J. Appl. Numer. Optim. 5 (2023), No. 3, pp. 335-348
Available online at http://jano.biemdas.com
https://doi.org/10.23952/jano.5.2023.3.04

AN INERTIAL FORWARD-BACKWARD-FORWARD ALGORITHM FOR SOLVING NON-CONVEX MIXED VARIATIONAL INEQUALITIES

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Dedicated to Professor Yair Censor on the Occasion of His 80th Birthday

Abstract. In this paper, we introduce a forward-backward-forward splitting algorithm with double inertial effects to approximate a solution of a non-convex mixed variational inequality problem. Our algorithm does not involve an on-line rule and one of the inertial factors is chosen to be non-positive. We give global convergence results of the iterative sequence generated by our algorithm. Some known results are recovered as special cases of our results. Numerical test is given to support the theoretical findings.

Keywords. Forward-backward-forward splitting; Non-convex variational inequalities; Proximal point algorithms; Wwo-step inertial extrapolation.

2020 Mathematics Subject Classification. 47H09, 65K15, 90C30.

1. Introduction

Let Ω be a non-empty subset of $\mathbb{R}^m$ with $F:\mathbb{R}^m \to \mathbb{R}^m$ a given mapping and $h:\Omega\to\mathbb{R}$. A mixed variational inequality (MVI, for short) is defined by

$$\text{find } x^* \in \Omega: \ \langle F(x^*), u-x^*\rangle + h(u) - h(x^*) \geq 0, \ \forall\, u\in\Omega. \tag{1.1}$$

Let $\mathscr{S}$ stand for the set of solutions of MVI (1.1). Special cases of MVI (1.1) were discussed in [7, 9, 18, 25, 26, 27] and the references therein.

If $h\equiv 0$ in (1.1), then MVI (1.1) reduces to the classical variational inequality:

$$\text{find } x^* \in \Omega: \ \langle F(x^*), u-x^*\rangle \geq 0, \ \forall\, u\in\Omega. \tag{1.2}$$

Also, if $F\equiv 0$ and h is convex in (1.1), then MVI (1.1) becomes a convex minimization problem [8, 22, 29, 35, 36]. However, when h in MVI (1.1) is non-convex function, then MVI (1.1) becomes harder to solve. This is because $\mathscr{S}=\emptyset$ might occur even when Ω is compact and convex (see [24, Example 3.1], [17, page 127] and [12] for more details).

The forward-backward-forward splitting algorithm [33] has been a desired algorithm in the literature for solving variational inequality problem (1.2). The method was shown to be effective

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Received November 15, 2022; Accepted August 16, 2023.

since only a projection onto Ω is needed at each iteration unlike the extragradient method [19]. The forward-backward-forward splitting method is given by

$$\begin{cases} y_s = P_\Omega(x_s - \lambda_s F(x_s)), \\ x_{s+1} = y_s - \lambda_s \left(F(y_s) - F(x_s)\right), \ s \geq 1. \end{cases} \tag{1.3}$$

Corresponding convergence results of forward-backward-forward algorithm (1.3) for solving convex variational inequality problem (1.2) were obtained in the literature when either the step-sizes $\lambda_s \in \left(0, \frac{1}{L}\right)$, L being the Lipschitz constant of F or when λ_s is obtained by using Armijo line search (see [33, (2.4)]) or λ_s is self-adaptively generated [5, 31, 32].

The weak convergence of the forward-backward-forward algorithm with one-step extrapolation for solving variational inequality problem (1.2) were studied in [3, 4, 6, 31, 32] and other related papers. In general, they considered:

$$\begin{cases} z_s = x_s + \theta(x_s - x_{s-1}), \\ y_s = P_\Omega(z_s - \lambda_s F(z_s)), \\ x_{s+1} = (1-\rho)z_s + \rho(y_s - \lambda_s(F(y_s) - F(z_s)), \ s \geq 1. \end{cases} \tag{1.4}$$

Boţ et al. in [6] obtained a weak convergence of (1.4) when $\theta \in [0,1)$ and $0 < \rho < \frac{2(1-\theta)^2}{(1+\mu)(2\theta^2-\theta+1)}$. Convergence results of (1.4) were obtained in [31] when $\rho = 1$ and $0 < a \leq \lambda_s \leq b < \frac{1}{L}$.

Recently, existence results for MVI (1.1) involving quasiconvex functions were obtained in [16, 17, 34], and this raises the quest for numerical iterative methods to solve non-convex MVI (1.1). Iterative processes for convex MVI (1.1) were obtained in [25, 26, 27], where the involved h is continuous and the methods proposed are either implicit methods or only conceptual methods with no numerical implementations. The proposed methods in [25, 26, 27] also require inner loops or two forward evaluations. Our approach in this paper is to propose and study a numerical method to solve non-convex MVI (1.1) for which h is a noncontinuous non-convex function which furthermore extends the results on minimization of quasiconvex functions obtained in [20, 28].

Quite recently, Grad and Lara [12] applied the Malitsky's Golden Ratio Algorithm [22] to solve non-convex MVI (1.1) and proposed the following method:

Algorithm 1 Golden Ratio Algorithm (GRA)

1: Choose $x_0, x_1 \in \Omega$ such that $x_0 \neq x_1, \phi = \frac{1+\sqrt{5}}{2}$, and $z_0 = x_1$.
2: If $x_{s+1} = x_s = z_s$, then STOP: $x_s \in \mathscr{S}$. Otherwise, go to Step 3.
3: Take $s = s+1$, and compute

$$\begin{aligned} z_s &= (1-\frac{1}{\phi})x_s + \frac{1}{\phi}z_{s-1}, \\ x_{s+1} &= \text{Prox}_{h+\iota_\Omega}\left(z_s - \frac{1}{\alpha}F(x_s)\right), \end{aligned} \tag{1.5}$$

and go to Step 2.

Grad and Lara [12] proved that $\{x_s\}$ and $\{z_s\}$ from Algorithm 1 converge to a solution of non-convex MVI (1.1).

Our Contributions.

