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EXISTENCE AND STABILITY TO VECTOR OPTIMIZATION PROBLEMS VIA IMPROVEMENT SETS

LAM QUOC ANH[1], PHAM THANH DUOC[2], VO THI MONG THUY[3,*]

[1]*Department of Mathematics, Teacher College, Can Tho University, Cantho, Vietnam*
[2] *Faculty of Information Technology, Can Tho University of Technology, Cantho, Vietnam*
[3]*Department of Mathematics, Tay Do University, Cantho, Vietnam*

Abstract. In this paper, we consider vector optimization problems via improvement sets and investigate their existence and the stability. More precisely, we introduce Benson weakly efficient solutions for non-convex vector optimization problems and investigate the existence of the solution via the scalarization method. Based on generalized convexlikeness and relaxed continuity properties of mappings, we formulate sufficient conditions of the (semi) continuity for solution mappings of parametric vector optimization problems. As an application, we investigate the special case of co-radiant vector optimization problems.

Keywords. Benson weakly efficient solution; Existence and stability conditions; Improvement set; Co-radiant set; Scalarization method.

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1. INTRODUCTION

The optimal theory is an important and interesting branch of mathematics, which is growing and has increasingly real applications in the fields of modern science, engineering, technology, and management. When studying the above fields, we are faced with the case of an objective function with vector values [1, 2, 3, 4]. From the needs of practical uses, there are various kinds of proper efficiency, such as the Edgeworth proper efficiency [5], the Pareto proper efficiency [6], the Geoffrion proper efficiency [7], the Borwein proper efficiency [8], the Benson proper efficiency [9], the Henig proper efficiency [10], and other efficiency [11, 12, 13, 14]. It is worth noting here that the main difference of the solutions is due to the different cones used as criteria for evaluating a variety of types of solutions. Therefore, the study of different types of ordered cones in vector spaces is a topic of great interest. Motivated by this, Chicco et al. [15] introduced the concept of improvement set E and a kind of optimality named as E-optimality in finite-dimensional spaces, where the ordering relation of E-optimality is given by an improvement set E, and E-optimality unifies some known concepts of exact and approximate solutions of vector optimization problems. Because of the important role of improvement sets, Gutiérrez et al. [16] extended the notion of E-optimal point to a general quasi-ordered linear space

*Corresponding author.
E-mail addresses: quocanh@ctu.edu.vn (L.Q. Anh), ptduoc@ctuet.edu.vn (P.T. Duoc), vtmthuy@tdu.edu.vn (V.T.M. Thuy).
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and derived optimal conditions for E-optimal solution of vector optimization problems soon. Since then, the topic of improvement sets has been extensively researched by the optimization community and were used in optimization problems with many important topics. For instance, Dhingra and Lalitha [17] discussed the optimization problems by using improvement set with set optimization criterion. In [18, 19], Oppezzi and Rossi established the lower and upper convergence results of E-optimal solution of a convergent set, presented several optimal conditions for the E-optimal solution, and discussed the existence and stability of E-optimal point in the context of optimization problems in infinite-dimensional spaces. Next, Gutiérrez et al. [20] investigated some characteristics of optimal solutions related to improvement sets in real linear spaces. With topics about the optimization problem via the improvement set, the researchers are also interested in investigating the stability. For example, by using the convergence of a sequence of sets in the sense of Wijsman, Zhao and Yang [2] addressed the convergence of efficiency and approximate efficiency by virtue of improvement sets in vector optimization, which unified and extended previously known results. Lalitha and Chatterjee [3] studied the lower and upper Painlevé–Kuratowski convergences of sequences of E-optimal and weak E-optimal solution sets to perturbed vector optimization problems under perturbations both of the feasible set and objective mapping. In [21], Mao et al. investigated the Hausdorff continuity of the solution mappings to parametric set optimization problems via improvement sets. As a result, improvement sets are increasingly important in investigating optimization problems, such as set optimization, vector optimization, and vector equilibrium problems.

Let us provide a brief overview of the Benson efficiency in vector optimization problem. In 1968, Geoffrion [7] suggested a slightly restricted definition of efficiency that eliminated efficient points of a certain anomalous type and lent itself to a more satisfactory characterization called proper efficiency. Then, in 1977, Borwein [8] proposed a generalization of the Geoffrion's concept of proper efficiency to the vector maximization problem in which the domination cone K is any nontrivial, closed convex cone. However, when K is the nonnegative orthant, solutions may exist which are proper according to Borwein's definition but improper by Geoffrion's definition. To rectify this situation, by strengthening Borwein's requirement for properness, Benson [9] proposed a definition of proper efficiency for the case when K is a nontrivial, closed convex cone which would coincide with the Geoffrion's definition when K is the nonnegative orthant. Besides, Benson presented the equivalence of his definition and Borwein's under an appropriate concavity assumption, and also compared properties of proper efficiency according to his proposed definition and according to Borwein's definition. Although these definitions yield identical extensions of Geoffrion's fundamental results, when "$K = \mathbb{R}^p_+$" all properties of proper efficiency as defined by Geoffrion hold under Benson's proposed definition, but not under Borwein's. From these results, Benson's definition seems preferable to Borwein's for developing a theory of proper efficiency when K is a nontrivial, closed convex cone. Thus, investigating the characteristic properties of Benson solutions in optimal problems is a great significance. More precisely, Sheng [22] established the Hahn-Banach theorem under weak Benson proper efficiency, from which the author proved the existence of weak Benson proper efficient subgradient and presented the optimality conditions of set-valued optimization with weak Benson proper efficient subgradient. Chen and Rong [23] characterized the Benson proper efficiency in terms of scalarization, Lagrange multiplier, saddle-point criterion, and duality under the assumption of generalized cone subconvexlikeness. In 2001, Yang et al. [24]

considered existence conditions of Benson proper efficient element of vector optimization problem based on the class of generalized convex set-valued functions, termed nearly-subconvexlike functions. Then, Xing et al. [25] investigated the dual problems under Benson efficiency of the constrained vector set-valued optimization. In the meantime, they established corresponding weakly and strongly dual theorems.

The Benson solution has also been considered widely in vector optimization problems via the improvement sets. In 2015, Zhao and Yang [26] gave a kind of proper efficiency, named as E-Benson proper efficiency, which unified some proper efficiency and approximate proper efficiency, was proposed via the improvement sets in vector optimization. In addition, the concept of E-subconvexlikeness of set-valued mappings was introduced via the improvement sets, an alternative theorem was proved and some scalarization theorems and Lagrange multiplier theorems of E-Benson proper efficiency were established for a vector optimization problem with set-valued mappings. Recently, Benson proper efficiency was suggested and researched for vector equilibrium problems. Chen et al. [27] proposed and studied the Benson proper efficiency of vector equilibrium problems via linear scalarization functions. The author introduced gerneralized concepts of monotonicity and convexity, and then by using them together with the linear scalarization approach, the authors established stability conditions for the Benson proper efficiency of vector equilibrium problem. After that, motivated by the work [27], Liang et al. [28] developed the Benson proper efficiency defined in [27] for set-valued vector equilibrium problem and also applied scalarization results to consider the connectedness conditions for this solution set. To the best of our knowledge, there have not been any works on the stability for the Benson proper efficiency mappings to vector optimization problems via improvement sets.

