

## CONVEXITY AND CONTINUITY OF SPECIFIC SET-VALUED MAPS AND THEIR EXTREMAL VALUE FUNCTIONS

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**Abstract.** In this paper, we study several classes of set-valued maps, which can be used in set-valued optimization and its applications, and their respective maximum and minimum value functions. The definitions of these maps are based on scalar-valued, vector-valued, and cone-valued maps. Moreover, we consider those extremal value functions which are obtained when optimizing linear functionals over the image sets of the set-valued maps. Such extremal value functions play an important role for instance for derivative concepts for set-valued maps or for algorithmic approaches in set-valued optimization. We formulate conditions under which the set-valued maps and their extremal value functions inherit properties like (Lipschitz-)continuity and convexity.

**Keywords.** Continuity; Convexity; Extremal value functions; Set-valued maps.

**2020 Mathematics subject classifications.** 26E25, 49J53, 54C60.

### 1. INTRODUCTION

Set-valued mappings arise in a vast variety in optimization theory and applications. Such maps are of interest, for example, in parametric and semi-infinite optimization [1], in bilevel optimization [2], and in robust optimization [3]. Important applications of set-valued maps can be found in the areas of socio economics [4], welfare economics [5], and finance [6].

In this paper we examine set-valued maps with special structures defined by scalar-valued, vector-valued, and cone-valued maps. Moreover, we study for these set-valued maps special types of extremal value functions obtained pointwise by minimizing or maximizing certain linear functionals over the image sets (see the forthcoming Definition 2.1). The extremal value functions defined in this way play an important role in the field of set-valued optimization, i.e., optimization problems with a set-valued objective function and where the image space is partially ordered by a given ordering cone. For a detailed introduction to set-valued optimization we refer to the book by Khan, Tammer and Zălinescu [7].

In the context of set-valued optimization there exist at least two approaches for optimality concepts. In the so called vector approach one is interested in nondominated points of the union of all image sets. In contrast to that, by using the so called set approach one defines

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Received April 22, 2022; Accepted October 28, 2022.

optimal elements by binary relations which compare image sets as a whole. Note that the latter approach is considered to be more realistic in many situations (cf. [8]). Such binary relations have been introduced in the literature in a wide variety. For an overview we refer to [9, 10]. Most prominent are the lower less relation, the upper less relation, and the set-less relation which is the combination of both.

According to [11], these binary relations can be characterized under certain convexity assumptions by infinitely many inequalities which make use of the above mentioned extremal value functions. Based on these characterizations, optimal solutions of set-valued optimization problems can be obtained by optimizing these extremal value functions. Additional investigations in this regard can be found in [12] and [13]. In addition to that, the extremal value functions can be used for the definition of set differences and based on this for directional derivatives of set-valued maps in the case of strictly convex and compact image sets [8] or, more general, in the case of convex and compact image sets [14]. These approaches allow numerical calculations and the formulation of necessary optimality conditions for set-valued optimization problems.

In the field of optimization in general (Lipschitz-)continuity and (quasi-)convexity of mathematical functions and mappings are of high relevance. Due to the reasons mentioned above we investigate in this paper under which assumptions on the scalar-valued, vector-valued, and cone-valued maps the respective set-valued maps and their extremal value functions fulfill these properties. Several of these results can be derived as special cases from results in the literature on parametric optimization, where they have been provided in a very general setting (cf. [15]). Others have already been examined in [16]. By using the specific structure and shifting to pre-image space  $\mathbb{R}^n$  and image space  $\mathbb{R}^m$  we simplify the results.

Additionally, we provide new results concerning the (Lipschitz-)continuity and  $u$ -type  $C$ -convexity of some of the above mentioned types of set-valued maps. By introducing a new concept of quasiconvexity that generalizes natural  $K$ -quasiconvexity in the sense of Tanaka [17], we are able to generalize a result that provides a sufficient condition for  $\ell$ -type  $C$ -convexity for one of the considered set-valued maps.

Moreover, by the structure of the paper we aim at providing a suitable overview for researches who want to apply optimization methods to the given set-valued maps. Therefore, we are interested in a condensed overview of relationships between the characterizing functions in the definitions of the set-valued maps, the set-valued maps themselves, and their respective extremal value functions.

The paper is organized as follows. In Section 2 we give the basic notations and definitions that we use throughout the paper. Section 3 contains the main results. First, we investigate continuity properties of the set-valued maps and their extremal value functions. Afterwards, we investigate Lipschitzianity and finally we study convexity properties. In Section 4 we give a short summary.

## 2. PRELIMINARIES

We start this section by introducing the notation we use in this paper. By  $\mathbb{N}$  we denote the natural numbers not including zero, by  $\mathbb{R}$  we denote the real numbers, and by  $\mathbb{R}_+$  we denote the nonnegative real numbers. Based on this we define  $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ . As usual, elements  $v$  of  $\mathbb{R}^m$  are considered as column vectors, and by  $v^\top$  we denote the transposed of  $v$ . The zero vector of a real linear space is denoted as  $0$ . In case it is not obvious to which space the  $0$

refers to, we denote the respective space in the index of the 0. For a set  $A \subseteq \mathbb{R}^m$  we denote its interior, closure, cardinality and convex hull by  $\text{int}(A)$ ,  $\text{cl}(A)$ ,  $|A|$  and  $\text{conv}(A)$ , respectively. Moreover, we denote by  $\|\cdot\|$  the Euclidean norm in  $\mathbb{R}^m$ . According to this we define for an element  $y^0 \in \mathbb{R}^m$  and  $\varepsilon \geq 0$  the sets

$$\mathbb{B}(y^0, \varepsilon) := \{y \in \mathbb{R}^m \mid \|y^0 - y\| < \varepsilon\} \quad \text{and} \quad \bar{\mathbb{B}}(y^0, \varepsilon) := \{y \in \mathbb{R}^m \mid \|y^0 - y\| \leq \varepsilon\}.$$

If  $y^0 = 0$  and  $\varepsilon = 1$ , we just write  $\mathbb{B}$  and  $\bar{\mathbb{B}}$  instead of  $\mathbb{B}(0, 1)$  and  $\bar{\mathbb{B}}(0, 1)$ , respectively. Furthermore, for  $A \subseteq \mathbb{R}^m$  and a point  $y^0 \in \mathbb{R}^m$  we denote the distance of  $y^0$  to  $A$  as

$$d(y^0, A) := \inf_{y \in A} \|y^0 - y\|.$$

Note that here and in the whole paper we use the convention  $\inf_{\emptyset} := \infty$  and  $\sup_{\emptyset} := -\infty$ . Finally, for  $k \in \mathbb{N}$  the set  $[k]$  is defined as  $[k] := \{i \in \mathbb{N} \mid i \leq k\}$ .

Now, we recall the main concepts that are used throughout this paper. First, recall that a nonempty set  $C \subseteq \mathbb{R}^m$  is a cone if  $\lambda k \in C$  holds for all  $k \in C$  and all  $\lambda \geq 0$ . A cone  $C$  is called pointed if  $C \cap (-C) = \{0\}$  and solid if  $\text{int}(C) \neq \emptyset$ . Moreover, a cone  $C$  is convex if and only if  $C + C = C$  and by  $C^*$  we denote the dual cone of  $C$  defined as

$$C^* := \{\ell \in \mathbb{R}^m \mid \ell^\top k \geq 0 \text{ for all } k \in C\}.$$

A cone  $C$  is called polyhedral if there exists  $B \in \mathbb{R}^{d \times m}$  such that  $C = \{y \in \mathbb{R}^m \mid By \leq_{\mathbb{R}_+^d} 0\}$ . For elements  $y^1, y^2 \in \mathbb{R}^m$  and a cone  $C \subseteq \mathbb{R}^m$  we define the following binary relation

$$y^1 \leq_C y^2 : \iff y^2 - y^1 \in C$$

and for a solid cone  $C$  we define

$$y^1 <_C y^2 : \iff y^2 - y^1 \in \text{int}(C).$$

The binary relations defined above can be generalized for sets  $A, B \subseteq \mathbb{R}^m$ . We define

$$A \preceq_C^\ell B : \iff B \subseteq A + C \quad \text{and} \quad A \preceq_C^u B : \iff A \subseteq B - C.$$

Obviously, for  $y^1, y^2 \in \mathbb{R}^m$  we have  $y^1 \leq_C y^2$  if and only if  $\{y^1\} \preceq_C^\ell \{y^2\}$  if and only if  $\{y^1\} \preceq_C^u \{y^2\}$ . The relation  $\preceq_C^\ell$  is the so called lower less relation and  $\preceq_C^u$  is the so called upper less relation. These two binary relations can be used when comparing sets in the context of set-valued optimization. See [10] for an overview on these and other set relations. In case  $C = \mathbb{R}_+^m$  we may omit the subscript  $C$ . For a given set-valued map  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$  we define its domain and its graph by

$$\text{dom}(F) := \{x \in \mathbb{R}^n \mid F(x) \neq \emptyset\}$$

and

$$\text{gph}(F) := \{(x, y) \in \text{dom}(F) \times \mathbb{R}^m \mid y \in F(x)\},$$

respectively. Furthermore, the epigraph of  $F$  with respect to a cone  $C \subseteq \mathbb{R}^m$  is defined as

$$\text{epi}(F, C) := \{(x, y) \in \text{dom}(F) \times \mathbb{R}^m \mid \exists z \in F(x) : z \leq_C y\}.$$

Unless stated otherwise, we work with the following setup throughout the rest of this paper.

**Assumption 2.1.** Let  $C \subseteq \mathbb{R}^m$  be a closed, pointed, solid and convex cone, let  $q \in \mathbb{R}^m$ , let  $g, g^1, g^2: \mathbb{R}^n \rightarrow \mathbb{R}^m$  be vector-valued functions with  $g^1(x) \leq_C g^2(x)$  for all  $x \in \mathbb{R}^n$ , let  $r: \mathbb{R}^n \rightarrow \mathbb{R}_+$  be a nonnegative scalar-valued function, let  $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map, let  $A \subseteq \mathbb{R}^m$  be a nonempty set, let  $P: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$  be a set-valued map such that  $P(x)$  is a closed convex cone for all  $x \in \mathbb{R}^n$  and let  $h: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^k$  be a vector-valued function. Then, we define the set-valued maps  $F_j: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$  for  $j \in [6]$  by

- $F_1(x) := [g(x), g(x) + q] := \{\lambda g(x) + (1 - \lambda)(g(x) + q) \mid \lambda \in [0, 1]\}$  (interval-type)
- $F_2(x) := [g^1(x), g^2(x)]_C := (\{g^1(x)\} + C) \cap (\{g^2(x)\} - C)$  (box-type)
- $F_3(x) := \{g(x)\} + r(x)\bar{\mathbb{B}}$  (tubular-type)
- $F_4(x) := \{L(x)\} + A$  (affinelike-type)
- $F_5(x) := \{g(x)\} + P(x)$  (cone-valued-type)
- $F_6(x) := \{y \in \mathbb{R}^m \mid h(x, y) \leq_{\mathbb{R}_+^k} 0\}$  (constraint-type)

for all  $x \in \mathbb{R}^n$ . Additionally, we assume that  $\text{dom}(F_6) \neq \emptyset$ .