- We apply the forward-backward-forward splitting algorithm with double extrapolation to solve a non-convex mixed variational inequality problem. We extend the usage of forward-backward-forward splitting methods from the convex variational inequality problem (1.2) studied in [5, 31, 32] to non-convex MVI (1.1).
- Instead of one-step inertial extrapolation, which was considered in several papers in [3, 4, 6, 31] for convex variational inequality problem (1.2), we further accelerate the forward-backward-forward splitting method by adding a two-step inertial extrapolation.
- Numerical experiments are given to show the benefits gained by considering a two-step inertial extrapolation instead of one-step inertial extrapolation.

Presentation. We arrange the paper as follows: Section 2 gives some definitions which we need in our convergence analysis. In Section 3, we examine our algorithm and global convergence results are given. We present numerical tests in Section 4 and final remarks in Section 5.

2. Preliminaries

In this section, we recall some basic notions in convex analysis which are used frequently in the sequel. Given a non-empty set $\Omega \subseteq \mathbb{R}^m$, the indicator function ι_Ω is defined by

$$\iota_\Omega(u) := \begin{cases} 0 \text{ if } u \in \Omega \\ +\infty \text{ otherwise.} \end{cases}$$

The effective domain of $h : \mathbb{R}^m \to \mathbb{R} \cup \{\pm\infty\}$ is defined by dom $h := \{w \in \mathbb{R} : h(w) < +\infty\}$. Recall that h is proper if $h(w) > -\infty$ for all $w \in \mathbb{R}^m$ and $\mathrm{dom} h \neq \emptyset$. We say that h with a convex domain is

(a) convex if, for given $u, v \in \mathrm{dom} h$,

$$h(tu + (1-t)v) \leq th(u) + (1-t)h(v) \quad \forall\, t \in [0,1];$$

(b) quasiconvex if, for given $u, v \in \mathrm{dom} h$,

$$h(\alpha u + (1-\alpha)v) \leq \max\{h(u), h(v)\} \;\; \forall\, \alpha \in [0,1].$$

Clearly, a convex function is quasiconvex.

Given a function $h : \mathbb{R}^m \to \mathbb{R} \cup \{\pm\infty\}$, the proximity operator with $\lambda > 0$ at $u \in \mathbb{R}^m$ is defined by $\mathrm{Prox}_{\lambda h} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ with

$$\mathrm{Prox}_{\lambda h}(u) = \arg\min_{v \in \mathbb{R}^m} \left\{ h(v) + \frac{1}{2\lambda} \|v - u\|^2 \right\}.$$

If $h : \Omega \to \mathbb{R} \cup \{\pm\infty\}$ with $\Omega \cap \mathrm{dom} h \neq \emptyset$ is a proper function and $f : \Omega \times \Omega \to \mathbb{R}$, then f is said to be

(a) monotone on Ω if, for all $u, v \in \Omega$ $f(u,v) + f(v,u) \leq 0$,
(b) h-pseudomonotone on Ω if, for all $u, v \in \Omega$

$$f(u,v) + h(v) - h(u) \geq 0 \Longrightarrow f(v,u) + h(u) - h(v) \leq 0.$$

It can be seen from the definition above that a monotone bifunction is a $h-$ pseudomonotone bifunction. However, the converse fails.

A new class of generalized convex functions called "prox-convex functions", which includes some classes of quasiconvex functions and weakly convex functions along other functions was introduced in [13].

Definition 2.1. ([13]) We say that a proper function $h:\mathbb{R}^m \to \mathbb{R}\cup\{\pm\infty\}$ with Ω closed subset of $\mathbb{R}^m$ and $\Omega \cap \mathrm{dom} h \neq \emptyset$ is said to be prox-convex on Ω (with prox-convex value ρ) if there exists $\rho > 0$ such that, for all $w \in \Omega$, $\mathrm{Prox}_{h+\iota_\Omega}(w) \neq \emptyset$, and

$$\bar{u} \in \mathrm{Prox}_{h+\iota_\Omega}(w) \Longrightarrow h(\bar{u}) - h(u) \leq \rho\langle \bar{u}-w, u-\bar{u}\rangle, \ \forall u \in \Omega. \tag{2.1}$$

Properties of prox-convex functions like the map $w \to \mathrm{Prox}_{h+\iota_K}(w)$ is single-valued and further details can be found in [13]. Also, the generalized convexity can be found in [11, 14].

The following lemma is useful in the sequel.

Lemma 2.1. *Suppose $u,v,w \in \mathbb{R}^m$ and $\theta,\beta \in \mathbb{R}$. Then*

$$\begin{aligned} & \|(1+\theta)u-(\theta-\beta)v-\beta w\|^2 \\ = \ & (1+\theta)\|u\|^2-(\theta-\beta)\|v\|^2-\beta\|w\|^2+(1+\theta)(\theta-\beta)\|u-v\|^2 \\ & +\beta(1+\theta)\|u-w\|^2-\beta(\theta-\beta)\|v-w\|^2. \end{aligned}$$

Also, the following identity is needed later.

$$\langle u-w, v-u\rangle = \frac{1}{2}\|w-v\|^2 - \frac{1}{2}\|u-w\|^2 - \frac{1}{2}\|v-u\|^2$$

for all $u,v,w \in \mathbb{R}^m$.

3. The Algorithm and Convergence Analysis

We present our algorithm for non-convex MVI (1.1) with its convergence analysis in this section.

3.1. **Proposed Method.** Before giving our proposed method, and analysing the convergence, the following convergence criteria are assumed.

Assumption 3.1. (A1) F is Lipschitz continuous Ω with constant $L > 0$;
(A2) h is lower semicontinuous and prox-convex on Ω with prox-convex value $\alpha > 0$;
(A3) F and h satisfy (cf. [12, 22, 30])

$$\langle F(y), y-\bar{x}\rangle + h(y) - h(\bar{x})\rangle \geq 0, \ \ \forall\, y \in \Omega, \ \forall\, \bar{x} \in \mathscr{S};$$

(A4) $\mathscr{S} \neq \emptyset$;
(A5) $\alpha > L$.

Remark 3.1. Assumption (A5) is weaker than assumption $\alpha > \frac{2L}{\phi}$, where $\phi = \frac{1+\sqrt{5}}{2}$ was required in [12].

