3.17) is nonempty, i.e., $E \neq \emptyset$. Take the weights δ_i^n $(i = 1,2,\ldots,N) > 0$ as in Algorithm 1. For any starting point $x_0 \in H_1$, let $\{x_n\}$ be the sequence generated by*

$$x_{n+1} = \sum_{i=1}^N \delta_i^n P_{C^*_{i,n}} \left((1-\alpha_n)\left(x_n - \tau_n T^*(I - P_{Q^*_n})Tx_n\right)\right)$$

where $\{\alpha_n\} \subset (0,1)$, $C^*_{i,n}$ is the ball given as in (3.1), Q^*_n is given as in (1.12), and the step-size τ_n is self-adaptively updated via

$$\tau_n := \frac{\rho_n \left\|(I - P_{Q^*_n})Tx_n\right\|^2}{\Theta_n^2}$$

where $\{\rho_n\} \subset (0,2)$ and

$$\Theta_n := \max\left\{1,\ \left\|T^*(I - P_{Q^*_n})Tx_n\right\|\right\}.$$

Suppose that $\{\rho_n\}$ and $\{\alpha_n\}$ satisfy the assumptions **(A1)** *and* **(A2)** *in Algorithm 1. Then, $\{x_n\}$ converges strongly to the minimum-norm element $x^* = P_E(0)$.*

4. Numerical Examples

In this section, we present two numerical examples to illustrate the performances of our proposed scheme. All testings are executed on a standard FUJITSUNOTEBOOK laptop with 11th Gen Intel(R) Core(TM) i7-1165G7 @ 2.80GHz 2.80 GHz with memory 16GB. The code is implemented in MATLAB R2022a.

Example 4.1. Consider $H=\mathbb{R}^3$, $H_1=\mathbb{R}^6$, $H_2=\mathbb{R}^9$, $H_3=\mathbb{R}^{12}$, and $H_4=\mathbb{R}^{15}$. Find a point $x^*\in\mathbb{R}^3$ such that $x^*\in\Omega:=C_1\cap C_2\cap C_3\cap T_1^{-1}(Q_1)\cap T_2^{-1}(Q_2)\cap T_3^{-1}(Q_3)\cap T_4^{-1}(Q_4)\neq\emptyset$, where

$$C_1=\{x\in\mathbb{R}^3:\|x-\mathbf{o}_1\|^2\le\mathbf{r}_1^2\},\ C_2=\{x\in\mathbb{R}^3:\|x-\mathbf{o}_2\|^2\le\mathbf{r}_2^2\},\ C_3=\{x\in\mathbb{R}^3:\|x-\mathbf{o}_3\|^2\le\mathbf{r}_3^2\},$$

$$Q_1=\{T_1x\in\mathbb{R}^6:\|T_1x-\mathbf{c}_1\|^2\le\rho_1^2\},\ Q_2=\{T_2x\in\mathbb{R}^9:\|T_2x-\mathbf{c}_2\|^2\le\rho_2^2\},$$

and

$$Q_3=\{T_3x\in\mathbb{R}^{12}:\|T_3x-\mathbf{c}_3\|^2\le\rho_3^2\},\ Q_4=\{T_4x\in\mathbb{R}^{15}:\|T_4x-\mathbf{c}_4\|^2\le\rho_4^2\},$$

where $\mathbf{o}_1,\mathbf{o}_2,\mathbf{o}_3\in\mathbb{R}^3$, $\mathbf{c}_1\in\mathbb{R}^6$, $\mathbf{c}_2\in\mathbb{R}^9$, $\mathbf{c}_3\in\mathbb{R}^{12}$, $\mathbf{c}_4\in\mathbb{R}^{15}$, $\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\rho_1,\rho_2,\rho_3,\rho_4\in\mathbb{R}$, and $T_1:\mathbb{R}^3\to\mathbb{R}^6$, $T_2:\mathbb{R}^3\to\mathbb{R}^9$, $T_3:\mathbb{R}^3\to\mathbb{R}^{12}$, $T_4:\mathbb{R}^3\to\mathbb{R}^{15}$.

