J. Appl. Numer. Optim. 5 (2023), No. 2, pp. 287-296
Available online at http://jano.biemdas.com
https://doi.org/10.23952/jano.5.2023.2.07

A NOTE ON THE SUBDIFFERENTIAL OF CONVEX MULTI-COMPOSITE FUNCTIONALS

MOHAMED LAGHDIR

Department of Mathematics, Faculty of Sciences, Chouaïb Doukkali University, El Jadida, Morocco

Abstract. This paper studies the general formulae for the subdifferential of a convex multi-composite function. Subdifferentiability is characterized under an adequate constraint qualification. This formulae is applied to establish optimality conditions for constrained convex minmax location problems.

Keywords. Multi-composed programming; Minmax location problems; Optimality conditions; Subdifferential calculus.

2020 Mathematics Subject Classification. 49J52, 49K27.

1. Introduction

In nondifferentiable convex optimization theory, it is well-known that, in addition to conjugate duality and the saddle point approach, the classical sum and composition rules of subdifferential calculus can be used to derive optimality conditions for a constrained convex composed optimization problems; see, e.g., [1, 2, 3, 4, 5]. In this paper, we are interested in developing a more general calculus rule of subdifferential of multi-composed convex functions. The origin of interest in such calculus rules comes from the recent contribution of Wanka and Wilfer [6] that introduced and examined the optimality conditions of the so-called multi-composed convex optimization problems via conjugate duality approach in 2016. For more details regarding this new branch of convex optimization, we refer to a recent book on multi-composed programming [7].

To attain our goal, a recent result due to Laghdir et al. [8] will be exploited in the direction to reduce the calculus of subdifferential of multi-composed convex function to that of sums of convex functions. It is important to note that such a calculus rule of subdifferentials needs a regularity condition or the so-called constraint qualification (like the Moreau-Rockafellar constraint qualification or the one of Attouch-Brézis [2, 5, 9], etc). So, an appropriate regularity condition, like Moreau-Rockafellar constraint qualification, is employed in this paper. We give an application to study optimality conditions for a constrained convex minmax location problems.

This paper is organized as follows. The remainder of this section is addressed to present some basic notations and definitions, used throughout this paper. In Section 2, we give some definitions and tools, which are necessary later. The main result of this paper is provided in Section 3. Finally, Section 4 is devoted to deriving optimality conditions for a constrained convex minmax location problems.

E-mail address: laghdirm@yahoo.fr.
Received March 29, 2023; Accepted May 9, 2023.

2. PRELIMINARIES

Let X_1 and X_2 be two real Banach spaces whose topological dual spaces are X_1^* and X_2^* paired in duality by $\langle .,. \rangle$ and equipped respectively with the weak-star topology $w(X_1^*, X_1)$ and $w(X_2^*, X_2)$. We denote by $\operatorname{int} C$ the topological interior of subset $C \subseteq X_1$ and we assume that X_2 is equipped with a convex cone $K \subseteq X_2$, which induces a partial preorder $\leqq_K$ by

$$\forall\, a, b \in X_2,\ a \leqq_K b \iff b - a \in K.$$

The positive polar cone of K is defined by

$$K^* := \{y^* \in Y^* : \langle y^*, y \rangle \geq 0,\ \forall\, y \in K\}.$$

To X_2, we attach an abstract maximal element denoted by $+\infty_{X_2}$ such that

$$y \leqq_K +\infty_{X_2}, \quad y + (+\infty_{X_2}) := (+\infty_{X_2}) + y := +\infty_{X_2},\ \forall y \in X_2 \cup \{+\infty_{X_2}\}$$

and

$$y^*(+\infty_{X_2}) := +\infty,\ t.(+\infty_{X_2}) := +\infty_{X_2},\ \forall y^* \in K^*,\ \forall t \geq 0.$$

Let $f : X_1 \to \mathbb{R} \cup \{+\infty\}$ be a function. Then f is said to be proper if its effective domain $\operatorname{dom} f := \{x \in X_1 : f(x) \in \mathbb{R}\} \neq \emptyset$, and it is said to be convex if $f(tx + (1-t)y) \leq t f(x) + (1-t) f(y)$ for all $x, y \in X_1$ and all $t \in [0,1]$. Moreover, function f is said to be lower semicontinuous if $\liminf_{y \to x} f(y) \geq f(x)$ for all $x \in X_1$. The conjugate function of f is defined by

$$f^* : X_1^* \to \mathbb{R} \cup \{\pm\infty\},\ f^*(x^*) := \sup_{x \in X_1} \{\langle x^*, x \rangle - f(x)\}.$$

In view of the definition of f^*, one has the so-called Young-Fenchel inequality

$$f^*(x^*) + f(x) \geq \langle x^*, x \rangle, \quad \forall\, (x, x^*) \in X \times X^*.$$

The subdifferential of f at $\bar{x} \in X_1$ is defined by

$$\partial f(\bar{x}) := \{x^* \in X_1^* : f(x) \geq f(\bar{x}) + \langle x^*, x - \bar{x} \rangle, \quad \forall x \in X\}.$$

The indicator function of a nonempty subset $C \subseteq X_1$ is defined by

$$\delta_C(x) := \begin{cases} 0, \text{ if } x \in C, \\ +\infty, \text{ otherwise} \end{cases}$$