Remark 3.2. Note that if $F = 0$, then $L = 0$ and $\alpha > L$ holds automatically from Condition (A5).

Along with the assumption above, we also use more the following assumption related to inertial parameters.

Assumption 3.2. We assume that the inertial parameters $\theta \in \left[0, \frac{1}{3}\right)$ and $\beta \in (-\infty, 0]$ satisfy the following conditions.

(i)
$$0 \leq \theta < \frac{\alpha - L}{3\alpha + L};$$
(ii)
$$\max\left\{2\theta\left(\frac{\alpha - L}{\alpha + L}\right) - (1-\theta), \frac{1}{2\alpha}\left[\theta(\alpha + L) - \frac{(\alpha - L)(1-\theta)^2}{1+\theta}\right]\right\} < \beta \leq 0;$$
(iii)
$$2\theta^2 L - \alpha(1-3\theta) + L(1-\theta) - \beta(4\alpha\theta + 3\alpha - L) + 2L\beta^2 < 0.$$

Now, we present the proposed method:

Algorithm 2 Non-convex Forward-Backward-Forward Method

1: Choose $\beta \in (-\infty, 0]$ and $\theta \in \left[0, \frac{1}{3}\right)$ such that Assumptions 3.2 are fulfilled. Pick $x_{-1}, x_0, x_1 \in \mathbb{R}^m$ and set $s = 1$.

2: Given x_{s-2}, x_{s-1} and x_s, compute x_{s+1} as follows:
$$\begin{cases} z_s = x_s + \theta(x_s - x_{s-1}) + \beta(x_{s-1} - x_{s-2}) \\ y_s = \text{Prox}_{h+\iota_\Omega}\left(z_s - \frac{1}{\alpha}F(z_s)\right) \\ x_{s+1} = y_s + \frac{1}{\alpha}(F(z_s) - F(y_s)). \end{cases} \tag{3.1}$$

3: Set $s \leftarrow s+1$, and **go to Step 2**.

We give the following remarks regarding Algorithm 2.

Remark 3.3.

(i) Unlike Algorithm 1, Algorithm 2 does not contain the golden ratio step but a recall of a previous step at the current iteration and a two-step inertial extrapolation.

(ii) If $F \equiv 0$, and $\theta = 0 = \beta$, our Algorithm 2 reduces to $x_{s+1} = \text{Prox}_{h+\iota_\Omega}(x_s)$ proposed in [13, Theorem 4.1] and also an extension of algorithms proposed in [20, 28] for minimization of quasiconvex functions.

(iii) If $h \equiv 0$, then Algorithm 2 reduces to [30, Algorithm 2.1].

3.2. **Convergence analysis.** In this subsection, we present the global convergence of Algorithm 2. The first lemma demonstrates the boundedness of $\{x_s\}$.

Lemma 3.1. *Let $\{x_s\}$ be generated by Algorithm 2. Then $\{x_s\}$ is bounded under Assumptions 3.2.*

Proof. Since
$$y_s = \text{Prox}_{h+\iota_\Omega}\left(z_s - \frac{1}{\alpha}F(z_s)\right),$$
we obtain by using (2.1) that, for all $y \in \Omega$,
$$h(y_s) - h(y) \leq \alpha\langle y_s - z_s + \frac{1}{\alpha}F(z_s), y - y_s\rangle. \tag{3.2}$$
Thus $0 \leq \alpha\langle y_s - z_s + \frac{1}{\alpha}F(z_s), y - y_s\rangle + h(y) - h(y_s)$ for all $y \in \Omega$. Replacing $y = \bar{x} \in \mathscr{S}$, we have from the last inequality that
$$0 \leq \alpha\langle y_s - z_s + \frac{1}{\alpha}F(z_s), \bar{x} - y_s\rangle + h(\bar{x}) - h(y_s). \tag{3.3}$$

Since F satisfies condition (A3), we obtain $\langle F(y), y-\bar{x}\rangle + h(y) - h(\bar{x}) \geq 0$ for all $y \in \Omega$. In particular, we have

$$\langle F(y_s), y_s - \bar{x}\rangle + h(y_s) - h(\bar{x}) \geq 0. \tag{3.4}$$

Combining (3.3) and (3.4), we arrive at

$$-\langle F(y_s), \bar{x} - y_s\rangle + h(y_s) - h(\bar{x}) + \alpha\langle y_s - z_s + \frac{1}{\alpha}F(z_s), \bar{x} - y_s\rangle + h(\bar{x}) - h(y_s) \geq 0.$$

Thus $\alpha\langle y_s - z_s + \frac{1}{\alpha}F(z_s) - \frac{1}{\alpha}F(y_s), \bar{x} - y_s\rangle \geq 0$. We have from the last inequality and the definition of $\{x_s\}$ that $\alpha\langle x_{s+1} - z_s, \bar{x} - y_s\rangle \geq 0$, which further implies that

$$\begin{aligned}
\langle x_{s+1} - \bar{x}, x_{s+1} - z_s\rangle &\leq \langle x_{s+1} - y_s, x_{s+1} - z_s\rangle \\
&= \|x_{s+1} - z_s\|^2 + \langle z_s - y_s, x_{s+1} - z_s\rangle \\
&= \|x_{s+1} - z_s\|^2 + \langle z_s - y_s, y_s + \frac{1}{\alpha}(F(z_s) - F(y_s)) - z_s\rangle \\
&= \|x_{s+1} - z_s\|^2 - \|z_s - y_s\|^2 + \frac{1}{\alpha}\langle z_s - y_s, F(z_s) - F(y_s)\rangle. \qquad (3.5)
\end{aligned}$$

In addition,

$$\|x_{s+1} - \bar{x}\|^2 - \|z_s - \bar{x}\|^2 + \|x_{s+1} - z_s\|^2 = 2\langle x_{s+1} - \bar{x}, x_{s+1} - z_s\rangle. \tag{3.6}$$

If we combine (3.5) and (3.6), then

$$\|x_{s+1} - \bar{x}\|^2 \leq \|z_s - \bar{x}\|^2 + \|x_{s+1} - z_s\|^2 - 2\|z_s - y_s\|^2 + \frac{2}{\alpha}\langle z_s - y_s, F(z_s) - F(y_s)\rangle. \tag{3.7}$$