Motivated by this research stream, in this paper, we aim to consider vector optimization problems via improvement sets and study the existence and stability for the concerning problems. First, we propose the Benson weakly efficient solutions to non-convex vector optimization problems and study the existence of solutions as well as their scalarization characterizations. Next, we use generalized convexity conditions proposed in [26, 29, 30, 31] to formulate sufficient conditions of the (semi) continuity of the Benson weakly efficient mappings to the parametric vector optimization problems via improvement sets. Finally, we apply the obtained results to co-radiant vector optimization problems as an application.

2. Preliminaries

Let $\mathbb{X}$ be a normed space, and let $\mathbb{Y}$ be a real locally convex topological linear space. Assume that $\mathbb{Y}^*$ is the dual space of $\mathbb{Y}$ and K is a pointed closed convex cone in $\mathbb{Y}$ with a nonempty interior ($\text{int}K \neq \emptyset$). The family of all nonempty subsets of $\mathbb{X}$ and the family of all nonempty subsets of $\mathbb{Y}$ are presented as $\mathbf{P}(\mathbb{X})$ and $\mathbf{P}(\mathbb{Y})$, respectively. The positive polar cone K^* of K is denoted by $K^* := \{\ell \in \mathbb{Y}^* : \ell(y) \geq 0, \forall y \in K\}$. Based on [13, P. 77], one has

$$\text{int}\, K = \{y \in \mathbb{Y} : \ell(y) > 0, \forall \ell \in K^* \setminus \{0_{\mathbb{Y}^*}\}\}. \tag{2.1}$$

Let B be a nonempty subset of $\mathbb{Y}$, and denote the closure of B by $\text{cl}B$. The cone hull of B is defined by $\text{cone}(B) := \{ta : t \geq 0, a \in B\}$.

Let us recall a version of separation theorem for convex sets, which is used in the next sections.

Lemma 2.1. [13, P. 74] *Let B and D be two nonempty convex subsets of a real topological linear space $\mathbb{Y}$ with $\operatorname{int} B \neq \emptyset$. Then, $D \cap \operatorname{int} B = \emptyset$ if and only if there are a linear functional $l \in \mathbb{Y}^* \setminus \{0_{\mathbb{Y}^*}\}$ and a real number α with $l(b) \leq \alpha \leq l(d)$ for all $b \in B$ and all $d \in D$, and $l(b) < \alpha$, for all $b \in \operatorname{int} B$.*

Lemma 2.2. [32, P. 22] *Assume that $C \subset \mathbb{Y}$ is a convex cone with $\operatorname{int} C \neq \emptyset$, and let $B \subset \mathbb{Y}$. Then, $\operatorname{cl}(B+C)$ is convex if and only if $B + \operatorname{int} C$ is convex.*

We now recall the notions and characterizations of upper semicontinuity and lower semicontinuity for set-valued mappings, which are used in the sequel.

Definition 2.1. [11, P. 51] A set-valued mapping $Q : \mathbb{X} \rightrightarrows \mathbb{Y}$ is said to be

(a) *upper semicontinuous (usc)* at $x_0 \in \mathbb{X}$ if, for any neighborhood U of $Q(x_0)$, there is a neighborhood N of x_0 such that $Q(N) \subset U$;
(b) *lower semicontinuous (lsc)* at $x_0 \in \mathbb{X}$ if, for any open subset U of $\mathbb{Y}$ with $Q(x_0) \cap U \neq \emptyset$, there exists a neighborhood N of x_0 such that for all $x \in N, Q(x) \cap U \neq \emptyset$;
(c) *continuous* at x_0 if it is both usc and lsc at x_0.

Lemma 2.3. [33, P. 37] *Let a set-valued mapping $Q : \mathbb{X} \rightrightarrows \mathbb{Y}$ be given. Then,*

(a) *Q is lsc at x_0 if and only if, for all $x_n \to x_0$ and $y_0 \in Q(x_0)$, there exists $y_n \in Q(x_n)$ such that $y_n \to y_0$.*
(b) *Q is lsc at x_0 if and only if, for all $x_n \to x_0$, then one has $Q(x_0) \subset \liminf Q(x_n)$, where $\liminf Q(x_n) := \{y_0 \in Y : \exists y_n \in Q(x_n), y_n \to y_0\}$.*

Lemma 2.4. [33, P. 41] *If $Q(x_0)$ is compact, then Q is usc at x_0 if and only if, for any sequence $\{x_n\}$ converging to x_0 and $y_n \in Q(x_n)$, there is a subsequence $\{y_{n_k}\}$ converging to $y_0 \in Q(x_0)$.*

Definition 2.2. [12, P. 22] A vector-valued mapping $g : \mathbb{X} \to \mathbb{Y}$ is said to be

(a) *K-lower semicontinuous* (K-lsc) at $x_0 \in \mathbb{X}$ if, for any neighborhood V of $g(x_0)$, there exists some neighborhood U of x_0 such that $g(x) \in V + K$ for all $x \in U$;
(b) *K-upper semicontinuous* (K-usc) at $x_0 \in \mathbb{X}$ if $-g$ is K-lsc at x_0;
(c) *K-continuous* at $x_0 \in \mathbb{X}$ if it is both K-usc and K-lsc at x_0.

Remark 2.1. When $\mathbb{Y} = \mathbb{R}$ and $K = \mathbb{R}_+$, the K-lower semicontinuity reduces to the ordinary lower semicontinuity. To be more specify, a function g is said to be lower semicontinuous at $x_0 \in \mathbb{X}$ if, for every real number $r < g(x_0)$, there exists some neighborhood U of x_0 such that $r < g(x)$ for all $x \in U$.

We are in a position to study the continuity properties of a composition function, which play important roles in our analysis.

Lemma 2.5. *Let $x_0 \in \mathbb{X}$, $\ell \in K^*$ be given, and let $f : \mathbb{X} \to \mathbb{Y}$ be a vector-valued mapping. Then,*

(a) *$\ell \circ f$ is lower semicontinuous at x_0 if f is K-lower semicontinuous at x_0;*
(b) *$\ell \circ f$ is upper semicontinuous at x_0 if f is K-upper semicontinuous at x_0;*
(c) *$\ell \circ f$ is continuous at x_0 if f is K-continuous at x_0.*

Chứng minh. In view of the same techniques in the proof, we only prove the first statement. By Remark 2.1, we need to present that, for each $\gamma \in \mathbb{R}$ with $\gamma < \ell(f(x_0))$, there exists some neighborhood U of x_0 such that $\gamma < \ell(f(x))$ for all $x \in U$. Since $\gamma < \ell(f(x_0))$ and ℓ is lower continuous at $f(x_0)$, we can choose a neighborhood V of $f(x_0)$ such that

$$\gamma < \ell(y), \quad \forall y \in V. \tag{2.2}$$

For the neighborhood V, there is a neighborhood U of x_0 such that $f(x) \in V + K$ for all $x \in U$ as f is K-lower semicontinuous at x_0. Then, for any $x \in U$, there is some $y \in V$ such that $f(x) \in y + K$. It follows from $\ell \in K^*$ that

$$\ell(y) \leq \ell(f(x)). \tag{2.3}$$

Combining (2.2) and (2.3), we obtain $\gamma < \ell(f(x)), \quad \forall x \in U$. Therefore, $\ell \circ f$ is lower semicontinuous at x_0. □

Definition 2.3. [16, P. 305] The set $E \in \mathbf{P}(\mathbb{Y})$ is called *an improvement set* with respect to (wrt) K if $0_{\mathbb{Y}} \notin E$ and $E + K = E$. The class of the improvement sets wrt K in $\mathbb{Y}$ is denoted by $\mathbf{I}_K$. It is clear that $K \setminus \{0_{\mathbb{Y}}\} \in \mathbf{I}_K$, $\operatorname{int} K \in \mathbf{I}_K$ and $\mathbb{Y} \setminus (-K) \in \mathbf{I}_K$.

