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# A CIRCUMCENTERED-REFLECTION METHOD FOR FINDING COMMON FIXED POINTS OF FIRMLY NONEXPANSIVE OPERATORS

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Honoring Professor Yair Censor on His 80th Birthday

**Abstract.** The circumcentered-reflection method (CRM) has been recently proposed as a methodology for accelerating several algorithms for solving the Convex Feasibility Problem (CFP), equivalent to finding a common fixed-point of the orthogonal projections onto a finite number of closed and convex sets. In this paper, we apply CRM to the more general Fixed Point Problem (denoted as FPP), consisting of finding a common fixed-point of operators belonging to a larger family of operators, namely firmly nonexpansive operators. We prove that, in this setting, CRM is globally convergent to a common fixed-point (supposing at least one exists). We also establish the linear convergence of the sequence generated by CRM applied to FPP under a not too demanding error bound assumption, and provide an estimate of the asymptotic constant. We provide solid numerical evidence of the superiority of CRM when compared to the classical Parallel Projections Method (PPM). Additionally, we present certain results of convex combination of orthogonal projections, of some interest on its own.

**Keywords.** Alternating projections; Circumcentered-reflection method; Common fixed points; Convergence rate; Firmly nonexpansive operators; Error bound.

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## 1. Introduction

We start by recalling the Convex Feasibility Problem (CFP), which consists of finding a point in the intersection of a finite number of closed convex subsets of $\mathbb{R}^n$. CFP is clearly equivalent to solving a finite system of convex inequalities in $\mathbb{R}^n$, and it can be also rephrased as the problem of finding a common fixed-point of the orthogonal projections onto such subsets. A natural extension of CFP is the problem of finding a common fixed-point of a finite set of operators other than orthogonal projections, but sharing some of their properties. A vast literature on the subject has been developed. We cite just a few references, namely [1, 2, 3, 4]. In this paper, we consider a particular generalization of orthogonal projections, namely *firmly nonexpansive operators*.

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We next define this family of operators, together with two related families.

**Definition 1.1.** An operator $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be:

i) nonexpansive when $\|T(x) - T(y)\| \le \|x - y\|$ for all $x, y \in \mathbb{R}^n$.
ii) nonexpansive plus when it is nonexpansive, and whenever $\|T(x) - T(y)\| = \|x - y\|$ it holds that $T(x) - T(y) = x - y$.
iii) firmly nonexpansive when

$$\|T(x) - T(y)\|^2 \le \|x - y\|^2 - \|(T(x) - T(y)) - (x - y)\|^2 \tag{1.1}$$

for all $x, y \in \mathbb{R}^n$.

It is immediate that firmly nonexpansive operators are nonexpansive plus, and nonexpansive plus operators are nonexpansive. It is well known and easy to prove that orthogonal projections onto closed and convex sets are firmly nonexpansive. The notation *nonexpansive plus* is not standard; we adopt it because of the analogy with copositive plus matrices.

Let $T_1, \ldots, T_m : \mathbb{R}^n \to \mathbb{R}^n$ be firmly nonexpansive operators. The problem of finding a common fixed-point of $T_1, \ldots, T_m$ (i.e., a point $\bar{x} \in \mathbb{R}^n$ such that $T_i(\bar{x}) = \bar{x}$ for all $i \in \{1, \ldots, m\}$) is denoted as FPP. The set of common fixed-points of the $T_i$'s is be denoted as $\mathrm{Fix}(T_1, \ldots, T_m)$. Two classical methods for FPP are the Sequential Projection Method (SPM) and the Parallel Projection Method (PPM), which can be traced back to [5] and [6], respectively, and are defined as follows. Consider the operators $\widehat{T}$ and $\overline{T} : \mathbb{R}^n \to \mathbb{R}^n$ given by $\widehat{T} = T_m \circ \cdots \circ T_1$ and $\overline{T} = \frac{1}{m} \sum_{i=1}^m T_i$. Starting from an arbitrary $x^0 \in \mathbb{R}^n$, SPM and PPM generate sequences $\{x^k\}$ given by $x^{k+1} = \widehat{T}(x^k)$ and $x^{k+1} = \overline{T}(x^k)$, respectively. When $\mathrm{Fix}(T_1, \ldots T_m) \neq \emptyset$ the sequences generated by both methods are known to be globally convergent to points belonging to a point in $\mathrm{Fix}(T_1, \ldots, T_m)$, i.e., to solve FPP. We refer to [7] for an in-depth study of these and other projections methods for FPP.

An interesting relation between SPM and PPM was found in [8]. Given firmly nonexpansive operators $T_1, \ldots, T_m : \mathbb{R}^n \to \mathbb{R}^n$, define the operator $\widetilde{T} : \mathbb{R}^{nm} \to \mathbb{R}^{nm}$ as $\widetilde{T}(x^1, \ldots, x^m) = (T_1(x^1), \ldots, T_M(x^m))$ with $x^i \in \mathbb{R}^n$ $(1 \le i \le m)$. It is rather immediate to check that $\widetilde{T}$ is firmly nonexpansive. Consider the set $\widetilde{U} = \{(x, \ldots, x) : x \in \mathbb{R}^n\} \subset \mathbb{R}^{nm}$, and let $P_{\widetilde{U}} : \mathbb{R}^{nm} \to \widetilde{U}$ be the orthogonal projection onto $\widetilde{U}$. Define $\{\bar{x}^k\} \subset \mathbb{R}^{nm}$ as the sequence resulting from applying SPM, as defined above, to the operators $\widetilde{T}, P_{\widetilde{U}}$, starting from a point $\bar{x}^0 = (x^0, \cdots, x^0) \in \widetilde{U}$, i.e., take $\bar{x}^{k+1} = P_{\widetilde{U}}(\widetilde{T}(\bar{x}^k))$. Clearly, $\bar{x}^k$ belongs to $\widetilde{U}$ for all $k$, so that we may write $\bar{x}^k = (x^k, \ldots, x^k)$ with $x^k \in \mathbb{R}^n$. It was proved in [8] that $x^{k+1} = \overline{T}(x^k)$, i.e., a step of SPM applied to two specific firmly nonexpansive operators in the product space $\mathbb{R}^{nm}$ is equivalent to a step of PPM in the original space $\mathbb{R}^n$. Thus, SPM with just two operators plays a sort of special role, and deserves a name of its own. We will call it the *Method of Alternating Projections* (MAP from now on). Observe that in the equivalence above one of the two sets in the product space, namely $\widetilde{U}$, is a linear subspace. This fact will be essential for the convergence of the Circumcentered-Reflection Method (CRM from now on), applied for solving FPP.

We reckon that the use of the word "projections" in the names of SPM, PPM, and MAP applied to FPP is an abuse of notation, since in general there are no projections involved in FPP. Indeed, they correspond to these methods applied to CFP, a particular case of FPP. We keep them because the structure of the methods applied to either CFP or FPP is basically the same.

We proceed to describe CRM. Take three non-collinear points $x, y, z \in \mathbb{R}^n$, and let $M$ be their affine hull. The *circumcenter* $\mathrm{circ}(x, y, z)$ is the center of the circle in $M$ passing through $x, y, z$ (or, equivalently, the point in $M$ equidistant from $x,y,z$). It is easy to check that $\operatorname{circ}(x,y,z)$ is well defined. Now we take two firmly nonexpansive operators $A,B:\mathbb{R}^n\to\mathbb{R}^n$ and define $Q=A\circ B$. Under adequate assumptions, the sequence $\{x^k\}\subset\mathbb{R}^n$ defined by

$$x^{k+1}=Q(x^k)=A(B(x^k)) \tag{1.2}$$

is expected to converge to a common fixed-point of $A$ and $B$. Note that, if $A,B$ are orthogonal projections onto convex sets $K_1,K_2$, then MAP turns out to be a special case of this iteration, and $\operatorname{Fix}(A,B)=K_1\cap K_2$. CRM can be seen as an acceleration technique for the sequence defined by (1.2). Define the reflection operators $A^R,B^R:\mathbb{R}^n\to\mathbb{R}^n$ as $A^R=2A-I, B^R=2B-I$, where $I$ stands for the identity operator in $\mathbb{R}^n$. The CRM operator $C:\mathbb{R}^n\to\mathbb{R}^n$ is defined as $C(x)=\operatorname{circ}(x,B^R(x),A^R(B^R(x)))$, i.e., the circumcenter of the points $x,B^R(x),A^R(B^R(x))$. The CRM sequence $\{x^k\}\subset\mathbb{R}^n$, starting at some $x^0\in\mathbb{R}^n$, is then defined as $x^{k+1}=C(x^k)$.

CRM was introduced in [9] and [10] and has been successfully applied for accelerating several methods for solving CFP, like MAP, PPM, and the Douglas-Rachford Method (DRM), outperforming all of them. It was further enhanced in [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]. CRM was shown in [10] to converge to a solution of CFP. In [12], it was proved that, under a not too demanding error bound condition, the sequences generated by MAP and CRM for solving CFP converge linearly, but the asymptotic constant for CRM is better than the one for MAP. This superiority was widely confirmed in the numerical experiences exhibited in [11].

Here, we will apply CRM for solving the FPP with firmly nonexpansive operators $T_1,\dots T_m:\mathbb{R}^n\to\mathbb{R}^n$ in the following way. We will apply it to two operators in $\mathbb{R}^{nm}$, namely $\widetilde{T}$ and $P_{\widetilde{U}}$ as defined above, starting from a point in $\widetilde{U}$. Note that, since $\widetilde{U}$ is a linear subspace, the operator $P_{\widetilde{U}}$ is affine.

The main purpose of this paper consists of establishing that CRM, when applied to FPP, is globally convergent, that linear convergence is achieved by both CRM and MAP under an error bound condition, and that CRM is computationally much faster than MAP, as corroborated by solid numerical evidence. We were not able to prove the superiority of CRM in terms of the asymptotic constant of linear convergence, but our numerical experiments suggest that a theoretical superiority is likely to hold. This issue is left as a subject for future research.

The paper is organized as follows. In Section 2, we present certain results, of some interest on its own, on convex combinations of orthogonal projections, which we take as a prototypical family of firmly nonexpansive operators (beyond orthogonal projections themselves). In Section 3, we prove global convergence of CRM applied for solving FPP. We prove in Section 4 that, under a reasonable error bound assumption, convergence of CRM applied for solving FPP is linear, and we provide as well an estimate of the asymptotic constant, which holds also for MAP. In Section 5, we present our numerical experiments, which show that CRM categorically outperforms PPM. In these experiments, we use the family of firmly nonexpansive operators studied in Section 2.

## 2. Some Properties of Firmly Nonexpansive Operators

We start with some elementary properties of nonexpansive plus and firmly nonexpansive operators (see Definition 1.1).