Because F is Lipschitz continuous with L, we deduce

$$\begin{aligned}
\|x_{s+1} - z_s\|^2 &= \|y_s - z_s\|^2 + \frac{2}{\alpha}\langle y_s - z_s, F(z_s) - F(y_s)\rangle + \frac{1}{\alpha^2}\|F(z_s) - F(y_s)\|^2 \\
&\leq \|y_s - z_s\|^2 + \frac{2}{\alpha}\langle y_s - z_s, F(z_s) - F(y_s)\rangle + \frac{L^2}{\alpha^2}\|z_s - y_s\|^2. \qquad (3.8)
\end{aligned}$$

From (3.7) and (3.8), we have

$$\|x_{s+1} - \bar{x}\|^2 \leq \|z_s - \bar{x}\|^2 - \left(1 - \frac{L^2}{\alpha^2}\right)\|y_s - z_s\|^2. \tag{3.9}$$

Observe that

$$\begin{aligned}
z_s - \bar{x} &= x_s + \theta(x_s - x_{s-1}) + \beta(x_{s-1} - x_{s-2}) - \bar{x} \\
&= (1+\theta)(x_s - \bar{x}) - (\theta - \beta)(x_{s-1} - \bar{x}) - \beta(x_{s-2} - \bar{x}).
\end{aligned}$$

Hence, by Lemma 2.1, we see that

$$\begin{aligned}
\|z_s - \bar{x}\|^2 &= \|(1+\theta)(x_s - \bar{x}) - (\theta - \beta)(x_{s-1} - \bar{x}) - \beta(x_{s-2} - \bar{x})\|^2 \\
&= (1+\theta)\|x_s - \bar{x}\|^2 - (\theta - \beta)\|x_{s-1} - \bar{x}\|^2 - \beta\|x_{s-2} - \bar{x}\|^2 \\
&\quad + (1+\theta)(\theta - \beta)\|x_s - x_{s-1}\|^2 + \beta(1+\theta)\|x_s - x_{s-2}\|^2 \\
&\quad - \beta(\theta - \beta)\|x_{s-1} - x_{s-2}\|^2.
\end{aligned} \tag{3.10}$$

Note that

$$2\theta\langle x_{s+1} - x_s, x_s - x_{s-1}\rangle \leq 2|\theta|\|x_{s+1} - x_s\|\|x_s - x_{s-1}\| = 2\theta\|x_{s+1} - x_s\|\|x_s - x_{s-1}\|$$

and hence

$$-2\theta\langle x_{s+1}-x_s, x_s-x_{s-1}\rangle \geq -2\theta\|x_{s+1}-x_s\|\|x_s-x_{s-1}\|. \tag{3.11}$$

On the other hand, we have

$$\begin{aligned}2\beta\langle x_{s+1}-x_s, x_{s-1}-x_{s-2}\rangle &= 2\langle \beta(x_{s+1}-x_s), x_{s-1}-x_{s-2}\rangle \\ &\leq 2|\beta|\|x_{s+1}-x_s\|\|x_{s-1}-x_{s-2}\|,\end{aligned}$$

i.e.,

$$-2\beta\langle x_{s+1}-x_s, x_{s-1}-x_{s-2}\rangle \geq -2|\beta|\|x_{s+1}-x_s\|\|x_{s-1}-x_{s-2}\|. \tag{3.12}$$

Similarly, we have $2\beta\theta\langle x_{s-1}-x_s, x_{s-1}-x_{s-2}\rangle \leq 2|\beta|\theta\|x_s-x_{s-1}\|\|x_{s-1}-x_{s-2}\|$, so

$$2\beta\theta\langle x_s-x_{s-1}, x_{s-1}-x_{s-2}\rangle \geq -2|\beta|\theta\|x_s-x_{s-1}\|\|x_{s-1}-x_{s-2}\|. \tag{3.13}$$

By (3.1), we have

$$\begin{aligned}\|x_{s+1}-z_s\| &\leq \frac{1}{\alpha}\|F(z_s)-F(y_s)\|+\|y_s-z_s\| \\ &\leq \frac{L}{\alpha}\|y_s-z_s\|+\|y_s-z_s\| \\ &= \left(1+\frac{L}{\alpha}\right)\|y_s-z_s\|.\end{aligned}$$

Therefore,

$$-\|y_s-z_s\|^2 \leq -\frac{1}{\left(1+\frac{L}{\alpha}\right)^2}\|x_{s+1}-z_s\|^2. \tag{3.14}$$

Substituting (3.14) into (3.9) gives

$$\|x_{s+1}-\bar{x}\|^2 \leq \|z_s-\bar{x}\|^2 - \frac{\left(1-\frac{L}{\alpha}\right)}{\left(1+\frac{L}{\alpha}\right)}\|x_{s+1}-z_s\|^2 = \|z_s-\bar{x}\|^2 - \left(\frac{\alpha-L}{\alpha+L}\right)\|x_{s+1}-z_s\|^2. \tag{3.15}$$