By definition,  $\text{dom}(F_i) = \mathbb{R}^n$  for  $i \in [5]$ . The set-valued maps  $F_j$ ,  $j \in [5]$  have been studied in [16]. There, certain continuity properties and  $\ell$ -type  $C$ -convexity (see the forthcoming Definition 3.6) have already been examined. Here, we want to further investigate properties of these set-valued mappings and also study set-valued maps of the type  $F_6$ . Thereby, we have a special interest in the relationship between properties of the function  $g, g^1, g^2, r, L, P, h$  and the set-valued maps  $F_j$ ,  $j \in [6]$ . Note that under additional assumptions it is possible to rewrite  $F_j$  for all  $j \in [5]$  in the form of  $F_6$ . However, by maintaining the structure given by the respective definitions it is possible to drop some assumptions and hence, obtain stronger results as it would be possible when using said reformulations.

As we mentioned in the introduction, in set-valued optimization it is often helpful to study the following minimum and maximum value functions of the according optimization problem; see [11, 12, 13]. This is particularly the case if the image sets of the set-valued mappings are convex. Hence, we investigate the convexity of these sets in the forthcoming Lemma 3.3.

**Definition 2.1.** Let  $\ell \in C^*$ . Then the minimum value function  $\varphi_{\min}^{F, \ell}: \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$  and the maximum value function  $\varphi_{\max}^{F, \ell}: \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$  of a set-valued map  $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$  are defined by

$$\varphi_{\min}^{F, \ell}(x) := \inf_{y \in F(x)} \ell^\top y \text{ for all } x \in \mathbb{R}^n,$$

$$\varphi_{\max}^{F, \ell}(x) := \sup_{y \in F(x)} \ell^\top y \text{ for all } x \in \mathbb{R}^n.$$

Note that these are special cases of extreme value functions, cf. [15].

### 3. MAIN RESULTS

First, we investigate some basic properties of the image sets  $F_j(x)$ ,  $j \in [6]$ . Therefore, we need the following concepts of continuity for scalar-valued functions (cf. [15, Chapter 2]).

**Definition 3.1.** A scalar-valued function  $f: \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$  is called

- (i) lower semicontinuous (l.s.c.) at  $x^0 \in \mathbb{R}^n$  if for every sequence  $(x^k)_{k \in \mathbb{N}} \subseteq \mathbb{R}^n$  with  $x^k \rightarrow x^0$  it holds

$$\inf\{\alpha \in \mathbb{R} \mid f(x^k) \leq \alpha \text{ for almost all } k \in \mathbb{N}\} \geq f(x^0).$$

- (ii) upper semicontinuous (u.s.c.) at  $x^0 \in \mathbb{R}^n$  if for every sequence  $(x^k)_{k \in \mathbb{N}} \subseteq \mathbb{R}^n$  with  $x^k \rightarrow x^0$  it holds

$$\sup\{\alpha \in \mathbb{R} \mid f(x^k) \geq \alpha \text{ for almost all } k \in \mathbb{N}\} \leq f(x^0).$$

- (iii) continuous at  $x^0 \in \mathbb{R}^n$  if it is l.s.c. and u.s.c. at  $x^0$ .

If a scalar-valued function  $f$  is l.s.c., u.s.c. or continuous at every point  $x \in \mathbb{R}^n$ , then  $f$  is called l.s.c., u.s.c. or continuous, respectively.

This allows us to formulate the following lemma investigating closedness and compactness of the image sets  $F_j(x)$ . The statements (i) to (iv) are given in [16, Proposition 3.1]. The proof of statement (v) is easy and hence omitted.

**Lemma 3.1.** *The following statements hold:*

- (i) *The set  $F_j(x)$  is closed for all  $x \in \mathbb{R}^n$  and  $j \in [5] \setminus \{4\}$ .*
- (ii) *If  $A$  is closed, then the set  $F_4(x)$  is closed for all  $x \in \mathbb{R}^n$ .*
- (iii) *The set  $F_j(x)$  is compact for all  $x \in \mathbb{R}^n$  and  $j \in [3]$ .*
- (iv) *If  $A$  is compact, then the set  $F_4(x)$  is compact for all  $x \in \mathbb{R}^n$ .*
- (v) *If  $h_i(x, \cdot) : \mathbb{R}^m \rightarrow \mathbb{R}$  is l.s.c. for an  $x \in \mathbb{R}^n$  and all  $i \in [k]$ , then the set  $F_6(x)$  is closed. Hence, if  $F_6(x)$  is additionally bounded, then  $F_6(x)$  is a compact set.*

Due to the structure provided by the set-valued maps  $F_j$ ,  $j \in [4]$  we can explicitly calculate their respective extremal value functions. This will be very useful for our investigations later on. For  $j = 5$  we can do this only under additional assumptions and for  $j = 6$  we can only formulate a sufficient condition such that  $\varphi_{\min}^{F_6, \ell}(x)$  and  $\varphi_{\max}^{F_6, \ell}(x)$  are attained for  $x \in \text{dom}(F_6)$ .

**Lemma 3.2.** *Let  $\ell \in C^*$  and  $x \in \mathbb{R}^n$ . Then, the infima  $\varphi_{\min}^{F_j, \ell}(x)$  and suprema  $\varphi_{\max}^{F_j, \ell}(x)$  for  $j \in [5]$  are attained and the following statements hold:*

- (i)  $\varphi_{\min}^{F_1, \ell}(x) = \ell^\top g(x) + \min\{0, \ell^\top q\}$  and  $\varphi_{\max}^{F_1, \ell}(x) = \ell^\top g(x) + \max\{0, \ell^\top q\}$ .
- (ii)  $\varphi_{\min}^{F_2, \ell}(x) = \ell^\top g^1(x)$  and  $\varphi_{\max}^{F_2, \ell}(x) = \ell^\top g^2(x)$ .
- (iii)  $\varphi_{\min}^{F_3, \ell}(x) = \ell^\top g(x) - \|\ell\|r(x)$  and  $\varphi_{\max}^{F_3, \ell}(x) = \ell^\top g(x) + \|\ell\|r(x)$ .
- (iv) *If  $A$  is compact, then it holds*

$$\varphi_{\min}^{F_4, \ell}(x) = \ell^\top L(x) + \min_{y \in A} \ell^\top y \quad \text{and} \quad \varphi_{\max}^{F_4, \ell}(x) = \ell^\top L(x) + \max_{y \in A} \ell^\top y.$$

- (v) *If  $P(x) \subseteq C$  or  $P(x) \subseteq -C$ , then it holds*

$$\varphi_{\min}^{F_5, \ell}(x) = \ell^\top g(x) \quad \text{or} \quad \varphi_{\max}^{F_5, \ell}(x) = \ell^\top g(x),$$

*respectively.*

- (vi) *If  $x \in \text{dom}(F_6)$ ,  $F_6(x)$  is bounded and  $h_i(x, \cdot)$  is l.s.c. for all  $i \in [k]$ , then  $\varphi_{\min}^{F_6, \ell}(x)$  and  $\varphi_{\max}^{F_6, \ell}(x)$  are attained.*

*Proof.* Note that  $\ell \in C^*$  implies  $\ell^\top y \geq 0$  for all  $y \in C$  and  $\ell^\top z \leq 0$  for all  $z \in -C$ . Based on that, the proofs of (i), (ii), (iv) and (v) are easy and hence omitted. For the proof of (iii) we refer to [18, Lemma 10.3.17]. Statement (vi) follows by Lemma 3.1 and the Weierstrass theorem.  $\square$

For examining convexity of the image sets  $F_j(x)$ ,  $j \in [6]$  we need the well-known and important concept of  $K$ -convexity of a vector-valued function w.r.t. a convex cone  $K$ .

**Definition 3.2.** Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a vector-valued function and  $K \subseteq \mathbb{R}^m$  be a convex cone. Then  $f$  is called  $K$ -convex if for all  $x^1, x^2 \in \mathbb{R}^n$  and all  $\lambda \in [0, 1]$  it holds

$$f(\lambda x^1 + (1 - \lambda)x^2) \leq_K \lambda f(x^1) + (1 - \lambda)f(x^2). \quad (3.1)$$

If  $M \subseteq \mathbb{R}^n$  is a convex set and (3.1) holds for all  $x^1, x^2 \in M$  and all  $\lambda \in [0, 1]$ , then  $f$  is called  $K$ -convex on  $M$ .

Note that  $f = (f_1, \dots, f_m)^\top: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is  $\mathbb{R}_+^m$ -convex (on a convex set  $M \subseteq \mathbb{R}^n$ ) if and only if  $f_i$  is convex (on  $M$ ) for all  $i \in [m]$ .

The following lemma examines convexity of the image sets  $F_j(x)$ ,  $j \in [6]$ . The statements (i) and (ii) are given in [16, Proposition 3.1]. The proof of statement (iii) is easy and hence omitted.

**Lemma 3.3.** *The following statements hold for all  $x \in \mathbb{R}^n$ :*

- (i) *The set  $F_j(x)$  is convex for all  $j \in [5] \setminus \{4\}$ .*
- (ii) *If  $A$  is convex, then the set  $F_4(x)$  is convex.*
- (iii) *If  $h(x, \cdot): \mathbb{R}^m \rightarrow \mathbb{R}^k$  is  $\mathbb{R}_+^k$ -convex, then  $F_6(x)$  is convex.*

**3.1. Continuity.** In this section we examine continuity properties of the set-valued maps  $F_j$ ,  $j \in [6]$  and their respective extremal value functions according to Definition 2.1. Thereby, we use the following concepts (cf. [15, Section 2.2], [19, Chapter 1] and [20, Definition 5.14]).

**Definition 3.3.** A set-valued map  $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$  is called

- (i) closed at  $x^0 \in \mathbb{R}^n$  if for all sequences  $(x^k)_{k \in \mathbb{N}} \subseteq \mathbb{R}^n$  and  $(y^k)_{k \in \mathbb{N}} \subseteq \mathbb{R}^m$  with

$$x^k \rightarrow x^0, \quad y^k \rightarrow y^0, \quad y^k \in F(x^k) \text{ for all } k \in \mathbb{N}$$

it holds  $y^0 \in F(x^0)$ .

- (ii) locally bounded at  $x^0 \in \mathbb{R}^n$  if there exist  $\delta, \varepsilon > 0$  such that for all  $x \in \mathbb{B}(x^0, \delta)$  it holds

$$F(x) \subseteq \varepsilon \mathbb{B}. \quad (3.2)$$

- (iii) locally compact at  $x^0 \in \mathbb{R}^n$  if there exists  $\delta > 0$  and a compact set  $\mathcal{K} \subseteq \mathbb{R}^m$  such that for all  $x \in \mathbb{B}(x^0, \delta)$  it holds

$$F(x) \subseteq \mathcal{K}.$$

- (iv) lower semicontinuous (l.s.c.) at  $x^0 \in \mathbb{R}^n$  if for each open set  $\Omega$  with  $\Omega \cap F(x^0) \neq \emptyset$  there exists a  $\delta > 0$  such that  $\Omega \cap F(x) \neq \emptyset$  holds for all  $x \in \mathbb{B}(x^0, \delta)$ .