For any $x\in\mathbb{R}^3$, we have $c_i(x)=\|x-\mathbf{o}_i\|^2-\mathbf{r}_i^2$ for $i=1,2,3$, and $q_j(T_jx)=\|T_jx-\mathbf{c}_j\|^2-\rho_j^2$ for $j=1,2,3,4$.

In what follows, the subgradients $\xi_{i,n}$ and $\eta_{j,n}$ of respectively $c_i(x_n)$ and $q_j(T_jx_n)$ can be calculated respectively at the points x_n and T_jx_n by $\xi_{i,n}(x_n)=2(x_n-\mathbf{o}_i)$ and $\eta_{j,n}(T_jx_n)=2(T_jx_n-\mathbf{c}_j)$. Thus, according to (3.1) and (3.2), the balls $C_{i,n}^*$ $(i=1,2,3)$ and $Q_{j,n}^*$ $(j=1,2,3,4)$, respectively of the sets C_i and Q_j can be easily determined at a point x_n and T_jx_n, respectively and the metric projections onto the balls $C_{i,n}^*$ $(i=1,2,3)$ and $Q_{j,n}^*$ $(j=1,2,3,4)$ can be easily calculated.

Now, we take the radii $\mathbf{r}_1=4,\mathbf{r}_2=5=\mathbf{r}_3$, $\rho_1=8,\rho_2=15,\rho_3=22$, and $\rho_4=18$. Then

$$T_1=\begin{pmatrix}-3.70&0.93&-1.45\\-2.75&-3.37&-45\\-1.50&3.38&-2.86\\-2.13&-3.32&-1.02\\4.27&0.02&-1.66\\-4.49&4.99&-2.70\end{pmatrix},\qquad T_2=\begin{pmatrix}4.36&4.32&3.30\\1.83&2.63&-2.10\\4.62&3.26&-0.97\\-0.62&0.73&3.62\\4.40&2.92&1.15\\-4.94&-1.71&4.91\\1.10&-2.76&-2.96\\3.01&-1.88&3.27\\-2.67&0.84&1.76\end{pmatrix},$$

$$T_3=\begin{pmatrix}-2.51&2.42&0.01\\-0.24&2.58&0.22\\-1.01&-1.11&-4.10\\0.99&-0.71&4.05\\3.00&4.56&3.84\\-3.95&0.73&-0.61\\3.21&3.50&2.82\\3.41&-2.24&-3.52\\-1.45&1.22&1.20\\-0.70&0.88&-2.39\\0.72&4.63&-.54\\2.01&-4.14&3.44\end{pmatrix},\qquad T_4=\begin{pmatrix}-3.04&1.32&1.53\\-1.96&4.85&-3.92\\-0.17&0.59&-4.64\\-1.62&4.34&1.18\\2.98&2.20&0.67\\4.87&-0.16&4.62\\-3.41&1.39&2.46\\-2.63&3.88&1.63\\2.02&-3.01&0.23\\-1.24&-1.05&-2.40\\4.74&4.92&4.62\\4.72&-0.98&0.40\\1.44&1.59&-4.70\\3.60&4.01&1.96\\-0.98&4.95&0.20\end{pmatrix},$$

and the centers

$$\mathbf{o}_1 = (0.4, 0.6, 0.6)^T,$$

$$\mathbf{o}_2 = (-0.4, -0.4, 0.1)^T,$$

$$\mathbf{o}_3 = (-0.3, 0.7, 0.6)^T,$$

$$\mathbf{c}_1 = (0.1, -0.5, 0.4, -0.5, -0.1, -0.2)^T,$$

$$\mathbf{c}_2 = (0.1, 1.0, 0.5, 1.0, -0.5, 0.1, -0.9, 0.5, 0.2)^T,$$

$$\mathbf{c}_3 = (0.7, 1.0, 0.9, -0.2, -1.0, 0.1, -0.6, -0.6, -0.3, -0.9, 0.5, 0.5)^T,$$