By (3.11), (3.12), (3.13), and Cauchy-Schwarz inequality, one has

$$\begin{aligned}\|x_{s+1}-z_s\|^2 &= \|x_{s+1}-x_s-\theta(x_s-x_{s-1})-\beta(x_{s-1}-x_{s-2})\|^2 \\ &= \|x_{s+1}-x_s\|^2 - 2\theta\langle x_{s+1}-x_s, x_s-x_{s-1}\rangle \\ &\quad -2\beta\langle x_{s+1}-x_s, x_{s-1}-x_{s-2}\rangle + \theta^2\|x_s-x_{s-1}\|^2 \\ &\quad +2\beta\theta\langle x_s-x_{s-1}, x_{s-1}-x_{s-2}\rangle + \beta^2\|x_{s-1}-x_{s-2}\|^2 \\ &\geq \|x_{s+1}-x_s\|^2 - 2\theta\|x_{s+1}-x_s\|\|x_s-x_{s-1}\| \\ &\quad -2|\beta|\|x_{s+1}-x_s\|\|x_{s-1}-x_{s-2}\| + \theta^2\|x_s-x_{s-1}\|^2 \\ &\quad -2|\beta|\theta\|x_s-x_{s-1}\|\|x_{s-1}-x_{s-2}\| + \beta^2\|x_{s-1}-x_{s-2}\|^2 \\ &\geq \|x_{s+1}-x_s\|^2 - \theta\|x_{s+1}-x_s\|^2 - \theta\|x_s-x_{s-1}\|^2 \\ &\quad -|\beta|\|x_{s+1}-x_s\|^2 - |\beta|\|x_{s-1}-x_{s-2}\|^2 + \theta^2\|x_s-x_{s-1}\|^2 \\ &\quad -|\beta|\theta\|x_s-x_{s-1}\|^2 - |\beta|\theta\|x_{s-1}-x_{s-2}\|^2 + \beta^2\|x_{s-1}-x_{s-2}\|^2 \\ &= (1-|\beta|-\theta)\|x_{s+1}-x_s\|^2 + (\theta^2-\theta-|\beta|\theta)\|x_s-x_{s-1}\|^2 \\ &\quad +(\beta^2-|\beta|-|\beta|\theta)\|x_{s-1}-x_{s-2}\|^2.\end{aligned} \tag{3.16}$$

Combining (3.10) and (3.16) into (3.15) and noting that $\beta \le 0$, we have

$$\begin{aligned}
\|x_{s+1}-\bar{x}\|^2 &\le (1+\theta)\|x_s-\bar{x}\|^2-(\theta-\beta)\|x_{s-1}-\bar{x}\|^2-\beta\|x_{s-2}-\bar{x}\|^2\\
&\quad+(1+\theta)(\theta-\beta)\|x_s-x_{s-1}\|^2+\beta(1+\theta)\|x_s-x_{s-2}\|^2\\
&\quad-\beta(\theta-\beta)\|x_{s-1}-x_{s-2}\|^2-\Big(\frac{\alpha-L}{\alpha+L}\Big)(1-|\beta|-\theta)\|x_{s+1}-x_s\|^2\\
&\quad-\Big(\frac{\alpha-L}{\alpha+L}\Big)(\theta^2-\theta-|\beta|\theta)\|x_s-x_{s-1}\|^2\\
&\quad-\Big(\frac{\alpha-L}{\alpha+L}\Big)(\beta^2-|\beta|-|\beta|\theta)\|x_{s-1}-x_{s-2}\|^2\\
&=(1+\theta)\|x_s-\bar{x}\|^2-(\theta-\beta)\|x_{s-1}-\bar{x}\|^2-\beta\|x_{s-2}-\bar{x}\|^2\\
&\quad+\Big((1+\theta)(\theta-\beta)-\Big(\frac{\alpha-L}{\alpha+L}\Big)(\theta^2-\theta-|\beta|\theta)\Big)\|x_s-x_{s-1}\|^2\\
&\quad+\beta(1+\theta)\|x_s-x_{s-2}\|^2-\Big(\frac{\alpha-L}{\alpha+L}\Big)(1-|\beta|-\theta)\|x_{s+1}-x_s\|^2\\
&\quad-\Big(\beta(\theta-\beta)+\Big(\frac{\alpha-L}{\alpha+L}\Big)(\beta^2-|\beta|-|\beta|\theta)\Big)\|x_{s-1}-x_{s-2}\|^2\\
&\le(1+\theta)\|x_s-\bar{x}\|^2-(\theta-\beta)\|x_{s-1}-\bar{x}\|^2-\beta\|x_{s-2}-\bar{x}\|^2\\
&\quad+\Big((1+\theta)(\theta-\beta)-\Big(\frac{\alpha-L}{\alpha+L}\Big)(\theta^2-\theta+\beta\theta)\Big)\|x_s-x_{s-1}\|^2\\
&\quad-\Big(\frac{\alpha-L}{\alpha+L}\Big)(1+\beta-\theta)\|x_{s+1}-x_s\|^2\\
&\quad-\Big(\beta(\theta-\beta)+\Big(\frac{\alpha-L}{\alpha+L}\Big)(\beta^2+\beta+\beta\theta)\Big)\|x_{s-1}-x_{s-2}\|^2.
\end{aligned}$$

By rearranging, we arrive at

$$\begin{aligned}
&\|x_{s+1}-\bar{x}\|^2-\theta\|x_s-\bar{x}\|^2-\beta\|x_{s-1}-\bar{x}\|^2+\Big(\frac{\alpha-L}{\alpha+L}\Big)(1+\beta-\theta)\|x_{s+1}-x_s\|^2\\
&\le\|x_s-\bar{x}\|^2-\theta\|x_{s-1}-\bar{x}\|^2-\beta\|x_{s-2}-\bar{x}\|^2\\
&\quad+\Big((1+\theta)(\theta-\beta)-\Big(\frac{\alpha-L}{\alpha+L}\Big)(\theta^2-2\theta+\beta\theta+\beta+1)\Big)\|x_s-x_{s-1}\|^2\\
&\quad-\Big(\beta(\theta-\beta)+\Big(\frac{\alpha-L}{\alpha+L}\Big)(\beta^2+\beta+\beta\theta)\Big)\|x_{s-1}-x_{s-2}\|^2\\
&\quad+\Big(\frac{\alpha-L}{\alpha+L}\Big)(1+\beta-\theta)\|x_s-x_{s-1}\|^2.
\end{aligned} \tag{3.17}$$

Define

$$\Lambda_s:=\|x_s-\bar{x}\|^2-\theta\|x_{s-1}-\bar{x}\|^2-\beta\|x_{s-2}-\bar{x}\|^2+\Big(\frac{\alpha-L}{\alpha+L}\Big)(1+\beta-\theta)\|x_s-x_{s-1}\|^2.$$