Lemma 2.6. [26, P. 741] *If $E \in \mathbf{I}_K$ is solid, then* $\operatorname{int} E = E + \operatorname{int} K = \operatorname{cl} E + \operatorname{int} K$.

Lemma 2.7. [31, P. 1290] *If $E \in \mathbf{I}_K$ and $\emptyset \neq B \subset \mathbb{Y}$, then* $\operatorname{cl}(\operatorname{cone}(B+E)) = \operatorname{cl}(\operatorname{cone}(B+\operatorname{int} E))$.

Next, we investigate some properties of the improvement sets.

Lemma 2.8. *For any nonempty subset B of $\mathbb{Y}$, one has* $\operatorname{int}(B+E) = B + \operatorname{int} E$.

Chứng minh. For any $y \in \operatorname{int}(B+E)$, there exists $\varepsilon > 0$ such that $y + \varepsilon \mathscr{B}_{\mathbb{Y}} \subset B + E$, where $\mathscr{B}_{\mathbb{Y}}$ is the closed unit ball of $\mathbb{Y}$. Taking arbitrarily $b \in \varepsilon \mathscr{B}_{\mathbb{Y}} \cap (-\operatorname{int} K)$, one concludes by Lemma 2.6 that

$$y \in -\varepsilon b + B + E \subset \operatorname{int} K + B + E \subset B + \operatorname{int} E,$$

which implies that $\operatorname{int}(B+E) \subset B + \operatorname{int} E$. Conversely, since $B + \operatorname{int} E$ is an open subset of $B+E$, one has $B + \operatorname{int} E \subset \operatorname{int}(B+E)$. □

Lemma 2.9. *Let B be a nonempty subset of $\mathbb{Y}$. Then,*

$$B \cap \operatorname{int} K = \emptyset \textit{ implies that } \operatorname{cl} B \cap \operatorname{int} K = \emptyset.$$

Chứng minh. If $\operatorname{cl} B \cap \operatorname{int} K \neq \emptyset$, then there is $y \in \operatorname{cl} B \cap \operatorname{int} K$. Since $y \in \operatorname{int} K$ and $\operatorname{int} K$ is open, there is a neighbourhood of zero V such that $y + V \subset \operatorname{int} K$. On the other hand, since $y \in \operatorname{cl} B$, we have $(y+V) \cap B \neq \emptyset$, which implies that $\operatorname{int} K \cap B \neq \emptyset$ which leads to a contradiction. □

Definition 2.4. [29, P. 408, 409] Let A be a nonempty subset of $\mathbb{X}$ and $x_1, x_2 \in A$. Then,

(a) the continuous mapping $\Gamma_{x_1,x_2} : [0,1] \to \mathbb{X}$ satisfying $\Gamma_{x_1,x_2}(0) = x_1$ and $\Gamma_{x_1,x_2}(1) = x_2$ is called an *arc* on A with endpoints x_1, x_2;
(b) A is said to be *arcwise connected* if, for each pair of points $x_1, x_2 \in A$, there is an arc Γ_{x_1,x_2} on A.

Definition 2.5. [29, P. 409] Let A be an arcwise connected subset of $\mathbb{X}$. A vector-valued mapping $g : \mathbb{X} \to \mathbb{Y}$ is said to be

(a) *arcwise connected K-convex* on A if, for all $x_1, x_2 \in A$, there exists an arc Γ_{x_1,x_2} on A such that

$$g(\Gamma_{x_1,x_2}(t)) \in (1-t)g(x_1) + tg(x_2) - K, \quad \forall t \in [0,1].$$

(b) *naturally arcwise connected K-quasiconvex* on A if, for all $x_1, x_2 \in A$, there exists an arc Γ_{x_1,x_2} on A such that, for each $t \in [0,1]$, there exists some $s \in]0,1[$ satisfying

$$g(\Gamma_{x_1,x_2}(t)) \in (1-s)g(x_1) + sg(x_2) - K.$$

Motivated by [26, 30, 31], we propose generalized convexlikeness properties of a vector-valued map as follows.

Definition 2.6. Let $A \in \mathbf{P}(\mathbb{X}), B \in \mathbf{P}(\mathbb{Y})$ be given. The vector-valued mapping $g : \mathbb{X} \to \mathbb{Y}$ is said to be

(a) *B-convexlike* on A if $g(A) + B$ is a convex set in $\mathbb{Y}$;
(b) *nearly B-convexlike* on A if $\mathrm{cl}(g(A) + B)$ is a convex set in $\mathbb{Y}$;
(c) *B-subconvexlike* on A if $g(A) + \mathrm{int} B$ is a convex set in $\mathbb{Y}$;
(d) *nearly B-subconvexlike* on A if $\mathrm{cl}(\mathrm{cone}(g(A) + B))$ is a convex set in $\mathbb{Y}$.

Remark 2.2. If $B = E \in \mathbf{I}_K$, then Definition 2.6 (d) is reduced to Definition 3.1 of [31] while Definition 2.6 (c) coincides with Definition 6.1 in [26]. Moreover, when $B = K$, the statements of Definition 2.6 are reduced to the concepts of convexlikeness via cones provided in [30].

We now finalize this section with a result that provides relationships between the concepts defined in the above definition.

Lemma 2.10. *Let $A \in \mathbf{P}(\mathbb{X})$, and let $B \in \mathbf{P}(\mathbb{Y})$ and $E \in \mathbf{I}_K$ be given. If E is convex, then the following statements hold true:*

(a) *every B-convexlike mapping on A is nearly B-convexlike on A;*
(b) *every E-subconvexlike mapping on A is nearly E-subconvexlike on A;*
(c) *every nearly K-convexlike mapping on A is nearly E-subconvexlike on A;*
(d) *every K-subconvexlike mapping on A is nearly E-subconvexlike on A.*

Chứng minh. (a) If g is B-convexlike on A, then $g(A) + B$ is convex. Hence, $\mathrm{cl}(g(A) + B)$ is also convex, and thus g is nearly B-convexlike on A.
(b) Because g is E-subconvexlike on A, $g(A) + \mathrm{int} E$ is convex. Then, $\mathrm{cl}(\mathrm{cone}(g(A) + \mathrm{int} E))$ is also a convex subset of $\mathbb{Y}$. Combining this with Lemma 2.7, we conclude that

$$\mathrm{cl}(\mathrm{cone}(g(A) + E)) = \mathrm{cl}(\mathrm{cone}(g(A) + \mathrm{int} E))$$

is convex. Therefore, g is nearly E-subconvexlike on A.
(c) Due to the near K-convexlikeness of g, we achieve the convexity of $\mathrm{cl}(g(A) + K)$. By employing Lemma 2.2, the set $g(A) + \mathrm{int} K$ is convex, which together with the convexity of E would imply that $g(A) + \mathrm{int} K + E$ is convex. Then, Lemmas 2.6 and 2.7 lead to

$$\begin{aligned} \mathrm{cl}(\mathrm{cone}(g(A) + E)) &= \mathrm{cl}(\mathrm{cone}(g(A) + \mathrm{int} E)) \\ &= \mathrm{cl}(\mathrm{cone}(g(A) + \mathrm{int} K + E)). \end{aligned}$$

Hence, g is nearly E-subconvexlike on A.
(d) In view of the K-subconvexlike of g, we obtain the convexity of $g(A) + \mathrm{int} K$. By using the same arguments as in the proof of (c), we also have conclusion (d). □

3. The Existence to Vector Optimization Problems via Improvement Sets

Let $\mathbb{X}, \mathbb{Y}, K$ be defined as in Section2, and let $E \in \mathbf{I}_K$ be fixed. We consider the following vector optimization problem:

$$\text{(VOP)} \quad \min \quad f(x) \quad \text{subject to} \quad x \in A,$$

where $f : \mathbb{X} \to \mathbb{Y}$ is a vector-valued mapping and A is a nonempty subset of $\mathbb{X}$.