and

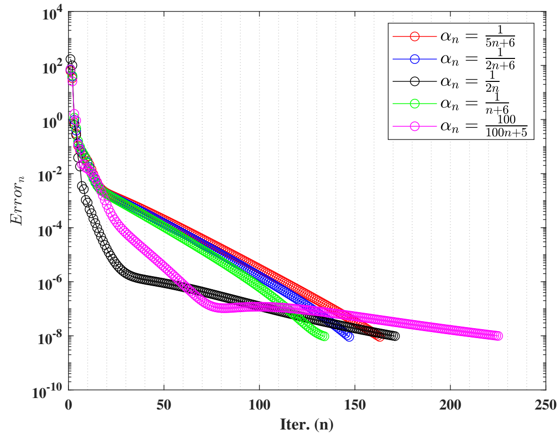
$$\mathbf{c}_4 = (0.1, -0.3, 0.7, 0.1, 0.9, 0.8, -0.3, 0.1, -0.3, 0.26, 0.6, 0.5, -0.7, 0.6, -0.9)^T.$$

The parameters choices in this example are: $\rho_n = \frac{n}{6n+1}$, $\delta_i^n = \frac{i}{6}, i = 1,2,3$, $\lambda_i = 0.95$, $\varpi_j = 0.5$, $\beta_1 = \frac{1}{10}$, $\beta_2 = \frac{1}{5}$, $\beta_3 = \frac{3}{10}$, and $\beta_4 = \frac{2}{5}$.

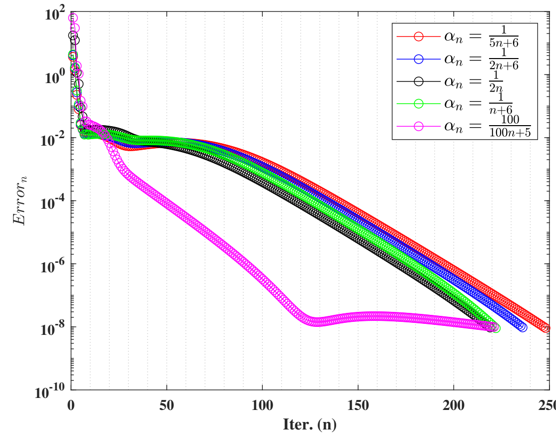
The stopping criteria that we take is $Error_n = \|x_{n+1} - x_n\|^2 < 10^{-8}$. All results are reported in Table 1 and Figure 1.