Let us demonstrate that $\Lambda_s \geq 0$ for all $s \geq 1$. Indeed,

$$\begin{aligned}\Lambda_s &= \|x_s - \bar{x}\|^2 - \theta\|x_{s-1} - \bar{x}\|^2 - \beta\|x_{s-2} - \bar{x}\|^2 + \left(\frac{\alpha - L}{\alpha + L}\right)(1+\beta-\theta)\|x_s - x_{s-1}\|^2 \\ &\geq \|x_s - \bar{x}\|^2 - 2\theta\|x_s - x_{s-1}\|^2 - 2\theta\|x_s - \bar{x}\|^2 \\ &\quad - \beta\|x_{s-2} - \bar{x}\|^2 + \left(\frac{\alpha - L}{\alpha + L}\right)(1+\beta-\theta)\|x_s - x_{s-1}\|^2 \\ &= (1-2\theta)\|x_s - \bar{x}\|^2 + \left[\left(\frac{\alpha - L}{\alpha + L}\right)(1+\beta-\theta) - 2\theta\right]\|x_s - x_{s-1}\|^2 - \beta\|x_{s-2} - \bar{x}\|^2.\end{aligned} \tag{3.18}$$

In view of $\theta < \frac{1}{2}$, $\beta \leq 0$, Assumption 3.2 (i) and (ii), and $2\theta\left(\frac{\alpha+L}{\alpha-L}\right) - (1-\theta) < \beta$, it follows from (3.18) that $\Lambda_s \geq 0$ for all $n \geq 1$. Furthermore, we derive from (3.17) that

$$\begin{aligned}\Lambda_{s+1} - \Lambda_s &\leq \left((1+\theta)(\theta-\beta) - \left(\frac{\alpha - L}{\alpha + L}\right)(\theta^2 - 2\theta + \beta\theta + \beta + 1)\right)\|x_s - x_{s-1}\|^2 \\ &\quad - \left(\beta(\theta-\beta) + \left(\frac{\alpha - L}{\alpha + L}\right)(\beta^2 + \beta + \beta\theta)\right)\|x_{s-1} - x_{s-2}\|^2 \\ &\quad = -\left((1+\theta)(\theta-\beta) - \left(\frac{\alpha - L}{\alpha + L}\right)(\theta^2 - 2\theta + \beta\theta + \beta + 1)\right)\Big(\|x_{s-1} - x_{s-2}\|^2 \\ &\quad - \|x_s - x_{s-1}\|^2\Big) + \left((1+\theta)(\theta-\beta) - \left(\frac{\alpha - L}{\alpha + L}\right)(\theta^2 - 2\theta + \beta\theta + \beta + 1)\right. \\ &\quad \left. - \beta(\theta-\beta) - \left(\frac{\alpha - L}{\alpha + L}\right)(\beta^2 + \beta + \beta\theta)\right)\|x_{s-1} - x_{s-2}\|^2 \\ &= c_1\left(\|x_{s-1} - x_{s-2}\|^2 - \|x_s - x_{s-1}\|^2\right) - c_2\|x_{s-1} - x_{s-2}\|^2,\end{aligned} \tag{3.19}$$

where

$$c_1 := -\left((\theta-\beta)(1+\theta) - \left(\frac{\alpha - L}{\alpha + L}\right)(\theta^2 - 2\theta + \beta\theta + \beta + 1)\right)$$

and

$$\begin{aligned}c_2 := -\Big((\theta-\beta)(1+\theta) - \left(\frac{\alpha - L}{\alpha + L}\right)(\theta^2 - 2\theta + \beta\theta + \beta + 1) \\ - \beta(\theta-\beta) - \left(\frac{\alpha - L}{\alpha + L}\right)(\beta^2 + \beta + \beta\theta)\Big).\end{aligned}$$

By Assumption 3.2 (ii), it holds that

$$\frac{1}{2\alpha}\left[\theta(\alpha+L) - \frac{(\alpha-L)(1-\theta)^2}{1+\theta}\right] < \beta.$$

As a result, $c_1 > 0$. Thus $c_2 > 0$ by Assumption 3.2 (iii). By (3.19), we have

$$\Lambda_{s+1} + c_1\|x_s - x_{s-1}\|^2 \leq \Lambda_s + c_1\|x_{s-1} - x_{s-2}\|^2 - c_2\|x_{s-1} - x_{s-2}\|^2. \tag{3.20}$$

Letting $\bar{\Lambda}_s := \Lambda_s + c_1\|x_{s-1} - x_{s-2}\|^2$, we have $\bar{\Lambda}_s \geq 0$ for all $s \geq 1$. Also, it follows from (3.20) that $\bar{\Lambda}_{s+1} \leq \bar{\Lambda}_s$. That is, $\{\bar{\Lambda}_s\}$ is decreasing and bounded from below. Thus $\lim_{s\to\infty} \bar{\Lambda}_s$ exists. Consequently, we obtain from (3.20) and the squeeze theorem that $\lim_{s\to\infty} c_2\|x_{s-1} - x_{s-2}\|^2 = 0$. Hence,

$$\lim_{s\to\infty}\|x_{s-1} - x_{s-2}\| = 0. \tag{3.21}$$

As a result,

$$\begin{aligned}\|x_{s+1}-z_s\| &= \|x_{s+1}-x_s-\theta(x_s-x_{s-1})-\beta(x_{s-1}-x_{s-2})\| \\ &\le \|x_{s+1}-x_s\|+\theta\|x_s-x_{s-1}\|+|\beta|\|x_{s-1}-x_{s-2}\|\to 0 \end{aligned} \tag{3.22}$$

as $s\to\infty$. By $\lim_{s\to\infty}\|x_{s+1}-x_s\|=0$, one has

$$\|x_s-z_s\|\le\|x_s-x_{s+1}\|+\|x_{s+1}-z_s\|\to 0,\quad s\to\infty. \tag{3.23}$$

From the fact that F is Lipschitz continuous one sees $\lim_{s\to\infty}\|F(x_{s+1})-F(x_s)\|=0$. By (3.21) and the existence of $\lim_{s\to\infty}\bar{\Lambda}_s$, we have that $\lim_{s\to\infty}\Lambda_s$ exists and hence $\{\Lambda_s\}$ is bounded. Now, since $\lim_{s\to\infty}\|x_{s-1}-x_s\|=0$, we have from the definition of Λ_s that

$$\lim_{s\to\infty}\left[\|x_s-\bar{x}\|^2-\theta\|x_{s-1}-\bar{x}\|^2-\beta\|x_{s-2}-\bar{x}\|^2\right] \text{ exists.} \tag{3.24}$$

Using the boundedness of $\{\Lambda_s\}$, we obtain from (3.18) that $\{x_s\}$ is bounded, so both $\{z_s\}$ and $\{y_s\}$ are also bounded. □

Our global convergence result for Algorithm 2 is given as following.