- (v) upper semicontinuous (u.s.c.) at  $x^0 \in \mathbb{R}^n$  if for each open set  $\Omega$  with  $F(x^0) \subseteq \Omega$  there exists a  $\delta > 0$  such that  $F(x) \subseteq \Omega$  holds for all  $x \in \mathbb{B}(x^0, \delta)$ .

- (vi) continuous at  $x^0 \in \mathbb{R}^n$  if it is l.s.c. and u.s.c. at  $x^0$ .

If a set-valued map  $F$  is closed, locally bounded, locally compact, l.s.c., u.s.c. or continuous at every point  $x^0 \in \mathbb{R}^n$ , then  $F$  is called closed, locally bounded, locally compact, l.s.c., u.s.c. or continuous, respectively. Moreover, if there exists  $\varepsilon > 0$  such that (3.2) holds for every  $x \in \mathbb{R}^n$ , then  $F$  is called globally bounded.

**Remark 3.1.** Obviously, a set-valued map  $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$  is locally bounded at  $x^0 \in \mathbb{R}^n$  if and only if  $F$  is locally compact at  $x^0$ . Still, we use both notions in order to maintain consistency with the results we cite.

Note that different continuity concepts for  $F_j$ ,  $j \in [5]$  have been studied for instance in [16, Proposition 3.4]. Here, we focus on l.s.c. and u.s.c. as these concepts allow us to make statements concerning the continuity of the extremal value functions.

The following proposition investigates continuity properties of the sum of set-valued maps that will be useful in later proofs.

**Proposition 3.1.** *Let  $F, G: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$  be set-valued maps and let  $F + G: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$  be defined by*

$$(F + G)(x) := F(x) + G(x).$$

*Then, the following statements hold:*

- (i) *If  $F$  and  $G$  are u.s.c. at  $x^0 \in \mathbb{R}^n$  and the image sets  $F(x^0)$  and  $G(x^0)$  are compact, then  $F + G$  is u.s.c. at  $x^0$ .*
- (ii) [21, Proposition 2.5.13] *If  $F$  and  $G$  are l.s.c. at  $x^0 \in \mathbb{R}^n$ , then  $F + G$  is l.s.c. at  $x^0$ .*

*Proof.* For the proof of (i) let  $F$  and  $G$  be u.s.c. at  $x^0 \in \mathbb{R}^n$  and the image sets  $F(x^0)$  and  $G(x^0)$  be compact. Furthermore, let  $\Omega \subseteq \mathbb{R}^m$  be an open set such that  $F(x^0) + G(x^0) \subseteq \Omega$  holds. Then, there exists  $\varepsilon > 0$  such that  $F(x^0) + G(x^0) + 2\varepsilon\mathbb{B} \subseteq \Omega$  holds, since  $F(x^0) + G(x^0)$  is compact and  $\Omega$  is open. Moreover, there exists  $\delta > 0$  such that  $F(x) \subseteq F(x^0) + \varepsilon\mathbb{B}$  and  $G(x) \subseteq G(x^0) + \varepsilon\mathbb{B}$  holds for all  $x \in \mathbb{B}(x^0, \delta)$ , since  $F$  and  $G$  are u.s.c. at  $x^0$  and  $F(x^0) + \varepsilon\mathbb{B}$  and  $G(x^0) + \varepsilon\mathbb{B}$  are open sets. Then, for all  $x \in \mathbb{B}(x^0, \delta)$  it holds  $F(x) + G(x) \subseteq F(x^0) + G(x^0) + 2\varepsilon\mathbb{B} \subseteq \Omega$  and we are done.  $\square$

For results considering the continuity of  $F_6$  we make use of additional concepts. Based on the definition of  $F_6$  we define the set-valued map  $F_6^0: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$  by

$$F_6^0(x) := \begin{cases} \{y \in \mathbb{R}^m \mid h(x, y) <_{\mathbb{R}_+^k} 0\} & \text{for all } x \in \text{dom}(F_6) \\ \emptyset & \text{for all } x \in \mathbb{R}^n \setminus \text{dom}(F_6) \end{cases}.$$

Additionally, we need the concept of strict quasiconvexity for scalar-valued functions. Recall that  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is called strictly quasiconvex if for all  $x^1, x^2 \in \mathbb{R}^n$  with  $f(x^1) < f(x^2)$  and all  $\lambda \in (0, 1)$  the inequality  $f(\lambda x^1 + (1 - \lambda)x^2) < f(x^2)$  holds. Note that convex functions are in particular strictly quasiconvex. The following theorem establishes relationships between the continuity of  $h$  and the continuity of  $F_6$ . The results are a consequence from [15, Theorem 3.1.1], [15, Theorem 3.1.2], [15, Theorem 3.1.5] and [15, Theorem 3.1.6] for our specific setting.

**Theorem 3.1.** *Let  $x^0 \in \text{dom}(F_6)$ . Then the following statements hold:*

- (i) *If the functions  $h_i$  are l.s.c. on  $\{x^0\} \times \mathbb{R}^m$  for all  $i \in [k]$ , then  $F_6$  is closed at  $x^0$ .*
- (ii) *If the functions  $h_i$  are l.s.c. on  $\{x^0\} \times \mathbb{R}^m$  for all  $i \in [k]$  and  $F_6$  is locally compact at  $x^0$ , then  $F_6$  is u.s.c. at  $x^0$ .*
- (iii) *If the functions  $h_i$  are u.s.c. on  $\{x^0\} \times \mathbb{R}^m$  for all  $i \in [k]$  and  $F_6(x^0) \subseteq \text{cl}(F_6^0(x^0))$ , then  $F_6$  is l.s.c. at  $x^0$ .*
- (iv) *If the functions  $h_i$  are u.s.c. on  $\{x^0\} \times \mathbb{R}^m$  for all  $i \in [k]$ , the functions  $h_i(x^0, \cdot)$  are strictly quasiconvex for all  $i \in [k]$  and  $F_6^0(x^0) \neq \emptyset$ , then  $F_6$  is l.s.c. at  $x^0$ .*

Note that the requirement of local compactness in statement (ii) is equivalent to the fact that  $F_6$  is locally bounded at  $x^0$  according to Remark 3.1. Also note that (iv) needs stronger assumptions on the  $h_i$  than (iii) to derive the same conclusion, but in practice the condition  $F_6^0(x^0) \neq \emptyset$  is easier to check than  $F_6(x^0) \subseteq \text{cl}(F_6^0(x^0))$ . The following theorem gives sufficient

conditions for the continuity of  $F_j$ ,  $j \in [6] \setminus \{5\}$  and l.s.c. of  $F_5$ . Note that u.s.c. of  $F_5$  cannot be guaranteed by Proposition 3.1 (i) due to the noncompactness of the image sets of  $P$ .

**Theorem 3.2.** *Let  $x^0 \in \mathbb{R}^n$ . Then the following statements hold:*

- (i) *If  $g$  is continuous at  $x^0$ , then  $F_1$  is continuous at  $x^0$ .*
- (ii) *If  $g^1$  and  $g^2$  are continuous at  $x^0$ ,  $g^1(x^0) <_C g^2(x^0)$  and  $C$  is polyhedral with  $\text{int}(C) = \{y \in \mathbb{R}^m \mid By <_{\mathbb{R}^d} 0\}$  for some  $B \in \mathbb{R}^{d \times m}$ , then  $F_2$  is continuous at  $x^0$ .*
- (iii) *If  $g$  and  $r$  are continuous at  $x^0$ , then  $F_3$  is continuous at  $x^0$ .*
- (iv) *If  $A$  is compact, then  $F_4$  is continuous at  $x^0$ .*
- (v) *If  $g$  is continuous at  $x^0$  and  $P$  is l.s.c. at  $x^0$ , then  $F_5$  is l.s.c. at  $x^0$ .*
- (vi) *If  $h_i$  is continuous on  $\{x^0\} \times \mathbb{R}^m$  and  $h_i(x^0, \cdot)$  is strictly quasiconvex for all  $i \in [k]$ ,  $F_6$  is locally bounded at  $x^0$  and  $F_6^0(x^0) \neq \emptyset$ , then  $F_6$  is continuous at  $x^0$ .*

*Proof.* First, recall that for a given vector-valued function  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$  the set-valued map  $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$  defined by  $F(x) := \{f(x)\}$  for all  $x \in \mathbb{R}^n$  is continuous at  $x^0 \in \mathbb{R}^n$  if and only if  $f$  is continuous at  $x^0$ . Furthermore, constant set-valued maps  $F$ , i.e., maps  $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$  with  $F(x) = M \subseteq \mathbb{R}^m$  for all  $x \in \mathbb{R}^n$  are continuous at every  $x \in \mathbb{R}^n$ . Finally, the sum of set-valued maps that are continuous at  $x^0 \in \mathbb{R}^n$  and have compact image sets is also continuous at  $x^0$  due to Proposition 3.1. This proves statement (iv). Statement (v) follows directly from Proposition 3.1 (ii). For the proof of (i) we make the additional observation that  $F_1(x^0) = \{g(x^0)\} + [0, q]$  and observe that  $[0, q]$  is compact. Statement (vi) is a direct consequence of Theorem 3.1.

For the proof of (ii) we can now apply (vi) and consider therefore the set-valued map  $F_6$  with the special setting  $h: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^{2d}$  defined by

$$h(x, y) := \begin{pmatrix} B(y - g^1(x)) \\ B(g^2(x) - y) \end{pmatrix}$$

for all  $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ . Then it holds  $F_2(x) = F_6(x)$  for all  $x \in \mathbb{R}^n$ . Moreover, we observe that  $h_i$  is continuous on  $\{x^0\} \times \mathbb{R}^m$  and  $h_i(x^0, \cdot)$  is strictly quasiconvex for all  $i \in [2d]$ . Furthermore, by using  $g^1(x^0) <_C g^2(x^0)$  we obtain  $\frac{1}{2}(g^1(x^0) + g^2(x^0)) \in F_6^0(x^0)$  and thus  $F_6^0(x^0) \neq \emptyset$ . Hence, it remains to show that  $F_6$  is locally bounded at  $x^0$ . In doing so we obtain by the continuity of  $g^1$  and  $g^2$  at  $x^0$  that there exists a  $\delta > 0$  such that for all  $x \in \mathbb{B}(x^0, \delta)$  it holds

$$\begin{aligned} F_6(x) = F_2(x) &= (\{g^1(x)\} + C) \cap (\{g^2(x)\} - C) \\ &\subseteq (\{g^1(x^0)\} + \bar{\mathbb{B}} + C) \cap (\{g^2(x^0)\} + \bar{\mathbb{B}} - C). \end{aligned}$$

Let  $M := (\{g^1(x^0)\} + \bar{\mathbb{B}} + C) \cap (\{g^2(x^0)\} + \bar{\mathbb{B}} - C)$  and assume that there exists a sequence  $(y^i)_{i \in \mathbb{N}} \subseteq M$  such that  $\|y^i\| \rightarrow \infty$ . Then there exists a subsequence  $(y^{i_j})_{j \in \mathbb{N}}$  such that

$$\frac{y^{i_j}}{\|y^{i_j}\|} \rightarrow v \neq 0 \text{ and } y^{i_j} \neq 0 \text{ for all } j \in \mathbb{N}.$$

However, since  $y^{i_j} \in M \setminus \{0\}$  for all  $j \in \mathbb{N}$  there exist  $\underline{b}^j, \bar{b}^j \in \bar{\mathbb{B}}$  such that

$$\frac{g^1(x^0) + \underline{b}^j}{\|y^{i_j}\|} \leq_C \frac{y^{i_j}}{\|y^{i_j}\|} \leq_C \frac{g^2(x^0) + \bar{b}^j}{\|y^{i_j}\|} \text{ for all } j \in \mathbb{N}.$$

By the closedness of  $C$  it follows  $0 \leq_C v \leq_C 0$  and by the pointedness of  $C$  this implies  $v = 0$ , which is a contradiction. Hence, our assumption is false and the defined set  $M$  is bounded and thus  $F_6$  is locally bounded at  $x^0$ .

