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STRONG CONVERGENCE THEOREM FOR SOLVING QUASI-MONOTONE VARIATIONAL INEQUALITY PROBLEMS

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Abstract. In this paper, we propose a Mann type self-adaptive Tseng's extragradient method for solving the classical variational inequality problem with a Lipschitz continuous and quasi-monotone mapping in a real Hilbert space. The strong convergence of the proposed algorithm is proven without the prior knowledge of the Lipschitz constant of the corresponding function. Finally, we give some numerical examples to illustrate the superiority of our proposed algorithm.

Keywords. Quasi-monotone mapping; Superiority; Tseng's extragradient method; Variational inequality problem.

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1. Introduction

Let $\mathscr{H}$ be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$, and let C be a nonempty, convex, and closed sset in $\mathscr{H}$. Fichera [8, 9] introduced the *variational inequality problem* (VIP), which consists of find a point $x^*\in C$ such that

$$\langle Ax^*, x-x^*\rangle \geq 0, \quad \forall x\in C, \tag{1.1}$$

where $A:\mathscr{H}\to\mathscr{H}$ is a single-valued operator. The *dual variational inequality problem* (DVIP) of VIP (1.1) is to find a point $x^*\in C$ such that

$$\langle Ax, x-x^*\rangle \geq 0, \quad \forall x\in C. \tag{1.2}$$

We denote the solution set of VIP (1.1) and DVIP (1.2) by S and S_D, respectively.

In view of the wide applications of the variational inequality problem in economics, mathematical programming, transportation, optimization, and other fields, it has attracted extensive attention; see, e.g., [2, 4, 5, 6, 14, 18, 23, 24] and the references therein.

Recently, various projected methods were introduced for solving the VIP (1.1), and the simplest among these methods is the gradient projection method:

$$x^{k+1} = P_C(x^k-\lambda Ax^k),$$

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where P_C denotes the metric projection from $\mathscr{H}$ onto C, and λ is the stepsize. It is known that the assumption that guarantees the convergence of this method is that operator A is L-Lipschitz continuous and α-strongly monotone (or inverse-strongly).

If we relax the strong monotonicity to the monotonicity, then this method may not converge. In order to deal with this situation, Korpelevich [13] proposed the extragradient method in finite dimensional Euclidean space $\mathbb{R}^n$:

$$\begin{cases} x^0 \in \mathbb{R}^n, \\ y^k = P_C(x^k - \lambda A x^k), \\ x^{k+1} = P_C(x^k - \lambda A y^k), \end{cases}$$

where $A : \mathbb{R}^n \to \mathbb{R}^n$ is a monotone and L-Lipschitz continuous operator and λ is in $(0, \frac{1}{L})$. Then, the sequence $\{x^k\}$ generated by this algorithm converges to an element of the solution set of VIP (1.1).

It is noted that the extragradient method needs to calculate the projection onto feasible set C twice in each iteration. As everyone knows, when C is a general closed and convex set, the evaluation of the projection operator onto set C is computationally expensive, which may seriously affect the computational efficiency of the extragradiet method. Therefore, numerous authors investigate how can one improve the extragradient method so that one only needs to calculate the projection onto C once in each iteration. To this end, Tseng [21] introduced the following famous extragradient method, Tseng's extragradient method,

$$\begin{cases} y^k = P_C(x^k - \lambda A x^k), \\ x^{k+1} = y^k - \lambda (A y^k - A x^k). \end{cases}$$

Recently, numerous scholars had studied VIP (1.1) with operator A being a pseudo-monotone operator [3, 7, 16, 17, 22]. It is known that the quasi-monotonicity is a more general than the pseudo-monotonicity. If A is pseudo-monotone and continuous, then $S_D = S$. If A is quasi-monotone and continuous, we only have $S_D \subseteq S$ (see [3]).

The problem of VIP (1.1) with A being quasi-monotone has attracted the attention of scholars [1, 12, 15, 19, 20]. For the practical problems in infinite dimensional spaces, one generally expects to obtain strong convergence results. However, most of the current algorithms for quasi-monotone variational inequality problems are only weakly convergent. In this paper, we introduce a Mann type self-adaptive Tseng's extragradient method for solving VIP (1.1) in real Hilbert spaces with A being quasi-monotone and L-Lipschitz continuous. The proposed algorithm does not need to know the Lipschitz constant of the mapping A. Under some conditions, we prove that the iterative sequence generated by the suggested algorithm converges to a solution of VIP (1.1) strongly. Some numerical experiments are provided to support the theoretical results.

The remainder of this paper is organized as follows. In Section 2, we recalls some preliminary results and lemmas for further use. In Section 3, the algorithm is given and its convergence is analyzed. In Section 4, some numerical examples are presented to illustrate the numerical behavior of the proposed algorithm and compare it with some existing ones. In the last section, Section 5, a brief summary is given.

2. PRELIMINARIES

The weak convergence of a sequence $\{x^k\}$ to x as $k \to \infty$ is denoted by $x^k \rightharpoonup x$ while the strong convergence of $\{x^k\}$ to x as $k \to \infty$ is denoted by $x^k \to x$.