Theorem 3.1. *Let Assumptions 3.2 be satisfied. Then $\{x_s\}$ generated by Algorithm 2 converges to a solution of MVI (1.1).*

Proof. By Lemma 3.1, we have that $\{x_s\}$ is bounded. Let x^* be an accumulating point of $\{x_s\}$. By (3.2), we have, for all $y\in\Omega$,

$$h(y_s)-h(y)\le\alpha\left\langle y_s-z_s+\frac{1}{\alpha}F(z_s),y-y_s\right\rangle. \tag{3.25}$$

From (3.9), we obtain

$$\begin{aligned}\left(1-\frac{L^2}{\alpha^2}\right)\|y_s-z_s\|^2 &\le \|z_s-\bar{x}\|^2-\|x_{s+1}-\bar{x}\|^2 \\ &= \Big(\|z_s-\bar{x}\|-\|x_{s+1}-\bar{x}\|\Big)\Big(\|z_s-\bar{x}\|+\|x_{s+1}-\bar{x}\|\Big) \\ &\le M^*\Big(\|z_s-\bar{x}\|-\|x_{s+1}-\bar{x}\|\Big) \\ &\le M^*\|x_{s+1}-z_s\|,\end{aligned}$$

where $M^*:=\sup_{s\ge1}(\|z_s-\bar{x}\|+\|x_{s+1}-\bar{x}\|)<\infty$ since both $\{x_s\}$ and $\{z_s\}$ are bounded. By Assumption 3.1 (A5) and (3.22), one derives that $\lim_{s\to\infty}\|y_s-z_s\|=0$, which implies that

$$\|x_{s+1}-y_s\|\le\|y_s-z_s\|+\|x_{s+1}-z_s\|\to 0, s\to\infty. \tag{3.26}$$

Because x^* is an accumulating point of $\{x_s\}$, by (3.23), it is also an accumulating point of $\{y_s\}$ and of $\{z_s\}$. Passing to the limit in (3.25), we have (note (3.26) and (3.21)) that

$$h(x^*)-h(y)\le\langle F(x^*),y-x^*\rangle,\quad\forall y\in\Omega.$$

Thus, $x^*\in\mathscr{S}$. Suppose now that there exist $\{x_{s_j}\}\subset\{x_s\}$ and $\{x_{s_m}\}\subset\{x_s\}$ such that $x_{s_j}\to x^{**}, j\to\infty$ and $x_{s_m}\to x^*, m\to\infty$. We claim that $x^{**}=x^*$. Observe that

$$2\langle x_s,x^*-x^{**}\rangle=\|x_s-x^{**}\|^2-\|x_s-x^*\|^2-\|x^{**}\|^2+\|x^*\|^2, \tag{3.27}$$

$$2\langle x_{s-1},x^*-x^{**}\rangle=\|x_{s-1}-x^{**}\|^2-\|x_{s-1}-x^*\|^2-\|x^{**}\|^2+\|x^*\|^2, \tag{3.28}$$

and

$$2\langle x_{s-2}, x^* - x^{**}\rangle = \|x_{s-2} - x^{**}\|^2 - \|x_{s-2} - x^*\|^2 - \|x^{**}\|^2 + \|x^*\|^2. \tag{3.29}$$

Therefore,

$$2\langle -\theta x_{s-1}, x^* - x^{**}\rangle = -\theta\|x_{s-1} - x^{**}\|^2 + \theta\|x_{s-1} - x^*\|^2 + \theta\|x^{**}\|^2 - \theta\|x^*\|^2 \tag{3.30}$$

and

$$2\langle -\beta x_{s-2}, x^* - x^{**}\rangle = -\beta\|x_{s-2} - x^{**}\|^2 + \beta\|x_{s-2} - x^*\|^2 + \beta\|x^{**}\|^2 - \beta\|x^*\|^2. \tag{3.31}$$

A combination of (3.27), (3.30), and (3.31) gives

$$\begin{aligned} 2\langle x_s - \theta x_{s-1} - \beta x_{s-2}, x^* - x^{**}\rangle &= \left(\|x_s - x^{**}\|^2 - \theta\|x_{s-1} - x^{**}\|^2 - \beta\|x_{s-2} - x^{**}\|^2\right) \\ &\quad -\left(\|x_s - x^*\|^2 - \theta\|x_{s-1} - x^*\|^2 - \beta\|x_{s-2} - x^*\|^2\right) \\ &\quad +(1-\theta-\beta)\left(\|x^*\|^2 - \|x^{**}\|^2\right). \end{aligned} \tag{3.32}$$

Using (3.24), it holds that $\lim_{s\to\infty}\left[\|x_s - x^*\|^2 - \theta\|x_{s-1} - x^*\|^2 - \beta\|x_{s-2} - x^*\|^2\right]$ exists and

$$\lim_{s\to\infty}\left[\|x_s - x^{**}\|^2 - \theta\|x_{s-1} - x^{**}\|^2 - \beta\|x_{s-2} - x^{**}\|^2\right]$$

exists. These imply from (3.32) that $\lim_{s\to\infty}\langle x_s - \theta x_{s-1} - \beta x_{s-2}, x^* - x^{**}\rangle$ exists. It follows that

$$\begin{aligned} \langle x^{**} - \theta x^{**} - \beta x^{**}, x^* - x^{**}\rangle &= \lim_{j\to\infty}\langle x_{s_j} - \theta x_{s_j-1} - \beta x_{s_j-2}, x^* - x^{**}\rangle \\ &= \lim_{s\to\infty}\langle x_s - \theta x_{s-1} - \beta x_{s-2}, x^* - x^{**}\rangle \\ &= \lim_{m\to\infty}\langle x_{s_m} - \theta x_{s_m-1} - \beta x_{s_m-2}, x^* - x^{**}\rangle \\ &= \langle x^* - \theta x^* - \beta x^*, x^* - x^{**}\rangle. \end{aligned}$$