TABLE 1. Results of Algorithm 1 with different choices of x_0 and α_n

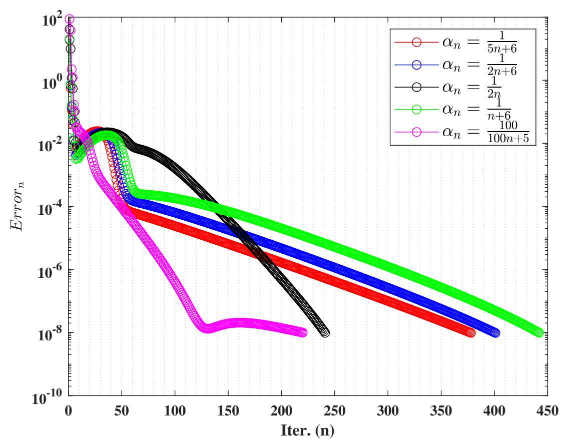
		$\alpha_n = \frac{1}{5n+6}$	$\alpha_n = \frac{1}{2n+6}$	$\alpha_n = \frac{1}{2n}$	$\alpha_n = \frac{1}{n+6}$	$\alpha_n = \frac{100}{100n+5}$
$x_0 = (1,1,1)^T$	Iter. (n)	163	147	171	134	225
	CPU(s)	0.009856	0.015708	0.015068	0.017095	0.017968
	$Error_n$	9.1247e-09	9.1508e-09	9.7497e-09	9.5981e-09	9.9332e-09
$x_0 = (-1,2,-2)^T$	Iter. (n)	248	236	219	222	221
	CPU(s)	0.012187	0.019565	0.017350	0.018336	0.017495
	$Error_n$	9.1511e-09	9.5716e-09	9.4844e-09	9.0429e-09	9.8820e-09
$x_0 = (-0.05,-0.01,-0.03)^T$	Iter. (n)	378	401	241	442	220
	CPU(s)	0.020337	0.019179	0.016844	0.018579	0.016722
	$Error_n$	9.7671e-09	9.7251e-09	9.8093e-09	9.7901e-09	9.9832e-09
$x_0 = (-1,-1,-1)^T$	Iter. (n)	261	248	156	251	223
	CPU(s)	0.017217	0.023305	0.016495	0.017885	0.017254
	$Error_n$	9.4467e-09	9.5806e-09	9.9104e-09	9.7798e-09	9.8821e-09
$x_0 = (1,1,-1)^T$	Iter. (n)	177	164	152	150	224
	CPU(s)	0.018949	0.017725	0.019414	0.017590	0.016947
	$Error_n$	9.6776e-09	9.4563e-09	9.2707e-09	9.2744e-09	9.9009e-09
$x_0 = (4,-2,-3)^T$	Iter. (n)	145	175	177	221	226
	CPU(s)	0.003837	0.016237	0.016613	0.017001	0.018756
	$Error_n$	9.5935e-09	9.7352e-09	9.9798e-09	9.9954e-09	9.8471e-09