Definition 2.1. Let $A : \mathscr{H} \to \mathscr{H}$ be an operator. Then

(a) A is called *L-Lipschitz continuous* with Lipschitz constant $L > 0$ if

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in \mathscr{H}.$$

(b) A is called *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in \mathscr{H}.$$

(c) A is called *pseudo-monotone* if

$$\langle Ax, y - x\rangle \geq 0 \Rightarrow \langle Ay, y - x\rangle \geq 0, \quad \forall x, y \in \mathscr{H}.$$

(d) A is called *quasi-monotone* if

$$\langle Ax, y - x\rangle > 0 \Rightarrow \langle Ay, y - x\rangle \geq 0, \quad \forall x, y \in \mathscr{H}.$$

(e) A is called *sequentially weakly continuous* if, for each sequence $\{x^k\}$, the fact that $\{x^k\}$ converges weakly to x implies that $\{Ax^k\}$ converges weakly to Ax.

In this paper, an important tool of our work is the projection. Let K be a nonempty, closed, and convex subset of $\mathscr{H}$. Recall that the *projection* from $\mathscr{H}$ onto K, denoted by P_K, is defined in such a way that, for each $x \in \mathscr{H}$, $P_K(x)$ is the unique point in K such that

$$\|x - P_K(x)\| = \min\{\|x - z\| : z \in K\}.$$

Lemma 2.1. [10] *Let K be a closed and convex subset of a real Hilbert spaces $\mathscr{H}$ and $x \in \mathscr{H}$. Then the following inequalities are true:*

(a) $\|P_K(x) - P_K(y)\|^2 \leq \langle P_K(x) - P_K(y), x - y\rangle, \quad \forall y \in \mathscr{H}.$

(b) $\|P_K(x) - y\|^2 \leq \|x - y\|^2 - \|x - P_K(x)\|^2, \quad \forall y \in K.$

(c) $\langle x - P_K(x), y - P_K(x)\rangle \leq 0, \quad \forall y \in K.$

Lemma 2.2. *The following statements hold in any real Hilbert space $\mathscr{H}$:*

(a) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in \mathscr{H}.$

(b) $\left\|\sum_{i=1}^{m} t_i x_i\right\|^2 = \sum_{i=1}^{m} t_i\|x_i\|^2 - \sum_{i\neq j} t_i t_j\|x_i - x_j\|^2$, *where $t_i \geq 0$ and $\sum_{i=1}^{m} t_i = 1$, for all $x_i \in \mathscr{H}$, $1 \leq i \leq m$.*

Lemma 2.3. [11] *Let $\{s^k\}$ be a nonnegative real sequence such that*

$$s^{k+1} \leq (1 - \alpha_k)s^k + \alpha_k b^k, \quad k \geq 0,$$

$$s^{k+1} \leq s^k - \eta^k + \nu^k, \quad k \geq 0,$$

where $\{\alpha_k\}$ is a sequence in $(0,1)$, $\{\eta^k\}$ is a sequence of nonnegative real numbers, and $\{b^k\}$ and $\{\nu^k\}$ are two sequences in $\mathbb{R}$ such that

(i) $\sum_{k=0}^{\infty} \alpha_k = 0,$

(ii) $\lim_{k\to\infty} \nu^k = 0,$

(iii) $\lim_{j\to\infty}\eta^{k_j}=0$ *implies* $\limsup_{j\to\infty} b^{k_j}\le 0$ *for any subsequence* $\{k_j\}\subset\{k\}$.

Then $\lim_{k\to\infty} s^k=0$.

3. Main Results

In this section, we propose a Mann type self-adaptive Tseng's extragradient algorithm for solving the VIP (1.1) when A is a quasi-monotone operator and demonstrate its strong convergence. In order to state the main results, we need the following assumptions:

Condition 3.1. $S_D\neq\emptyset$.

Condition 3.2. Operator A is sequentially weakly continuous on C and L-Lipschitz continuous on $\mathscr{H}$ with constant $L>0$.

Condition 3.3. Operator A is quasi-monotone on $\mathscr{H}$.

Condition 3.4. Let $\{\varepsilon_k\}$ be a positive sequence such that $\lim_{k\to\infty}\dfrac{\varepsilon_k}{\alpha_k}=0$, where $\{\alpha_k\}\subset(0,1)$ is with the restrictions that $\sum_{k=1}^{\infty}\alpha_k=\infty$ and $\lim_{k\to\infty}\alpha_k=0$. Let $\{\beta_k\}\subset(a,b)\subset(0,1-\alpha_k)$ for positive real numbers a and b.

3.1. **Algorithm.** By combining Tseng's extragradient method and Mann's iterative method, we introduce a Mann type self-adaptive Tseng's extragradient algorithm.

Algorithm 1 (A Mann type self-adaptive Tseng's extragradient algorithm)

Step 0. Give $\theta>0$, $\lambda_1>0$, and $\mu\in(0,1)$. Choose a nonnegative real sequence $\{\xi_k\}$ such that $\sum_{k=1}^{\infty}\xi_k<+\infty$. Let $x^0,x^1\in\mathscr{H}$ be arbitrary.