Motivated by [22], we propose a solution concept of the reference problem via the improvement set E as follows.

Definition 3.1. The vector $x_0 \in A$ is called *a Benson weakly efficient solution* to (VOP) with respect to E, written as $x_0 \in \mathrm{WBEff}(A, f)$ if $\mathrm{cl}(\mathrm{cone}(f(A) - f(x_0) + E)) \cap (-\mathrm{int} K) = \emptyset$.

Example 3.1. Let $\mathbb{X} = \mathbb{R}, \mathbb{Y} = \mathbb{R}^2, K = \mathbb{R}^2_+, A = [0,2], E = (1,1) + \mathbb{R}^2_+$, and let the mapping $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f(x) = (x,x)$. Then, $f(A) = \{(x,x) \in \mathbb{R}^2 : 0 \le x \le 2\}$. For each $x_0 \in A$, one has

$$f(A) - f(x_0) + E = \{(x - x_0 + 1, x - x_0 + 1) : 0 \le x \le 2\} + \mathbb{R}^2_+.$$

If $x_0 \in \mathrm{WBEff}(A, f)$, then, for all $t \ge 0$ and $x \in A$, $t(x - x_0 + 1) \ge 0$, which implies that $x_0 \le 1$.

Conversely, for $x_0 \le 1$, one has $x_0 \le x + 1, \quad \forall x \in [0,2]$, which means that $x - x_0 + 1 \ge 0$ for all $x \in [0,2]$. So, for all $x \in [0,2]$, we obtain

$$\left(\{(x - x_0 + 1, x - x_0 + 1) : 0 \le x \le 2\} + \mathbb{R}^2_+\right) \cap (-\mathrm{int}\,\mathbb{R}^2_+) = \emptyset.$$

Therefore, $x_0 \in \mathrm{WBEff}(A, f)$. As a result, $\mathrm{WBEff}(A, f) = [0,1]$.

For each $\ell \in \mathbb{Y}^*$, we consider the following set

$$S_\ell := \{x \in A : \ell(f(z)) + \ell(e) \ge \ell(f(x)) \text{ for all } (z,e) \in A \times E\},$$

and discuss the intimate associations between it and $\mathrm{WBEff}(A, f)$.

Lemma 3.1. *Let $E \in \mathbf{I}_K$ be given. Then, $S_{\hat{\ell}} \subset \mathrm{WBEff}(A, f)$, for all $\hat{\ell} \in K^* \setminus \{0_{\mathbb{Y}^*}\}$.*

Chứng minh. Let $\bar{x} \in S_{\hat{\ell}}$ be arbitrary, that is,

$$\hat{\ell}(f(x)) + \hat{\ell}(e) \ge \hat{\ell}(f(\bar{x})), \quad \forall (x,e) \in A \times E. \tag{3.1}$$

If $\bar{x} \notin \mathrm{WBEff}(A, f)$, then

$$\mathrm{cl}(\mathrm{cone}(f(A) - f(\bar{x}) + E)) \cap (-\mathrm{int} K) \neq \emptyset.$$

By using Lemma 2.9, we derive

$$\mathrm{cone}(f(A) - f(\bar{x}) + E) \cap (-\mathrm{int} K) \neq \emptyset.$$

For any $y \in \mathrm{cone}(f(A) - f(\bar{x}) + E) \cap (-\mathrm{int} K)$, it follows from (2.1) that

$$\hat{\ell}(y) < 0, \tag{3.2}$$

as $y \in (-\mathrm{int} K)$.

On the other hand, $y \in \mathrm{cone}(f(A) - f(\bar{x}) + E)$, there exist $(\hat{z}, \hat{e}) \in A \times E$ and $\hat{t} > 0$ such that $y = \hat{t}(f(\hat{z}) - f(\bar{x}) + \hat{e})$. Combining this with (3.2) and the linear property of $\widehat{\ell}$, we have

$$\hat{\ell}(f(\hat{z})) - \hat{\ell}(f(\bar{x})) + \hat{\ell}(\hat{e}) < 0,$$

or equivalently $\hat{\ell}(f(\hat{z})) + \hat{\ell}(\hat{e}) < \hat{\ell}(f(\bar{x}))$ which contradicts (3.1). Thus $\bar{x} \in \mathrm{WBEff}(A, f)$. $\square$

We discuss sufficient conditions of the existence of the Benson weakly efficient solutions of non-convex vector optimization problems via the linear scalarization function and the cone-lower semicontinuity property of the objective functions.

Theorem 3.1. *Assume that*

(i) $E\cap(-\operatorname{int}K)=\emptyset$;
(ii) *A is compact*;
(iii) *f is K-lower semicontinuous on A.*

Then, $\mathrm{WBEff}(A,f)$ *is nonempty. Moreover, if f is nearly E-subconvexlike on A, then*

$$\mathrm{WBEff}(A,f)=\bigcup_{\ell\in K^*\setminus\{0_{\mathbb{Y}^*}\}} S_\ell.$$

Chứng minh. By (i) and Lemma 2.1, there exist $\ell_0\in\mathbb{Y}^*\setminus\{0_{\mathbb{Y}^*}\}$ and $\gamma\in\mathbb{R}$ such that

$$\ell_0(e)\geq\gamma\geq-\ell_0(k), \tag{3.3}$$

for all $e\in E,k\in K$. If there is some $\bar{k}\in K$ such that $\ell_0(\bar{k})<0$, then (3.3) leads to

$$\gamma\geq-\ell_0(\bar{k})>0. \tag{3.4}$$

On the other hand, we have $\alpha\bar{k}\in K$ for all $\alpha\geq 0$ as K is a cone. Applying (3.3) again, we obtain

$$\gamma\geq-\ell_0(\alpha\bar{k})=\alpha(-\ell_0(\bar{k})),\quad\forall\alpha\geq 0 \tag{3.5}$$

By employing (3.4) and (3.5), we see a contradiction due to the existence of γ. Hence, $\ell_0(k)\geq 0$ for all $k\in K$, and so $\ell_0\in K^*\setminus\{0_{\mathbb{Y}^*}\}$. Letting $k\to 0$ in (3.3), we achieve $\ell_0(e)\geq 0$ for all $e\in E$. By the cone-lower semicontinuity of f and the continuity of ℓ_0, Lemma 2.5 yields that $\ell_0\circ f$ is lower semicontinuous, which together with the compactness of A implies that S_{ℓ_0} is nonempty. Combining this with Lemma 3.1, we conclude that $\mathrm{WBEff}(A,f)$ is nonempty. Then, for any $\bar{x}\in\mathrm{WBEff}(A,f)$, one has

$$\mathrm{cl}(\mathrm{cone}(f(A)-f(\bar{x})+E))\cap(-\operatorname{int}K)=\emptyset.$$

Since f is nearly E-subconvexlike on A, the set $\mathrm{cl}(\mathrm{cone}(f(A)-f(\bar{x})+E))$ is convex. Applying Lemma 2.1, there exist $\bar{\ell}\in\mathbb{Y}^*\setminus\{0_{\mathbb{Y}^*}\}$ and $\bar{\gamma}\in\mathbb{R}$ such that

$$\bar{\ell}(f(x))-\bar{\ell}(f(\bar{x}))+\bar{\ell}(e)\geq\bar{\gamma}\geq-\alpha\bar{\ell}(k), \tag{3.6}$$

for all $(x,e)\in A\times E,k\in K,\alpha>0$. By using the same arguments as above, one also has $\bar{\ell}(k)\geq 0$ for all $k\in K$, and so $\bar{\ell}\in K^*\setminus\{0_{\mathbb{Y}^*}\}$. Letting $\alpha\to 0$ in (3.6), we obtain

$$\bar{\ell}(f(x))+\bar{\ell}(e)\geq\bar{\ell}(f(\bar{x})),\quad\forall(x,e)\in A\times E,$$

and thus $\bar{x}\in S_{\bar{\ell}}\subset\bigcup_{\ell\in K^*\setminus\{0_{\mathbb{Y}^*}\}} S_\ell$. Then, Lemma 3.1 helps us to finish the proof. □

Now, let us provide an example to illustrate the applicability of Theorem 3.1.