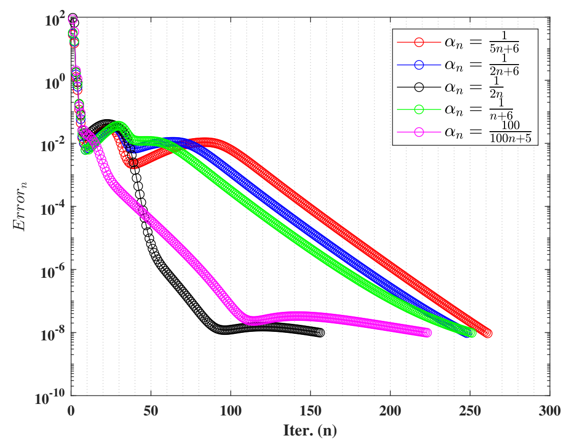
(a) $x_0 = (1,1,1)^T$



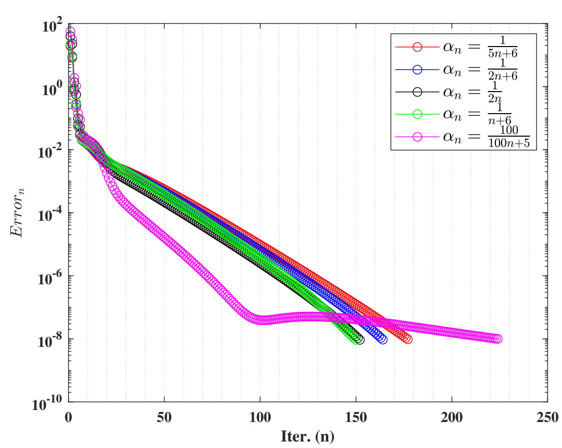
(b) $x_0 = (-1,2,-2)^T$



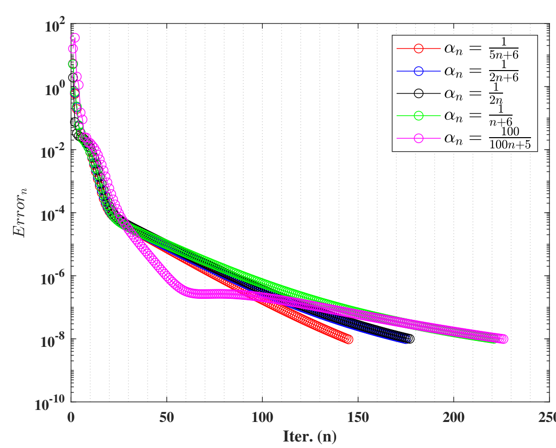
(c) $x_0 = (-0.05,-0.01,-0.03)^T$



(d) $x_0 = (-1,-1,-1)^T$



(e) $x_0 = (1,1,-1)^T$



(f) $x_0 = (4,-2,-3)^T$

FIGURE 1. Iter. (n) vs $Error_n$, experimental results of Algorithm 1 for different choices of x_0 and different values of α_n

Example 4.2. Consider $H=\mathbb{R}^4$, $H_1=\mathbb{R}^3$, $H_2=\mathbb{R}^6$, $H_3=\mathbb{R}^9$, $H_4=\mathbb{R}^{12}$, and $H_5=\mathbb{R}^{15}$. Consider the sets C_i and Q_j are ellipsoids in $\mathbb{R}^n$ defined by

$$C_1=\{x\in\mathbb{R}^4:(x-z_1)^TD_1(x-z_1)\le \mathbf{r}_1\},\ C_2=\{x\in\mathbb{R}^4:(x-z_2)^TD_2(x-z_2)\le \mathbf{r}_2\},$$

$$C_3=\{x\in\mathbb{R}^4:(x-z_3)^TD_3(x-z_3)\le \mathbf{r}_3\},\ C_4=\{x\in\mathbb{R}^4:(x-z_4)^TD_4(x-z_4)\le \mathbf{r}_4\},$$

$$Q_1=\{T_1x\in\mathbb{R}^3:(T_1x-w_1)^TP_1(T_1x-w_1)\le\rho_1\},\ Q_2=\{T_2x\in\mathbb{R}^6:(T_2x-w_2)^TP_2(T_2x-w_2)\le\rho_2\},$$

$$Q_3=\{T_3x\in\mathbb{R}^9:(T_3x-w_3)^TP_3(T_3x-w_3)\le\rho_3\},\ Q_4=\{T_4x\in\mathbb{R}^{12}:(T_3x-w_4)^TP_4(T_4x-w_4)\le\rho_4\},$$

and

$$Q_5=\{T_5x\in\mathbb{R}^{15}:(T_5x-w_5)^TP_5(T_5x-w_5)\le\rho_5\},$$

where each $D_i\in\mathbb{R}^{4\times4}$, $P_1\in\mathbb{R}^{3\times3}$, $P_2\in\mathbb{R}^{6\times6}$, $P_3\in\mathbb{R}^{9\times9}$, $P_4\in\mathbb{R}^{12\times12}$, and $P_5\in\mathbb{R}^{15\times15}$ are positive definite matrices, $z_i\in\mathbb{R}^4$, $w_1\in\mathbb{R}^3$, $w_2\in\mathbb{R}^6$, $w_3\in\mathbb{R}^9$, $w_4\in\mathbb{R}^{12}$, $w_5\in\mathbb{R}^{15}$, each $\mathbf{r}_i,\rho_j>0$, and $T_1:\mathbb{R}^4\to\mathbb{R}^3$, $T_2:\mathbb{R}^4\to\mathbb{R}^6$, $T_3:\mathbb{R}^4\to\mathbb{R}^9$, $T_4:\mathbb{R}^4\to\mathbb{R}^{12}$, $T_5:\mathbb{R}^4\to\mathbb{R}^{15}$ are bounded linear operators.