The normal cone of C at $\bar{x}$ is defined by

$$N_C(\bar{x}) := \partial \delta_C(\bar{x}) = \{x_1^* \in X^* : \langle x^*, x - \bar{x} \rangle \leq 0,\ \forall x \in C\}.$$

Let X_3 be another real Banach space partially ordered by a nonempty and convex cone $Q \subseteq X_3$. The mapping $g : X_2 \to X_3 \cup \{+\infty_{X_3}\}$ is called proper if its effective domain $\operatorname{dom} g := \{y \in X_2 : g(y) \in X_3\} \neq \emptyset$ and it is called Q-epi closed (resp. Q-convex) if its epigraph

$$\operatorname{epi} g := \{(y, z) \in X_2 \times X_3 : g(y) \leqq_Q z\}$$

is a closed (resp. convex) subset of $X_2 \times X_3$. The mapping g is said to be (K, Q)-nondecreasing on $A \subseteq X_2$ if

$$\forall\, x, y \in A,\ x \leqq_K y \implies g(x) \leqq_Q g(y).$$

In particular, we call it K-nondecreasing on $A \subseteq X_2$ when $X_3 = \mathbb{R}$ and $Q = \mathbb{R}_+$. Let $h : X_1 \to X_2 \cup \{+\infty_{X_2}\}$ be another vector mapping. The composed vector mapping $g \circ h : X_1 \longrightarrow X_3 \cup \{+\infty_{X_3}\}$ is defined by

$$(g \circ h)(x) := \begin{cases} g(h(x)), & \text{if } x \in \operatorname{dom} h, \\ +\infty_{X_3}, & \text{otherwise.} \end{cases}$$

Let us note that if $g : X_2 \to X_3 \cup \{+\infty_{X_3}\}$ is Q-convex and (K,Q)-nondecreasing on $\operatorname{dom} g$ and h is K-convex with $h(\operatorname{dom} h) \subseteq \operatorname{dom} g$, then $g \circ h$ is Q-convex.

3. Main Results

This section is devoted to the calculus rule of the subdifferential for multi-composition function.

Let X and X_i be real Banach spaces, $i = 0, ..., m$ $(m \geq 2)$. Moreover, we assume that X_i is partially ordered by the nonempty convex cone $K_i \subseteq X_i$, $i = 0, ..., m$. Let $f + \varphi \circ \psi + g \circ h_1 \circ h_2 \circ ... \circ h_m : X \to \mathbb{R} \cup \{+\infty\}$ be a multi-composed function defined on X, where

- $f : X \to \mathbb{R} \cup \{+\infty\}$ is proper, convex, and lower semicontinuous,
- $\varphi : X_0 \to \mathbb{R} \cup \{+\infty\}$ is proper, convex, K_0-nondecreasing on $\operatorname{dom}\varphi$, and lower semicontinuous,
- $\psi : X \to X_0 \cup \{+\infty_{X_0}\}$ is proper, K_0-convex, K_0-epi closed, and $\psi(\operatorname{dom}\psi) \subseteq \operatorname{dom}\varphi$,
- $g : X_1 \to \mathbb{R} \cup \{+\infty\}$ is proper, convex, K_1-nondecreasing on $\operatorname{dom} g$, and lower semicontinuous,
- $h_1 : X_2 \to X_1 \cup \{+\infty_{X_1}\}$ is proper, K_1-convex, (K_2, K_1)-nondecreasing on $\operatorname{dom} h_1$, K_1-epi closed, and $h_1(\operatorname{dom} h_1) \subseteq \operatorname{dom} g$,
- $h_i : X_{i+1} \to X_i \cup \{+\infty_{X_i}\}$ is proper, K_i-convex, (K_{i+1}, K_i)-nondecreasing on $\operatorname{dom} h_i$, K_i-epi closed, and $h_i(\operatorname{dom} h_i) \subseteq \operatorname{dom} h_{i-1}$, $i = 2, ..., m-1$,
- $h_m : X \to X_m \cup \{+\infty_{X_m}\}$ is proper, K_m-convex, K_m-epi closed, and $h_m(\operatorname{dom} h_m) \subseteq \operatorname{dom} h_{m-1}$,
- $\operatorname{dom} f \cap \psi^{-1}(\operatorname{dom}\varphi) \cap \operatorname{dom}\psi \cap (h_m^{-1} \circ h_{m-1}^{-1} \circ ... \circ h_1^{-1})(\operatorname{dom} g) \cap \operatorname{dom} h_m \neq \emptyset$,
- $\varphi(+\infty_{X_0}) = +\infty$, $g(+\infty_{X_1}) = +\infty$, $h_i(+\infty_{X_{i+1}}) = +\infty_{X_i}$, $i = 1, ..., m-1$.

Remark 3.1. Under the above assumptions, the multi-composed function $f + \varphi \circ \psi + g \circ h_1 \circ h_2 \circ ... \circ h_m : X \to \mathbb{R} \cup \{+\infty\}$ is convex.