Example 3.2. Let $\mathbb{X}=\mathbb{Y}=\mathbb{R}^2, A=\{(x_1,x_2)\in\mathbb{R}^2: x_1^2+x_2^2\leq 1\}$, $K=\mathbb{R}^2_+$, $E=(0.1,0.1)+\mathbb{R}^2_+$ and $f:\mathbb{R}^2\to\mathbb{R}^2$ be defined by $f(x)=\left(x_1^2+x_2^2,x_1^2+x_2^2\right)$ for all $x=(x_1,x_2)\in\mathbb{R}^2$. It is clear that the conditions (i)-(iii) of Theorem 3.1 are satisfied. Hence, $\mathrm{WBEff}(A,f)$ is nonempty. Furthermore, we have

$$f(A)=\left\{(x_1^2+x_2^2,x_1^2+x_2^2)\in\mathbb{R}^2: x_1^2+x_2^2\leq 1\right\},$$

and, for each $\bar{x} = (\bar{x}_1, \bar{x}_2) \in A$,

$f(A) - f(\bar{x}) + E = \big\{ \big((x_1^2 + x_2^2) - (\bar{x}_1^2 + \bar{x}_2^2) + 0.1, (x_1^2 + x_2^2) - (\bar{x}_1^2 + \bar{x}_2^2) + 0.1 \big) : x_1^2 + x_2^2 \leq 1 \big\} + \mathbb{R}^2_+.$

If $\bar{x} \in \mathrm{WBEff}(A, f)$, then, for all $t \geq 0$ and $x \in A$,

$$t \left((x_1^2 + x_2^2) - (\bar{x}_1^2 + \bar{x}_2^2) + 0.1 \right) \geq 0,$$

and then $\bar{x}_1^2 + \bar{x}_2^2 \leq 0.1$.

Conversely, for any $\bar{x} = (\bar{x}_1, \bar{x}_2)$ with $\bar{x}_1^2 + \bar{x}_2^2 \leq 0.1$, we always have

$$\bar{x}_1^2 + \bar{x}_2^2 \leq 0.1 + (x_1^2 + x_2^2), \quad \forall (x_1, x_2) \in A.$$

Hence, for all $(x_1, x_2) \in A$, we achieve

$\big(\{ \big((x_1^2 + x_2^2) - (\bar{x}_1^2 + \bar{x}_2^2) + 0.1, (x_1^2 + x_2^2) - (\bar{x}_1^2 + \bar{x}_2^2) + 0.1 \big) : x_1^2 + x_2^2 \leq 1 \} + \mathbb{R}^2_+ \big) \cap (-\operatorname{int} \mathbb{R}^2_+) = \emptyset.$

As a result, $\bar{x} \in \mathrm{WBEff}(A, f)$. Therefore, $\mathrm{WBEff}(A, f) = \big\{ (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 0.1 \big\}$.

4. Continuity of Solution Mappings to Vector Optimization Problems via Improvement Sets

Let $\mathbb{X}, \mathbb{Y}, K, A$ be defined as in Section 3, and let $\mathbb{W}$ be a normed space and $E \in \mathbf{I}_K$ be given. The aim of this section is to discuss the (semi) continuity of a solution mapping of the following parametric vector optimization problem:

$$\text{(PVOP)} \quad \min \quad f(x, p) \quad \text{subject to} \quad (x, p) \in A \times P,$$

where $f : \mathbb{X} \times \mathbb{W} \to \mathbb{Y}$ is a vector-valued mapping and P is a nonempty subset of $\mathbb{W}$.

Motivated by Definition 3.1, we also define the Benson weakly efficient solution of (PVOP) for each $p \in P$, as follows

$$\mathrm{WBEff}(A, f)(p) = \{ x_0 \in A : \mathrm{cl}(\mathrm{cone}(f(A, p) - f(x_0, p) + E)) \cap (-\operatorname{int} K) = \emptyset \}.$$

Now, we present a result used to discuss conditions of the (semi) continuity of the Benson weakly efficient solution mappings of the parametric vector optimization problems.

Theorem 4.1. *Assume that*

(i) $E \cap (-\operatorname{int} K) = \emptyset$;

(ii) *A is compact*;

(iii) *f is K-continuous on $A \times P$*;

Then, for any $\ell \in K^* \setminus \{0_{\mathbb{Y}^*}\}$, the map

$$p \mapsto S_\ell(p) := \{ x \in A : \ell(f(z, p) + e) \geq \ell(f(x, p)) \text{ for all } (z, e) \in A \times E \} \tag{4.1}$$

is upper semicontinuous with nonempty compact values on P.

Chứng minh. By using the same techniques as in the proof of Theorem 3.1, we see that $S_\ell(p)$ is nonempty for all $\ell \in K^* \setminus \{0_{\mathbb{Y}^*}\}$. Taking $p_0 \in P$ arbitrarily, we need to prove that S_ℓ is usc at p_0. Suppose to the contrary that S_ℓ is not usc at p_0. It means that we can find some open set U with $S_\ell(p_0) \subset U$ and a sequence $\{p_n\}$ converging to p_0 such that, for each $n \in \mathbb{N}$, there exists $x_n \in S_\ell(p_n) \setminus U$. By $x_n \in A$ and the compactness of A, we can assume that $\{x_n\}$ converges to $x_0 \in A$. If $x_0 \notin S_\ell(p_0)$, then there is an element $(z_0, e_0) \in A \times E$ such that

$$\ell(f(z_0, p_0) + e_0) < \ell(f(x_0, p_0)). \tag{4.2}$$

On the other hand, in view of $x_n \in S_\ell(p_n)$, we have $\ell(f(z_0,p_n)+e_0) \geq \ell(f(x_n,p_n))$. Combining this with the continuity of ℓ and f, we obtain $\ell(f(z_0,p_0)+e_0) \geq \ell(f(x_0,p_0))$, which contradicts (4.2). Hence, $x_0 \in S_\ell(p_0) \subset U$, which is impossible as $x_n \notin U$ for all n. For the compactness of $S_\ell(p_0)$, we only need to present that it is a closed subset of the compact set A. For any sequence $\{x_n\} \subset S_\ell(p_0)$ converging to x_0, one sees that $x_0 \in A$ as $x_n \in A$ and A is closed. Because $x_n \in S_\ell(p_0)$, we obtain $\ell(f(z,p_0)+e) \geq \ell(f(x_n,p_0))$ for all $(z,e) \in A \times E$. By the continuity of f and ℓ, we have $\ell(f(z,p_0)+e) \geq \ell(f(x_0,p_0))$ for all $(z,e) \in A \times E$, or equivalently $x_0 \in S_\ell(p_0)$. This implies that $S_\ell(p_0)$ is closed, and so it is compact. □

Employing the above result, we formulate sufficient conditions of the upper semicontinuity of $\mathrm{WBEff}(A,f)$.