**Proposition 2.1.** *i) Compositions of nonexpansive plus operators are nonexpansive plus.*
*ii) Convex combinations of firmly nonexpansive operators are firmly nonexpansive.*

*Proof.* i) Suppose that $S$ and $T$ are nonexpansive plus operators. Then

$$\|S(T(x)) - S(T(y))\| \leq \|T(x) - T(y)\| \leq \|x - y\|, \tag{2.1}$$

by nonexpansiveness of $S$ and $T$. If $\|S(T(x)) - S(T(y))\| = \|x - y\|$, then the equality holds throughout (2.1) so that, using the "plus" property of $S, T$, we have $S(T(x)) - S(T(y)) = T(x) - T(y) = x - y$, which establishes the result.

ii) Take firmly nonexpansive operators $T_1 \ldots, T_m$ and nonnegative scalars $\alpha_1, \ldots, \alpha_m$ such that $\sum_{i=1}^m \alpha_i = 1$. Let $\overline{T} = \sum_{i=1}^m \alpha_i T_i$. We next prove that $\overline{T}$ is firmly nonexpansive.

Note that (1.1) is equivalent to

$$\|T(x) - T(y)\|^2 \leq \langle T(x) - T(y), x - y \rangle. \tag{2.2}$$

It suffices to check that $\overline{T}$ satisfies (2.2), and we proceed to do so

$$\begin{aligned}
\left\|\overline{T}(x) - \overline{T}(y)\right\|^2 &= \left\| \sum_{i=1}^m \alpha_i (T_i(x) - T_i(y)) \right\|^2 \\
&\leq \sum_{i=1}^m \alpha_i \|T_i(x) - T_i(y)\|^2 \\
&\leq \sum_{i=1}^m \alpha_i \langle T_i(x) - T_i(y), x - y \rangle \\
&= \left\langle \sum_{i=1}^m \alpha_i \left(T_i(x) - T_i(y)\right), x - y \right\rangle \\
&= \langle \overline{T}(x) - \overline{T}(y), x - y \rangle,
\end{aligned}$$

using the convexity of $\|\cdot\|^2$ in the first inequality and the fact that the $T_i$'s satisfy (2.2) in the second one.

□

For an operator $T : \mathbb{R}^n \to \mathbb{R}^n$, we denote as $F(T)$ the set of its fixed points, i.e., $F(T) = \{x \in \mathbb{R}^n : T(x) = x\}$ (we comment that $\mathrm{Fix}(\cdot, \cdot)$ denotes the set of common fixed points of two or more operators). We will also need the following "acute angle" property of firmly nonexpasive operators.

**Proposition 2.2.** *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a firmly nonexpansive operator. Then $0 \geq \langle T(x) - y, T(x) - x \rangle$ for all $x \in \mathbb{R}^n$ and all $y \in F(T)$.*

*Proof.* It is an immediate result from (1.1). □

We continue by stating, for future reference, some elementary and well known properties of orthogonal projections onto closed and convex sets.

Let $\subset \mathbb{R}^n$ be closed and convex. The *orthogonal projection* $P_C : \mathbb{R}^n \to C$ is defined as $P_C(x) = \operatorname{argmin}_{y \in C} \|x - y\|$.

**Proposition 2.3.** *If $C \subset \mathbb{R}^n$ is closed and convex, then*

*i)* $z = P_C(x)$ *if and only if* $\langle x - z, y - z \rangle \leq 0$ *for all* $x \in \mathbb{R}^n$ *and all* $y \in C$.

*ii)* $P_C$ *is firmly nonexpansive.*

*iii)* $F(P_C) = C$.

*iv) Take* $x \in \mathbb{R}^n$ *and let* $z = P_C(x)$. *Then,* $P_C(z + \alpha(x - z)) = P_C(x)$ *for all* $\alpha \geq 0$.

*v) Define* $h : \mathbb{R}^n \to \mathbb{R}$ *as* $h(x) = \|x - P_C(x)\|^2$. *Then* $h$ *is continuously differentiable and* $\nabla h(x) = 2(x - P_C(x))$.

*Proof.* They are elementary. □

It is worthwhile to comment at this point that the composition of two firmly nonexpansive operators may fail to be firmly nonexpansive: consider $A = \{(x_1,x_2) \in \mathbb{R}^2 : x_2 = 0\}$, $B = \{(x_1,x_2) \in \mathbb{R}^2 : x_2 = x_1\}$. $P_A$ and $P_B$ are firmly nonexpansive by Proposition 2.3(ii), but its composition $P_A \circ P_B$ fails to satisfy (2.2) with $x = (0,0)$ and $y = (2,-1)$.

We next present some properties of the set of fixed points of combinations of orthogonal projections. They have been proved in, for example, [23] and [24], but we include the proof for the sake of completeness. From now on, for $C \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, $\operatorname{dist}(x,C)$ will denote the Euclidean distance between $x$ and $C$.

**Proposition 2.4.** *Consider closed and convex sets* $C_1 \dots, C_m \subset \mathbb{R}^n$ *and nonnegative scalars* $\alpha_1, \dots, \alpha_m$ *such that* $\sum_{i=1}^m \alpha_i = 1$. *Denote* $P_i = P_{C_i}$ *and let* $\overline{P} = \sum_{i=1}^m \alpha_i P_i$. *Define* $g : \mathbb{R}^n \to \mathbb{R}$ *as* $g(x) = \sum_{i=1}^m \alpha_i \|x - P_i(x)\|^2 = \sum_{i=1}^m \alpha_i \operatorname{dist}(x,C_i)^2$ *and let* $C = \cap_{i=1}^m C_i$. *Then,*

*i)* $F(\overline{P}) = \{x \in \mathbb{R}^n : \nabla g(x) = 0\}$, *i.e., since* $g$ *is convex, the set of fixed points of* $\overline{P}$ *(if nonempty) is precisely the set of minimizers of* $g$.

*ii) If* $C \neq \emptyset$, *then* $F(\overline{P}) = C$.

*Proof.* i) By Proposition 2.3(v),

$$\nabla g(x) = 2\sum_{i=1}^m \alpha_i (x - P_i(x)) = 2\left(x - \sum_{i=1}^m \alpha_i P_i(x)\right) = 2(x - \overline{P}(x)),$$

so that $\nabla g(x) = 0$ iff $x = \overline{P}(x)$ iff $x \in F(\overline{P})$.

ii) Clearly, $C \subset F(\overline{P})$. For the converse inclusion note that when $C \neq \emptyset$, we have $g(x) = 0$ for all $x \in C$ so that the minimum value of $g$ is indeed 0, and the set of minimizers of $g$ coincides with the set of its zeroes, which is $C$, because $g(x) > 0$ whenever $x \notin C$. The result follows then from item (i).

□

The next result provides a more accurate description of the set $F(\overline{P})$ when $m = 2$, i.e., for the case of a convex combination of the orthogonal projections onto two closed and convex sets.

Let $A, B \subset \mathbb{R}^n$ be two closed sets. Take $\alpha \in (0,1)$ and $\overline{P} = (1-\alpha)P_A + \alpha P_B$. Define $D \subset A \times B$ as $D = \{(x,y) \in A \times D : \|x - y\| = \operatorname{dist}(A,B)\}$. $S_A, S_B$ will denote the projections of $D$ onto $A, B$, respectively, i.e., $S_A = \{x \in A : \exists y \in B \text{ with } (x,y) \in D\}$ and $S_B = \{y \in B : \exists x \in A \text{ with } (x,y) \in D\}$. In other words, $D$ consist of the pairs in $A \times B$, which realize the distance between $A$ and $B$, $S_A$ is the set of points in $A$, which realize the distance to $B$, and $S_B$ is the set of points in $B$, which realize the distance to $A$. We remark that $D$ may be empty; take for instance $A = \{(x_1,x_2) \in \mathbb{R}^2 : x_2 \le 0\}$, $B = \{(x_1,x_2) \in \mathbb{R}^2 : x_2 \ge e^{x_1}\}$.

**Proposition 2.5.** *With the notation above,*

*i) for all* $(x,y), (x,', y') \in D$, *it holds that* $x - y = x' - y'$;

*ii) taking* $(x,y) \in D$, $\alpha \in (0,1)$ *and defining* $w = (1-\alpha)x + \alpha y$, *it holds that* $P_A(w) = x, P_B(w) = y$.

*iii)* $F(\overline{P}) = \{w = (1-\alpha)x + \alpha y : (x,y) \in D\}$.

*Proof.* i) Since, for any $(x,y) \in D$, the pair $(x,y)$ realizes the distance between $A$ and $B$, it follows that $P_B(x) = y$ and $P_A(y) = x$ for all $(x,y) \in D$. Hence $P_A(P_B(x)) = x$ for all $x \in S_A$. So, for all $(x,y),(x',y') \in D$, we have

$$\left\|x - x'\right\| = \left\|P_A(P_B(x)) - P_A(P_B(x'))\right\| \le \left\|P_B(x) - P_B(x')\right\| \le \left\|x - x'\right\|, \tag{2.3}$$

due to Proposition 2.3(ii). It follows that the equality holds throughout 2.3, and since $P_A \circ P_B$ is nonexpansive plus by Proposition 2.1(i), because both $P_A$ and $P_B$ are firmly nonexpansive (and so nonexpansive plus) by Proposition 2.3(ii), we conclude from Definition 1.1(ii) that $x - x' = P_B(x) - P_B(x') = y - y'$, which implies that $x - y = x' - y'$.

ii) Take $(x,y) \in D$ so that $x \in A$. Then $w = y + (1-\alpha)(x-y) = P_B(x) + (1-\alpha)(x - P_B(x))$. Since $1 - \alpha > 0$, it follows from Proposition 2.4(iv) that $P_B(w) = y$. A similar argument establishes that $P_A(w) = x$.

iii) Take $w = (1-\alpha)x + \alpha y$ with $(x,y) \in D$. By (ii), $w = (1-\alpha)P_A(w) + \alpha P_B(w) = \overline{P}(w)$. Hence, $w \in F(\overline{P})$ so that $\{w = (1-\alpha)x + \alpha y : (x,y) \in D\} \subset F(\overline{P})$. For the converse inclusion, consider any $x \in F(\overline{P})$, i.e.,

$$x = (1-\alpha)P_A(x) + \alpha P_B(x). \tag{2.4}$$

Let $\delta = \operatorname{dist}(A,B)$ and $\eta = \|P_A(x) - P_B(x)\|$. It suffices to check that $(P_A(x), P_B(x)) \in D$, i.e.,

$$\eta = \delta. \tag{2.5}$$

From (2.4), we have

$$\|x - P_A(x)\| = \alpha \|P_B(x) - P_A(x)\| = \alpha\eta,$$

and

$$\|x - P_B(x)\| = (1-\alpha)\|P_B(x) - P_A(x)\| = (1-\alpha)\eta,$$

which imply that

$$\begin{aligned} g(x) &= (1-\alpha)\|x - P_A(x)\|^2 + \alpha\|x - P_B(x)\|^2 \\ &= [(1-\alpha)\alpha^2 + \alpha(1-\alpha)^2]\eta^2 \\ &= (1-\alpha)\alpha\eta^2. \end{aligned} \tag{2.6}$$