Finally, for the proof of statement (iii), note that the compactness of the image sets of the set-valued map  $\hat{F}_3: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$  defined by  $\hat{F}_3(x) := r(x)\bar{\mathbb{B}}$  for all  $x \in \mathbb{R}^n$  is obvious. Hence, in view of the above and Proposition 3.1, it suffices to show the continuity of  $\hat{F}_3$  at  $x^0$ . To verify the upper semicontinuity of  $\hat{F}_3$  at  $x^0$  let  $\Omega \subseteq \mathbb{R}^m$  be an open set with  $\hat{F}_3(x^0) \subseteq \Omega$ . By compactness of  $\hat{F}_3(x^0)$  and openness of  $\Omega$  there exists  $\varepsilon > 0$  such that  $\hat{F}_3(x^0) + \varepsilon\bar{\mathbb{B}} \subseteq \Omega$ . Due to the continuity of the nonnegative scalar-valued function  $r$  at  $x^0$  there exists  $\delta > 0$  such that  $r(x) \in \{r(x^0)\} + [-\varepsilon, \varepsilon]$  for all  $x \in \mathbb{B}(x^0, \delta)$ . By  $|r(x^0) - \varepsilon| \leq r(x^0) + \varepsilon$  we conclude

$$\hat{F}_3(x) = r(x)\bar{\mathbb{B}} \subseteq (r(x^0) + \varepsilon)\bar{\mathbb{B}} = r(x^0)\bar{\mathbb{B}} + \varepsilon\bar{\mathbb{B}} = \hat{F}_3(x^0) + \varepsilon\bar{\mathbb{B}} \subseteq \Omega$$

for all  $x \in \mathbb{B}(x^0, \delta)$ .

To prove the lower semicontinuity of  $\hat{F}_3$  at  $x^0$  let  $\Omega \subseteq \mathbb{R}^m$  be an open set with  $\Omega \cap \hat{F}_3(x^0) \neq \emptyset$ . Hence, there exist  $y^0 = r(x^0)b^0 \in \hat{F}_3(x^0)$  with  $b^0 \in \bar{\mathbb{B}}$  and there exists  $\varepsilon > 0$  such that  $\{y^0\} + \varepsilon\bar{\mathbb{B}} \subseteq \Omega$ . Again, due to the continuity of  $r$  at  $x^0$ , there exists  $\delta > 0$  such that  $r(x) \in \{r(x^0)\} + [-\varepsilon, \varepsilon]$  for all  $x \in \mathbb{B}(x^0, \delta)$  and by  $|r(x^0) - \varepsilon| \leq r(x^0) + \varepsilon$  we conclude

$$r(x)b^0 \in (\{r(x^0)\} + [-\varepsilon, \varepsilon])b^0 \subseteq \{r(x^0)b^0\} + \varepsilon\bar{\mathbb{B}} = \{y^0\} + \varepsilon\bar{\mathbb{B}} \subseteq \Omega$$

for all  $x \in \mathbb{B}(x^0, \delta)$ . Together with  $r(x)b^0 \in \hat{F}_3(x)$  we obtain  $\Omega \cap \hat{F}_3(x) \neq \emptyset$  for all  $x \in \mathbb{B}(x^0, \delta)$ .  $\square$

Since we have to impose several assumptions on  $h$  in order to derive sufficient conditions for properties of  $F_6$ , we introduce the following example. With this we show that the assumptions we make on  $h$  are not too restrictive.

**Example 3.1.** Let  $n = 1$ ,  $m = 2$ ,  $C = \mathbb{R}_+^2$  and let  $f: \mathbb{R} \rightarrow \mathbb{R}^2$  be an arbitrary continuously differentiable function. We define  $M := \text{conv}(\{(0,0)^\top, (1,2)^\top, (2,1)^\top\})$  and set

$$\begin{aligned} F_6(x) &= \{f(x)\} + M = \left\{ f(x) + z \in \mathbb{R}^2 \mid \begin{pmatrix} (1,1)z - 3 \\ (-2,1)z \\ (1,-2)z \end{pmatrix} \leq_{\mathbb{R}_+^3} 0 \right\} \\ &= \left\{ y \in \mathbb{R}^2 \mid \begin{pmatrix} (1,1)(y - f(x)) - 3 \\ (-2,1)(y - f(x)) \\ (1,-2)(y - f(x)) \end{pmatrix} \leq_{\mathbb{R}_+^3} 0 \right\} \end{aligned}$$

for all  $x \in \mathbb{R}$ . Hence, for obtaining a representation of  $F_6$  with the help of a function  $h$ , we set  $h: \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^3$  with

$$h(x, y) := \begin{pmatrix} (1,1)(y - f(x)) - 3 \\ (-2,1)(y - f(x)) \\ (1,-2)(y - f(x)) \end{pmatrix}$$

for all  $(x, y) \in \mathbb{R} \times \mathbb{R}^2$ . Then for every  $i \in [3]$  and every  $x^0 \in \mathbb{R}$ , the function  $h_i$  is continuous on  $\{x^0\} \times \mathbb{R}^2$  and  $h_i(x^0, \cdot)$  is strictly quasiconvex. Moreover, by construction,  $F_6$  is locally bounded at  $x^0$  and  $F_6^0(x^0) \neq \emptyset$ . By Theorem 3.2 (vi) the set-valued map  $F_6$  is continuous.

Next, we investigate the continuity of the extremal value functions. Therefore, we need the following lemma (cf. [15, Theorem 4.2.2], [15, Theorem 4.2.3]).

**Lemma 3.4.** *Let  $\ell \in C^*$ . Then the following statements hold:*

- (i) *If  $F_6$  is l.s.c. at  $x^0 \in \text{dom}(F_6)$ , then the functions  $\varphi_{\min}^{F_6, \ell}$  and  $\varphi_{\max}^{F_6, \ell}$  are u.s.c. at  $x^0$ .*

(ii) If  $F_6(x^0)$  is a closed set for  $x^0 \in \text{dom}(F_6)$  and  $F_6$  is u.s.c. at  $x^0$ , then the functions  $\varphi_{\min}^{F_6, \ell}$  and  $\varphi_{\max}^{F_6, \ell}$  are l.s.c. at  $x^0$ .

The following theorem gives sufficient conditions for the continuity of  $\varphi_{\min}^{F_j, \ell}$  and  $\varphi_{\max}^{F_j, \ell}$ ,  $j \in [6]$ . The proof of statements (i) to (v) follows from Lemma 3.2. The proof of statement (vi) follows directly from Lemma 3.1 (v), Theorem 3.2 (vi) and Lemma 3.4.

**Theorem 3.3.** *Let  $x^0 \in \mathbb{R}^n$  and  $\ell \in C^*$ . Then the following statements hold:*

- (i) *If  $g$  is continuous at  $x^0$ , then  $\varphi_{\min}^{F_1, \ell}$  and  $\varphi_{\max}^{F_1, \ell}$  are continuous at  $x^0$ .*
- (ii) *If  $g^1$  or  $g^2$  are continuous at  $x^0$ , then  $\varphi_{\min}^{F_2, \ell}$  or  $\varphi_{\max}^{F_2, \ell}$  are continuous at  $x^0$ , respectively.*
- (iii) *If  $g$  and  $r$  are continuous at  $x^0$ , then  $\varphi_{\min}^{F_3, \ell}$  and  $\varphi_{\max}^{F_3, \ell}$  are continuous at  $x^0$ .*
- (iv) *If  $A$  is compact, then  $\varphi_{\min}^{F_4, \ell}$  and  $\varphi_{\max}^{F_4, \ell}$  are continuous at  $x^0$ .*
- (v) *If  $g$  is continuous at  $x^0$  and there exists  $\delta > 0$  such that  $P(x) \subseteq C$  holds for all  $x \in \mathbb{B}(x^0, \delta)$ , then  $\varphi_{\min}^{F_5, \ell}$  is continuous at  $x^0$ . If  $g$  is continuous at  $x^0$  and there exists  $\delta > 0$  such that  $P(x) \subseteq -C$  holds for all  $x \in \mathbb{B}(x^0, \delta)$ , then  $\varphi_{\max}^{F_5, \ell}$  is continuous at  $x^0$ .*
- (vi) *If  $h_i$  is continuous on  $\{x^0\} \times \mathbb{R}^m$  and  $h_i(x^0, \cdot)$  is strictly quasiconvex for all  $i \in [k]$ ,  $F_6$  is locally bounded at  $x^0$  and  $F_6^0(x^0) \neq \emptyset$ , then  $\varphi_{\min}^{F_6, \ell}$  and  $\varphi_{\max}^{F_6, \ell}$  are continuous at  $x^0$ .*

According to [22, Lemma 3.4] the cone-valued map  $P$  is u.s.c. at  $x^0$  if and only if there is a neighbourhood  $U$  of  $x^0$  such that  $P(x) \subseteq P(x^0)$  for all  $x \in U$ . Hence, for  $P$  u.s.c. at  $x^0$ , the condition  $P(x^0) \subseteq C$  already guarantees that there exists  $\delta > 0$  with  $P(x) \subseteq C$  for all  $x \in \mathbb{B}(x^0, \delta)$ .

**3.2. Lipschitz-continuity.** In this section we investigate Lipschitzian properties of the set-valued maps  $F_j$ ,  $j \in [6]$  and their impact on the respective minimum and maximum value functions. We start with some basic concepts and definitions (cf. [19, Definition 1.40]).