To achieve the goal of this section, we consider the following auxiliary functions

$$F: \begin{array}{ccc} X \times \prod_{k=0}^m X_k & \to & \mathbb{R} \cup \{+\infty\} \\ (x, x_0, x_1, ..., x_m) & \mapsto & f(x) \end{array}, \quad \Phi: \begin{array}{ccc} X \times \prod_{k=0}^m X_k & \to & \mathbb{R} \cup \{+\infty\} \\ (x, x_0, x_1, ..., x_m) & \mapsto & \varphi(x_0) \end{array}$$

$$\Psi: \begin{array}{ccc} X \times \prod_{k=0}^m X_k & \to & \mathbb{R} \cup \{+\infty\} \\ (x, x_0, x_1, ..., x_m) & \mapsto & \delta_{\operatorname{epi}\psi}(x, x_0) \end{array}, \quad G: \begin{array}{ccc} X \times \prod_{k=0}^m X_k & \to & \mathbb{R} \cup \{+\infty\} \\ (x, x_0, x_1, ..., x_m) & \mapsto & g(x_1) \end{array}$$

$$H_i: \begin{array}{ccc} X \times \prod_{k=0}^m X_k & \to & \mathbb{R} \cup \{+\infty\} \\ (x, x_0, x_1, ..., x_m) & \mapsto & \delta_{\operatorname{epi} h_i}(x_{i+1}, x_i), \end{array} \quad (i = 1, ..., m-1)$$

$$H_m: \begin{array}{ccc} X \times \prod_{k=0}^m X_k & \to & \mathbb{R} \cup \{+\infty\} \\ (x, x_0, x_1, ..., x_m) & \mapsto & \delta_{\operatorname{epi} h_m}(x, x_m) \end{array}$$

and also introduce the following regularity condition

$$(\mathscr{RC}) \left| \begin{array}{l} \exists\, (x, x_1, ..., x_{m-1}) \in X \times \prod_{k=1}^{m-1} X_m \text{ such that } x \in \operatorname{dom} f \cap \operatorname{dom}\psi \cap \operatorname{dom} h_m, \\ \psi(x) \in \operatorname{int} \operatorname{dom}\varphi,\ x_1 \in \operatorname{dom} g,\ (x_{i+1}, x_i) \in \operatorname{int} \operatorname{epi} h_i,\ i = 1, ..., m-2, \text{ and} \\ (h_m(x), x_{m-1}) \in \operatorname{int} \operatorname{epi} h_{m-1}. \end{array} \right.$$

Remark 3.2. (1) Let us note that F, Φ, Ψ, G, and H_i, $i = 1,...,m$, are all convex and lower semicontinuous.

(2) In the absence of g and $h_1,...,h_m$, $(\mathscr{RC})$ becomes the classical Moreau-Rockafellar constraint qualification

$$(\mathscr{MR})\Big|\ \exists\, x \in \operatorname{dom} f \cap \operatorname{dom}\psi \text{ such } \psi(x) \in \operatorname{int}\operatorname{dom}\varphi.$$

The following lemmas play a crucial role in our investigation.

Lemma 3.1 ([8])**.** *Assume that* $\bar{x} \in \operatorname{dom} f \cap \psi^{-1}(\operatorname{dom}\varphi) \cap \operatorname{dom}\psi \cap (h_m^{-1} \circ h_{m-1}^{-1} \circ ... \circ h_1^{-1})(\operatorname{dom} g) \cap \operatorname{dom} h_m$, $\bar{x}_m := h_m(\bar{x})$, $\bar{x}_{m-1} := h_{m-1}(\bar{x}_m)$, ..., $\bar{x}_1 := h_1(\bar{x}_2)$ *and* $\bar{x}_0 := \psi(\bar{x})$. *Then*

$$x^* \in \partial(f + \varphi \circ \psi + g \circ h_1 \circ h_2 \circ ... \circ h_m)(\bar{x})$$
$$\iff (x^*, 0, 0, ..., 0) \in \partial(F + \Phi + \Psi + G + \sum_{i=1}^{m} H_i)(\bar{x}, \bar{x}_0, \bar{x}_1, ..., \bar{x}_m).$$

Lemma 3.2. *Let* $\bar{x} \in \operatorname{dom} f \cap \psi^{-1}(\operatorname{dom}\varphi) \cap \operatorname{dom}\psi \cap (h_m^{-1} \circ h_{m-1}^{-1} \circ ... \circ h_1^{-1})(\operatorname{dom} g) \cap \operatorname{dom} h_m$, $\bar{x}_m = h_m(\bar{x})$, $\bar{x}_{m-1} = h_{m-1}(\bar{x}_m), ..., \bar{x}_1 = h_1(\bar{x}_2)$ *and* $\bar{x}_0 = \psi(\bar{x})$. *Under condition* $(\mathscr{RC})$, *one has*

$$\partial(F + \Phi + \Psi + G + \sum_{i=1}^{m} H_i)(\bar{x}, \bar{x}_0, \bar{x}_1, ..., \bar{x}_m)$$
$$= \partial(F + \Psi + G + H_m)(\bar{x}, \bar{x}_0, \bar{x}_1, ..., \bar{x}_m)$$
$$+ \partial\Phi(\bar{x}, \bar{x}_0, \bar{x}_1, ..., \bar{x}_m) + \sum_{i=1}^{m-1} \partial H_i(\bar{x}, \bar{x}_0, \bar{x}_1, ..., \bar{x}_m).$$