Now take any pair $(u,v) \in D$ so that $\|u - v\| = \delta$, and let $w = (1-\alpha)u + \alpha v$. By item (ii), $u = P_A(w), v = P_B(w)$,

$$\|w - P_A(w)\| = \alpha\|P_B(w) - P_A(w)\| = \alpha\|u - v\| = \alpha\delta,$$

and

$$\|w - P_B(w)\| = (1-\alpha)\|P_B(w) - P_A(w)\| = (1-\alpha)\|u - v\| = (1-\alpha)\alpha\delta.$$

Hence,

$$\begin{aligned} g(w) &= (1-\alpha)\|w - P_A(w)\|^2 + \alpha\|w - P_B(w)\|^2 \\ &= [(1-\alpha)\alpha^2 + \alpha(1-\alpha)^2]\delta^2 \\ &= (1-\alpha)\alpha\delta^2. \end{aligned} \tag{2.7}$$

By Proposition 2.4(i), $x$ is a minimizer of $g$ so that $g(x) \le g(w)$, which implies, in view of (2.6) and (2.7) and the fact that $\alpha \in (0,1)$, that $\eta \le \delta$. On the other hand, $\eta =$

$\|P_A(x) - P_B(x)\|$ with $P_A(x) \in A, P_B(x) \in B$ so that $\eta \geq \operatorname{dist}(A,B) = \delta$. We conclude that (2.5) holds, and the result is established. □

Now we deal with the main result of this section, which we describe below. The prototypical examples of firmly nonexpansive operators are the orthogonal projections onto closed and convex sets. Proposition 2.1(ii) provides a larger class of firmly nonexpansive operators, namely the convex combinations of orthogonal projections. It is therefore relevant to check that the second class is indeed larger, i.e., generically, the convex combinations of orthogonal projections are not orthogonal projections themselves. We will prove that this is indeed the case when the intersection of the convex sets is nonempty. However, when this intersection is empty, a convex combination of orthogonal projections may be itself an orthogonal projection. We will establish a necessary and sufficient condition for this situation to occur for the case of two convex sets.

**Proposition 2.6.** *Consider closed and convex sets $C_1 \ldots, C_m \subset \mathbb{R}^n$ and positive scalars $\alpha_1, \ldots, \alpha_m$ such that $\sum_{i=1}^m \alpha_i = 1$. Denote $C = \cap_{i=1}^m C_i$, $P_i = P_{C_i}$ and let $\overline{P} = \sum_{i=1}^m \alpha_i P_i$. Assume that $C \neq \emptyset$. If there exists $E \subset \mathbb{R}^n$ such that $\overline{P} = P_E$, then $E = C_1 = \cdots = C_m$.*

*Proof.* By Propositions 2.4(ii) and 2.3(iii), one has

$$C = F(\overline{P}) = F(P_E) = E. \tag{2.8}$$

Take $x \in C_i$. Let $\ell = \operatorname{argmax}_{1 \leq j \leq m}\{\|x - P_j(x)\|\}$ and $w = \sum_{j=1}^m \alpha_j P_j(x) = \overline{P}(x) = P_E(x)$ so that $w \in \operatorname{Im}(P_E) = E = C$, using (2.8). Hence $w \in C_\ell$. It follows that

$$\begin{aligned}
\|x - P_\ell(x)\| \leq \|x - w\| = \left\| \sum_{i=j}^m \alpha_j (x - P_j(x)) \right\| &\leq \sum_{j=1}^m \alpha_j \|x - P_j(x)\| \\
&= \sum_{j=1, j\neq i}^m \alpha_j \|x - P_j(x)\| \leq \sum_{j=1, j\neq i}^m \alpha_j \|x - P_\ell(x)\| \\
&= \left( \sum_{j=1, j\neq i}^m \alpha_j \right) \|x - P_\ell(x)\| = (1 - \alpha_i) \|x - P_\ell(x)\|,
\end{aligned} \tag{2.9}$$

using the convexity of $\|\cdot\|$ in the first inequality, the fact that $x \in C_i$ in the second equality, and the definition of $\ell$ in the second inequality. It follows from (2.9) that $\alpha_i \|x - P_\ell(x)\| \leq 0$ so that $\|x - P_\ell(x)\| = 0$. Since $0 \leq \|x - P_j(x)\| \leq \|x - P_\ell(x)\|$ for all $j$, from the definition of $\ell$, we conclude that $\|x - P_j(x)\| = 0$ for all $j$, i.e., $x \in C_j$. Since $x$ is an arbitrary point in $C_i$, we have that $C_i \subset C_j$ for all $i, j$, i.e., $C_1 = \cdots = C_m$. The result follows immediately from (2.8). □

Next, we fully characterize the situation for the case of 2 convex sets. For $A \subset \mathbb{R}^n$, we denote the affine hull of $A$ as $\operatorname{aff}(A)$.

**Proposition 2.7.** *Take closed and convex sets $A, B \subset \mathbb{R}^n$ and $\alpha \in (0,1)$. Define $\overline{P} = (1-\alpha)P_A + \alpha P_B$. Then, there exists a nonempty, closed, and convex set $E \subset \mathbb{R}^n$ such that $\overline{P} = P_E$ if and only if there exists $c \in \operatorname{aff}(A)^\perp$ such that $B = A + c$.*

*Proof.* We start with the "only if" statement. We claim that the result holds with $E = A + \alpha c$. First, we prove that $P_B(x) = P_A(x) + c$ for all $x \in \mathbb{R}^n$. Let $z = P_A(x) + c$. By Proposition 2.3(i), it suffices to prove that $\langle x-z, y-z\rangle \leq 0$ for all $x \in \mathbb{R}^n$ and all $y \in B = A+c$, that is, for all $y \in A$, we have

$$\begin{aligned} 0 &\geq \langle x-z, y+c-z\rangle \\ &= \langle x-P_A(x)-c, y+c-P_A(x)-c\rangle \\ &= \langle x-P_A(x), y-P_A(x)\rangle - \langle c, y-P_A(x)\rangle \\ &= \langle x-P_A(x), y-P_A(x)\rangle, \end{aligned} \tag{2.10}$$

using in the last equality the facts that $c \in \operatorname{aff}(A)^{\perp}$ and $y, P_A(x) \in A$ so that $y-P(A) \in \operatorname{aff}(A)$. Hence, $\langle c, y-P_A(x)\rangle = 0$. Note that $0 \leq \langle x-P_A(x), y-P_A(x)\rangle$ by Proposition 2.3(i) so that the inequality in (2.10) holds, and hence we have proved that $P_B(x) = P_A(x)+c$ for all $x \in \mathbb{R}^n$. It follows that $\overline{P} = (1-\alpha)P_A + \alpha P_B = (1-\alpha)P_A + \alpha P_A + \alpha c = P_A + \alpha c$.

Now, the same argument, used to prove that $P_{A+c} = P_A + c$, allows us to conclude that $P_A + \alpha c = P_{A+\alpha c}$ so that $(1-\alpha)P_A + \alpha P_B = P_E$ with $E = A + \alpha c$.

Now we prove the "if" statement. First, we must identify the appropriate vector $c$. By assumption, $\overline{P} = P_E$ so that $F(\overline{P}) = E \neq \emptyset$ by Proposition 2.3(iii). It follows that $D$, as defined in Proposition 2.5, is nonempty. We take any pair $(u,v) \in D$ and take $c = v-u$. By Proposition 2.5(i), $c$ does not depend on the chosen pair $(u,v)$. We must prove that $B = A+c$, and we first claim that

$$S_B = S_A + c, \tag{2.11}$$

with $S_A, S_B$ as in Proposition 2.5. Take $u \in S_B$ so that there exists $v \in S_A$ such that $(u,v) \in D$. Hence $v = u + (v-u) = x + c$, which shows that $v \in S_A + c$. Therefore, $S_B \subset S_A + c$. Reversing the roles of $A, B$, we obtain the reverse inclusion, and then (2.11) holds.

We next show that the assumption $\overline{P} = P_E$ implies that $A = S_A, B = S_B$. Take any $x \in A$. We must prove that $x$ realizes the distance to $B$. Let $z = \overline{P}(x) = (1-\alpha)P_A(x) + \alpha P_B(x) = (1-\alpha)x + \alpha P_B(x)$. It follows from Proposition 2.3(iv) that $P_B(z) = P_B(x)$. Note that

$$(1-\alpha)(x-P_B(x)) = z - P_B(x) = z - P_B(z). \tag{2.12}$$

Now $z = \overline{P}(x) = P_E(x)$ so that $z \in E = F(P_E) = F(\overline{P})$. By Proposition 2.5(ii) and (iii), $z = (1-\alpha)P_A(z) + \alpha P_B(z)$ with $(P_A(z), P_B(z)) \in D$. It follows that

$$z - P_B(z) = (1-\alpha)(P_A(z) - P_B(z)). \tag{2.13}$$

Since $\alpha \in (0,1)$, we conclude from (2.12) and (2.13) that $x - P_B(x) = P_A(z) - P_B(z)$ so that, in view of the fact that $(P_A(z), P_B(z)) \in D$,

$$\operatorname{dist}(x,B) = \|x - P_B(x)\| = \|P_A(z) - P_B(z)\| = \operatorname{dist}(A,B).$$

We have proved that $x$ realizes the distance between $A$ and $B$, i.e., $x \in S_A$. Since $x$ is an arbitrary point in $A$, we have $A \subset S_A \subset A$ so that $A = S_A$. By the same token, $B = S_B$. In view of (2.11), we have that $B = A + c$.

It only remains to be verified that $c \in \operatorname{aff}(A)^{\perp}$. Let $\operatorname{relint}(A)$ be the relative interior of $A$ (i.e., the interior of $A$ with respect to $\operatorname{aff}(A)$). Take any $x \in \operatorname{relint}(A)$ and any $z \in \operatorname{aff}(A)$. Since $x \in \operatorname{relint}(A)$, there exists $\varepsilon > 0$ such that both $x + \varepsilon(z-x)$ and $x - \varepsilon(z-x)$ belong to $A$. Since $x \in A = S_A$, we obtain from Proposition 2.5(ii) that $x = P_A(v)$ for some $v \in S_B$ and $c = v - P_B(v) = v - x$ so that, by Proposition 2.3(i), $\langle c, y-x\rangle = \langle v - P_B(v), y - P_B(v)\rangle \leq 0$ for all $y \in A$. Taking first $y = x + \varepsilon(z-x)$ and then $y = x - \varepsilon(z-x)$, we conclude that $\varepsilon\langle c, z-x\rangle \leq 0$, $-\varepsilon\langle c, z-x\rangle \leq 0$, implying that $\langle c, z-y\rangle = 0$ for all $z \in \operatorname{aff}(A)$. Hence, $c \in \operatorname{aff}(A)^{\perp}$, which completes the proof. $\square$

**Corollary 2.1.** *Assume that any of the equivalent statements in Proposition 2.7 hold and that $A \neq B$. Then $A$ has empty interior and $A \cap B = \emptyset$.*

*Proof.* Since $B = A + c$ and $A \neq B$, we have $c \neq 0$. Since $c \in \operatorname{aff}(A)^{\perp}$, we obtain that $\operatorname{aff}(A) \neq \mathbb{R}^n$, i.e., $\operatorname{aff}(A)$ is not full dimensional. Hence, $A$ has empty interior.