**Definition 3.4.** Let  $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$  be a set-valued map and  $(x^0, y^0) \in \text{gph}(F)$ . Then  $F$  is called

- (i) Lipschitz-like at  $(x^0, y^0)$  if there exist  $L, \delta, \varepsilon > 0$  such that for all  $x^1, x^2 \in \mathbb{B}(x^0, \delta)$  it holds

$$F(x^1) \cap \mathbb{B}(y^0, \varepsilon) \subseteq F(x^2) + L\|x^2 - x^1\| \mathbb{B}.$$

- (ii) Lipschitzian at  $x^0$  if there exist  $L, \delta > 0$  such that for all  $x^1, x^2 \in \mathbb{B}(x^0, \delta)$  it holds

$$F(x^1) \subseteq F(x^2) + L\|x^2 - x^1\| \mathbb{B}.$$

Furthermore, a vector-valued function  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is called Lipschitz-continuous at  $x^0 \in \mathbb{R}^n$  if the set-valued map  $x \mapsto \{f(x)\}$  is Lipschitzian at  $x^0$ .

Please note that a set-valued map  $F$  that is Lipschitzian at  $x^0 \in \text{dom}(F)$  is also Lipschitz-like at every  $(x^0, y^0) \in \text{gph}(F)$ . For the proof of Lipschitzianity of  $F_6$  we will need some additional concepts (cf. [23, Section 2]).

**Definition 3.5.** Let  $(x^0, y^0) \in \text{gph}(F_6)$ . We define the set of indices of active inequality constraints at  $(x^0, y^0)$  as

$$I(x^0, y^0) := \{i \in [k] \mid h_i(x^0, y^0) = 0\}.$$

Furthermore, if  $h_i$  is continuous and  $\nabla_y h_i: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$  exists and is continuous for all  $i \in [k]$ , then the set-valued map  $F_6$

- (i) satisfies the Mangasarian-Fromovitz constraint qualification (MFCQ) at  $y^0$  w.r.t.  $F_6(x^0)$  if there exists no  $\lambda \geq 0_{|I(x^0, y^0)|}$ ,  $\lambda \neq 0$  such that

$$\sum_{i \in I(x^0, y^0)} \lambda_i \nabla_y h_i(x^0, y^0) = 0.$$

- (ii) satisfies the Relaxed Constant Rank Constraint Qualification (RCRCQ) at  $(x^0, y^0)$  if for any  $J \subseteq I(x^0, y^0)$  there exist  $\delta, \varepsilon > 0$  such that for all  $(x, y) \in \mathbb{B}(x^0, \delta) \times \mathbb{B}(y^0, \varepsilon)$  it holds

$$\text{rank}\{\nabla_y h_i(x, y) \mid i \in J\} = \text{rank}\{\nabla_y h_i(x^0, y^0) \mid i \in J\}.$$

- (iii) is called R-regular at  $(x^0, y^0)$  if there exist  $M, \delta, \varepsilon > 0$  such that for all  $(x, y) \in \mathbb{B}(x^0, \delta) \times \mathbb{B}(y^0, \varepsilon)$  it holds

$$d(y, F_6(x)) \leq M \max(\{0\} \cup \{h_i(x, y) \mid i \in [k]\}).$$

For relations between the constraint qualifications given above and to other constraint qualifications we refer the reader to [24].

The following Lemma from [23] establishes sufficient conditions for  $F_6$  being Lipschitz-like and the minimum and maximum value functions being Lipschitz continuous.

**Lemma 3.5.** *Let  $(x^0, y^0) \in \text{gph}(F_6)$  and  $\ell \in C^*$ . Then the following statements hold:*

- (i) [23, Theorem 4.2] *If  $F_6$  is l.s.c. at  $x^0$  and satisfies (RCRCQ) at  $(x^0, y^0)$ , then  $F_6$  is R-regular at  $(x^0, y^0)$ .*
- (ii) [23, Theorem 5.1] *If  $F_6$  is R-regular at  $(x^0, y^0)$ , then  $F_6$  is Lipschitz-like at  $(x^0, y^0)$ .*
- (iii) [23, Theorem 5.4] *If  $h_i$  is Lipschitz-continuous for all  $i \in [k]$ ,  $F_6$  is locally bounded and  $F_6$  is R-regular at  $(x^0, y^0)$ , then  $\phi_{\min}^{F_6, \ell}$  and  $\phi_{\max}^{F_6, \ell}$  are Lipschitz-continuous at  $x^0$ .*

The following lemma allows us to derive Lipschitzianity of  $F_6$  from  $F_6$  being Lipschitz-like (cf. [19, Theorem 1.42]).

**Lemma 3.6.** *Let  $x^0 \in \text{dom}(F_6)$  and  $F_6$  be locally compact and closed at  $x^0$ . Then,  $F_6$  is Lipschitzian at  $x^0$  if and only if  $F_6$  is Lipschitz-like at  $(x^0, y^0)$  for every  $y^0 \in F(x^0)$ .*

The following theorem gives sufficient conditions for the Lipschitzianity of  $F_j$ ,  $j \in [6]$ .

**Theorem 3.4.** *Let  $x^0 \in \mathbb{R}^n$ . Then the following statements hold:*

- (i) *If  $g$  is Lipschitz-continuous at  $x^0$ , then  $F_1$  is Lipschitzian at  $x^0$ .*
- (ii) *If  $g^1$  and  $g^2$  are continuously differentiable,  $g^1(x) <_C g^2(x)$  for all  $x \in \mathbb{R}^n$  and  $C$  is polyhedral with  $\text{int}(C) = \{y \in \mathbb{R}^m \mid By <_{\mathbb{R}_+^d} 0\}$  for some  $B \in \mathbb{R}^{d \times m}$ , then  $F_2$  is Lipschitzian at  $x^0$ .*
- (iii) *If  $g$  and  $r$  are Lipschitz-continuous at  $x^0$ , then  $F_3$  is Lipschitzian at  $x^0$ .*
- (iv) *The set-valued map  $F_4$  is Lipschitzian at  $x^0$ .*
- (v) *If  $g$  is Lipschitz-continuous at  $x^0$  and  $P$  is Lipschitzian at  $x^0$ , then  $F_5$  is Lipschitzian at  $x^0$ .*
- (vi) *Let  $x^0 \in \text{dom}(F_6)$  and let at least one of the following conditions be fulfilled:*
  - *$F_6$  satisfies (RCRCQ) at  $(x^0, y^0)$  for all  $y^0 \in F_6(x^0)$ ,  $F_6^0(x^0) \neq \emptyset$  and the functions  $h_i(x^0, \cdot)$  are strictly quasiconvex for all  $i \in [k]$ .*
  - *$F_6$  satisfies (MFCQ) at  $y^0$  w.r.t.  $F_6(x^0)$  for all  $y^0 \in F_6(x^0)$ .*

If additionally  $h_i$  is continuously differentiable for all  $i \in [k]$  and  $F_6$  is locally bounded at  $x^0$ , then  $F_6$  is Lipschitzian at  $x^0$ .

*Proof.* First, we observe that the sum of two set-valued maps that are Lipschitzian at  $x^0$  is also Lipschitzian at  $x^0$  and that a set-valued map  $F$  that is constant is Lipschitzian at every  $x \in \mathbb{R}^n$ . This proves (i), (iv) and (v).

For the proof of (vi) we have that  $F_6$  is locally bounded at  $x^0$  by assumption, due to Remark 3.1 also locally compact at  $x^0$ , and closed at  $x^0$  from Theorem 3.1. Now, assume that the first condition is fulfilled. Then,  $F_6$  is l.s.c. at  $x^0$  due to Theorem 3.1. According to Lemma 3.5 the map  $F_6$  is Lipschitz-like at  $(x^0, y^0)$  for all  $y^0 \in F_6(x^0)$ . From Lemma 3.6 we obtain that  $F_6$  is Lipschitzian at  $x^0$ .

If the second condition is fulfilled, then the assumptions of [19, Corollary 4.39] are fulfilled for every  $y^0 \in F(x^0)$  and, as a consequence, we obtain that  $F_6$  is Lipschitz-like at  $(x^0, y^0)$  for every  $y^0 \in F_6(x^0)$ . With Lemma 3.6 we conclude that  $F_6$  is Lipschitzian at  $x^0$ .

For the proof of (ii) we apply (vi). Therefore, and analogously to the proof of Theorem 3.2, we define  $h: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^{2d}$  by

$$h(x, y) := \begin{pmatrix} B(y - g^1(x)) \\ B(g^2(x) - y) \end{pmatrix}$$

for all  $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$  and have  $F_2(x) = F_6(x)$  for all  $x \in \mathbb{R}^n$ . From our assumptions it follows that  $h_i$  is continuously differentiable and that  $h_i(x^0, \cdot)$  is strictly quasiconvex for all  $i \in [2d]$ . Analogously to the proof of Theorem 3.2 one can show that  $F_6$  is locally bounded at  $x^0$  and that  $F_6^0(x^0) \neq \emptyset$ . Finally, it is easy to see that (RCRCQ) is fulfilled at  $(x^0, y^0)$  for all  $y^0 \in F_6(x^0)$ .

For the proof of statement (iii) it remains to show that the set-valued map  $\hat{F}_3$  defined analogously to the proof of Theorem 3.2 is Lipschitzian at  $x^0$ . Therefore we observe that due to the Lipschitz-continuity of  $r$  at  $x^0$  there exist  $L, \delta > 0$  such that for all  $x^1, x^2 \in \mathbb{B}(x^0, \delta)$  it holds

$$\hat{F}_3(x^1) = r(x^1)\bar{\mathbb{B}} \subseteq (r(x^2) + L\|x^2 - x^1\|)\bar{\mathbb{B}} = r(x^2)\bar{\mathbb{B}} + L\|x^2 - x^1\|\bar{\mathbb{B}} = \hat{F}_3(x^2) + L\|x^2 - x^1\|\bar{\mathbb{B}}$$

and we are done.  $\square$

**Remark 3.2.** (i) The assumption of Theorem 3.4 (v) concerning  $P$  is very strong. Assume the map  $P$  is Lipschitzian at  $x^0$ . Then, by [22, Theorem 3.8], there exist some  $\delta > 0$  such that  $P$  is constant on  $\mathbb{B}(x^0, \delta)$ .

(ii) If, on the other hand,  $F_5$  is Lipschitzian at  $x^0$  and  $g$  is Lipschitz-continuous at  $x^0$ , then  $P(x) = F_5(x) + \{-g(x)\}$  for all  $x \in \mathbb{R}^n$ . Hence,  $P$  is Lipschitzian at  $x^0$ , since it is the sum of set-valued maps that are Lipschitzian at  $x^0$ . Again, this would imply that  $P$  is constant on  $\mathbb{B}(x^0, \delta)$  for some  $\delta > 0$ .

We investigate the assumptions of Theorem 3.4 concerning  $h$  in the following example.

**Example 3.2** (Continuation of Example 3.1). For all  $(x, y) \in \mathbb{R} \times \mathbb{R}^2$  we obtain the pairwise linearly independent vectors

$$\nabla_y h_1(x, y) = (1, 1)^\top, \quad \nabla_y h_2(x, y) = (-2, 1)^\top \quad \text{and} \quad \nabla_y h_3(x, y) = (1, -2)^\top.$$

Let now  $(x^0, y^0) \in \text{gph}(F_6)$  be arbitrarily chosen. Then by  $|I(x^0, y^0)| \leq 2$  we conclude that  $F_6$  satisfies (MFCQ) at  $y^0$  w.r.t.  $F_6(x^0)$ . Moreover, since the vectors  $\nabla_y h_i(x, y)$ ,  $i \in [3]$  are constant for all  $(x, y) \in \mathbb{R} \times \mathbb{R}^2$ , it is also easy to see that  $F_6$  satisfies the (RCRCQ) at  $(x^0, y^0)$ . Thus, by Theorem 3.4 (vi)  $F_6$  is Lipschitzian at  $x^0$ .