Theorem 4.2. *Assume that all conditions in Theorem 4.1 hold true. If the mapping f is nearly E-subconvexlike in the first component on A, then the solution mapping* $\mathrm{WBEff}(A,f)$ *is nonempty-valued and upper semicontinuous on P.*

Chứng minh. By employing the same arguments in the proof of Theorem 3.1, the Benson weakly efficient solution set $\mathrm{WBEff}(A,f)$ is nonempty, and

$$\mathrm{WBEff}(A,f)(p) = \bigcup_{\ell \in K^* \setminus \{0_{\mathbb{Y}^*}\}} S_\ell(p), \quad \forall p \in P. \tag{4.3}$$

If the mapping $\mathrm{WBEff}(A,f)$ is not usc at p_0, then we can find some open neighborhood U_0 of $\mathrm{WBEff}(A,f)(p_0)$ and a sequence $\{p_n\}$ converging to p_0 such that, for each n, there is $x_n \in \mathrm{WBEff}(A,f)(p_n)$, but $x_n \notin U_0$. In view of (4.3), for each x_n, we can choose $\ell_n \in K^* \setminus \{0_{\mathbb{Y}^*}\}$ such that $x_n \in S_{\ell_n}(p_n)$, which means that

$$\ell_n(f(z,p_n)+e-f(x_n,p_n)) \geq 0, \quad \forall (z,e) \in A \times E,$$

which together with $\ell_n \in K^* \setminus \{0_{\mathbb{Y}^*}\}$ implies that

$$f(z,p_n)+e-f(x_n,p_n) \notin -\operatorname{int} K, \tag{4.4}$$

Since f is continuous on $A \times P$ and the complement set $\mathbb{Y} \setminus (-\operatorname{int} K)$ is closed, (4.4) leads to

$$f(z,p_0)+e-f(x_0,p_0) \notin -\operatorname{int} K, \quad \forall (z,e) \in A \times E.$$

Then, there exists some $\ell \in K^* \setminus \{0_{\mathbb{Y}^*}\}$ such that

$$\ell(f(z,p_0)+e-f(x_0,p_0)) \geq 0, \quad \forall (z,e) \in A \times E.$$

This together with (4.1) and (4.3) implies that

$$x_0 \in S_\ell(p_0) \subset \mathrm{WBEff}(A,f)(p_0) \subset U_0,$$

which contradicts the fact that $x_n \notin U_0$ for all n. We finish the proof. □

Motivated by the Remark 3.3 in [17], in the rest of this section, we consider a case of $E = k_0 + K$, where $k_0 \in \operatorname{int} K$, and study sufficient conditions of the continuity of the Benson weakly efficient solution mappings of (PVOP) via this improvement set.

Theorem 4.3. *Assume that*

(i) *A is compact and arcwise connected*;
(ii) *f is K-continuous on $A \times P$*;
(iii) *for each $p \in P$, $f(.,p)$ is naturally arcwise connected K-quasiconvex on A.*

Then, for any $\ell \in K^* \setminus \{0_{\mathbb{Y}^*}\}$, S_ℓ is nonempty-valued and continuous on P.

Chứng minh. Let $\ell \in K^* \setminus \{0_{\mathbb{Y}^*}\}$ be given. Due to Theorem 4.1, S_ℓ is nonempty-valued and upper semicontinuous on P, and so we only need to show that S_ℓ is lower semicontinuous on P. Set

$$\widehat{S}_\ell(p) = \{x \in A : \ell(f(z,p)+e) > \ell(f(x,p)) \text{ for all } (z,e) \in A \times E\}.$$

Because $E = k_0 + K$, for each $e \in E$, there exists some $k \in K$ such that $e = k_0 + k$, and so

$$\ell(e) = \ell(k_0 + k) > 0, \quad \forall e \in E. \tag{4.5}$$

Since f is K-lower semicontinuous and ℓ is continuous, Lemma 2.5 implies that $\ell \circ f$ is lower semicontinuous on A. Hence, $\ell \circ f$ achieves the minimal value over the compact subset A. Combining this with (4.5), we obtain that $\widehat{S}_\ell$ is nonempty. Let $p_0 \in P$ be arbitrary. Suppose that $\widehat{S}_\ell$ is not lsc at p_0, namely there are an element x_0 in $\widehat{S}_\ell(p_0)$ and a sequence $\{p_n\}$ converging to p_0 such that, for all $\{x_n\}$ with $x_n \in \widehat{S}_\ell(p_n), x_n \nrightarrow x_0$. Then, there is a subsequence $\{p_m\}$ of $\{p_n\}$ such that $x_0 \notin \widehat{S}_\ell(p_m)$ for all m, which means that there exist $(z_m, e_m) \in A \times E$,

$$\ell(f(z_m, p_m)) + \ell(e_m) \leq \ell(f(x_0, p_m)). \tag{4.6}$$

Since $E = k_0 + K$, for each m, there exists $k_m \in K$ such that $e_m = k_0 + k_m$, which together with (4.6) implies that

$$\ell(f(z_m, p_m)) + \ell(k_0 + k_m) \leq \ell(f(x_0, p_m)), \quad k_m \in K,$$

and so

$$\ell(f(z_m, p_m)) + \ell(k_0) \leq \ell(f(x_0, p_m)), \tag{4.7}$$

as $\ell(k_m) \geq 0$. In view of the compactness of A, we can assume that $z_m \to z_0 \in A$. From the continuity of ℓ and f, (4.7) leads to $\ell(f(z_0, p_0)) + \ell(k_0) \leq \ell(f(x_0, p_0))$, which contradicts the fact that $x_0 \in \widehat{S}_\ell(p_0)$. Thus, $\widehat{S}_\ell$ is lsc at p_0.

Next, let $\bar{x} \in S_\ell(p_0)$ and $x_1 \in \widehat{S}_\ell(p_0)$ be arbitrary. Since $f(\cdot, p_0)$ is naturally arcwise connected K-quasiconvex on A, there exists an arc $\Gamma_{\bar{x},x_1}$ on A such that for each $t \in [0,1]$, we can find some $s \in]0,1[$,

$$(1-s)f(\bar{x}, p_0) + sf(x_1, p_0) \in f(\Gamma_{\bar{x},x_1}(t), p_0) + K.$$

Therefore,

$$f(\Gamma_{\bar{x},x_1}(t), p_0) = (1-s)f(\bar{x}, p_0) + sf(x_1, p_0) - k, \quad \text{for some } k \in K,$$

which together with $\ell(k) \geq 0$ implies that

$$\ell(f(\Gamma_{\bar{x},x_1}(t), p_0)) \leq (1-s)\ell(f(\bar{x}, p_0)) + s\ell(f(x_1, p_0)). \tag{4.8}$$

It follows from $\bar{x} \in S_\ell(p_0), x_1 \in \widehat{S}_\ell(p_0)$ and (4.8) that

$$\begin{aligned} \ell(f(\Gamma_{\bar{x},x_1}(t), p_0)) &< (1-s)[\ell(f(x,p_0)) + \ell(e)] + s[\ell(f(x,p_0)) + \ell(e)] \\ &< \ell(f(x,p_0)) + \ell(e), \end{aligned}$$

for all $(x,e) \in A \times E$. Consequently, $\Gamma_{\bar{x},x_1}(t) \in \widehat{S}_\ell(p_0)$ for all $t \in [0,1]$. Combining this with $\Gamma_{\bar{x},x_1}(t) \to \bar{x}$ when $t \to 0$, we achieve $\bar{x} \in \mathrm{cl}\widehat{S}_\ell(p_0)$, and so $S_\ell(p_0) \subset \mathrm{cl}\widehat{S}_\ell(p_0)$. By the lower semicontinuity of $\widehat{S}_\ell$ at p_0, we have

$$S_\ell(p_0) \subset \mathrm{cl}\widehat{S}_\ell(p_0) \subset \liminf \widehat{S}_\ell(p_n) \subset \liminf S_\ell(p_n),$$

and so S_ℓ is lsc at p_0. □

Next, we present the continuity conditions of the Benson weakly efficient solution mappings via the following theorem.