For the second statement, we assume that $A \cap B \neq \emptyset$ and take $x \in A \cap B$. Since $B = A + c$, we have $x = x' + c$ with $x' \in A$ so that $\|c\|^2 = \langle c, x - x' \rangle = 0$ because $c \in \operatorname{aff}(A)^{\perp}$ and $x, x' \in A$ so that $x - x' \in \operatorname{aff}(A)$. It follows that $c = 0$, and the resulting contradiction entails the result. $\square$

We mention that the second statement of the corollary follows also from Proposition 2.6.

The "only if" statement of Proposition 2.7 can be easily generalized to the case of $m$ convex sets; unfortunately we do not have at this point a proof for the much more interesting generalization of the "if" statement. The following corollary contains the generalization of the "only if" statement.

**Corollary 2.2.** *Consider closed and convex sets $C_1 \ldots, C_m \subset \mathbb{R}^n$ and nonnegative scalars $\alpha_1, \ldots, \alpha_m$ such that $\sum_{i=1}^m \alpha_i = 1$. Denote $P_i = P_{C_i}$ and let $\overline{P} = \sum_{i=1}^m \alpha_i P_i$. Take $\beta_2, \ldots, \beta_m \in \mathbb{R}, c \in \operatorname{aff}(C_1)^{\perp}$, and assume that $C_i = C_1 + \beta_i c$ for $i = 2, \cdots, m$. Define $\bar{\beta} = \sum_{i=2}^m \alpha_i \beta_i, E = C_1 + \bar{\beta} c$. Then $\overline{P} = P_E$.*

*Proof.* The argument used in the proof of Proposition 2.7 shows that $P_i(x) = P_1(x) + \beta_i c$ for $i = 2, \ldots, m$, and all $x \in \mathbb{R}^n$ so that $\overline{P}(x) = P_1(x) + \bar{\beta} c$ for all $x \in \mathbb{R}^n$. The same argument then shows that $P_1 + \bar{\beta} c = P_E$. $\square$

## 3. Convergence of CRM Applied to FPP

In this section, we establish the convergence of CRM applied to finding a point in $\operatorname{Fix}(T, P_U)$, where $T : \mathbb{R}^n \to \mathbb{R}^n$ is firmly nonexpansive and $P_U : \mathbb{R}^n \to \mathbb{R}^n$ is the orthogonal projection onto an affine manifold $U \subset \mathbb{R}^n$. As explained in Section 1, through Pierra's formalism in the product space $\mathbb{R}^{nm}$, this result entails convergence of CRM applied to finding a point in $\operatorname{Fix}(T_1, \ldots, T_m)$, where $T_i : \mathbb{R}^n \to \mathbb{R}^n$ is firmly nonexpansive for $1 \leq i \leq m$.

Our convergence analysis for CRM requires comparing the CRM and the MAP sequences so that we start by proving convergence of the second one, defined as

$$z^{k+1} = P_U(T(z^k)), \tag{3.1}$$

starting at some $z^0 \in \mathbb{R}^n$. This is a classical result, but we include it for the sake of self-containment. We start with the following intermediate result.

**Proposition 3.1.** *For all $x \in \mathbb{R}^n$ and all $y \in \operatorname{Fix}(T, P_U)$, it holds that*

$$\|P_U(T(x)) - y\|^2 \leq \|x - y\|^2 - \|P_U(x) - x\|^2 - \|P_U(T(x)) - T(x)\|^2. \tag{3.2}$$

*Proof.* By firm nonexpansiveness of $P_U$, we have

$$\|P_U(x) - y\|^2 \leq \|x - y\|^2 - \|P_U(x) - x\|^2 \tag{3.3}$$

for all $x \in \mathbb{R}^n$ due to the fact that $u \in U$. Substituting $T(x)$ for $x$ in (3.3), we obtain

$$\|P_U(T(x)) - y\|^2 \leq \|T(x) - y\|^2 - \|P_U(T(x)) - T(x)\|^2. \tag{3.4}$$

Since $T$ is firmly nonexpansive, we have

$$\|T(x) - y\|^2 \leq \|x - y\|^2 - \|T(x) - x\|^2. \tag{3.5}$$

Now combining (3.4) with (3.5), we obtain

$$\|P_U(T(x))-y\|^2 \le \|x-y\|^2 - \|T(x)-x\|^2 - \|P_U(T(x))-T(x)\|^2 \tag{3.6}$$

which implies the result. □

Using Proposition 3.1, we obtain the convergence of $\{z^k\}$ by using the classical argument for MAP applied to CFP.

**Proposition 3.2.** *If* $\mathrm{Fix}(T,P_U)\neq\emptyset$*, then the sequence* $\{z_k\}$ *defined by* (3.1) *converges to a point* $\bar{z}\in\mathrm{Fix}(T,P_U)$.

*Proof.* Take any $y\in\mathrm{Fix}(T,P_U)$. By (3.1), $z^{k+1}=PU(T(z^k))$. Using (3.3), we have

$$\left\|z^{k+1}-y\right\|^2 \le \left\|z^k-y\right\|^2 - \left\|P_U(T(z^k))-T(z^k)\right\|^2 - \left\|T(z^k)-z^k\right\|^2 \le \left\|z^k-y\right\|^2. \tag{3.7}$$

It follows from (3.7) that $\left\|z^{k+1}-y\right\|^2 \le \left\|z^k-y\right\|$ for all $k\in\mathbb{N}$ so that $\{z^k\}$ is bounded and $\{\left\|z^k-y\right\|\}$ is nonincreasing and nonnegative, therefore convergent.

Hence, rewriting (3.7) as

$$\left\|P_U(T(z^k))-T(z^k)\right\|^2 + \left\|T(z^k)-z^k\right\|^2 \le \left\|z^k-y\right\|^2 - \left\|z^{k+1}-y\right\|^2,$$

we conclude that

$$\lim_{k\to\infty}\left\|T(z^k)-z^k\right\|=0. \tag{3.8}$$

Let $\bar{z}$ be a cluster point of the bounded sequence $\{z^k\}$. Taking limits in (3.8) along a subsequence converging to $\bar{z}$ and using the continuity of $T$, which results from its nonexpansiveness, we have that $T(\bar{z})=\bar{z}$. Since $z^k\in U$ for all $k\in\mathbb{N}$, by (3.1), we have that $\bar{z}\in U$ so that $\bar{z}\in\mathrm{Fix}(T,P_U)$. Taking now $y=\bar{z}$ in (3.7), we conclude that $\{\left\|z^k-\bar{z}\right\|\}$ is convergent. Since a subsequence of this sequence converges to 0, the whole sequence $\{\left\|z^k-\bar{z}\right\|\}$ converges to 0, i.e., $\lim_{k\to\infty}z^k=\bar{z}\in\mathrm{Fix}(T,P_U)$. □

Now we proceed to the convergence analysis of CRM applied to FPP. Let $T:\mathbb{R}^n\to\mathbb{R}^n$ be a firmly nonexpansive operator, $U\subset\mathbb{R}^n$ an affine manifold, and $P_U:\mathbb{R}^n\to\mathbb{R}^n$ the orthogonal projection onto $U$. We assume that $\mathrm{Fix}(T,P_U)\neq\emptyset$. We denote as $R,R_U$ the reflection operators related to $T,P_U$ respectively, i.e., $R(x)=2T(x)-x, R_U(x)=2P_U(x)-x$. We define $C:\mathbb{R}^n\to\mathbb{R}^n$ as the CRM operator, i.e., $C(z)=\mathrm{circ}\{z,R(z),R_U(R(z))\}$, where "circ" denotes the circumcenter of three points, as defined in Section 1. We also define $S:\mathbb{R}^n\to\mathbb{R}^n$ as $S(x)=P_U(T(x))$ so that $S$ can be seen the MAP operator.

We will prove that, starting from any initial point $x^0\in U$, the sequence $\{x^k\}$ generated by CRM, defined as $x^{k+1}=C(x^k)$, converges to a point in $\mathrm{Fix}(T,P_U)$.

Our convergence analysis is close to the one in [12] for CRM applied to CFP, but with several differences, resulting from the fact that $T$ is an arbitrary firmly nonexpansive operator, rather than the orthogonal projection onto a convex set. One of differences is the use of the following property of circumcenters, which will substitute for a specific property of orthogonal projections.

**Proposition 3.3.** *For all* $x\in\mathbb{R}^n$, $\langle x-T(x),C(x)-T(x)\rangle=0$.

*Proof.* By the definition of the reflection, for all $x \in \mathbb{R}^n$,

$$T(x) = \frac{1}{2}(R(x) + x). \tag{3.9}$$

By the definition of circumcenter, for all $x \in \mathbb{R}^n$,

$$\|C(x) - x\|^2 = \|C(x) - R(x)\|^2. \tag{3.10}$$

Expanding (3.10) and rearranging, we have

$$2\langle x - R(x), C(x)\rangle = \|x\|^2 - \|R(x)\|^2. \tag{3.11}$$

Subtracting $2\langle x - R(x), T(x)\rangle$ from both sides of (3.11) and using (3.9), we obtain

$$\begin{aligned} 4\langle x - T(x), C(x) - T(x)\rangle &= 2\langle x - R(x), C(x) - T(x)\rangle \\ &= \|x\|^2 - \|R(x)\|^2 - 2\langle x - R(x), T(x)\rangle \\ &= \|x\|^2 - \|R(x)\|^2 - \langle x - R(x), x + R(x)\rangle = 0, \end{aligned}$$

which implies the result. □

Next, we establish a basic property of the circumcenter, which ensures that the CRM sequence, starting at a point in $U$, remains in $U$.

**Proposition 3.4.** *If $z \in U$, then $C(z) \in U$.*

*Proof.* We consider three cases. If $R(z) \in U$, then $R_U(R(z)) = R(z)$, where $z, R(z), R_U(R(z)) \in U$ so that the affine hull of these three points is contained in $U$. Since by definition $C(z)$ belongs to this affine hull, the result holds. If $z = P_U(R(z))$, then the affine hull of $\{z, R(z), R_U(R(z))\}$ is the line determined by $z$ and $R(z)$, and $C(z) = \operatorname{circ}\{z, R(z), R_U(R(z))\} = P_U(R(z)) = z \in U$ so that the result holds. Assume that $z \neq P_U(R(z))$ and that $R(z) \notin U$. We claim that $C(z)$ belongs to the line passing through $z$ and $P_U(R(z))$. Observe that, since $\|C(z) - R(z)\| = \|C(z) - R_U(R(z))\|$, $C(z)$ belongs to the hyperplane orthogonal to $R(z) - R_U(R(z))$ passing through $\frac{1}{2}(R(z), R_U(R(z))) = P_U(R(z))$, say $H$. On the other hand, $C(z)$ belongs to the affine manifold $E$ spanned by $z, R(z), R_U(z)$. So, $C(z) \in E \cap U$. Since $R(z) \notin U$, $\dim(E \cap U) < \dim(E) \leq 2$. Note that $P_U(z) = \frac{1}{2}(R(z) + R_U(R(z))) = P_U(R(z))$ belongs to $E$. Hence the line through $z, P_U(R(z))$, say $L$, is contained in $E$, and by a dimensionality argument, we conclude that $L = E$. Since $C(z) \in E$, we obtain that $C(z) \in L$. Since $z, P_U(R(z))$ belong to $U$, we have that $C(z) \in L \subset U$. This completes the proof. □

We continue with an important intermediate result.