The next lemma shows that if a set-valued map  $F$  is Lipschitzian at  $x^0$ , so are its respective extremal value functions for every  $\ell \in C^*$ .

**Lemma 3.7.** *Let  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$  be a set-valued map,  $x^0 \in \text{dom}(F)$  and  $\ell \in C^*$ . If  $F$  is Lipschitzian at  $x_0$ , then  $\varphi_{\min}^{F,\ell}$  and  $\varphi_{\max}^{F,\ell}$  are Lipschitz-continuous at  $x^0$ .*

*Proof.* Let  $L, \delta > 0$  such that  $F(x^1) \subseteq F(x^2) + L\|x^2 - x^1\|\bar{\mathbb{B}}$  holds for all  $x^1, x^2 \in \mathbb{B}(x^0, \delta)$ . Then, for every  $x^1, x^2 \in \mathbb{B}(x^0, \delta)$  we obtain

$$\begin{aligned} \varphi_{\min}^{F,\ell}(x^2) - \varphi_{\min}^{F,\ell}(x^1) &\leq \inf_{y \in F(x^2)} \ell^\top y - \inf_{y \in F(x^2) + L\|x^2 - x^1\|\bar{\mathbb{B}}} \ell^\top y \\ &= -L\|x^2 - x^1\| \inf_{y \in \bar{\mathbb{B}}} \ell^\top y = L\|\ell\|\|x^2 - x^1\|. \end{aligned}$$

Analogously, we get  $\varphi_{\min}^{F,\ell}(x^1) - \varphi_{\min}^{F,\ell}(x^2) \leq L\|\ell\|\|x^1 - x^2\|$  and the respective statements for  $\varphi_{\max}^{F,\ell}$ .  $\square$

Hence, the sufficient conditions for the Lipschitzianity of the  $F_j$ ,  $j \in [6]$  from Theorem 3.4 are also sufficient for the Lipschitz-continuity of their respective extremal value functions.

**Remark 3.3.** For  $F_2$  it suffices that  $g^1$  and  $g^2$  are Lipschitz-continuous at  $x^0 \in \mathbb{R}^n$  in order for  $\varphi_{\min}^{F_2,\ell}$  and  $\varphi_{\max}^{F_2,\ell}$  to be Lipschitz-continuous at  $x^0$ . This follows from Lemma 3.2.

**3.3. Convexity.** In this section we investigate convexity properties of  $F_j$ . The following concepts for convexity of set-valued maps can be found in [25]. The notion of  $\ell$ -type  $C$ -convexity is also known as ( $C$ -)convexity in the literature, see, for instance, [26] and [27, Section 2.1].

**Definition 3.6.** A set-valued map  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$  is called

(i)  $\ell$ -type  $C$ -convex, if for all  $x^1, x^2 \in \text{dom}(F)$  and all  $\lambda \in (0, 1)$  it holds

$$F(\lambda x^1 + (1 - \lambda)x^2) \preceq_C^\ell \lambda F(x^1) + (1 - \lambda)F(x^2).$$

(ii)  $u$ -type  $C$ -convex, if for all  $x^1, x^2 \in \text{dom}(F)$  and all  $\lambda \in (0, 1)$  it holds

$$F(\lambda x^1 + (1 - \lambda)x^2) \preceq_C^u \lambda F(x^1) + (1 - \lambda)F(x^2).$$

Based on this and according to [28] we denote by  $\mathcal{C}_\ell(\mathbb{R}^n, \mathbb{R}^m, C)$  and  $\mathcal{C}_u(\mathbb{R}^n, \mathbb{R}^m, C)$  the classes of all  $\ell$ -type  $C$ -convex set-valued maps and all  $u$ -type  $C$ -convex set-valued maps, respectively. Note that the  $\ell$ -type  $C$ -convexity of a set-valued map  $F$  implies the convexity of  $\text{dom}(F)$ . In contrast to this the  $u$ -type  $C$ -convexity of a set-valued map  $F$  does not necessarily imply the convexity of  $\text{dom}(F)$ .

In the following we derive sufficient conditions such that  $F_j$  is  $\ell$ -type  $C$ -convex. For  $j \in [5]$  we can rely on known results from the literature. For  $j = 6$  we will use the following lemma which gives a characterization for the  $\ell$ -type  $C$ -convexity of  $F$  (cf. [29, Lemma 14.8])

**Lemma 3.8.** *It holds  $F \in \mathcal{C}_\ell(\mathbb{R}^n, \mathbb{R}^m, C)$  if and only if  $\text{epi}(F, C)$  is a convex set.*

We use the Gerstewitz's functional, see for instance [7, Section 5.2], in order to derive a sufficient condition for the convexity of  $\text{epi}(F_6, C)$ . For an element  $k^0 \in \text{int}(C)$  and  $M \subseteq \mathbb{R}^m$  the Gerstewitz's functional  $\rho_{M, k^0} : \mathbb{R}^m \rightarrow \bar{\mathbb{R}}$  is defined as

$$\rho_{M, k^0}(z) := \inf\{t \in \mathbb{R} \mid z \in tk^0 + M\}$$

for all  $z \in \mathbb{R}^m$ . Since we are only interested in the case  $M = -C$ , we set  $\rho_{k^0} := \rho_{-C, k^0}$ .

In the following definition we introduce the well-known concept of generalized monotonicity.

**Definition 3.7.** A functional  $\varphi: \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$  is called  $C$ -monotone, if  $y^1, y^2 \in \mathbb{R}^m$  and  $y^1 \leq_C y^2$  imply  $\varphi(y^1) \leq \varphi(y^2)$ .

The following proposition (cf. [7, Section 5.2]) states some useful properties of the Gerstewitz's functional which we use in later proofs.

**Proposition 3.2.** Let  $k^0 \in \text{int}(C)$ . Then the following statements hold:

- (i) The functional  $\rho_{k^0}$  is finite-valued and continuous.
- (ii) The functional  $\rho_{k^0}$  is convex.
- (iii) The functional  $\rho_{k^0}$  is  $C$ -monotone.

Based on the Gerstewitz's functional we define the *lower inner function*, see [30, Definition 3.4],  $\phi_C: \text{dom}(F_6) \times \mathbb{R}^m \rightarrow \mathbb{R}$  by

$$\phi_C(x, y) := \min_{z \in F_6(x)} \rho_{k^0}(z - y)$$

for all  $(x, y) \in \text{dom}(F_6) \times \mathbb{R}^m$ . In order to assure well definedness of  $\phi_C(x, y)$  for all  $(x, y) \in \text{dom}(F_6) \times \mathbb{R}^m$  a sufficient condition would be compactness of  $F_6(x)$ . Also, since we want to investigate convexity properties of  $F_6$ , it makes sense to assume convexity of  $\text{dom}(F_6)$ . Therefore, we introduce the following additional assumption.

**Assumption 3.1.** Let  $k^0 \in \text{int}(C)$ , let  $\text{dom}(F_6)$  be a convex set, let  $h_i(x, \cdot)$ ,  $i \in [k]$  be l.s.c. at all  $x \in \text{dom}(F_6)$ , and let  $F_6$  be locally bounded.

Under this additional assumption the minimum in the definition of  $\phi_C$  is attained since  $F_6(x)$  is nonempty by Assumption 2.1 and bounded by Assumption 3.1 and closed by Lemma 3.1 for every  $x \in \text{dom}(F_6)$  and  $\rho_{k^0}$  is continuous due to Proposition 3.2.

We obtain a sufficient condition for the  $\ell$ -type  $C$ -convexity of  $F_6$  in the following lemma.

**Lemma 3.9.** Let Assumption 3.1 be fulfilled. Furthermore, let  $\phi_C$  be quasiconvex, i.e.,

$$\phi_C(\lambda(x^1, y^1) + (1 - \lambda)(x^2, y^2)) \leq \max\{\phi_C(x^1, y^1), \phi_C(x^2, y^2)\}$$

for all  $(x^1, y^1), (x^2, y^2) \in \text{dom}(F_6) \times \mathbb{R}^m$  and all  $\lambda \in [0, 1]$ , then  $F_6 \in \mathcal{C}_\ell(\mathbb{R}^n, \mathbb{R}^m, C)$ .

*Proof.* For the epigraph  $\text{epi}(F_6, C)$  of  $F_6$  it holds

$$\begin{aligned} \text{epi}(F_6, C) &= \{(x, y) \in \text{dom}(F_6) \times \mathbb{R}^m \mid y \in F_6(x) + C\} \\ &= \{(x, y) \in \text{dom}(F_6) \times \mathbb{R}^m \mid \exists z \in F_6(x) \exists k \in C : y = z + k\} \\ &= \{(x, y) \in \text{dom}(F_6) \times \mathbb{R}^m \mid \exists z \in F_6(x) : y - z \in C\} \\ &= \{(x, y) \in \text{dom}(F_6) \times \mathbb{R}^m \mid \exists z \in F_6(x) : z - y \in -C\} \\ &= \{(x, y) \in \text{dom}(F_6) \times \mathbb{R}^m \mid \exists z \in F_6(x) : \rho_{k^0}(z - y) \leq 0\} \\ &= \{(x, y) \in \text{dom}(F_6) \times \mathbb{R}^m \mid \min_{z \in F_6(x)} \rho_{k^0}(z - y) \leq 0\} \\ &= \{(x, y) \in \text{dom}(F_6) \times \mathbb{R}^m \mid \phi_C(x, y) \leq 0\}. \end{aligned}$$

Now, let  $(x^1, y^1), (x^2, y^2) \in \text{epi}(F_6, C)$  and  $\lambda \in [0, 1]$ . Then it holds  $\phi_C(x^1, y^1) \leq 0$  and  $\phi_C(x^2, y^2) \leq 0$  and we obtain

$$\phi_C(\lambda(x^1, y^1) + (1 - \lambda)(x^2, y^2)) \leq \max\{\phi_C(x^1, y^1), \phi_C(x^2, y^2)\} \leq 0$$

by the quasiconvexity of  $\phi_C$ . Thus  $\text{epi}(F_6, C)$  is a convex set and  $F_6 \in \mathcal{C}_\ell(\mathbb{R}^n, \mathbb{R}^m, C)$  follows by Lemma 3.8.  $\square$

Lemma 3.9 motivates the formulation of a sufficient condition for  $h$  such that the function  $\phi_C$  is quasiconvex. Therefore, we introduce the following convexity concept, which is inspired by and generalizes the concept of natural  $K$ -quasiconvexity by Tanaka introduced in [17].