Proof. Observe that $(\mathscr{RC})$ implies the existence of $(x, \psi(x), x_1, ..., x_{m-1}, h_m(x)) \in X \times \prod_{k=0}^m X_k$, which is in $\operatorname{dom} F = \operatorname{dom} f \times \prod_{k=0}^m X_k$, $\operatorname{dom}\Psi = \Lambda_{(X,X_0)}^{-1}(\operatorname{epi}\psi)$, $\operatorname{dom} G = X \times X_0 \times \operatorname{dom} g \times \prod_{k=2}^m X_k$, $\operatorname{dom} H_m = \Lambda_{(X,X_m)}^{-1}(\operatorname{epi} h_m)$, $\operatorname{int}\operatorname{dom}\Phi = X \times \operatorname{int}\operatorname{dom}\varphi \times \prod_{k=1}^m X_k$, and $\Lambda_{(X_{i+1},X_i)}^{-1}(\operatorname{int}\operatorname{epi} h_i) \subseteq \operatorname{int}\operatorname{dom} H_i = \operatorname{int}\Lambda_{(X_{i+1},X_i)}^{-1}(\operatorname{epi} h_i)$, $i = 1, ..., m-1$, where $\Lambda_{(X,X_0)}$, $\Lambda_{(X,X_m)}$, and $\Lambda_{(X_{i+1},X_i)}$, $i = 1, ..., m-1$, are continuous mappings defined by

$$\begin{array}{llll} \Lambda_{(X_{i+1},X_i)}: & X \times \prod_{k=0}^m X_k & \to & X_{i+1} \times X_i \ \ (i = 1, ..., m-1) \\ & (x, x_0, x_1, ..., x_m) & \mapsto & (x_{i+1}, x_i), \end{array}$$

$$\begin{array}{llll} \Lambda_{(X,X_0)}: & X \times \prod_{k=0}^m X_k & \to & X \times X_0 \\ & (x, x_0, x_1, ..., x_m) & \mapsto & (x, x_0), \end{array}$$

and

$$\begin{array}{llll} \Lambda_{(X,X_m)}: & X \times \prod_{k=0}^m X_k & \to & X \times X_m \\ & (x, x_0, x_1, ..., x_m) & \mapsto & (x, x_m). \end{array}$$

Thus, by [1, Theorem V.2], we obtain the desired statement. □

Lemma 3.3. *(1) Let* $(\bar{x}, \bar{x}_0, \bar{x}_1, ..., \bar{x}_m) \in \operatorname{dom} F \cap \operatorname{dom}\Psi \cap \operatorname{dom} G \cap \operatorname{dom} H_m$. *Then*

$$(x^*, x_0^*, x_1^*, ..., x_m^*) \in \partial(F + \Psi + G + H_m)(\bar{x}, \bar{x}_0, \bar{x}_1, ..., \bar{x}_m)$$
$$\iff \begin{cases} x^* \in \partial(f - x_0^* \circ \psi - x_m^* \circ h_m)(\bar{x}), \\ -x_0^* \in K_0^*,\ \langle -x_0^*, \bar{x}_0 - \psi(\bar{x})\rangle = 0, \\ x_1^* \in \partial g(\bar{x}_1),\ x_k^* = 0,\ k = 2, ..., m-1, \\ -x_m^* \in K_m^*,\ \langle -x_m^*, \bar{x}_m - h_m(\bar{x})\rangle = 0. \end{cases}$$

(2) Let $(\bar{x},\bar{x}_0,\bar{x}_1,\dots,\bar{x}_m)\in \operatorname{dom}\Phi$. *Then*

$$\partial\Phi(\bar{x},\bar{x}_0,\bar{x}_1,\dots,\bar{x}_m)=\{0\}\times\partial\varphi(\bar{x}_0)\times\{0\}\times\dots\times\{0\}.$$

(3) Let $i\in\{1,\dots,m-1\}$ *and* $(\bar{x},\bar{x}_0,\bar{x}_1,\dots,\bar{x}_m)\in \operatorname{dom}H_i$. *Then*

$$(x^*,x_0^*,x_1^*,\dots,x_m^*)\in\partial H_i(\bar{x},\bar{x}_0,\bar{x}_1,\dots,\bar{x}_m)$$

$$\iff \begin{cases} x^*=0,\ x_k^*=0,\ k\in\{0,\dots,m\}\backslash\{i,i+1\},\\ -x_i^*\in K_i^*,\ \langle -x_i^*,\bar{x}_i-h_i(\bar{x}_{i+1})\rangle=0,\\ x_{i+1}^*\in\partial(-x_i^*\circ h_i)(\bar{x}_{i+1}).\end{cases}$$