**Proposition 3.5.** *Consider the operators $C, S : \mathbb{R}^n \to \mathbb{R}^n$ defined above. Then $S(x)$ belongs to the segment between $x$ and $C(x)$ for all $x \in U$.*

*Proof.* Let $E$ denote the affine manifold spanned by $x, R(x)$, and $R_U(R(x))$. From the definition, the circumcenter of these three points, namely $C(x)$, belongs to $E$. We claim that $S(x)$ also belongs to $E$. We proceed to prove the claim. Since $U$ is an affine manifold, $P_U$ is an affine operator so that $P_U(\alpha x + (1-\alpha)x') = \alpha P_U(x) + (1-\alpha)P_U(x')$ for all $\alpha \in \mathbb{R}$ and all $x, x' \in \mathbb{R}^n$. Thus $R_U(R(x)) = 2P_U(R(x)) - R(x)$ so that

$$P_U(R(x)) = \frac{1}{2}(R_U(R(x)) + R(x)). \tag{3.12}$$

On the other hand, using the affinity of $P_U$, the definition of $S$, and the assumption that $x \in U$, we have

$$P_U(R(x)) = P_U(2T(x) - x) = 2P_U(T(x)) - P_U(x) = 2S(x) - x \tag{3.13}$$

so that

$$S(x) = \frac{1}{2}\left(P_U(R(x)) + x\right). \tag{3.14}$$

Combining (3.12) and (3.14), we have

$$S(x) = \frac{1}{2}x + \frac{1}{4}R_U(R(x)) + \frac{1}{4}R(x),$$

i.e., $S(x)$ is a convex combination of $x, R_U(R(x))$, and $R(x)$. Since these three points belong to $E$, the same holds for $S(x)$ and the claim holds.

We observe now that $x \in U$ by assumption, $S(x) \in U$ by definition, and $C(x) \in U$ by Proposition 3.4. Now we consider three cases: if $\dim(E \cap U) = 0$, then $x, S(x)$ and $C(x)$ coincide and the result holds trivially. If $\dim(E \cap U) = 2$, then $E \subset U$ so that $R(x) \in U$. Hence, $R_U(R(x)) = R(x)$, in which case $C(x)$ is the midpoint between $x$ and $R(x)$, which is precisely $T(x)$. Hence, $T(x) \in U$ so that $S(x) = P_U(T(x)) = B(x) = C(x)$, implying that $S(x)$ and $C(x)$ coincide, and the result holds trivially. The interesting case is the remaining one, i.e., $\dim(E \cap U) = 1$. In this case $x, S(x)$, and $C(x)$ lie in a line so that we can write $C(x) = x + \eta(S(x) - x)$ with $\eta \in \mathbb{R}$, and it suffices to prove that $\eta \geq 1$. By the definition of $\eta$, we have

$$\|C(x) - x\| = |\eta| \, \|T(x) - x\|. \tag{3.15}$$

Since $C(x) \in U$, the nonexpansiveness of $P_U$ implies that

$$\|C(x) - R(x)\| \geq \|C(x) - P_U(R(x))\|. \tag{3.16}$$

Then

$$\begin{aligned} \|C(x) - x\| &= \|C(x) - R(x)\| \\ &\geq \|C(x) - P_U(R(x))\| \\ &= \|(C(x) - x) - (P_U(R(x)) - x)\| \\ &= \|\eta\,(S(x) - x) - 2\,(S(x) - x)\| \\ &= |\eta - 2| \, \|S(x) - x\|, \end{aligned} \tag{3.17}$$

using the definition of the circumcenter in the first equality, (3.16) in the inequality, and the definition of $\eta$ and $S$ in the third equality. Combining (3.15) and (3.17), we have

$$|\eta| \, \|S(x) - x\| \geq |\eta - 2| \, \|S(x) - x\|,$$

implying that $|\eta| \geq |2 - \eta|$, which holds only when $\eta \geq 1$. This completes the proof. □

We continue with a key result for the convergence analysis of CRM, comparing the behavior of the CRM and the MAP operators. Again the argument in this proof differs from the case of CRM applied to MAP, presented in [12].

**Proposition 3.6.** *With the notation of Proposition 3.5 for all $y \in \operatorname{Fix}(T, P_U)$ and all $z \in U$, it holds that*

*i)* $\|C(z) - y\| \leq \|S(z) - y\|$,
*ii)* $\operatorname{dist}(C(z), \operatorname{Fix}(T, P_U)) \leq \operatorname{dist}(S(z), \operatorname{Fix}(T, P_U))$,

*Proof.* i) Take $z \in U, y \in \operatorname{Fix}(T, P_U)$. If $z \in F(T)$, then the result follows trivially because $P_U(T(z)) = z = C(z)$ and there is nothing to prove. So, we assume that $z \in U \setminus F(T)$. We claim that

$$\|P_U(T(z)) - z\| \leq \|T(z) - z\| \leq \|C(z) - z\|. \tag{3.18}$$

For proving the first inequality in (3.18), we conclude, from the fact that $z \in U$ and an elementary property of orthogonal projections, that

$$\|P_U(T(z)) - z\| \leq \|T(z) - z\|. \tag{3.19}$$

Since $R(z) = 2T(z) - z$, we obtain that

$$\|R(z) - z\| = 2\,\|T(z) - z\|. \tag{3.20}$$

Using (3.19) and (3.20), we have

$$\begin{aligned} \|T(z) - z)\| &= \frac{1}{2}\,\|(R(z) - C(z) + C(z) - z)\| \\ &\leq \frac{1}{2}\,(\|R(z) - C(z)\| + \|C(z) - z\|) \\ &= \frac{1}{2}\,(\|z - C(z)\| + \|C(z) - z\|) \\ &= \|C(z) - z\|, \end{aligned} \tag{3.21}$$

where the third equality holds because $C(z)$ is equidistant from $z, R(z)$, and $R_U(R(z))$. The claim then follows from (3.18) and (3.21). By Proposition 3.5, $T(z)$ belongs to the segment between $z$ and $C(z)$, i.e., there exists $\alpha \in [0,1]$ such that $S(z) = \alpha C(z) + (1-\alpha)z$ and $\alpha < 1$ because $z \notin F(T)$ so that

$$S(z) - C(z) = \frac{1-\alpha}{\alpha}(z - S(z)). \tag{3.22}$$

Note that

$$\begin{aligned} &\langle z - S(z), C(z) - y\rangle \\ &= \langle z - T(z), C(z) - T(z)\rangle + \langle z - T(z), T(z) - y\rangle + \langle T(z) - S(z), C(z) - y\rangle. \end{aligned} \tag{3.23}$$

Now we look at the three terms in the right hand side of (3.23). The first one vanishes as a consequence of Proposition 3.3. The third one vanishes because $S(z) = P_U(T(z))$, and $U$ is an affine manifold so that $T(z) - S(z)$ is orthogonal to any vector in $U$, as is the case for $C(z) - y$, since $y \in U$ by assumption and $C(z) \in U$ by Proposition 3.4. The second term is nonnegative by Proposition 2.2. Hence, it follows from (3.23) that

$$\langle z - S(z), C(z) - y\rangle \geq 0. \tag{3.24}$$

Now, (3.24) together with (3.22) gives us

$$\langle S(z) - C(z), y - C(z)\rangle = \frac{1-\alpha}{\alpha}\langle z - S(z), y - C(z)\rangle \leq 0. \tag{3.25}$$

It follows from (3.25) that $\|C(z) - y\| \leq \|S(z) - y\|$ for all $y \in \operatorname{Fix}(T, P_U)$ and all $z \in U$, establishing (i).

ii) Let $\bar{z},\hat{z} \in \operatorname{Fix}(T,P_U)$ realize the distance from $C(z),S(z)$ to $\operatorname{Fix}(T,P_U)$, respectively. Then, in view of (i), we have

$$\begin{aligned}\operatorname{dist}(C(z),\operatorname{Fix}(T,P_U)) = \|C(z)-\bar{z}\| \leq \|C(z)-\hat{z}\| \leq \|S(z)-\hat{z}\| \\ = \operatorname{dist}(S(z),\operatorname{Fix}(T,P_U))\end{aligned}$$

proving (ii).

□

Next, we complete the convergence analysis of CRM applied to FPP. Here again, the proofline differs from the one given in [12], where a specific property of orthogonal projections was used to characterize $C(z)$ as the projection onto a certain set, which does not work when $T$ is an arbitrary firmly nonexpansive operator.

**Theorem 3.1.** *Let $T:\mathbb{R}^n \to \mathbb{R}^n$ be a firmly nonexpansive operator and $U \subset \mathbb{R}^n$ an affine manifold. Assume that $\operatorname{Fix}(T,P_U) \neq \emptyset$. Let $\{x^k\}$ be the sequence generated by CRM for solving FPP$(T,P_U)$, i.e., $x^{k+1} = C(x^k)$. If $x^0 \in U$, then $\{x^k\}$ is contained in $U$ and converges to a point in $\operatorname{Fix}(T,P_U)$.*

*Proof.* The fact that $\{x^k\} \subset U$ results from invoking Proposition 3.4 in an inductive way, starting with the assumption that $x^0 \in U$. Take any $y \in \operatorname{Fix}(T,P_U)$, Then

$$\left\|x^{k+1}-y\right\|^2 = \left\|C(x^k)-y\right\|^2 \leq \left\|S(x^k)-y\right\|^2 \leq \left\|x^k-y\right\|^2 - \left\|S(x^k)-x^k\right\|^2, \tag{3.26}$$

where the first inequality follows from Proposition 3.6(i), and the second one follows from Proposition 3.1 since $P_U(x^k) = x^k$ by Proposition 3.4 and $S = P_U \circ T$. (3.26) says that $\{x^k\}$ is Fejér monotone with respect to $\operatorname{Fix}(T,P_U)$, and the remainder of the proof is standard. By (3.26), $\{x^k\}$ is bounded and $\{\left\|x^k-y\right\|\}$ is nonincreasing and nonnegative, hence convergent, for all $y \in \operatorname{Fix}(T,P_U)$. It also follows from (3.26) that

$$\lim_{k\to\infty} S(x^k) - x^k = 0. \tag{3.27}$$

Let $\bar{x}$ be any cluster point of $\{x^k\}$. Taking limits in (3.27) along a subsequence converging to $\bar{x}$, we conclude that $S(\bar{x}) = \bar{x}$, i.e., $\bar{x} \in F(S) = \operatorname{Fix}(T,P_U)$ so that all cluster points of $\{x^k\}$ belong to $\operatorname{Fix}(T,P_U)$. Looking now (3.26) with $\bar{x}$ substituting for $y$, we have that $\{\left\|x^k-\bar{x}\right\|\}$ is a nonincreasing sequence with a subsequence converging to 0 so that the whole sequence $\{\left\|x^k-\bar{x}\right\|\}$ converges to 0. It follows that $\bar{x}$ is the unique cluster point of $\{x^k\}$ so that $\lim_{k\to\infty} x^k = \bar{x} \in \operatorname{Fix}(T,P_U)$. □

For future reference, we state the Fejér monotonicity of $\{x^k\}$ with respect to $\operatorname{Fix}(T,P_U)$ as a corollary.