Step 1. Given the current iterates x^{k-1} and x^k, set

$$w^k=x^k+\theta_k(x^k-x^{k-1}),$$

where

$$\theta_k=\begin{cases}\min\left\{\dfrac{\varepsilon_k}{\|x^k-x^{k-1}\|},\theta\right\}, & \text{if } x^k\neq x^{k-1},\\ \theta, & \text{otherwise.}\end{cases}$$

Step 2. Compute

$$y^k=P_C(w^k-\lambda_k Aw^k).$$

If $x^k=y^k$, then stop, and y^k is a solution to Problem 1.1. Otherwise, go to **Step 3.**

Step 3. Compute

$$z^k=y^k-\lambda_k(Ay^k-Aw^k),$$
$$x^{k+1}=(1-\alpha_k-\beta_k)w^k+\beta_k z^k.$$

Update

$$\lambda_{k+1}=\begin{cases}\min\left\{\dfrac{\mu\|w^k-y^k\|}{\|Aw^k-Ay^k\|},\lambda_k+\xi_k\right\}, & \text{if } Aw^k\neq Ay^k,\\ \lambda_k+\xi_k, & \text{otherwise.}\end{cases}$$

Set $k:=k+1$, and go to **Step 1.**

Remark 3.1. It follows from Algorithm 1 that

$$\lim_{k\to\infty}\frac{\theta_k}{\alpha_k}\|x^k-x^{k-1}\|=0.$$

In fact, whether $x^k=x^{k-1}$ or $x^k\neq x^{k-1}$, the definition of $\{\theta_k\}$ implies that $\theta_k\|x^k-x^{k-1}\|\leq\varepsilon_k$ for all $k\geq 1$. Combining it with $\lim_{k\to\infty}\frac{\varepsilon_k}{\alpha_k}=0$, we have

$$\lim_{k\to\infty}\frac{\theta_k}{\alpha_k}\|x^k-x^{k-1}\|\leq\lim_{k\to\infty}\frac{\varepsilon_k}{\alpha_k}=0.$$

Lemma 3.1. *Let $\{\lambda_k\}$ be the sequence generated by Algorithm 1. Then, $\lim_{k\to\infty}\lambda_k=\lambda$, where $\lambda\in\left[\min\{\frac{\mu}{L},\lambda_0\},\lambda_0+p\right]$ and $p=\sum_{k=0}^{\infty}\xi_k$.*

Proof. The proof is similar to the result in Liu [15], so we omit it. □

3.2. **Convergence analysis.** To establish the convergence of Algorithm 1, we give a key lemma first.

Lemma 3.2. *Let $\{x^k\}$ be a sequence generated by Algorithm 1 and Conditions (3.1)-(3.3) hold. For $p\in S_D$, the following inequality hold:*

$$\|z^k-p\|^2\leq\|w^k-p\|^2-\left(1-\mu^2\frac{\lambda_k^2}{\lambda_{k+1}^2}\right)\|y^k-w^k\|^2.$$

Proof. Since $p\in S_D$ and $z^k=y^k-\lambda_k(Ay^k-Aw^k)$, it is easy to see from Algorithm 1 that

$$\begin{aligned}\|z^k-p\|^2=&\|y^k-p\|^2+\lambda_k^2\|Ay^k-Aw^k\|^2-2\lambda_k\langle y^k-p,Ay^k-Aw^k\rangle\\=&\|y^k-w^k\|^2+\|w^k-p\|^2+2\langle y^k-w^k,w^k-p\rangle\\&+\lambda_k^2\|Ay^k-Aw^k\|^2-2\lambda_k\langle y^k-p,Ay^k-Aw^k\rangle\\=&\|w^k-p\|^2-\|y^k-w^k\|^2+2\langle y^k-w^k,y^k-p\rangle\\&+\lambda_k^2\|Ay^k-Aw^k\|^2-2\lambda_k\langle y^k-p,Ay^k-Aw^k\rangle.\end{aligned}$$

From Lemma 2.1, the definition of y^k implies that $\langle y^k-w^k+\lambda_kAw^k,y^k-p\rangle\leq 0$. Combining the above two formulas, one has

$$\|z^k-p\|^2\leq\|w^k-p\|^2-\|y^k-w^k\|^2+\lambda_k^2\|Ay^k-Aw^k\|^2-2\lambda_k\langle Ay^k,y^k-p\rangle. \tag{3.1}$$

It follows from $p\in S_D$ that $\langle Ax,x-p\rangle\geq 0$ for all $x\in C$. Then, $\langle Ay^k,y^k-p\rangle\geq 0$. Applying the inequality above to (3.1), we obtain

$$\|z^k-p\|^2\leq\|w^k-p\|^2-\|y^k-w^k\|^2+\lambda_k^2\|Ay^k-Aw^k\|^2. \tag{3.2}$$

Using the definition of λ_{k+1}, we arrive at

$$\|Aw^k-Ay^k\|\leq\frac{\mu}{\lambda_{k+1}}\|w^k-y^k\|.$$

Then, (3.2) implies

$$\|z^k-p\|^2\leq\|w^k-p\|^2-\left(1-\mu^2\frac{\lambda_k^2}{\lambda_{k+1}^2}\right)\|y^k-w^k\|^2.$$

This completes the proof. □

Lemma 3.3. *Let $\{w^k\}$ and $\{y^k\}$ be a sequence generated by Algorithm 1 and assume that Conditions (3.1)-(3.3) hold. If there exists a subsequence $\{w^{k_j}\}$ of $\{w^k\}$ converging weakly to x^* and $\lim_{j\to\infty}\|w^{k_j}-y^{k_j}\|=0$, then $x^*\in S_D$ or $Ax^*=0$.*

Proof. From $w^{k_j}\rightharpoonup x^*$ as $j\to\infty$ and $\lim_{j\to\infty}\|w^{k_j}-y^{k_j}\|=0$, it follows that $y^{k_j}\rightharpoonup x^*$ as $j\to\infty$. Now we divide the proof into two cases.