**Definition 3.8.** Let  $M \subseteq \mathbb{R}^s$  be a convex set, let  $D \subseteq \mathbb{R}^s$  and  $K \subseteq \mathbb{R}^k$  be convex cones and let  $f: M \rightarrow \mathbb{R}^k$  be a function. Then  $f$  is called  $(D, K)$ -quasiconvex-like on  $M$  if

$$\forall v^1, v^2 \in M, \lambda \in [0, 1] \exists v \in M \exists t \in [0, 1] : (v \leq_D \lambda v^1 + (1 - \lambda)v^2) \wedge (f(v) \leq_K t f(v^1) + (1 - t)f(v^2)).$$

First, we would like to establish a relation of this convexity concept to the well-known concepts of  $K$ -convexity and quasiconvexity concepts.

**Definition 3.9.** Let  $M \subseteq \mathbb{R}^s$  be a convex set,  $K \subseteq \mathbb{R}^k$  be a convex cone and let  $f: M \rightarrow \mathbb{R}^k$  be a function. Then  $f$  is called

- properly  $K$ -quasiconvex on  $M$  (in the sense of Ferro [31]) if

$$\forall v^1, v^2 \in M, \lambda \in [0, 1] : f(\lambda v^1 + (1 - \lambda)v^2) \leq_K f(v^1) \vee f(\lambda v^1 + (1 - \lambda)v^2) \leq_K f(v^2).$$

- natural  $K$ -quasiconvex on  $M$  (in the sense of Tanaka [17]) if

$$\forall v^1, v^2 \in M, \lambda \in [0, 1] \exists t \in [0, 1] : f(\lambda v^1 + (1 - \lambda)v^2) \leq_K t f(v^1) + (1 - t)f(v^2).$$

We observe that the above definitions generalize the standard concept of quasiconvexity in the case of scalar-valued functions  $f: \mathbb{R}^s \rightarrow \mathbb{R}$  with  $K = \mathbb{R}_+$ . Moreover, the statements of the following lemma follow directly from the definitions.

**Lemma 3.10.** Let  $M \subseteq \mathbb{R}^s$  be a convex set, let  $K \subseteq \mathbb{R}^k$  be a convex cone and let  $f: M \rightarrow \mathbb{R}^k$  be a function. Then the following statements hold:

- If  $f$  is  $K$ -convex on  $M$  or properly  $K$ -quasiconvex on  $M$ , then  $f$  is natural  $K$ -quasiconvex.
- If  $f$  is natural  $K$ -quasiconvex on  $M$ , then  $f$  is  $(D, K)$ -quasiconvex-like on  $M$  for every cone  $D \subseteq \mathbb{R}^s$ .
- The function  $f$  is natural  $K$ -quasiconvex on  $M$  if and only if  $f$  is  $(\{0\}, K)$ -quasiconvex-like on  $M$ .

The following example investigates  $(D, K)$ -quasiconvex-like functions with preimage space  $\mathbb{R}$ .

**Example 3.3.** Let  $f: \mathbb{R} \rightarrow \mathbb{R}^k$  be an arbitrary function. Then,  $f$  is  $(D, K)$ -quasiconvex-like on  $\mathbb{R}$  for all cones  $D \subseteq \mathbb{R}$  with  $D \neq \{0\}$  and all cones  $K \subseteq \mathbb{R}^k$ . To verify this we note that  $D \in \{\mathbb{R}_+, -\mathbb{R}_+, \mathbb{R}\}$ . Now, let  $v^1, v^2 \in \mathbb{R}$  and  $\lambda \in [0, 1]$ . Without loss of generality we may assume that  $v^1 \leq_{\mathbb{R}_+} v^2$ . If  $D \in \{\mathbb{R}_+, \mathbb{R}\}$ , then we set  $v := v^1$  and  $t := 1$ . Otherwise, we set  $v := v^2$  and  $t := 0$ . In both cases we get  $v \leq_D \lambda v^1 + (1 - \lambda)v^2$  and  $f(v) \leq_K t f(v^1) + (1 - t)f(v^2)$  since  $0 \in K$ .

Now, using the concept of  $(D, K)$ -quasiconvex-like functions we establish a connection between  $h$  and the  $\ell$ -type  $C$ -convexity of  $F_6$ . We also state results investigating the  $\ell$ -type  $C$ -convexity of  $F_j$ ,  $j \in [5]$ , which are taken from [16, Proposition 3.5].

**Theorem 3.5.** *The following statements hold:*

- (i) *If  $g$  is  $C$ -convex, then  $F_1 \in \mathcal{C}_\ell(\mathbb{R}^n, \mathbb{R}^m, C)$ .*
- (ii) *If  $g^1$  is  $C$ -convex, then  $F_2 \in \mathcal{C}_\ell(\mathbb{R}^n, \mathbb{R}^m, C)$ .*
- (iii) *If  $g$  is  $C$ -convex and  $r$  is concave, then  $F_3 \in \mathcal{C}_\ell(\mathbb{R}^n, \mathbb{R}^m, C)$ .*
- (iv) *If  $A + C$  is convex, then  $F_4 \in \mathcal{C}_\ell(\mathbb{R}^n, \mathbb{R}^m, C)$ .*
- (v) *If  $g$  is  $C$ -convex and  $P \in \mathcal{C}_\ell(\mathbb{R}^n, \mathbb{R}^m, C)$ , then  $F_5 \in \mathcal{C}_\ell(\mathbb{R}^n, \mathbb{R}^m, C)$ .*
- (vi) *If Assumption 3.1 is fulfilled and  $h$  is  $(D, K)$ -quasiconvex-like on  $\text{dom}(F_6) \times \mathbb{R}^m$  for  $D := \{0_{\mathbb{R}^n}\} \times C$  and  $K = \mathbb{R}_+^k$ , then  $F_6 \in \mathcal{C}_\ell(\mathbb{R}^n, \mathbb{R}^m, C)$ .*

*Proof.* For the proof of (vi) it suffices by Lemma 3.9 to show that  $\phi_C$  is quasiconvex. Therefore, let  $(x^1, y^1), (x^2, y^2) \in \text{dom}(F_6) \times \mathbb{R}^m$  and  $\lambda \in [0, 1]$  be given. It holds that  $\phi_C(x^1, y^1) = \rho_{k^0}(z^1 - y^1)$  for some  $z^1 \in F_6(x^1)$  and  $\phi_C(x^2, y^2) = \rho_{k^0}(z^2 - y^2)$  for some  $z^2 \in F_6(x^2)$ . Now, we set  $v^1 := (x^1, z^1)$  and  $v^2 := (x^2, z^2)$ . Since  $h$  is  $(D, \mathbb{R}_+^k)$ -quasiconvex-like on  $\text{dom}(F_6) \times \mathbb{R}^m$  for  $D = \{0_{\mathbb{R}^n}\} \times C$ , there exist  $v = (\bar{x}, \bar{z})$  and  $t \in [0, 1]$  with

$$v \leq_D \lambda v^1 + (1 - \lambda)v^2 \quad \text{and} \quad h(v) \leq_{\mathbb{R}_+^k} t h(v^1) + (1 - t)h(v^2).$$

Hence,  $\bar{x} = \lambda x^1 + (1 - \lambda)x^2$ ,  $\bar{z} \leq_C \lambda z^1 + (1 - \lambda)z^2$  and

$$h(\lambda x^1 + (1 - \lambda)x^2, \bar{z}) = h(\bar{x}, \bar{z}) = h(v) \leq_{\mathbb{R}_+^k} t h(v^1) + (1 - t)h(v^2) \leq_{\mathbb{R}_+^k} 0.$$

This implies  $\bar{z} \in F_6(\lambda x^1 + (1 - \lambda)x^2)$ . Since  $\rho_{k^0}$  is  $C$ -monotone and convex (see Proposition 3.2) and hence quasiconvex, we have

$$\begin{aligned} \rho_{k^0}(\bar{z} - (\lambda y^1 + (1 - \lambda)y^2)) &\leq \rho_{k^0}(\lambda z^1 + (1 - \lambda)z^2 - (\lambda y^1 + (1 - \lambda)y^2)) \\ &= \rho_{k^0}(\lambda(z^1 - y^1) + (1 - \lambda)(z^2 - y^2)) \\ &\leq \max\{\rho_{k^0}(z^1 - y^1), \rho_{k^0}(z^2 - y^2)\} \\ &= \max\{\phi_C(x^1, y^1), \phi_C(x^2, y^2)\}. \end{aligned}$$

Since  $\bar{z} \in F_6(\lambda x^1 + (1 - \lambda)x^2)$ , we have

$$\begin{aligned} \phi_C(\lambda(x^1, y^1) + (1 - \lambda)(x^2, y^2)) &= \min_{z \in F_6(\lambda x^1 + (1 - \lambda)x^2)} \rho_{k^0}(z - (\lambda y^1 + (1 - \lambda)y^2)) \\ &\leq \rho_{k^0}(\bar{z} - (\lambda y^1 + (1 - \lambda)y^2)) \\ &\leq \max\{\phi_C(x^1, y^1), \phi_C(x^2, y^2)\}. \end{aligned}$$

This shows the quasiconvexity of  $\phi_C$  and concludes the proof of (vi). □

**Remark 3.4.** The assumption of Theorem 3.5 (v) concerning  $P$  is rather strong. According to [22, Lemma 2.3] the condition  $P \in \mathcal{C}_\ell(\mathbb{R}^n, \mathbb{R}^m, C)$  implies  $P(x) + C = P(\mathbb{R}^n) + C$  for all  $x \in \mathbb{R}^n$ .

The following example provides a setting for which the assumptions of Theorem 3.5 (vi) concerning  $h$  are satisfied.

**Example 3.4** (Continuation of Example 3.1). First, we recall that  $F_6$  fulfills all the conditions to satisfy Assumption 3.1. Under the additional assumption that  $f$  is of the type  $f(x) = Bx + b$  for some  $B \in \mathbb{R}^{2 \times 1}$ ,  $b \in \mathbb{R}^2$ , the functions  $h_i$ ,  $i \in [3]$ , are affine-linear. Hence,  $h$  is  $\mathbb{R}_+^3$ -convex and, by Lemma 3.10,  $h$  is  $(\{0\} \times \mathbb{R}_+^2, \mathbb{R}_+^3)$ -quasiconvex-like on  $\mathbb{R} \times \mathbb{R}^2$ . By Theorem 3.5 we obtain  $F_6 \in \mathcal{C}_\ell(\mathbb{R}^n, \mathbb{R}^m, C)$ .