**Corollary 3.1.** *With the notation of Theorem 3.1, $\left\|x^{k+1}-y\right\|^2 \leq \left\|x^k-y\right\|^2 - \left\|S(x^k)-x^k\right\|^2$ for all $y \in \operatorname{Fix}(T,P_U)$ and all $k \in \mathbb{N}$.*

*Proof.* The result follows from (3.26). □

## 4. Linear Convergence of CRM Applied to FPP under an Error Bound Condition

In [12], when dealing with CFP with two convex sets, namely $K,U$, the following *global error bound*, which we will call **EB1**, was considered:

**EB1**: There exists $\bar{\omega} > 0$ such that $\operatorname{dist}(x,K) \geq \bar{\omega}\operatorname{dist}(K \cap U)$ for all $x \in U$.

Let us comment on the connection between **EB1** and other notions of error bounds which have been introduced in the past, all of them related to regularity assumptions imposed on the solutions of certain problems. If the problem at hand consists of solving $H(x) = 0$ with a smooth $H : \mathbb{R}^n \to \mathbb{R}^m$, a classical regularity condition demands that $m = n$ and the Jacobian matrix of $H$ is nonsingular at a solution $x^*$, in which case, Newton's method, for instance, is known to enjoy superlinear or quadratic convergence. This condition implies local uniqueness of the solution $x^*$. For problems with nonisolated solutions, a less demanding assumption is the notion of *calmness* (see [25], Chapter 8, Section F), which requires that

$$\frac{\|H(x)\|}{\operatorname{dist}(x, S^*)} \geq \omega \tag{4.1}$$

for all $x \in \mathbb{R}^n \setminus S^*$ and some $\omega > 0$, where $S^*$ is the solution set, i.e., the set of zeros of $H$. Calmness, also called upper-Lipschitz continuity (see [26]), is a classical example of error bound, and it holds in many situations, e.g., when $H$ is affine, by virtue of Hoffman's Lemma, (see [27]). It implies that the solution set is locally a Riemannian manifold (see [28]), and it has been used for establishing superlinear convergence of Levenberg-Marquardt methods in [29].

When dealing with convex feasibility problems, it seems reasonable to replace the numerator of (4.1) by the distance from $x$ to some of the convex sets, as was done in, for instance, [12], giving rise to **EB1**. In [12], it was proved that, under **EB1**, MAP converges linearly, with asymptotic constant bounded above by $\sqrt{1-\bar{\omega}^2}$, and that CRM also converges linearly, with a better upper bound for the asymptotic constant, namely $\sqrt{(1-\bar{\omega}^2)/(1+\bar{\omega}^2)}$. In this section, we will prove that in the FPP case both sequences converge linearly, with asymptotic constant bounded by $\sqrt{1-\bar{\omega}^2}$.

In the case of FPP, dealing with a firmly nonexpansive $T : \mathbb{R}^n \to \mathbb{R}^n$, and an affine manifold $U \subset \mathbb{R}^n$, the appropriate error bound turns out to be:

**EB**: There exists $\omega > 0$ such that $\|x - T(x)\| \geq \omega \operatorname{dist}(x, \operatorname{Fix}(T, P_U))$ for all $x \in U$.

We mention here that it suffices to consider an error bound less demanding than **EB**, namely a local one, where the inequality above is requested to hold only for points in $U \cap V$, where $V$ is a given set, e.g., a ball around the limit of the sequence generated by the algorithm, assumed to be convergent. An error bound of this type was used in [11]. We refrain to do so just for the sake of a simpler exposition.

**Proposition 4.1.** *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a firmly nonexpansive operator, $U \subset \mathbb{R}^n$ an affine manifold, and $C, S : \mathbb{R}^n \to \mathbb{R}^n$ the CRM and the MAP operators, respectively. Assume that* $\operatorname{Fix}(T, P_U) \neq \emptyset$ *and that* **EB** *holds. Then*

$$\operatorname{dist}(C(x), \operatorname{Fix}(T, P_U))^2 \leq \operatorname{dist}(S(x), \operatorname{Fix}(T, P_U))^2 \leq (1-\omega^2) dist(x, \operatorname{Fix}(T, P_U))^2, \tag{4.2}$$

*for all $x \in U$, with $\omega$ as in* **EB**.

*Proof.* First note that if $x \in F(T)$, then (4.2) holds trivially so that we assume from now on that $T(x) \neq x$. Take any $y \in \operatorname{Fix}(T, P_U)$. Since $T$ is firmly nonexpansive and $y \in F(T)$, we have

$$\|x-y\|^2 \geq \|T(x)-T(y)\|^2 + \|(x-y)-(T(x)-T(y))\|^2 = \|T(x)-y\|^2 + \|x-T(x)\|^2, \tag{4.3}$$

We take now a specific point in $\operatorname{Fix}(T, P_U)$, namely $\bar{y} = P_{\operatorname{Fix}(T, P_U)}(x)$, and rewrite **EB** as

$$\|x-T(x)\|^2 \geq \omega^2 \|x-\bar{y}\|^2. \tag{4.4}$$

Combining (4.3) and (4.4), we have

$$\|x-\bar{y}\|^2 \geq \|x-T(x)\|^2 + \|T(x)-\bar{y}\|^2 \geq \omega^2 \|x-\bar{y}\|^2 + \|T(x)-\bar{y}\|^2. \tag{4.5}$$

Rearranging (4.5), we conclude that

$$(1-\omega^2)\|x-\bar{y}\|^2 \geq \|T(x)-\bar{y}\|^2. \tag{4.6}$$

From an elementary property of orthogonal projections and $\bar{y} \in U$, we have $\langle T(x)-S(x), \bar{y}-T(x)\rangle = \langle T(x)-P_U(T(x)), \bar{y}-T(x)\rangle \leq 0$. Hence,

$$\|T(x)-\bar{y}\|^2 \geq \|T(x)-S(x)\|^2 + \|S(x)-\bar{y}\|^2. \tag{4.7}$$

Let $\hat{y} = P_{\mathrm{Fix}(T,P_U)}(S(x))$. From (4.6) and (4.7), we obtain

$$(1-\omega^2)\|x-\bar{y}\|^2 \geq \|T(x)-\bar{y}\|^2 \geq \|T(x)-S(x)\|^2 + \|S(x)-\bar{y}\|^2 \geq \|S(x)-\bar{y}\|^2 \geq \|S(x)-\hat{y}\|^2, \tag{4.8}$$

where the second inequality holds by (4.7), and the last one follows from the definition of orthogonal projection. From (4.8), we conclude, recalling the definitions of $\bar{y}, \hat{y}$, that

$$\mathrm{dist}(S(x), \mathrm{Fix}(T,P_U))^2 \leq (1-\omega^2)\mathrm{dist}(x, \mathrm{Fix}(T,P_U))^2, \tag{4.9}$$

which shows that the second inequality in (4.2) holds. Next, we look at the first one. Let $\tilde{y} = P_{\mathrm{Fix}(T,P_U)}(C(x))$. We have that

$$\|C(x)-\tilde{y}\|^2 \leq \|C(x)-\hat{y}\|^2 \leq \|S(x)-\hat{y}\|^2 \leq \|S(x)-\bar{y}\|^2 \leq (1-\omega^2)\|x-\bar{y}\|^2, \tag{4.10}$$

where the first and the third inequality hold by the definition of the orthogonal projection, the second one from Proposition 3.6(i) and the last one holds by (4.8). Note that the first inequality in (4.2) follows immediately from (4.10) due to the definitions of $\tilde{y}, \bar{y}$. □

**Corollary 4.1.** *Under the assumptions of Proposition 4.1, let $\{z^k\}, \{x^k\}$ be the sequences generated by MAP and CRM respectively, for solving FPP$(T,P_U)$, i.e., $z^{k+1} = S(z^k)$, and $x^{k+1} = C(x^k)$, starting from some $z^0 \in \mathbb{R}^n$ and $x^0 \in U$. Then the scalar sequences $\{a^k\}, \{b^k\}$, defined as $a^k = \mathrm{dist}(z^k, \mathrm{Fix}(T,P_U))$ and $b^k = \mathrm{dist}(x^k, \mathrm{Fix}(T,P_U))$, converge Q-linearly to zero with asymptotic constants bounded above by $\sqrt{1-\omega^2}$, with $\omega$ as in* **EB**.

*Proof.* It follows from (4.2) that, for all $x \in U$,

$$\mathrm{dist}(S(x), \mathrm{Fix}(T,P_U))^2 \leq (1-\omega^2)\mathrm{dist}(x, \mathrm{Fix}(T,P_U))^2, \tag{4.11}$$

and that, for all $z \in U$,

$$\mathrm{dist}(C(x), \mathrm{Fix}(T,P_U))^2 \leq (1-\omega^2)\mathrm{dist}(x, \mathrm{Fix}(T,P_U))^2, \tag{4.12}$$

In view of the definitions of $\{x^k\}$ and $\{z^k\}$ and remembering that both sequences are contained in $U$, by Proposition 3.4 in the case of $\{x^k\}$ and by definition of $S$ in the case of $\{z^k\}$, we obtain from (4.11) and (4.12) that

$$\frac{\mathrm{dist}(z^{k+1}, \mathrm{Fix}(T,P_U))}{\mathrm{dist}(z^k, \mathrm{Fix}(T,P_U))} \leq \sqrt{1-\omega^2}, \tag{4.13}$$

and

$$\frac{\mathrm{dist}(x^{k+1}, \mathrm{Fix}(T,P_U))}{\mathrm{dist}(x^k, \mathrm{Fix}(T,P_U))} \leq \sqrt{1-\omega^2}. \tag{4.14}$$

The result follows from (4.13) and (4.14) immediately. □

Note that the results of Corollary 4.1 do not entail immediately that the sequences $\{x^k\},\{z^k\}$ themselves converge linearly; a sequence $\{y^k\}$ may converge to a point $y \in M \subset \mathbb{R}^n$ in such a way that $\{\operatorname{dist}(y^k,M)\}$ converges linearly to 0, but $\{y^k\}$ itself converges sublinearly. Take for instance $M = \{(s,0) \in \mathbb{R}^2\}$ and $y^k = \left(1/k, 2^{-k}\right)$. This sequence converges to $0 \in M$, $\operatorname{dist}(y^k,M) = 2^{-k}$ converges linearly to 0 with asymptotic constant equal to $1/2$, but the first component of $y^k$ converges to 0 sublinearly. Hence, the same holds for the sequence $\{y^k\}$. The following well known lemma establishes that this situation cannot occur when $\{y^k\}$ is Fejér monotone with respect to $M$, i.e., $\left\|y^{k+1} - y\right\| \le \left\|y^k - y\right\|$ for all $y \in M$.