Hence, $(1-\theta-\beta)\|x^* - x^{**}\|^2 = 0$. Therefore, $x^* = x^{**}$. As a consequence, every accumulation point of $\{x_s\}$ is a solution to MVI (1.1). □

Remark 3.4. We can replace **Step 2** in Algorithm 2 to have an algorithm with a constant step size in the backward step. In this case, Algorithm 2 becomes

$$\left\{\begin{array}{l} z_s = x_s + \theta(x_s - x_{s-1}) + \beta(x_{s-1} - x_{s-2}) \\ y_s = \mathrm{Prox}_{\lambda\alpha h + \iota_\Omega}(z_s - \lambda F(z_s)) \\ x_{s+1} = y_s + \lambda\left(F(z_s) - F(y_s)\right) \end{array}\right. \tag{3.33}$$

where $\lambda\alpha h$ is assumed prox-convex with prox-convex value $\alpha > 0$ and $\lambda \in (0, \frac{1}{L})$. Following similar arguments as in the proof of Theorem 3.1, we can obtain similar convergence results of (3.33).

Remark 3.5. In the convex setting with $\beta = 0$, Algorithm 2 reduces to [31, Algorithm 3.1].

4. Numerical Experiments

In this section, we give a numerical test to compare Algorithm 2 to Algorithm 1 (proposed in [12]). All codes were written in MATLAB R2020b and performed on a PC Desktop Intel(R) Core(TM) i7-6600U CPU @ 3.00GHz 3.00 GHz, RAM 32.00 GB.

Oligopolistic Equilibrium Problem: We examine same oligopolistic equilibrium model given in [12, 18, 23]. Let us consider a mixed variational inequality problem modelled as 5 companies with the cost functions given as

$$\begin{cases} \varphi_1 : [0,2] \to \mathbb{R}, & \varphi_1(u) = -u^2 - u, \\ \varphi_2 : \mathbb{R} \to \mathbb{R}, & \varphi_2(u) = u^2, \\ \varphi_3 : [1,2] \to \mathbb{R}, & \varphi_3(u) = 5u + \ln(1+10u), \\ \varphi_4 : \mathbb{R} \to \mathbb{R}, & \varphi_4(u) = \begin{cases} \frac{u^2}{2}, & \text{if} \quad |u| \le 1, \\ |u| - \frac{1}{2}, & \text{otherwise}, \end{cases} \\ \varphi_5(u) : [1,2] \to \mathbb{R}, & \varphi_5(u) = 8 - u^3. \end{cases}$$

As explained in [12], cost functions φ_1, φ_3 and φ_5 are prox-convex with constant α for any $\alpha > 0$, while φ_2 and φ_4 are convex. More details on these cost functions can be found in [1, 2, 12, 15, 23].

In this numerical test, let $F(u) = Au$, $u \in \mathbb{R}^5$ with $A \in \mathbb{R}^{5\times 5}$ a real symmetry positive semidefinite matrix and $h(u_1, ..., u_5) = \sum_{i=1}^{5} \varphi_i(u_i)$. Furthermore, matrix A is randomly generated and scaled in order to have $L = 1$. The proximity operator of (the separable function) h has as components the ones of the involved functions, which are known (cf. [10, 12, 21]). By choosing $\alpha = 2, \theta = 0.14$, and $\beta = -0.04$, Assumption 3.2 is satisfied, and we now choose $x_0 = x_{-1} = (0.1, 2, 2, 2, 0.1)$ for Algorithm 2 (denoted by iFBF), with stopping condition

$$\text{Error} = \max\{\|x_{s+1} - z_s\|, \|x_s - x_{s-1}\|\} \le \varepsilon.$$

As in [12], for Algorithm 1 (denoted by GoldenRatio), we take $x_0 = (0.1, 2, 2, 2, 0.1)$ and $z_0 = x_1 = (0, 0, 1.9, 0, 0)$ with stopping condition

$$\text{Error} = \max\{\|x_{s+1} - x_s\|, \|x_s - z_s\|\} \le \varepsilon,$$

where $\varepsilon = 10^{-4}$. Figure 1 clearly demonstrates that iFBF (Algorithm 2) outperforms GoldenRatio-Algorithm (Algorithm 1).

5. Conclusion

We demonstrated that the forward-backward-forward splitting algorithm with the two-step inertial extrapolation can be adapted to solve non-convex mixed variational inequalities. Global convergence of the iterative sequence generated by the proposed method was established and some numerical illustrations were also presented. We demonstrated the benefits of considering two-step inertial extrapolation compared to one-step inertial extrapolation considered in several related papers on (convex) variational inequalities via numerical experiments. In the future project, we aim to study the proposed method with a combination of inertia and corrections terms.

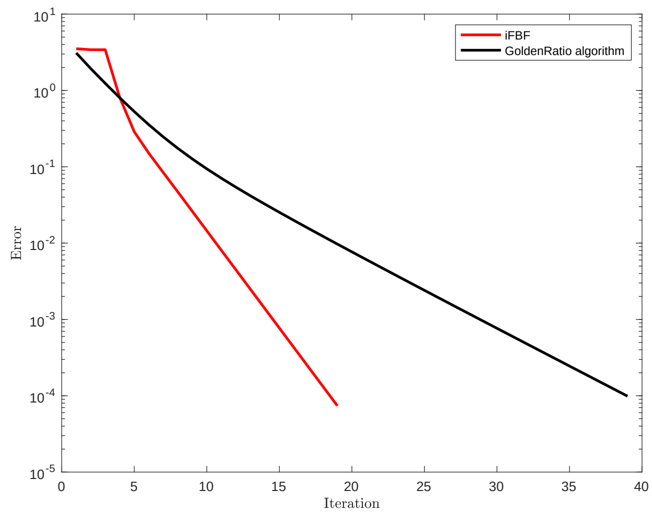

FIGURE 1. Performance of iFBF (red) and Golden-Ratio (black) Algorithms

Acknowledgments

This work was supported by the Vietnam Ministry of Education and Training under project B2023-SPK-02.

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