Proof. (1) Let $(\bar{x},\bar{x}_0,\bar{x}_1,\dots,\bar{x}_m)\in \operatorname{dom}F\cap\operatorname{dom}\Psi\cap\operatorname{dom}G\cap\operatorname{dom}H_m$. Then, by a simple computation, one can easily check that, for all $(x^*,x_0^*,x_1^*,\dots,x_m^*)\in X^*\times\prod_{k=0}^m X_k^*$, $(F+\Psi+G+H_m)^*(x^*,x_0^*,x_1^*,\dots,x_m^*)=(f-x_0^*\circ\psi-x_m^*\circ h_m)^*(x^*)+g^*(x_1^*)$, if $-x_0^*\in K_0^*$, $x_k^*=0$, $k=2,\dots,m-1$, $-x_m^*\in K_m^*$, and $(F+\Psi+G+H_m)^*(x^*,x_0^*,x_1^*,\dots,x_m^*)=+\infty$, otherwise. Therefore, it follows that

$$(x^*,x_0^*,x_1^*,\dots,x_m^*)\in\partial(F+\Psi+G+H_m)(\bar{x},\bar{x}_0,\bar{x}_1,\dots,\bar{x}_m)$$

$$\iff\begin{cases}[(f-x_0^*\circ\psi-x_m^*\circ h_m)^*(x^*)+(f-x_0^*\circ\psi-x_m^*\circ h_m)(\bar{x})-\langle x^*,\bar{x}\rangle]\\ +[\langle -x_0^*,\bar{x}_0-\psi(\bar{x})\rangle]+[g^*(x_1^*)+g(\bar{x}_1)-\langle x_1^*,\bar{x}_1\rangle]+[\langle -x_m^*,\bar{x}_m-h_m(\bar{x})\rangle]=0,\\ -x_0^*\in K_0^*,\ x_k^*=0,\ k=2,\dots,m-1,\ -x_m^*\in K_m^*.\end{cases}\tag{3.1}$$

By observing that $\langle -x_0^*,\bar{x}_0-\psi(\bar{x})\rangle\geq 0$ and $\langle -x_m^*,\bar{x}_m-h_m(\bar{x})\rangle\geq 0$ and using the Young-Fenchel inequality, we deduce that

$$(3.1)\iff\begin{cases}x^*\in\partial(f-x_0^*\circ\psi-x_m^*\circ h_m)(\bar{x}),\\ -x_0^*\in K_0^*,\ \langle -x_0^*,\bar{x}_0-\psi(\bar{x})\rangle=0,\\ x_1^*\in\partial g(\bar{x}_1),\ x_k^*=0,\ k=2,\dots,m-1,\\ -x_m^*\in K_m^*,\ \langle -x_m^*,\bar{x}_m-h_m(\bar{x})\rangle=0.\end{cases}$$

This completes the proof. For (2) and (3), we refer the reader to [8, Lemma 4.5]. □

Now, we state the main result of this work.

Theorem 3.1. *Let* $\bar{x}\in\operatorname{dom}f\cap\psi^{-1}(\operatorname{dom}\varphi)\cap\operatorname{dom}\psi\cap(h_m^{-1}\circ h_{m-1}^{-1}\circ\dots\circ h_1^{-1})(\operatorname{dom}g)\cap\operatorname{dom}h_m$, $\bar{x}_m=h_m(\bar{x})$, $\bar{x}_{m-1}=h_{m-1}(\bar{x}_m),\dots,\bar{x}_1=h_1(\bar{x}_2)$, *and* $\bar{x}_0=\psi(\bar{x})$. *Under condition* $(\mathscr{RC})$,

$$\partial(f+\varphi\circ\psi+g\circ h_1\circ h_2\circ\dots\circ h_m)(\bar{x})$$

$$=\bigcup_{\substack{x_0^*\in K_0^*\cap\partial\varphi(\bar{x}_0),\ z_0^*\in K_1^*\cap\partial g(\bar{x}_1),\\ z_i^*\in K_{i+1}^*\cap\partial(z_{i-1}^*\circ h_i)(\bar{x}_{i+1}),\ i=1,\dots,m-1}}\partial(f+x_0^*\circ\psi+z_{m-1}^*\circ h_m)(\bar{x}).$$

Proof. Clearly, by Lemma 3.1 and Lemma 3.2, one sees that $x^*\in\partial(f+\varphi\circ\psi+g\circ h_1\circ h_2\circ\dots\circ h_m)(\bar{x})$ if and only if

$$(x^*,0,0,\dots,0)\in\partial(F+\Psi+G+H_m)(\bar{x},\bar{x}_0,\bar{x}_1,\dots,\bar{x}_m)$$
$$+\partial\Phi(\bar{x},\bar{x}_0,\bar{x}_1,\dots,\bar{x}_m)+\sum_{i=1}^{m-1}\partial H_i(\bar{x},\bar{x}_0,\bar{x}_1,\dots,\bar{x}_m),$$

which yields that there exist

$$\begin{cases} (u^*, u_0^*, u_1^*, ..., u_m^*) \in \partial(F + \Psi + G + H_m)(\bar{x}, \bar{x}_0, \bar{x}_1, ..., \bar{x}_m), \\ (v^*, v_0^*, v_1^*, ..., v_m^*) \in \partial\Phi(\bar{x}, \bar{x}_0, \bar{x}_1, ..., \bar{x}_m), \\ (w^{i*}, w_0^{i*}, w_1^{i*}, ..., w_m^{i*}) \in \partial H_i(\bar{x}, \bar{x}_0, \bar{x}_1, ..., \bar{x}_m),\ i = 1, ..., m-1, \end{cases}$$

such that

$$\begin{cases} u^* + v^* + \sum_{i=1}^{m-1} w^{i*} = x^*, \\ u_0^* + v_0^* + \sum_{i=1}^{m-1} w_0^{i*} = 0, \\ u_1^* + v_1^* + \sum_{i=1}^{m-1} w_1^{i*} = 0, \\ u_2^* + v_2^* + \sum_{i=1}^{m-1} w_2^{i*} = 0, \\ \vdots \\ u_m^* + v_m^* + \sum_{i=1}^{m-1} w_m^{i*} = 0. \end{cases}$$