**Lemma 4.1.** *Consider $M \subset \mathbb{R}^n$, $\{y^k\} \subset \mathbb{R}^n$. Assume that $\{y^k\}$ is Fejér monotone with respect to $M$, and that $\operatorname{dist}(y^k,M)$ converges R-linearly to 0. Then $\{y^k\}$ converges R-linearly to some point $y^* \in M$, with asymptotic constant bounded above by the asymptotic constant of $\{\operatorname{dist}(y^k,M)\}$.*

*Proof.* See, e.g., [12, Lemma 1]. □

Next, we show that the sequences $\{x^k\}$ and $\{z^k\}$ are R-linearly convergent under Assumption **EB** with asymptotic constants bounded by $\sqrt{1-\omega^2}$, where $\omega$ is the **EB** parameter.

**Theorem 4.1.** *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a firmly nonexpansive operator and $U \subset \mathbb{R}^n$ be an affine manifold. Assume that $\operatorname{Fix}(T,P_U) \neq \emptyset$ and condition* **EB** *Holds. Consider the sequences $\{z^k\},\{x^k\}$ generated by MAP and CRM, respectively, for solving $\operatorname{Fix}(T,P_U)$, i.e., $x^{k+1} = S(x^k)$ and $z^{k+1} = C(z^k)$, starting from some $z^0 \in \mathbb{R}^n$ and some $x^0 \in U$. Then both sequences converge R-linearly to points in $\operatorname{Fix}(T,P_U)$ with asymptotic constants bounded above by $\sqrt{1-\omega^2}$, with $\omega$ as in assumption* **EB**.

*Proof.* By Corollary 4.1, both scalar sequences $a^k = \operatorname{dist}(z^k,\operatorname{Fix}(T,P_U))$ and $b^k = \operatorname{dist}(x^k,\operatorname{Fix}(T,P_U))$ are Q-linearly convergent to 0 with asymptotic constant bounded above by $\sqrt{1-\omega^2} < 1$, and hence R-linearly convergent to 0, with the same asymptotic constant. By Corollary 3.1, the sequence $\{x^k\}$ is Fejér monotone with respect to $\operatorname{Fix}(T,P_U)$, and the same holds for the sequence $\{z^k\}$ due to (3.7). By Theorem 3.1, both sequences converge to points in $\operatorname{Fix}(T,P_U)$. Finally, by Lemma 4.1, both sequences converge R-linearly convergent to their limit points in the intersection, with asymptotic constants bounded by $\sqrt{1-\omega^2}$. □

We mention that in [12] we showed that for CFP under EB, CRM achieves an asymptotic constant of linear convergence better than MAP. We have not been able to prove such superiority in the case of FPP. However, the numerical results exhibited in Section 5 strongly suggest that the asymptotic constant of CRM is indeed better than the MAP one. The task of establishing such theoretical superiority is left as an open problem.

## 5. Numerical Experiments

We report here numerical comparisons between CRM and PPM for solving FPP with $p$ firmly nonexpansive operators.

All operators in this section belong to the family studied in Section 2, i.e., they are convex combinations of orthogonal projections onto a finite number of closed and convex sets with nonempty intersection. In view of Proposition 2.4(ii), these operators are ensured to have fixed points. Hence, in view of Proposition 2.6, they are not orthogonal projections themselves.

The construction of the problems is as follows: for each instance, we choose randomly a number $r \in \{3,4,5\}$ ($r$ is the number of convex sets in the convex combination). Then we sample values

$\lambda_1,\ldots,\lambda_r \in (0,1)$ with uniform distribution. We define $\mu_i = \lambda_i/(\sum_{\ell=1}^r)$, and we take the firmly nonexpansive operator $T$ as $T = \sum_{i=1}^r \mu_i P_{\mathscr{E}_i}$, where $\mathscr{E}_i$ is an ellipsoid and $P_{\mathscr{E}_i}$ is the orthogonal projection onto it.

The ellipsoid $\mathscr{E}_i$ is of the form $\mathscr{E}_i := \{x \in \mathbb{R}^n : g_i(x) \leq 0\}$, where $g_i : \mathbb{R}^n \to \mathbb{R}$ is given as $g_i(x) = x^t A_i x + 2(b^i)^t x - \alpha_i$, with $A_i \in \mathbb{R}^{n\times n}$ symmetric positive definite, $b^i \in \mathbb{R}^n$ and $0 < \alpha_i \in \mathbb{R}$. Each matrix $A_i$ is of the form $A_i = \gamma I + B_i^\top B_i$, with $B_i \in \mathbb{R}^{n\times n}$, $\gamma \in \mathbb{R}_{++}$, where $I$ stands for the identity matrix. The matrix $B_i$ is a sparse matrix sampled from the standard normal distribution with sparsity density $p = 2n^{-1}$ and each vector $b^i$ is sampled from the uniform distribution between $[0,1]$. We then choose each $\alpha_i$ so that $\alpha_i > (b^i)^\top A b^i$, which ensures that 0 belongs to every $\mathscr{E}_i$, so that the intersection of the ellipsoids is nonempty. As explained above, this ensures that each instance of FPP has solutions.

In order to compute the projection onto the ellipsoids we use a version of the Alternating Direction Method of Multipliers (ADMM) suited for this purpose; see [30]. The stopping criterion for ADMM is as follows: we stop the ADMM iterative process when the norm of the difference between 2 consecutive ADMM iterates is less than $10^{-8}$. We also fix a maximum number of 10 000 ADMM iterations.

For CRM, we use Pierra's product space reformulation, as explained in Section 1. We implement PPM directly from its definition (see Section 1). The stopping criterion for both CRM and PPM is similar to the one for the ADMM subroutine, but with a different tolerance: the iterative process stops when the norm of the difference between 2 consecutive CRM or PPM iterates is less than $10^{-6}$. The maximum number of iterations is fixed at 50 000 for both algorithms.

The experiments consists of solving, with CRM and PPM, 250 instances of FPP selected as follows. We consider the following values for the dimension $n$: $\{10,30,50,100,200\}$, and for each $n$ we take $p$ firmly nonexpansive operators with $p \in \{10,25,50,100,200\}$. For each of these 25 pairs $(n,p)$, we randomly generate 10 instances of FPP with the above explained procedure.

The initial point $x^0$ is of the form $(\eta,\ldots,\eta) \in \mathbb{R}^n$, with $\eta < 0$ and $|\eta|$ sufficiently large so as to guarante that $x^0$ is far from all the ellipsoids.

The computational experiments were carried out on an Intel Xeon W-2133 3.60GHz with 32GB of RAM running Ubuntu 20.04. We implemented all experiments in Julia programming language v1.6 (see [31]). The codes of our experiments are fully available at: `https://github.com/Mirza-Reza/FPP`

We report in Table 1 the following descriptive statistics for CRM and PPM: mean, maximum (max), minimum (min) and standard deviation (std) for iteration count (it) and CPU time in seconds (CPU (s)). In particular, the ratio of the CPU time (in average for all instances) of PPM with respect to CRM is 7.69, meaning that CRM is, on the average, almost eight times faster that PPM.

We report next similar statistics, but separately for each dimension $n$. Looking at Table 2, we observe that the CPU time for PPM grows linearly with the dimension $n$, while the growth of the CRM CPU time is somewhat higher than linear. As a consequence, the superiority of CRM over PPM, measured in terms of the quotient between the PPM CPU time and the CRP CPU time, is slightly decreasing with $n$: it goes from a ratio of 9.17 for $n = 10$ to a ratio of 7.56 for $n = 200$. This said, it is clear that CRM vastly outperforms PPM in terms of CPU time for all the values of $n$ tested in our experiments.

TABLE 1. Statistics for all instances, reporting number of iterations and CPU time

| Method | | mean | max | min | std |
|---|---|---|---|---|---|
| CRM | it | 144.288 | 554 | 23 | 95.2581 |
| | CPU(s) | 14.6048 | 120.3020 | 0.2729 | 22.4890 |
| PPM | it | 5977.352 | 25000 | 209 | 6385.9388 |
| | CPU(s) | 112.3315 | 1085.9685 | 1.2483 | 190.3078 |

TABLE 2. Statistics for instances of each dimension $n$, reporting number of iterations and CPU time

| Method | | mean | max | min | std |
|---|---|---|---|---|---|
| CRM | it | 141.84 | 512 | 28 | 99.102 847 587 745 93 |
| $n = 10$ | CPU(s) | 2.3247 | 6.8150 | 0.2729 | 1.9756 |
| PPM | it | 6024.54 | 19163 | 209 | 6425.574 393 655 403 |
| $n = 10$ | CPU(s) | 21.3369 | 92.19569 | 1.2483 | 22.4132 |
| CRM | it | 153.5 | 526 | 46 | 92.215 020 468 468 15 |
| $n = 30$ | CPU(s) | 4.6989 | 16.6523 | 0.7607 | 4.1754 |
| PPM | it | 5608.44 | 18353 | 500 | 5956.758 800 421 585 |
| $n = 30$ | CPU(s) | 42.9296 | 174.9737 | 2.9861 | 46.2969 |
| CRM | it | 129.5 | 469 | 23 | 91.705 234 310 806 93 |
| $n = 50$ | CPU(s) | 6.8152 | 17.1668 | 1.0480 | 5.0391 |
| PPM | it | 5288.52 | 24680 | 423 | 5548.505 204 971 876 |
| $n = 50$ | CPU(s) | 53.3709 | 222.7307 | 3.5744 | 55.7054 |
| CRM | it | 152.04 | 399 | 28 | 84.192 389 204 725 63 |
| $n = 100$ | CPU(s) | 15.5937 | 41.2581 | 1.9661 | 12.4246 |
| PPM | it | 7224.42 | 21978 | 540 | 7663.860 453 035 403 |
| $n = 100$ | CPU(s) | 114.4037 | 428.8247 | 6.3108 | 108.4765 |
| CRM | it | 144.56 | 554 | 42 | 105.724 388 860 848 95 |
| n=200 | CPU(s) | 43.5915 | 120.3019 | 5.0157 | 34.3053 |
| PPM | it | 5740.84 | 22378 | 370 | 5948.570 740 472 034 |
| n=200 | CPU(s) | 329.6167 | 1085.9685 | 19.0842 | 315.8783 |

Next, we report in the next table similar statistics, but separately for problems involving $p$ firmly nonexpansive operators, for each value of $p$. Table 3 indicates that both the CRM and the PPM CPU time grow slightly less that linearly in $p$, the number of firmly nonexpansive operators in each instance of FPP, but the growth in both cases seems to become linear for $p \geq 50$. Consistently with this behavior, the ratio between the PPM CPU time and the the CRM CPU time is about 3 for $p = 10, 25$ and about 8 for $p = 50, 100, 200$. Again, for all values of $p$, CRM turns out to be highly better than PPM in terms of CPU time.