Case 1. We have $Ax^*=0$ if $\limsup_{j\to\infty}\|Ay^{k_j}\|=0$.

From $\limsup_{j\to\infty}\|Ay^{k_j}\|=0$, we have $\lim_{j\to\infty}\|Ay^{k_j}\|=\liminf_{j\to\infty}\|Ay^{k_j}\|=0$. Since $\{y^{k_j}\}$ converges weakly to x^* and A is sequentially weakly continuous on C, it follows that $\{Ay^{k_j}\}$ converges weakly to Ax^*. By the sequentially weakly lower semicontinuity of the norm, we obtain

$$0\le\|Ax^*\|\le\liminf_{j\to\infty}\|Ay^{k_j}\|=0,$$

which implies $Ax^*=0$.

Case 2. It holds that if $\limsup_{j\to\infty}\|Ay^{k_j}\|>0$, then $x^*\in S_D$.

Observe $\limsup_{j\to\infty}\|Ay^{k_j}\|>0$. Without loss of generality, we set $\lim_{j\to\infty}\|Ay^{k_j}\|=M>0$. It follows that there exists a $N_0\in\mathbb{N}$ such that $\|Ay^{k_j}\|>\frac{M}{2}$ for all $j\ge N_0$. From the definition of y^k, it is easy to see that $\langle y^{k_j}-w^{k_j}+\lambda_{k_j}Aw^{k_j},z-y^{k_j}\rangle\ge0$ for all $z\in C$, which in turn implies

$$\langle w^{k_j}-y^{k_j},z-y^{k_j}\rangle\le\lambda_{k_j}\langle Aw^{k_j},z-y^{k_j}\rangle.$$

It follows that

$$\frac{1}{\lambda_{k_j}}\langle w^{k_j}-y^{k_j},z-y^{k_j}\rangle-\langle Aw^{k_j}-Ay^{k_j},z-y^{k_j}\rangle\le\langle Ay^{k_j},z-y^{k_j}\rangle.$$

In the formula above, letting $j\to\infty$, and using the facts that $\{y^{k_j}\}$ is bounded, $\lim_{j\to\infty}\lambda_{k_j}=\lambda>0$, and $\lim_{j\to\infty}\|y^{k_j}-w^{k_j}\|\to0$, we see that

$$0\le\liminf_{j\to\infty}\langle Ay^{k_j},z-y^{k_j}\rangle\le\limsup_{j\to\infty}\langle Ay^{k_j},z-y^{k_j}\rangle<+\infty. \tag{3.3}$$

If $\limsup_{j\to\infty}\langle Ay^{k_j},z-y^{k_j}\rangle>0$, then there exists a subsequence $\{y^{k_{j_n}}\}$ such that $\lim_{n\to\infty}\langle Ay^{k_{j_n}},z-y^{k_{j_n}}\rangle>0$. Consequently, there exists $N_1\in\mathbb{N}$ such that $\langle Ay^{k_{j_n}},z-y^{k_{j_n}}\rangle>0$ for all $n\ge N_1$. By the quasimonotonicity of A, we obtain, for all $n\ge N_1$, $\langle Az,z-y^{k_{j_n}}\rangle\ge0$. Letting $n\to\infty$, we have $x^*\in S_D$. If $\limsup_{j\to\infty}\langle Ay^{k_j},z-y^{k_j}\rangle=0$, we deduce from (3.3) that $\lim_{j\to\infty}\langle Ay^{k_j},z-y^{k_j}\rangle=0$. Let $\varepsilon_j=|\langle Ay^{k_j},z-y^{k_j}\rangle|+\frac{1}{j+1}$. Then, it follows that

$$\langle Ay^{k_j},z-y^{k_j}\rangle+\varepsilon_j>0. \tag{3.4}$$

Set $v^{k_j}=\frac{Ay^{k_j}}{\|Ay^{k_j}\|^2}$ for all $j\ge N_0$. Then, it is easy to see that $\langle Ay^{k_j},v^{k_j}\rangle=1$. Consequently, from (3.4), we have that $\langle Ay^{k_j},z+\varepsilon_jv^{k_j}-y^{k_j}\rangle>0$. By the quasimonotonicity of A, we have

$$\langle A(z+\varepsilon_jv^{k_j}),z+\varepsilon_jv^{k_j}-y^{k_j}\rangle\ge0,$$

which in turn yields

$$\begin{aligned}
&\langle Az, z+\varepsilon_j v^{k_j}-y^{k_j}\rangle\\
=&\langle Az-A(z+\varepsilon_j v^{k_j}), z+\varepsilon_j v^{k_j}-y^{k_j}\rangle+\langle A(z+\varepsilon_j v_{k_j}), z+\varepsilon_j v^{k_j}-y^{k_j}\rangle\\
\geq&\langle Az-A(z+\varepsilon_j v^{k_j}), z+\varepsilon_j v^{k_j}-y^{k_j}\rangle\\
\geq&-\|Az-A(z+\varepsilon_j v^{k_j})\|\|z+\varepsilon_j v^{k_j}-y^{k_j}\|\\
\geq&-\varepsilon_j L\|v^{k_j}\|\|z+\varepsilon_j v^{k_j}-y^{k_j}\|\\
=&-\varepsilon_j\frac{L}{\|Ay^{k_j}\|}\|z+\varepsilon_j v^{k_j}-y^{k_j}\|\\
\geq&-\varepsilon_j\frac{2L}{M}\|z+\varepsilon_j v^{k_j}-y^{k_j}\|,\quad\forall j\geq N_0.
\end{aligned}$$