Theorem 4.4. *Assume that*

(i) *A is compact and arcwise connected*;

(ii) *f is K-continuous on $A \times P$*;

(iii) *for each $p \in P$, $f(.,p)$ is naturally arcwise connected K-quasiconvex as well as nearly E-subconvexlike on A.*

Then, $\mathrm{WBEff}(A,f)$ *is nonempty-valued and continuous on P.*

Chứng minh. Employing Theorem 4.2, $\mathrm{WBEff}(A,f)$ is nonempty-valued and upper continuous on P, and hence we only prove that $\mathrm{WBEff}(A,f)$ is lower semicontinuous on P. Since $f(.,p)$ is nearly E-subconvexlike on A for all $p \in P$, Theorem 3.1 helps us obtain

$$\mathrm{WBEff}(A,f)(p) = \bigcup_{\ell \in K^* \setminus \{0_{\mathbb{Y}^*}\}} S_\ell(p). \tag{4.9}$$

Because of Theorem 4.3, mapping S_ℓ is lower semicontinuous on P. For all $p_0 \in P$ and open subset U in $\mathbb{X}$ satisfying $\mathrm{WBEff}(A,f)(p_0) \cap U \neq \emptyset$, by (4.9), we can find $\ell_0 \in K^* \setminus \{0_{\mathbb{Y}^*}\}$ such that $S_{\ell_0}(p_0) \cap U \neq \emptyset$. Since S_{ℓ_0} is lower semicontinuous at p_0, there exists a neighborhood V of p_0 such that $S_{\ell_0}(p) \cap U \neq \emptyset, \quad \forall p \in V$, which is equivalent to

$$\mathrm{WBEff}(A,f)(p) \cap U \neq \emptyset, \quad \forall p \in V.$$

It follows that $\mathrm{WBEff}(A,f)$ is lsc at p_0. □

Based on Theorem 4.4, we obtain the following corollary.

Corollary 4.1. *Assume that the conditions (i)-(ii) in Theorem 4.4 are satisfied, and assume further that $f(.,p)$ is arcwise connected K-convex on A for each $p \in P$. Then,* $\mathrm{WBEff}(A,f)$ *is nonempty-valued and continuous on P.*

Chứng minh. In view of Theorem 4.4, we only need to present that $f(.,p)$ is nearly E-subconvexlike on A. Since $E = k_0 + K$, we have $f(A,p) + E = f(A,p) + k_0 + K$ for all $p \in P$. Hence it is sufficient to prove that $f(A,p) + K$ is convex. Let y_1, y_2 be arbitrary in $f(A,p) + K$. Then, there are $x_1, x_2 \in A$ such that $y_1 \in f(x_1,p) + K$ and $y_2 \in f(x_2,p) + K$. These together with the arcwise connected K-convexity of f imply that there exists an arc Γ_{x_1,x_2} on A such that, for all $t \in [0,1]$,

$$\begin{aligned}(1-t)y_1 + ty_2 &\in (1-t)\left(f(x_1,p) + K\right) + t\left(f(x_2,p) + K\right) \\ &\in f(\Gamma_{x_1,x_2}(t),p) + K \subset f(A,p) + K.\end{aligned}$$

Hence, we conclude that $f(A,p) + E)$ is convex, so is $\mathrm{cl}(\mathrm{cone}(f(A,p) + E))$. The proof is complete. □

Remark 4.1. Up to our knowledge now, there have not been any works on the stability of the Benson weakly solution mappings of the vector optimization problems via the improvement sets, and so the results presented in this section are entirely new. Moreover, when studying the stability of solution mappings of many optimization models, such as vector equilibrium problems and vector optimization problems [34, 35], and set optimization problems [21], hypotheses related to the convexity properties of objective functions and constrained sets as key assumptions were used, so the obtained results are difficult to apply to bilevel optimization models as the convexity of solution sets of vector optimization models is not easy to task. Herein, the convexity conditions are weakened by the connectedness ones, and hence our approach is different from the existence one, and the obtained results of this paper have potential applications to bilevel optimization models.

The following example is given to illustrate a case in which Corollary 4.1 can apply while the results of [21, 34, 35] cannot use as the convexity conditions are violated.

Example 4.1. Let $\mathbb{X}=\mathbb{Y}=\mathbb{R}^2, \mathbb{W}=\mathbb{R}, A=\{(x_1,x_2)\in\mathbb{R}^2 : 0\le x_1\le 1, 0\le x_2\le 1\}$, $K=\mathbb{R}^2_+$, $E=(0.5,0.5)+\mathbb{R}^2_+$, $P=[-1,1]$, and $f:\mathbb{R}^2\times\mathbb{R}\to\mathbb{R}^2$ be defined by

$$f(x,p)=2^p\left(x_1^2x_2^2, x_1^2x_2^2\right),\quad \forall x=(x_1,x_2)\in\mathbb{R}^2.$$

Then, the conditions (i)-(iii) of Theorem 4.4 hold true. In order to apply Corollary 4.1, we need only to check the arcwise connected K-convexity of $f(\cdot,p)$. For each $\bar{x}=(\bar{x}_1,\bar{x}_2)$ and $\hat{x}=(\hat{x}_1,\hat{x}_2)$ in A, we consider the arc $\Gamma_{\bar{x},\hat{x}}:[0,1]\to A$ defined by

$$\Gamma_{\bar{x},\hat{x}}(t)=\begin{cases}(1-2t)\bar{x}, \text{ if } 0\le t\le 0.5,\\ (2t-1)\hat{x}, \text{ if } 0.5<t\le 1.\end{cases}$$

Then, for each $t\in[0,1]$, we prove

$$f(\Gamma_{\bar{x},\hat{x}}(t),p)\in(1-t)f(\bar{x},p)+tf(\hat{x},p)-K. \tag{4.10}$$