By virtue of Lemma 3.3, we assert that $x^* \in \partial(f + \varphi \circ \psi + g \circ h_1 \circ h_2 \circ ... \circ h_m)(\bar{x})$ if and only if there exist

$$\begin{cases} u^* \in \partial(f - u_0^* \circ \psi - u_m^* \circ h_m)(\bar{x}),\ -u_0^* \in K_0^*,\ u_1^* \in \partial g(\bar{x}_1),\ -u_m^* \in K_m^*, \\ v_0^* \in \partial\varphi(\bar{x}_0),\ -w_i^{i*} \in K_i^*,\ w_{i+1}^{i*} \in \partial(-w_i^{i*} \circ h_i)(\bar{x}_{i+1}),\ i = 1, ..., m-1, \end{cases} \tag{3.2}$$

such that

$$\begin{cases} u^* = x^*, \\ v_0^* + u_0^* = 0, \\ u_1^* + w_1^{1*} = 0, \\ w_2^{1*} + w_2^{2*} = 0, \\ \vdots \\ w_{m-1}^{(m-2)*} + w_{m-1}^{(m-1)*} = 0, \\ w_m^{(m-1)*} + u_m^* = 0. \end{cases} \tag{3.3}$$

By taking $x_0^* := -u_0^*$ and $w_1^{0*} := u_1^*$, one concludes that

$$(3.2) \text{ and } (3.3) \iff \begin{cases} x^* \in \partial(f + x_0^* \circ \psi + w_m^{(m-1)*} \circ h_m)(\bar{x}), \\ x_0^* \in K_0^* \cap \partial\varphi(\bar{y}_0), \\ w_1^{0*} \in K_1^* \cap \partial g(\bar{x}_1), \\ w_{i+1}^{i*} \in K_{i+1}^* \cap \partial(w_i^{(i-1)*} \circ h_i)(\bar{x}_{i+1}),\ i = 1, ..., m-1. \end{cases}$$

We deduce the desired statement by taking $z_0^* := w_1^{0*}$ and $z_i^* := w_{i+1}^{i*}$, $i = 1,...,m-1$. Therefore, the proof is complete. □

By taking $K_i = X_i$, $h_i \equiv 0$ $(i \in \{1,2...m\})$ and $g \equiv 0$ in Theorem 3.1, we obtain the classical sum rule as well as the classical composition rule in convex subdifferential calculus; see, e.g., [2].

Corollary 3.1. *Let* $\bar{x} \in \operatorname{dom} f \cap \psi^{-1}(\operatorname{dom}\varphi) \cap \operatorname{dom}\psi$ *and* $\bar{x}_0 = \psi(\bar{x})$. *If condition* $(\mathscr{RC})$ *holds, then*

$$\partial(f+\varphi\circ\psi)(\bar{x}) = \bigcup_{x_0^*\in K_0^*\cap\partial\varphi(\bar{x}_0)} \partial(f+x_0^*\circ\psi)(\bar{x}).$$

4. Applications

In this section, we apply Theorem 3.1 to the optimality conditions of a constrained convex minmax location problem with perturbed minimal time functions and set-up costs. Such problems were recently investigated via conjugate duality approach; see [10] for more details.

Let X and Y be two real Banach spaces, where Y is partially ordered by a nonempty closed convex cone K. We consider now a constrained convex minmax location problems with perturbed minimal time functions and set-up costs denoted by $(\mathscr{MLP})$ and defined as follows

$$(\mathscr{MLP}) \qquad \inf_{\substack{x\in S\\ h(x)\leqq_K 0}} \max_{1\le i\le n} \left\{ l_i(\mathscr{T}^{C_i}_{\Omega_i,f_i}(x)) + a_i \right\},$$

where

- $a_1,...,a_n$ are positive set-up costs,
- $S, C_i \subseteq X$ are nonempty, closed and convex with $0 \in \operatorname{int} C_i$, $i = 1,...,n$,
- $\Omega_i \subseteq X$ is nonempty, convex and compact, $i = 1,...,n$,
- $h : X \to Y \cup \{+\infty_Y\}$ is proper, K-convex and K-epi closed,
- $\mathscr{T}^{C_i}_{\Omega_i,f_i} : X \to \mathbb{R}\cup\{+\infty\}$ is a perturbed minimal time function defined by

$$\mathscr{T}^{C_i}_{\Omega_i,f_i}(x) := \inf_{y\in X, z\in\Omega} \{\gamma_{C_i}(x-y-z) + f_i(y)\},$$

  with $f_i : X \to \mathbb{R}\cup\{+\infty\}$ is proper, positive, convex and lower semicontinuous function and $\gamma_{C_i} : X \to \mathbb{R}\cup\{+\infty\}$ is the well-known Minkowski functional of C_i defined by

$$\gamma_{C_i}(x) := \begin{cases} \inf\{\lambda > 0 : x \in \lambda C_i\}, \text{ if } \{\lambda > 0 : x \in \lambda C_i\} \neq \emptyset, \\ +\infty, \text{ otherwise}, \ i = 1,...,n, \end{cases}$$

- $l_i : \mathbb{R} \to \mathbb{R}\cup\{+\infty\}$ with $l_i(x) \ge 0$, if $x \ge 0$, $l_i(x) = +\infty$, otherwise, is a proper, convex, lower semicontinuous and nondecreasing function on $\mathbb{R}_+$, $i = 1,...,n$,
- $S \cap h^{-1}(-K) \cap \operatorname{dom} h \neq \emptyset$.