Our aim is to find a point $x^*\in\mathbb{R}^4$ such that $x^*\in\Omega:=\left(\cap_{i=1}^4 C_i\right)\cap\left(\cap_{j=1}^5 T_j^{-1}\left(Q_j\right)\right)\neq\emptyset$. Observe that an ellipsoid is a closed and convex set that can be represented as a sublevel set of a particular convex function; see [6]. Indeed, define $c_i:\mathbb{R}^4\to\mathbb{R}$ by $c_i(x)=\frac{1}{2}\left[(x-z)^TD_i(x-z)-\mathbf{r}_i\right]$. Then $C_i=\{x\in\mathbb{R}^4:c_i(x)\le0\}$ is a level set of c_i. It is easy to verify that $\nabla c_i(x)=D_i(x-z)$. Furthermore, it can be easily seen that

$$\|\nabla c_i(x)-\nabla c_i(y)\|=\|D_i(x-z)-D_i(y-z)\|=\|D_i(x-y)\|\le\|D_i\|\|x-y\|,\forall x,y\in\mathbb{R}^4$$

which further implies that ∇c_i is a $\|D_i\|$-Lipschitz continuous mapping. Similarly, each Q_j is a sublevel set of convex function.

Thus, according to (3.1) and (3.2), balls $C_{i,n}^*$ $(i=1,2,3,4)$ and $Q_{j,n}^*$ $(j=1,2,3,4,5)$ respectively of the sets C_i and Q_j can be easily determined at a point x_n and T_jx_n, respectively and the metric projections onto the balls $C_{i,n}^*$ $(i=1,2,3,4)$ and $Q_{j,n}^*$ $(j=1,2,3,4,5)$, can be easily calculated. We take $\mathbf{r}_1=9$, $\mathbf{r}_2=16$, $\mathbf{r}_3=30$, $\mathbf{r}_4=36$, $\rho_1=36$, $\rho_2=100$, $\rho_3=400$, $\rho_4=225$, $\rho_5=256$, $z_1=(0.4,0.6,0.5,0.6)^T$, $z_2=(0.4,0.4,0.1,0.5)^T$, $z_3=(0.3,0.7,0.6,0.5)^T$, $z_4=(0.3,0.7,0.6,0.5)^T$, $w_1=(0.1,0.4,0.1)^T$, $w_2=(0.1,0.5,0.4,0.5,0.1,0.2)^T$,

$$w_3=(0.1,1.0,0.5,1.0,0.5,0.1,0.9,0.5,0.2)^T,$$

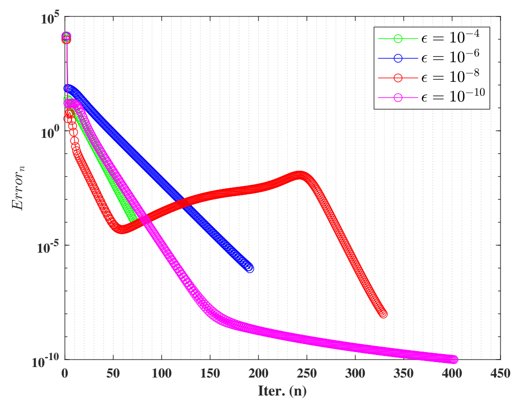
$$w_4=(0.7,1.0,0.9,0.2,1.0,0.1,0.6,0.6,0.3,0.9,0.5,0.5)^T,$$