There are two cases for considering.
Case 1. If $0\le t\le 0.5$, then

$$\begin{aligned}f(\Gamma_{\bar{x},\hat{x}}(t),p)&=f((1-2t)\bar{x},p)=2^p\left((1-2t)^4(\bar{x}_1\bar{x}_2)^2,(1-2t)^4(\bar{x}_1\bar{x}_2)^2\right)\\&\in 2^p\left((1-t)(\bar{x}_1\bar{x}_2)^2,(1-t)(\bar{x}_1\bar{x}_2)^2\right)-K\\&\in(1-t)2^p\left((\bar{x}_1\bar{x}_2)^2,(\bar{x}_1\bar{x}_2)^2\right)+t2^p\left((\hat{x}_1\hat{x}_2)^2,(\hat{x}_1\hat{x}_2)^2\right)-K\\&\in(1-t)f(\bar{x},p)+tf(\hat{x},p)-K,\end{aligned}$$

and hence (4.10) holds.
Case 2. If $0.5<t\le 1$, then

$$\begin{aligned}f(\Gamma_{\bar{x},\hat{x}}(t),p)&=f((2t-1)\hat{x},p)=2^p\left((2t-1)^4(\hat{x}_1\hat{x}_2)^2,(2t-1)^4(\hat{x}_1\hat{x}_2)^2\right)\\&\in 2^p\left(t(\hat{x}_1\hat{x}_2)^2,t(\hat{x}_1\hat{x}_2)^2\right)-K\\&\in(1-t)2^p\left((\bar{x}_1\bar{x}_2)^2,(\bar{x}_1\bar{x}_2)^2\right)+t2^p\left((\hat{x}_1\hat{x}_2)^2,(\hat{x}_1\hat{x}_2)^2\right)-K\\&\in(1-t)f(\bar{x},p)+tf(\hat{x},p)-K,\end{aligned}$$

and consequently (4.10) satisfies. Therefore, by Corollary 4.1, $\mathrm{WBEff}(A,f)$ is nonempty-valued and continuous on P. However, the vector-valued mapping f is not convex in the first component as for $x=(0.5,0.5), z=(1,0), t=0.5$. Hence,

$$\frac{1}{2}(f(x)+f(z))=2^p\left(\frac{1}{32},\frac{1}{32}\right)<2^p\left(\frac{9}{256},\frac{9}{256}\right)=f\left(\frac{1}{2}x+\frac{1}{2}z\right).$$

Therefore, the results in [21, 34, 35] cannot work.

5. A Practical Case in Optimization

Let $\mathbb{X}, \mathbb{Y}, \mathbb{W}, A, P$ and f be defined as in Section 4. In this section, we apply the results obtained in Theorems 4.2, 4.4, and Corollary 4.1 to address the (semi) continuity of the Benson weakly efficient solution mappings of (PVOP) with respect to co-radiant sets.

Definition 5.1. [20, P. 167] A nonempty subset R of $\mathbb{Y}$ is said to be *co-radiant* if for all $r \in R$ and $\alpha \geq 1$, $\alpha r \in R$.

The co-radiant set is a useful tool for studying approximate efficiency in optimization problems. To be more precise, via the co-radiant set, Gutiérrez et al. [36] introduced a new kind of approximate solutions to unify several existing approximate solutions, and the authors also established nonlinear scalarization results for the unified approximate solutions. Motivated by [36], Gao et al. [37] introduced a variety of properly efficient solutions for vector optimization problems and discussed their relationships as well as optimality conditions for these efficient solutions via nonlinear scalarization method. Then, based on the Hiriart-Urruty orient distance, Zhao et al. [38] provided nonlinear scalarization functions for vector optimization problems via co-radiant sets, and then by employing this function, the authors obtained many improvement versions of the results in [37] for Benson properly efficient solutions of concerning problems. Next, Sayadi-bander et al. [39] introduced and studied properties of Bishop-Phelps co-radiant sets and their duality, and then by using these results the authors established characterization properties for approximate efficient points via separation conditions for co-radiant sets. Recently, Gao and Xu [40] suggested several proper efficient solutions for multiobjective optimization problems involving co-radiant sets such as Benson proper efficient solutions, Borwein proper efficient solutions, proximal efficient solutions, Benson efficient solutions, super proper efficient solutions, Henig global proper efficient solutions. Then, by using the linear scalarization method, the authors have succeeded in studying optimality conditions and relations of these proper efficiencies. For generalized settings and applications of co-radiant sets, we refer the reader to typical works [41, 42, 43, 44, 45] and the references therein.

Motivated and inspired by above observations, in this section, we aim to propose the Benson weakly efficient solution of parametric vector optimization problems with respect to the co-radiant sets and study the existence and stability of solutions.

Let $R \subset K \setminus \{0_{\mathbb{Y}}\}$ be a convex and solid co-radiant given set. We consider the following parametric optimization problem (PVOP):

$$\text{(PVOP)} \quad \min \quad f(x,p) \quad \text{subject to} \quad (x,p) \in A \times P.$$

Based on the ideas of [20, 22, 36], we propose concepts of efficiency of (PVOP) via the co-radiant set R as follows.

Definition 5.2. For each $p \in P$, a vector $x_0 \in A$ is called *a Benson weakly efficient solution* of (PVOP) with respect to R, written $x_0 \in$cr-WBEff$(A, f)(p)$ if $\mathrm{cl}(\mathrm{cone}(f(A,p) - f(x_0,p) + R)) \cap (-\mathrm{int}C) = \emptyset$, where $C = \mathrm{cone}(R)$.

Now, we apply the obtained results of the previous sections to discuss the corresponding qualitative properties of the mapping cr-WBEff$(A, f)(p)$.

Corollary 5.1. *Assume that*

(i) *A is compact;*

(ii) *f is C-continuous on $A \times P$*;

(iii) *for each $p \in P$, $f(\cdot, p)$ is nearly R-subconvexlike on A.*

Then, cr-WBEff(A, f) *is nonempty-valued and upper semicontinuous on P.*

Chứng minh. In order to apply Theorem 4.2, we first prove that R is an improvemnet set in $\mathbb{Y}$ with respect to C. It follows from $R \subset K \setminus \{0_{\mathbb{Y}}\}$ that the vector zero does not belong to the set R. For any $y \in R + C$, there exist $r \in R$ and $c \in C$ such that $y = r + c$, which together with $C = \text{cone}(R)$ implies that we can find $t \in \mathbb{R}^+, \hat{r} \in R$ such that

$$y = r + t\hat{r} = (1+t)\left(\frac{1}{1+t}r + \frac{t}{1+t}\hat{r}\right) \in R,$$

as R is a convex co-radiant set. Thus, we have $R + C \subset R$, and so $R + C = R$.

Next, we check that $R \cap (-\operatorname{int} C) = \emptyset$. Since $R \subset K \setminus \{0_{\mathbb{Y}}\}$ and K is a cone,

$$\operatorname{cone} R = C \subset \operatorname{cone} K = K. \tag{5.1}$$

Consequently, $C \cap (-C) = \{0\}$ as K is a pointed cone. Hence, by $R \subset C$, we have $R \cap (-\operatorname{int} C) = \emptyset$. Therefore, all conditions of Theorem 4.2 are satisfied, and so the conclusions of Corollary 5.1 are followed from the mentioned theorem. □

By using the techniques given in the proof of Corollary 5.1, we also derive the following results from Theorem 4.4 and Corollary 4.1.

Corollary 5.2. *Assume that*

(i) *A is compact and arcwise connected*;

(ii) *f is C-continuous on $A \times P$*;

(iii) *for each $p \in P$, $f(\cdot, p)$ is naturally arcwise connected C-quasiconvex as well as nearly R-subconvexlike on A.*

Then, cr-WBEff(A, f) *is nonempty-valued and continuous on P.*

Corollary 5.3. *Assume that*

(i) *A is compact and arcwise connected*;

(ii) *f is C-continuous on $A \times P$*;

(iii) *for each $p \in P$, $f(\cdot, p)$ is arcwise connected C-convex on A*;

Then, cr-WBEff(A, f) *is nonempty-valued and continuous on P.*

To the best of our knowledge, until now there have not been any works on solvability and stability of the Benson weakly efficient solutions of vector optimization problems with respect to the co-radiant sets, and hence the results of this section are new.

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