In order to obtain optimality conditions for problem $(\mathscr{MLP})$, we transform it as an unconstrained convex multi-composed optimization problem by setting $X_0 = Y$, $K_0 = K$, $X_1 = X_2 := \mathbb{R}^n$, $K_1 = K_2 := \mathbb{R}^n_+$ and introducing the following auxillary functions

- $g : \mathbb{R}^n \to \mathbb{R}$ defined by

$$g(x_1,...,x_n) := \max_{1\le i\le n} \{|x_i^+|\},\ x_i^+ := \max\{0,x_i\},\ i = 1,...,n,$$

- $h_1 : \mathbb{R}^n \to \mathbb{R}^n \cup \{+\infty_{\mathbb{R}^n}\}$ defined by

$$h_1(x_1,...,x_n) := \begin{cases} (l_1(x_1)+a_1,...,l_n(x_n)+a_n), \text{ if } (x_1,...,x_n) \in \mathbb{R}^n_+, \\ +\infty_{\mathbb{R}^n}, \text{ otherwise}, \end{cases}$$

- $h_2 : X \to \mathbb{R}^n$ defined by

$$h_2(x) := \left(\mathscr{T}^{C_1}_{\Omega_1,f_1}(x),..., \mathscr{T}^{C_n}_{\Omega_n,f_n}(x) \right).$$

Thus problem $(\mathscr{MLP})$ can be written equivalently as an unconstrained convex multi-composed optimization problem

$$(\mathscr{MLP}) \quad \inf_{x\in X} (\delta_S + \delta_{-K} \circ h + g \circ h_1 \circ h_2)(x).$$

Remark 4.1. Note that the decomposition of the objective function of problem $(\mathscr{MLP})$ is completely different from that of [10], that is, the special construction of g, h_1 and h_2.

Remark 4.2. It is clear that g is proper, convex, continuous, and $\mathbb{R}^n_+$-nondecreasing (see [8, Remark 5.2]). In addition, one can easily see that h_1 is proper, $\mathbb{R}^n_+$-convex, $(\mathbb{R}^n_+,\mathbb{R}^n_+)$-nondecreasing on $\text{dom} h_1 = \mathbb{R}^n_+$ and $\mathbb{R}^n_+$-epi closed with $h_1(\text{dom} h_1) \subseteq \mathbb{R}^n_+$. On other hand, as $\mathscr{T}^{C_1}_{\Omega_1,f_1},..., \mathscr{T}^{C_n}_{\Omega_n,f_n}$ are all finite, positive, convex, and continuous (see [10, Theorem 2.1]), one can immediately deduce that h_2 is proper, $\mathbb{R}^n_+$-convex, and $\mathbb{R}^n_+$-epi closed with $h_2(\text{dom} h_2) = h_2(X) \subseteq \mathbb{R}^n_+$. In addition, since S and $-K$ are closed and convex, it is immediate that δ_C and δ_{-K} are proper, convex, lower semicontinuous, and δ_{-K} is K-nondecreasing on Y (see [2, Lemma 5.1]).

Lemma 4.1. *(1) Let $(x_1,...,x_n) \in \mathbb{R}^n$. Then*

$$\partial g(x_1,...,x_n) = \left\{ (x_1^*,...,x_n^*) \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i^* \leq 1 \text{ and } \max_{1\leq i\leq n} \{x_i^+\} = \sum_{i=1}^n x_i^* x_i \right\}.$$

(2) Let $x \in X$ and $i \in \{1,...,n\}$. Then

$$\partial \mathscr{T}^{C_i}_{\Omega_i,f_i}(x) = \mathbb{S}^i(x) := \partial \gamma_{C_i}(x-y_i-z_i) \cap \partial f_i(y_i) \cap N_{\Omega_i}(z_i),$$

where $(y_i,z_i) \in \text{dom} f_i \cap \Omega_i$ such that $\mathscr{T}^{C_i}_{\Omega_i,f_i}(x) = \gamma_{C_i}(x-y_i-z_i) + f_i(y_i)$.

Proof. (1) See [8, Lemma 5.3].
(2) See [10, Remark 2.3]. □

Remark 4.3. Since Theorem 3.1 is employed to derive the main result of this section, we point out that, for the fulfillment of the regularity $(\mathscr{RC})$, we may ask that problem $(\mathscr{MLP})$ satisfies the following assumption

$$(\mathscr{H}) \left| \; \exists\, x \in S \cap \text{dom} h \text{ such that } h(x) \in -\text{int}\, K \text{ and } \mathscr{T}^{C_i}_{\Omega_i,f_i}(x) > 0,\ i = 1,...,n. \right.$$

Now, we are in a position to provide optimality conditions for problem $(\mathscr{MLP})$ via the multi-composition rule in Theorem 3.1.