$$w_5=(0.1,0.3,0.7,0.1,0.9,0.8,0.3,0.1,0.3,0.26,0.6,0.5,0.7,0.6,0.9)^T,$$

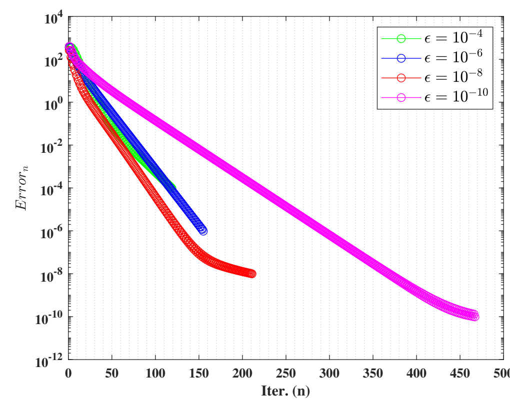
$D_i=diag(z_i)$ $(i=1,2,3,4)$, and $P_j=diag(w_j)$ $(j=1,2,3,4,5)$. The elements of the representing matrices T_j are randomly generated in the closed interval $[-5,5]$. We also fix the parameters sequences as $\rho_n=\frac{1}{6n+1}$, $\alpha_n=\frac{1}{5n+6}$, $\delta_i^n=\frac{i}{10}$, for $i=1,2,3,4$, $\lambda_i=0.05$, $\varpi_j=1.08$, and $\beta_j=\frac{j}{15}$ for $j=1,2,\ldots,5$. The stopping criteria that we take is $Error_n=\|x_{n+1}-x_n\|^2<\varepsilon$ for small enough $\varepsilon>0$. The results are reported in Table 2 and Figure 2.

TABLE 2. Results of Algorithm 1 with different choices of x_0 and ε

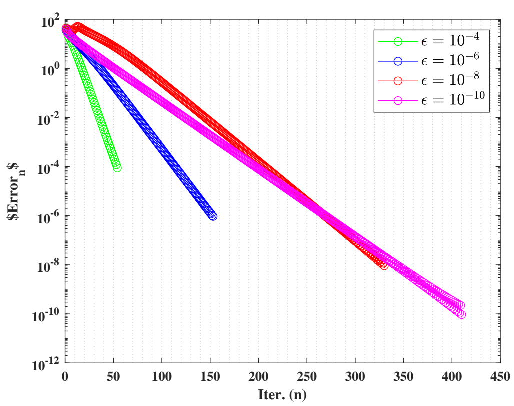
		$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-6}$	$\varepsilon = 10^{-8}$	$\varepsilon = 10^{-10}$
$x_0 = (1,1,1,1)^T$	Iter. (n)	73	191	329	402
	CPU(s)	0.029053	0.044395	0.007795	0.044373
	$Error_n$	8.9121e-05	9.3344e-07	9.7178e-09	9.9979e-11
$x_0 = (2,-1,-1,2)^T$	Iter. (n)	119	155	297	467
	CPU(s)	0.023579	0.024190	0.027815	0.030634
	$Error_n$	9.7235e-05	9.7152e-07	9.6760e-09	9.7687e-11
$x_0 = (3,-10,2,-4)^T$	Iter. (n)	54	153	330	410
	CPU(s)	0.022961	0.022139	0.009270	0.029090
	$Error_n$	8.8726e-05	9.3973e-07	9.1314e-09	9.3456e-11
$x_0 = (-0.5,-0.1,-0.3,-0.4)^T$	Iter. (n)	110	137	295	374
	CPU(s)	0.024781	0.014833	0.018609	0.022720
	$Error_n$	9.1706e-05	9.2732e-07	9.2888e-09	9.9338e-11



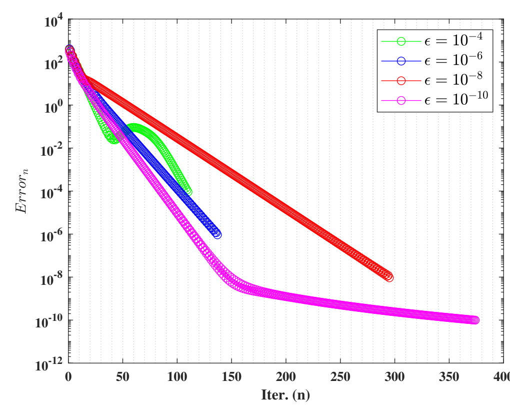
(a) $x_0 = (1,1,1,1)^T$



(b) $x_0 = (2,-1,-1,2)^T$



(c) $x_0 = (3,-10,2,-4)^T$



(d) $x_0 = (-0.5,-0.1,-0.3,-0.4)^T$

FIGURE 2. Iter. (n) vs $Error_n$, experimental results of Algorithm 1 for different choices of x_0 and different values of ε

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