Letting $j\to\infty$ in the inequality above, and applying the fact that $\lim_{j\to\infty}\varepsilon_j=0$ and the boundedness of $\{\|z+\varepsilon_j v^{k_j}-y^{k_j}\|\}$, we obtain $\langle Az, z-x^*\rangle\geq 0$ for all $z\in C$, which implies that $x^*\in S_D$. This completes the proof. □

Theorem 3.1. *Assume that Conditions (3.1)-(3.4) hold. Let the sequence $\{x^k\}$ be generated by Algorithm 1. If $\omega_w(x^k)\cap\{x\,|\,Ax=0\}=\emptyset$, then $\{x^k\}$ converges strongly to $p\in S_D$ with $\|p\|=$* $\min\{\|z\|: z\in S_D\}$.

Proof. **Claim 1.** We prove that $\{x^k\}$ is a bounded sequence. It is easy to see

$$\|w^k-p\|=\|x^k+\theta_k(x^k-x^{k-1})-p\|\leq\|x^k-p\|+\theta_k\|x^k-x^{k-1}\|\quad\forall p\in S_D.$$

From Remark 3.1, we know that there exists $N_0\in\mathbb{N}$ and $M_1>0$ such that, for all $k\geq N_0$,

$$\|w^k-p\|\leq\|x^k-p\|+\alpha_k M_1.$$

Since $\lim_{k\to\infty}\left(1-\mu^2\frac{\lambda_k^2}{\lambda_{k+1}^2}\right)=1-\mu^2>0$, then there exists $N_1\in\mathbb{N}$ such that, for all $k\geq N_1$, $\|z^k-p\|\leq\|w^k-p\|$. It follows by setting $N=\max\{N_0,\,N_1\}$ that

$$\begin{aligned}
\|x^{k+1}-p\|=&\|(1-\alpha_k-\beta_k)w^k+\beta_k z^k-p\|\\
\leq&(1-\alpha_k-\beta_k)\|w^k-p\|+\beta_k\|z^k-p\|+\alpha_k\|p\|\\
\leq&(1-\alpha_k)\|w^k-p\|+\alpha_k\|p\|\\
\leq&(1-\alpha_k)\|x^k-p\|+\alpha_k(\|p\|+M_1)\\
\leq&\max\left\{\|x^k-p\|,\|p\|+M_1\right\}\\
&\vdots\\
\leq&\max\left\{\|x^N-p\|,\|p\|+M_1\right\},\ \forall k\geq N,
\end{aligned}$$

which implies that $\{x^k\}$ is bounded. Thus sequences $\{w^k\}$ and $\{z^k\}$ are also bounded.

Claim 2. It holds there exists $N_2 \in \mathbb{N}$ and $M_3 > 0$ such that, for all $k \geq N_2$,

$$\begin{aligned}\|x^{k+1}-p\|^2 \leq& \|x^k-p\|^2+\alpha_k\left(3M_2\frac{\theta_k}{\alpha_k}\|x^k-x^{k-1}\|+\|p\|^2\right)\\&-\beta_k\left(1-\mu^2\frac{\lambda_k^2}{\lambda_{k+1}^2}\right)\|y^k-w^k\|^2-\beta_k M_3\|w^k-z^k\|^2,\end{aligned}$$

where $p \in S_D$, and $M_2 = \sup_{k\in\mathbb{N}}\{\|x^k-p\|, \theta_k\|x^k-x^{k-1}\|\}$. From Lemma 2.2 (b), the definition of x^k implies that

$$\begin{aligned}&\|x^{k+1}-p\|^2\\=&(1-\alpha_k-\beta_k)\|w^k-p\|^2+\beta_k\|z^k-p\|^2+\alpha_k\|p\|^2\\&-\beta_k(1-\alpha_k-\beta_k)\|w^k-z^k\|^2-\alpha_k(1-\alpha_k-\beta_k)\|w^k\|^2-\alpha_k\beta_k\|z^k\|^2\\\leq&(1-\alpha_k-\beta_k)\|w^k-p\|^2+\beta_k\|z^k-p\|^2+\alpha_k\|p\|^2-\beta_k(1-\alpha_k-\beta_k)\|w^k-z^k\|^2.\end{aligned}$$