Theorem 4.1. *Let $\bar{x} \in S \cap h^{-1}(-K) \cap \text{dom} h$ and assume that problem $(\mathscr{MLP})$ satisfies the assumption $(\mathscr{H})$. Then $\bar{x}$ is an optimal solution of $(\mathscr{MLP})$ if and only if there exist $y^* \in K^*$, $(\lambda_1,...,\lambda_n) \in \mathbb{R}^n_+$, and $(\alpha_1,...,\alpha_n) \in \mathbb{R}^n_+$ such that*

(i) $0 \in \partial(\delta_S + y^* \circ h)(\bar{x}) + \sum_{i=1}^n \alpha_i \mathbb{S}^i(\bar{x}),$

(ii) $\alpha_i \in \partial\Big(\lambda_i l_i\Big)\Big(\mathscr{T}^{C_i}_{\Omega_i,f_i}(\bar{x})\Big)$, $i = 1,\dots,n$,

(iii) $\sum_{i=1}^{n} \lambda_i \leq 1$, $\max_{1\leq i\leq n}\Big\{l_i\Big(\mathscr{T}^{C_i}_{\Omega_i,f_i}(\bar{x})\Big)+a_i\Big\} = \sum_{j=1}^{n}\lambda_j\Big[l_j\Big(\mathscr{T}^{C_j}_{\Omega_j,f_j}(\bar{x})\Big)+a_j\Big]$,

(iv) $\langle y^*, h(\bar{x})\rangle = 0$.

Proof. Observe that $\bar{x}$ is an optimal solution to $(\mathscr{MLP})$ if and only if

$$0 \in \partial(\delta_S + \delta_{-K}\circ h + g\circ h_1 \circ h_2)(\bar{x}).$$

On other hand, Remark 4.2 and assumption $(\mathscr{H})$ demonstrate that functions $f := \delta_S$, $\varphi := \delta_{-K}$, $\psi := h$, g, h_1 and h_2 satisfy all the assumptions of Theorem 3.1. Hence, by applying the multi-composition rule in Theorem 3.1, we assert that there exist $y^* \in K^*$, $(\lambda_1,\dots,\lambda_n) \in \mathbb{R}^n_+$ and $(\alpha_1,\dots,\alpha_n) \in \mathbb{R}^n_+$ such that

$$\begin{cases} 0 \in \partial\Big(\delta_S + y^*\circ h + \sum_{i=1}^{n}\alpha_i \mathscr{T}^{C_i}_{\Omega_i,f_i}\Big)(\bar{x}), & (4.1)\\ (\alpha_1,\dots,\alpha_n) \in \partial\Big(\sum_{i=1}^{n}\lambda_i[l_i(.)+a_i]\Big)\Big(\mathscr{T}^{C_1}_{\Omega_1,f_1}(\bar{x}),\dots,\mathscr{T}^{C_n}_{\Omega_n,f_n}(\bar{x})\Big), & (4.2)\\ (\lambda_1,\dots,\lambda_n) \in \partial g\Big(l_1\Big(\mathscr{T}^{C_1}_{\Omega_1,f_1}(\bar{x})\Big)+a_1,\dots,l_n\Big(\mathscr{T}^{C_n}_{\Omega_n,f_n}(\bar{x})\Big)+a_n\Big), & (4.3)\\ y^* \in N_{-K}(h(\bar{x})). \end{cases}$$

Since functions $\mathscr{T}^{C_1}_{\Omega_1,f_1},\dots,\mathscr{T}^{C_n}_{\Omega_n,f_n}$ are all finites and continuous and $\partial\mathscr{T}^{C_i}_{\Omega_i,f_i}(\bar{x}) = \mathbb{S}^i(\bar{x})$, $i = 1,\dots,n$ (see Lemma 4.1), we have

$$(4.1) \iff 0 \in \partial(\delta_S + y^*\circ h)(\bar{x}) + \sum_{i=1}^{n}\alpha_i\partial\mathscr{T}^{C_i}_{\Omega_i,f_i}(\bar{x}) = \partial(\delta_S + y^*\circ h)(\bar{x}) + \sum_{i=1}^{n}\alpha_i\mathbb{S}^i(\bar{x}).$$

By using [5, Corollary 2.4.5] and Lemma 4.1, respectively, it follows that

$$(4.2) \iff \alpha_i \in \partial\Big(\lambda_i l_i\Big)\Big(\mathscr{T}^{C_i}_{\Omega_i,f_i}(\bar{x})\Big), i = 1,\dots,n,$$

and

$$(4.3) \iff \sum_{i=1}^{n}\lambda_i \leq 1,\ \max_{1\leq i\leq n}\Big\{l_i\Big(\mathscr{T}^{C_i}_{\Omega_i,f_i}(\bar{x})\Big)+a_i\Big\} = \sum_{j=1}^{n}\lambda_j\Big[l_j\Big(\mathscr{T}^{C_j}_{\Omega_j,f_j}(\bar{x})\Big)+a_j\Big].$$

Finally, it results by [11, Lemma 2.1] that

$$(4.3) \iff \langle y^*, h(\bar{x})\rangle = 0.$$

Hence, the proof is complete. □

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