TABLE 3. Statistics for instances of FPP problems with $p$ firmly nonexpansive operators, reporting number of iterations and CPU time

| **Method** | | **mean** | **max** | **min** | **std** |
|---|---|---|---|---|---|
| fneCRM | `it` | 91.0 | 263 | 28 | 50.174495513158874 |
| $p = 10$ | `CPU(s)` | 2.8569 | 13.1619 | 0.2729 | 2.8807 |
| PPM | `it` | 1316.68 | 6765 | 209 | 1264.5452849147 |
| $p = 10$ | `CPU(s)` | 13.1578 | 50.8767 | 1.2483 | 11.4271 |
| CRM | `it` | 113.7 | 469 | 36 | 83.5955142337195 |
| $p = 25$ | `CPU(s)` | 6.5062 | 45.2021 | 0.6664 | 8.9416 |
| PPM | `it` | 2865.92 | 14617 | 650 | 2651.0789036918536 |
| $p = 25$ | `CPU(s)` | 34.6541 | 242.2805 | 2.9785 | 47.5093 |
| CRM | `it` | 128.8 | 331 | 23 | 76.92437845052763 |
| $p = 50$ | `CPU(s)` | 10.43880 | 46.8045 | 1.3100 | 11.8677 |
| PPM | `it` | 4949.42 | 25000 | 870 | 5531.401251364793 |
| $p = 50$ | `CPU(s)` | 88.4859 | 602.8599 | 6.5347 | 125.0821 |
| CRM | `it` | 166.28 | 526 | 49 | 91.18882387661331 |
| $p = 100$ | `CPU(s)` | 18.8065 | 70.6532 | 2.4265 | 20.1719 |
| PPM | `it` | 7077.46 | 25000 | 1586 | 4970.777481279966 |
| $p = 100$ | `CPU(s)` | 143.0699 | 729.1966 | 12.0125 | 171.2874 |
| CRM | `it` | 221.66 | 554 | 88 | 105.57890130134903 |
| $p = 200$ | `CPU(s)` | 34.4157 | 120.3019 | 4.7277 | 35.5202 |
| PPM | `it` | 13677.28 | 25000 | 4015 | 6856.3914 |
| $p = 200$ | `CPU(s)` | 282.2900 | 1085.9685 | 31.8832 | 295.7094 |

Finally, we exhibit the performance profile, in the sense of [32], for all the instances. Again, the superiority of CRM with respect to PPM is fully corroborated.

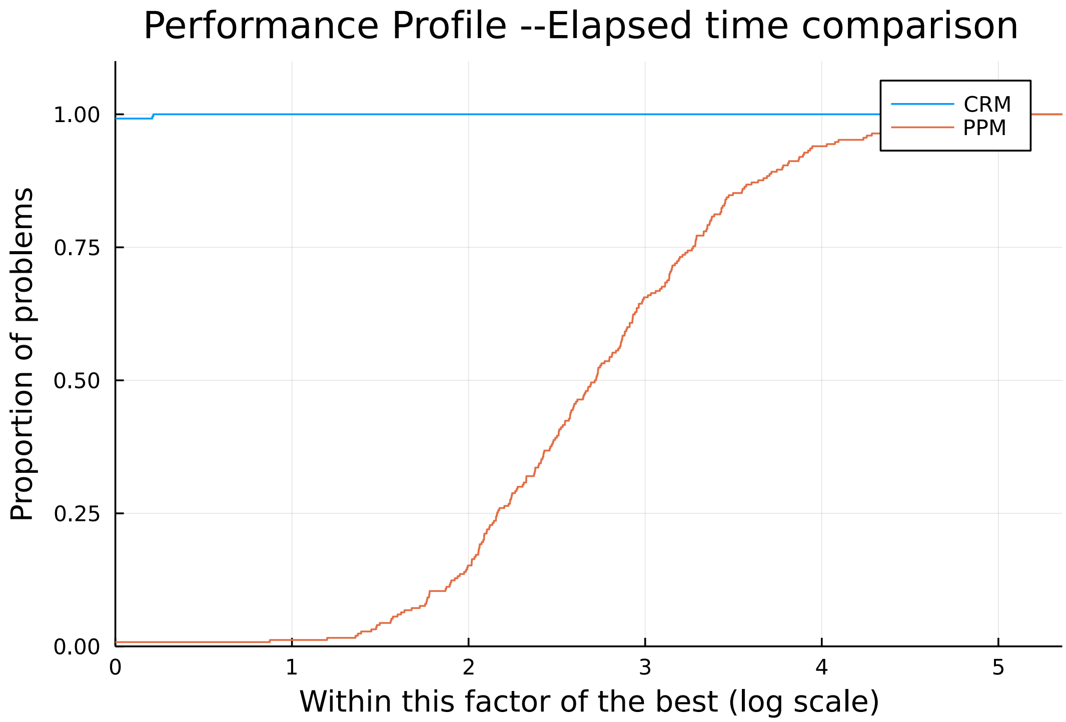


FIGURE 1. Performance profile of experiments with ellipsoidal feasibility – CRM vs PPM

## REFERENCES


[1] Y. Censor, A. Segal, The split common fixed point problem for directed operators, J. Convex Anal. 16 (2009), 587-600.

[2] A. Moudafi, A note on the split common fixed-point problem for quasi nonexpansive operators, Nonlinear Anal. 74 (2011) 4083-4087.

[3] X. Qin, S.Y. Cho, H. Zhou, Common fixed points of a pair of non-expansive mappings with applications to convex feasibility problems, Glasgow Math. J. 52 (2010), 341-252.

[4] J. Zhao, S. He, Alternating Mann iterative algorithms for the split common fixed-point problem of quasi-nonexpansive maps, Fixed Point Theory Appl. 2013 (2013), 288.

[5] S. Kaczmarz, Angenaherte Auflösung von Systemen linearer Gleichungen, Bulletin de l'Académie Polonaise des Sciences et Lettres, A-35 (1937), 355-357.

[6] G. Cimmino, Calcolo approssimato per le soluzione dei sistemi de equazioni lineari, La Ricerca Scientifica II-16 (1938) 326-333.

[7] Y. Censor, S. Zenios, Parallel Optimization: Theory, Algorithms and Applications, Oxford University Press, New York, 1998.

[8] G. Pierra, Decomposition through formalization in a product space, Math. Program. 28 (1983), 96-115.

[9] R. Behling, J.Y. Bello Cruz, L.R. Santos, On the linear convergence of the circumcentered-reflection method, Oper. Res. Let. 46 (2018), 159-162.

[10] R. Behling, J.Y. Bello Cruz, L.R. Santos, Circumcentering the Douglas-Rachford method, Numer. Algo. 78 (2018), 759-776.

[11] G. Araújo, R. Arefidamghani, R. Behling, Y. Bello Cruz, A.N. Iusem, L.-R. Santos, Circumcentering approximate reflexions for solving the convex feasibility problem, Fixed Point Theory Algo. Sci. Eng. 2022 (2022), 1.

[12] R. Arefidamghani, R. Behling, J.Y. Bello Cruz, A.N. Iusem, L.R. Santos, The circumcentered-projection method achieves better rates than alternating projections, Comput. Optim. Appl. 79 (2021), 507-530.

[13] H.H. Bauschke, H. Ouyang, X. Wang, On circumcenters of finite sets in Hilbert spaces, Linear Nonlinear Anal. 4 (2018) 271-295.
[14] H.H. Bauschke, H Ouyang, X. Wang, Circumcentered methods induced by isometries, Vietnam J. Math. 48 (2020), 471–508.
[15] H.H. Bauschke, H. Ouyang, X. Wang, On circumcenter mappings induced by nonexpansive operators, Pure Appl. Funct. Anal. 6 (2021), 257-288.
[16] H.H. Bauschke, H. Ouyang, X. Wang, On the linear convergence of circumcentered isometry methodsm, Numer. Algo. 87 (2021), 268-297.
[17] H.H. Bauschke, H. Ouyang, X. Wang, Best Approximation mappings in Hilbert spaces, (2021), https://doi.org/10.1007/s10107-021-01718-y.
[18] R. Behling, J.Y. Bello Cruz, L.R. Santos, The block-wise circumcentered- reflection method, Comput. Optim. Appl. 76 (2020), 675-699.
[19] R. Behling, J.Y. Bello Cruz, L.R. Santos, On the Circumcentered-reflection method for the convex feasibility problem, Numer. Algo. 86 (2021), 1475–1494.
[20] N. Dizon, J. Hogan, S.B. Lindstrom, Circumcentering reflection methods for nonconvex feasibility problems, arXiv 1910.04384, (2019).
[21] N. Dizon, J. Hogan, S.B. Lindstrom, Centering projection methods for wavelet feasibility problem, arXiv: 2005.05687, (2020)
[22] H. Ouyang, Finite convergence of locally proper circumcentered methods, arXiv: 2011.13521 [math], (2020).
[23] A.R. De Pierro, A.N. Iusem, A parallel projections method for finding a common point of a family of convex sets, Pesquisa Operacional 5 (1985), 1-20.
[24] A.N. Iusem, A.R. De Pierro, On the set of weighted least squares solutions of systems of linear inequalities, Comment. Math. Univ. Carolin. 25 (1984) 667-678.
[25] R.T. Rockafellar,R.J-B. Wets, Variational Analysis, Springer, Berlin, 1998.
[26] S.M. Robinson, Stability theory for systems of inequalities, Part II: Differentiable nonlinear systems, SIAM J. Numer. Anal. 13 (1976), 497-513.
[27] A.J. Hoffman, On approximate solutions of systems of linear equations, J. Res. Nat, Bureau Standards 49 (1952), 263-265.
[28] R. Behling, A.N. Iusem, The effect of calmness on the solution of systems of nonlinear equations, Math. Program. 137 (2013), 155-165.
[29] C. Kanzow, N. Yamashita, M. Fukushima, Levenberg-Marquardt methods with strong local convergence properties for solving nonlinear equations with convex constraints, J. Comput. Appl. Math. 172 (2004), 375-397.
[30] Z. Jia, X. Cai, D. Han, Comparison of several fast algorithms for projection onto an ellipsoid, J. Comput. Appl. Math. 319 (2017), 320–337.
[31] J. Bezanson, A. Edelman, S. Karpinski, V.B. Shah, Julia: A fresh approach to numerical computing, SIAM Rev. 59 (2017), 65–98.
[32] E.D. Dolan, J.J. Moré, Benchmarking optimization software with performance profiles, Math. Program. 91 (2002), 201–213.