It is easy to see that

$$\begin{aligned}\|w^k-p\|^2 \leq& \|x^k-p\|^2+2\theta_k\|x^k-p\|\|x^k-x^{k-1}\|+\theta_k^2\|x^k-x^{k-1}\|^2\\\leq& \|x^k-p\|^2+3M_2\theta_k\|x^k-x^{k-1}\|,\end{aligned}\tag{3.5}$$

where $M_2 = \sup_{k\in\mathbb{N}}\{\|x^k-p\|, \theta_k\|x^k-x^{k-1}\|\}$. From this and Lemma 3.2, we arrive at

$$\begin{aligned}&\|x^{k+1}-p\|^2\\\leq&(1-\alpha_k-\beta_k)\|w^k-p\|^2+\beta_k\|w^k-p\|^2-\beta_k\left(1-\mu^2\frac{\lambda_k^2}{\lambda_{k+1}^2}\right)\|y^k-w^k\|^2\\&+\alpha_k\|p\|^2-\beta_k(1-\alpha_k-\beta_k)\|w^k-z^k\|^2\\\leq&\|w^k-p\|^2-\beta_k\left(1-\mu^2\frac{\lambda_k^2}{\lambda_{k+1}^2}\right)\|y^k-w^k\|^2+\alpha_k\|p\|^2-\beta_k(1-\alpha_k-\beta_k)\|w^k-z^k\|^2\\\leq&\|x^k-p\|^2+\alpha_k\left(3M_2\frac{\theta_k}{\alpha_k}\|x^k-x^{k-1}\|+\|p\|^2\right)\\&-\beta_k\left(1-\mu^2\frac{\lambda_k^2}{\lambda_{k+1}^2}\right)\|y^k-w^k\|^2-\beta_k(1-\alpha_k-\beta_k)\|w^k-z^k\|^2.\end{aligned}$$

Using Condition (3.4), we know that there exists $N_2 \in \mathbb{N}$ and $M_3 > 0$ such that

$$\begin{aligned}\|x^{k+1}-p\|^2 \leq& \|x^k-p\|^2+\alpha_k\left(3M_2\frac{\theta_k}{\alpha_k}\|x^k-x^{k-1}\|+\|p\|^2\right)\\&-\beta_k\left(1-\mu^2\frac{\lambda_k^2}{\lambda_{k+1}^2}\right)\|y^k-w^k\|^2-\beta_k M_3\|w^k-z^k\|^2, \quad \forall k \geq N_2.\end{aligned}$$

Claim 3. It holds

$$\|x^{k+1}-p\|^2 \leq (1-\alpha_k)\|x^k-p\|^2+\alpha_k b^k,$$

where $p \in S_D$, and $b^k = 3M_2 \frac{\theta_k}{\alpha_k}\|x^k - x^{k-1}\| + 2\beta_k\|z^k - w^k\|\|x^{k+1} - p\| + 2\langle p, p - x^{k+1}\rangle$. Using the definition of x^{k+1}, we know

$$x^{k+1} = (1-\alpha_k-\beta_k)w^k + \beta_k z^k = (1-\beta_k)w^k + \beta_k z^k - \alpha_k w^k. \tag{3.6}$$

Letting $u^k = (1-\beta_k)w^k + \beta_k z^k$, we see that $u^k - w^k = \beta_k(z^k - w^k)$. Applying this equation to (3.6), we find that there is

$$x^{k+1} = (1-\alpha_k)u^k + \alpha_k(u^k - w^k) = (1-\alpha_k)u^k + \alpha_k\beta_k(z^k - w^k).$$

From this and Lemma 2.2 (a), it is easy to see

$$\begin{aligned}\|x^{k+1} - p\|^2 =& \|(1-\alpha_k)(u^k - p) + \alpha_k(\beta_k(z^k - w^k) - p)\|^2\\ \leq& (1-\alpha_k)\|u^k - p\|^2 + 2\alpha_k\langle \beta_k(z^k - w^k) - p, x^{k+1} - p\rangle\\ \leq& (1-\alpha_k)\|u^k - p\|^2 + 2\alpha_k\beta_k\langle z^k - w^k, x^{k+1} - p\rangle - 2\alpha_k\langle p, x^{k+1} - p\rangle.\end{aligned} \tag{3.7}$$

By Lemma 3.2, we obtain

$$\|u^k - p\|^2 \leq (1-\beta_k)\|w^k - p\|^2 + \beta_k\|z^k - p\|^2 \leq \|w^k - p\|^2.$$

Applying this inequality and (3.5) to (3.7), we obtain

$$\begin{aligned}\|x^{k+1} - p\|^2 \leq& (1-\alpha_k)\|w^k - p\|^2 + 2\alpha_k\beta_k\|z^k - w^k\|\|x^{k+1} - p\| - 2\alpha_k\langle p, x^{k+1} - p\rangle\\ \leq& (1-\alpha_k)\left(\|x^k - p\|^2 + 3M_2\theta_k\|x^k - x^{k-1}\|\right) + 2\alpha_k\beta_k\|z^k - w^k\|\|x^{k+1} - p\|\\ &- 2\alpha_k\langle p, x^{k+1} - p\rangle\\ \leq& (1-\alpha_k)\|x^k - p\|^2 + \alpha_k\Big(3M_2\frac{\theta_k}{\alpha_k}\|x^k - x^{k-1}\| + 2\beta_k\|z^k - w^k\|\|x^{k+1} - p\|\\ &+ 2\langle p, p - x^{k+1}\rangle\Big).\end{aligned}$$

Claim 4. We have that sequence $\{\|x^k - p\|^2\}$ converges to zero for all $p \in S_D$. Setting

$$\eta^k = \beta_k\left(\left(1 - \mu^2\frac{\lambda_k^2}{\lambda_{k+1}^2}\right)\|y^k - w^k\|^2 + M_3\|w^k - z^k\|^2\right),$$

and

$$v^k = \alpha_k\left(3M_2\frac{\theta_k}{\alpha_k}\|x^k - x^{k-1}\| + \|p\|^2\right),$$

we find that Claim 2 can be rewritten as

$$\|x^{k+1} - p\|^2 \leq \|x^k - p\|^2 - \eta^k + v^k,\ \forall k \geq N_2.$$

From Remark 3.1 and Condition (3.4), we know that $\sum_{k=0}^{\infty}\alpha_k = 0$ and $\lim_{k\to\infty} v^k = 0$. By Claim 3 and Lemma 2.3, in order to complete the proof via Lemma 2.3, we just need to prove that $\lim_{j\to\infty}\eta^{k_j} = 0$ implies that $\limsup_{j\to\infty} b^{k_j} \leq 0$ for any subsequence $\{k_j\} \subset \{k\}$. From the boundedness of $\{x^k\}$ and $\lim_{k\to\infty}\frac{\theta_k}{\alpha_k}\|x^k - x^{k-1}\| = 0$, we just need to prove $\limsup_{j\to\infty}\|z^{k_j} - w^{k_j}\| \leq 0$ and $\limsup_{j\to\infty}\langle p, p - x^{k_j+1}\rangle \leq 0$. In fact, using $\lim_{j\to\infty}\eta^{k_j} = 0$, we obtain that

$$\lim_{j\to 0}\|y^{k_j} - w^{k_j}\| = 0, \quad \lim_{j\to 0}\|w^{k_j} - z^{k_j}\| = 0. \tag{3.8}$$

According the definition of w^k, we have

$$\lim_{j\to 0}\|x^{k_j}-w^{k_j}\| = \lim_{j\to 0}\alpha_{k_j}\frac{\theta_{k_j}}{\alpha_{k_j}}\|x^{k_j}-x^{k_j-1}\| = 0. \tag{3.9}$$

On the other hand, we see

$$\lim_{j\to 0}\|x^{k_j+1}-w^{k_j}\| = \lim_{j\to 0}\alpha_{k_j}\|w^{k_j}\| + \lim_{j\to 0}\beta_{k_j}\|w^{k_j}-z^{k_j}\| = 0.$$

This together with (3.9), we obtain

$$\lim_{j\to\infty}\|x^{k_j+1}-x^{k_j}\| = 0. \tag{3.10}$$

Since sequence $\{x^{k_j}\}$ is bounded, it follows that there exists a subsequence $\{x^{k_{j_i}}\}$ of $\{x^{k_j}\}$, which converges weakly to some $x^* \in \mathscr{H}$. Without loss of generality, we can assume that $x^{k_j} \rightharpoonup x^*$. Then, it follows that

$$\limsup_{j\to\infty}\langle p, p-x^{k_j}\rangle = \lim_{j\to\infty}\langle p, p-x^{k_j}\rangle = \langle p, p-x^*\rangle.$$

Because $\|w^{k_j}-x^{k_j}\| \to 0$, we know w^{k_j} converges weakly to x^*. From $\|w^{k_j}-y^{k_j}\| \to 0$ and Lemma 3.3, we have $x^* \in S_D$ or $Ax^* = 0$. By $\omega_w(x^k) \cap \{x \,|\, Ax = 0\} = \emptyset$, we have $Ax^* \neq 0$ and $x^* \in S_D$. Observe that $\|p\| = \min\{\|z\| : z \in S_D\}$, that is, $p = P_{S_D}0$. If $x^* \in S_D$, we obtain

$$\limsup_{j\to\infty}\langle p, p-x^{k_j}\rangle = \langle p, p-x^*\rangle \le 0. \tag{3.11}$$

Combining (3.10) and (3.11), we have

$$\limsup_{j\to\infty}\langle p, p-x^{k_j+1}\rangle = \limsup_{j\to\infty}\langle p, p-x^{k_j}\rangle = \langle p, p-x^*\rangle \le 0. \tag{3.12}$$

Consequently, it follows from (3.8), (3.12), Claim 3, and Lemma 2.3 that $\lim_{k\to\infty}\|x^k-p\| = 0$. $\square$

4. Numerical Experiments

In this section, we provide three numerical examples to test the proposed algorithm and compare it with the Algorithm 3.2 in [1], which proves the practicability of our proposed algorithm. All the codes were written in Matlab (R2016a) and run on PC with Intel(R) Core(TM) i3-370M Processor 2.40 GHz.

Take $\theta = 0.3$, $\lambda_1 = 1$, $\mu = 0.6$, and $\alpha_k = \frac{4}{k}$ in Algorithm 1 and Algorithm 3.2 in [1]. Choose $\xi_k = \rho_k = \frac{1}{k^2}$ in Algorithm 3.2 in [1] and $\varepsilon_k = \xi_k = \frac{1}{k^2}$, $\beta_k = 0.9(1-\alpha_k)$ in Algorithm 1.

Example 4.1. [15] Let $C = [-1, 1]$ and

$$Ax = \begin{cases} |2x-4|, & x > 1, \\ x^2+1, & x \in [-1,1], \\ -2x, & x < -1. \end{cases}$$

Then A is quasi-monotone and Lipschitz continuous and $S_D = S = \{-1\}$.

Let $x^0 = x^1 = 0.9$ and take Error$= \|x^k+1\| \le 10^{-2}$ as the stopping criterion in Figure 1. The numerical result is described in Figure 1.

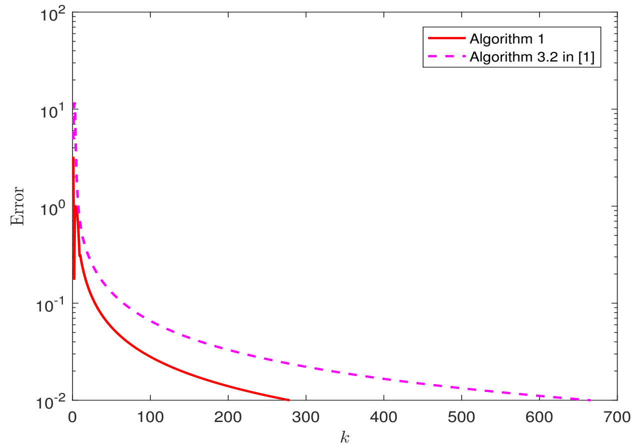

FIGURE 1. Comparison results of this algorithms in example 4.1.

Example 4.2. [1] Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$A(x_1, x_2) = (-x_1 e^{x^2}, x_2)$$

and $C = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1, 0 \leq x_1\}$. Then, $(1,0)^T \in S_D$ and $S = \{(1,0)^T, (0,0)^T\}$.

It can easily be verified that all the conditions of Algorithms 1 and Algorithm 3.2 in [1] are satisfied.

The initial point is randomly selected. Take $k = 5000$ as the stopping criterion and Error= $\|x^k - x^*\|$, where $x^* = (1,0)^T$. The numerical result is described in Figure 2.

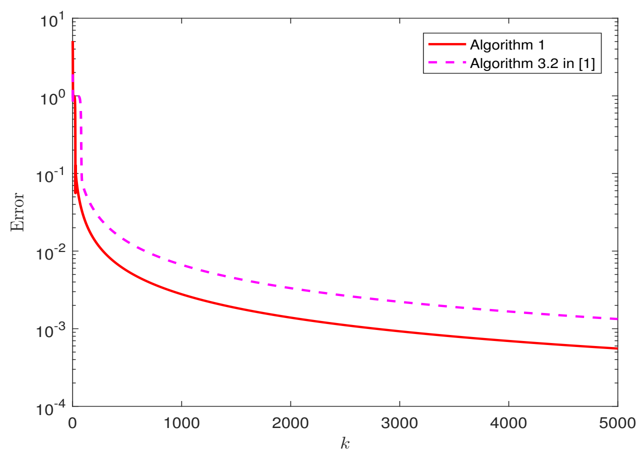

FIGURE 2. Comparison results of this algorithms in example 4.2.

Finally, we consider an example in infinite dimensional space.

Example 4.3. [20] Let

$$\mathscr{H} = \left\{ x = (x_1, x_2, \ldots, x_i, \ldots) : \sum_{i=1}^{\infty} | x_i |^2 < +\infty \right\}.$$

Take $C_q = \{x \in \mathscr{H} : \|x\| \leq q\}$ and

$$A_p(x) = (p - \|x\|)x,$$

where $p, q \in \mathbb{R}$ is such that $p > q > \frac{p}{2} > 0$. Then A is quasi-monnotone and Lipschitz continuous. Furthermore, $S_D = S = \{0\}$.

We take $p = 3$ and $q = 2$ and choose the initial values $x^0 = e^t$ and $x^1 = \sin(2\pi t^2)$. Take $k = 500$ as the stopping criterion and Error$= \|x^k\|$. The numerical result is described in Figure 3.

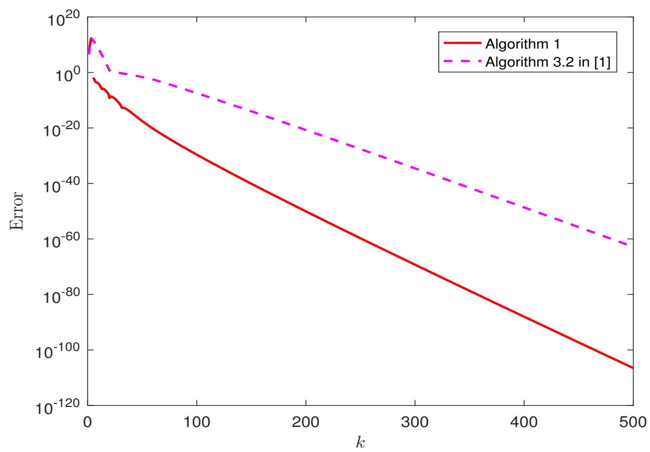

FIGURE 3. Comparison results of this algorithms in example 4.3.

The numerical results in Figure 1, Figure 2, and Figure 3 illustrate that the performance of Algorithm 1 is better than Algorithm 3.2 in [1].

5. CONCLUSION

In this paper, we introduced a Mann type self-adaptive Tseng's extragradient method for solving the variational inequality problem in real Hilbert spaces with A being a quasi-monotone and Lipschitz contiuous mapping. In this method, we does not need to know the Lipschitz constant of the operator A. We proved that the sequence $\{x^k\}$ generated by the proposed algorithm strongly converges to some soluiton. Finally, three numerical examples demonstrated that the proposed algorithm is better than an existing algorithm.

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