It is easy to see that in order to achieve  $\ell$ -type  $C$ -convexity for the set-valued map  $F_6$  it is also sufficient that  $h_i$  is quasiconvex for all  $i \in [k]$ . However, this assumption turns out to be much stronger than needed. Because in the mentioned case the function  $\bar{h}: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$  defined by  $\bar{h}(x, y) := \max_{i \in [k]} h_i(x, y)$  would also be quasiconvex. As a consequence  $\bar{h}$  would be natural  $K$ -quasiconvex for  $K = \mathbb{R}_+$  and thus  $\bar{h}$  would be  $(D, \mathbb{R}_+)$ -quasiconvex-like for every cone  $D \subseteq \mathbb{R}^n \times \mathbb{R}^m$  (see Lemma 3.10). In particular, for every cone  $\hat{C} \subseteq \mathbb{R}^m$  the function  $\bar{h}$  is  $(\{0_{\mathbb{R}^n}\} \times \hat{C}, \mathbb{R}_+)$ -quasiconvex-like. So we can apply Theorem 3.5 with  $h = \bar{h}$  and  $C = \hat{C}$  and obtain  $F_6 \in \mathcal{C}_\ell(\mathbb{R}^n, \mathbb{R}^m, \hat{C})$ . Since  $\hat{C}$  was arbitrarily chosen the function  $F_6$  is  $\ell$ -type  $\hat{C}$ -convex for every cone  $\hat{C} \subseteq \mathbb{R}^m$  in case of quasiconvex  $h_i$ , which is a much stronger statement than we intended to obtain. This promotes the usage of a weaker convexity concept for  $h$ , which we gave by  $(D, K)$ -quasiconvex-likeness.

Note that the results from Theorem 3.5 instantly yield sufficient conditions for the convexity of  $\varphi_{\min}^{F_j, \ell}$ ,  $j \in [6]$ . The reason is a strong relationship between  $\ell$ -type  $C$ -convexity of a set-valued map  $F$  and the convexity of  $\varphi_{\min}^{F, \ell}$ . Similar interdependencies also hold for the  $u$ -type  $C$ -convexity of  $F$  and the convexity of  $\varphi_{\max}^{F, \ell}$ .

**Proposition 3.3.** *The following statements hold:*

- (i) [26, Theorem 2.1] *If  $F(x) + C$  is closed and convex for all  $x \in \text{dom}(F)$ , then  $F \in \mathcal{C}_\ell(\mathbb{R}^n, \mathbb{R}^m, C)$  if and only if  $\varphi_{\min}^{F, \ell}$  is convex for all  $\ell \in C^*$ .*
- (ii) [32, Theorem 3.2] *If  $\text{dom}(F)$  is convex and  $F(x) - C$  is closed and convex for all  $x \in \text{dom}(F)$ , then  $F \in \mathcal{C}_u(\mathbb{R}^n, \mathbb{R}^m, C)$  if and only if  $\varphi_{\max}^{F, \ell}$  is convex for all  $\ell \in C^*$ .*

Note that  $\ell$ -type  $C$ -convexity of a set-valued map  $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$  implies the convexity of  $F(x) + C$  for all  $x \in \mathbb{R}^n$  ([22, Lemma 2.2]), while for  $F \in \mathcal{C}_u(\mathbb{R}^n, \mathbb{R}^m, C)$  the set  $F(x) - C$  is not necessarily convex. Also, in [32, Example 3.1] it is shown that for Theorem 3.3 (ii) the additional assumption that  $\text{dom}(F)$  is convex cannot be omitted. Thus,  $u$ -type  $C$ -convexity seems to be a more difficult property to deal with. This is also underlined by the following example. There, we show that for the  $u$ -type  $C$ -convexity of  $F_6$  even strong assumptions on the functions  $h_i$  are not sufficient. Even if all functions  $h_i$  are affine,  $F_6$  might not be  $u$ -type  $C$ -convex.

**Example 3.5.** Let  $n = 1$ ,  $m = 2$ ,  $k = 8$ ,  $C = \mathbb{R}_+^2$  and  $h: \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^8$ ,  $h(x, y) = Q^1 x + Q^2 y + c$  for all  $(x, y) \in \mathbb{R} \times \mathbb{R}^2$ , where

$$Q^1 = \frac{1}{2} \begin{pmatrix} e^4 \\ -e^4 \end{pmatrix}, \quad Q^2 = \begin{pmatrix} 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 \end{pmatrix}^\top, \quad c = -e^8.$$

Thereby,  $e^p$  for  $p \in \mathbb{N}$  denotes the all-one-vector of dimension  $p$ , i.e.,  $e^p \in \mathbb{R}^p$  with  $e_i^p = 1$  for all  $i \in [p]$ .

Now, we set  $x^1 = 1$ ,  $x^2 = -1$  and  $\lambda = \frac{1}{2}$ . Then, it is easy to check that

$$\begin{aligned} F_6(\lambda x^1 + (1 - \lambda)x^2) &= F_6(0) = [-1, 1]^2, \\ F_6(1) = F_6(x^1) &= F_6(x^2) = F_6(-1) = [-\frac{1}{2}, \frac{1}{2}]^2. \end{aligned}$$

Therefore, we have

$$F_6(\lambda x^1 + (1 - \lambda)x^2) = [-1, 1]^2 \not\subseteq [-\frac{1}{2}, \frac{1}{2}]^2 - C = \lambda F_6(x^1) + (1 - \lambda)F_6(x^2) - C.$$

Hence,  $F_6$  is not  $u$ -type  $C$ -convex.

This example suggests that it may be difficult to even find a function  $h$  such that  $F_6$  is  $u$ -type  $C$ -convex. However, since  $F_6$  has such a general structure, we can set, for example,  $h(x, y) := \|g(x) - y\| - r(x)$  for all  $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ . That way it holds  $F_6 = F_3$  and we obtain a  $u$ -type  $C$ -convex map  $F_6$  if we choose  $g$  and  $r$  in a way such that  $F_3$  is  $u$ -type  $C$ -convex. That the latter is possible is stated in the following theorem, which formulates sufficient conditions in order to achieve  $u$ -type  $C$ -convexity for  $F_j$ ,  $j \in [5]$ .

**Theorem 3.6.** *The following statements hold:*

- (i) *If  $g$  is  $C$ -convex, then  $F_1 \in \mathcal{C}_u(\mathbb{R}^n, \mathbb{R}^m, C)$ .*
- (ii) *If  $g^2$  is  $C$ -convex, then  $F_2 \in \mathcal{C}_u(\mathbb{R}^n, \mathbb{R}^m, C)$ .*
- (iii) *If  $g$  is  $C$ -convex and  $r$  is convex, then  $F_3 \in \mathcal{C}_u(\mathbb{R}^n, \mathbb{R}^m, C)$ .*
- (iv)  *$F_4 \in \mathcal{C}_u(\mathbb{R}^n, \mathbb{R}^m, \{0\})$  and thus  $F_4 \in \mathcal{C}_u(\mathbb{R}^n, \mathbb{R}^m, C)$ .*
- (v) *If  $g$  is  $C$ -convex and  $P \in \mathcal{C}_u(\mathbb{R}^n, \mathbb{R}^m, C)$ , then  $F_5 \in \mathcal{C}_u(\mathbb{R}^n, \mathbb{R}^m, C)$ .*

*Proof.* (i) Let  $x^1, x^2 \in \mathbb{R}^n$ ,  $\lambda \in [0, 1]$  and  $y \in F_1(\lambda x^1 + (1 - \lambda)x^2)$ . Then, there exists  $t \in [0, 1]$  such that  $y = tg(\lambda x^1 + (1 - \lambda)x^2) + (1 - t)(g(\lambda x^1 + (1 - \lambda)x^2) + q)$ . Since  $g$  is  $C$ -convex, it holds that

$$\begin{aligned} y &\leq_C t(\lambda g(x^1) + (1 - \lambda)g(x^2)) + (1 - t)(\lambda g(x^1) + (1 - \lambda)g(x^2) + q) \\ &= \lambda(tg(x^1) + (1 - t)(g(x^1) + q)) + (1 - \lambda)(tg(x^2) + (1 - t)(g(x^2) + q)) \\ &\in \lambda F_1(x^1) + (1 - \lambda)F_1(x^2). \end{aligned}$$

This implies  $y \in \lambda F_1(x^1) + (1 - \lambda)F_1(x^2) - C$ . As  $y$  was chosen arbitrarily, we have

$$F_1(\lambda x^1 + (1 - \lambda)x^2) \subseteq \lambda F_1(x^1) + (1 - \lambda)F_1(x^2) - C.$$

(ii) Let  $x^1, x^2 \in \mathbb{R}^n$  and  $\lambda \in [0, 1]$ . Since  $g^1(x) \leq_C g^2(x)$  for all  $x \in \mathbb{R}^n$  and  $C$  is pointed and convex, we have

$$\begin{aligned} &\lambda F_2(x^1) + (1 - \lambda)F_2(x^2) - C \\ &= (\{\lambda g^1(x^1)\} + C) \cap (\{\lambda g^2(x^1)\} - C) \\ &\quad + (\{(1 - \lambda)g^1(x^2)\} + C) \cap (\{(1 - \lambda)g^2(x^2)\} - C) - C \\ &\supseteq ((\{\lambda g^2(x^1)\} + C) \cap (\{\lambda g^2(x^1)\} - C) \\ &\quad + (\{(1 - \lambda)g^2(x^2)\} + C) \cap (\{(1 - \lambda)g^2(x^2)\} - C)) - C \\ &= (\{\lambda g^2(x^1)\} + \{(1 - \lambda)g^2(x^2)\}) - C \\ &= \{\lambda g^2(x^1) + (1 - \lambda)g^2(x^2)\} - C. \end{aligned}$$

Since  $g^2$  is  $C$ -convex, it holds  $\lambda g^2(x^1) + (1 - \lambda)g^2(x^2) - c = g^2(\lambda x^1 + (1 - \lambda)x^2)$  for some  $c \in C$ . Using the convexity of  $C$ , this implies

$$\begin{aligned}
 & \{\lambda g^2(x^1) + (1 - \lambda)g^2(x^2)\} - C \\
 &= \{\lambda g^2(x^1) + (1 - \lambda)g^2(x^2)\} - C - C \\
 &\supseteq \{\lambda g^2(x^1) + (1 - \lambda)g^2(x^2)\} - \{c\} - C \\
 &= \{g^2(\lambda x^1 + (1 - \lambda)x^2)\} - C \\
 &\supseteq (\{g^2(\lambda x^1 + (1 - \lambda)x^2)\} - C) \cap (\{g^1(\lambda x^1 + (1 - \lambda)x^2)\} + C) \\
 &= F_2(\lambda x^1 + (1 - \lambda)x^2)
 \end{aligned}$$

and thus  $F_2(\lambda x^1 + (1 - \lambda)x^2) \subseteq \lambda F_2(x^1) + (1 - \lambda)F_2(x^2) - C$ .

(iii) Let  $x^1, x^2 \in \mathbb{R}^n$  and  $\lambda \in [0, 1]$ . Moreover, let

$$\begin{aligned}
 y &\in F_3(\lambda x^1 + (1 - \lambda)x^2) \\
 &= \{g(\lambda x^1 + (1 - \lambda)x^2)\} + r(\lambda x^1 + (1 - \lambda)x^2)\bar{\mathbb{B}}.
 \end{aligned}$$

By the  $C$ -convexity of  $g$  and by the convexity of  $r$  it holds

$$\{g(\lambda x^1 + (1 - \lambda)x^2)\} \subseteq \lambda \{g(x^1)\} + (1 - \lambda)\{g(x^2)\} - C$$

and

$$r(\lambda x^1 + (1 - \lambda)x^2)\bar{\mathbb{B}} \subseteq (\lambda r(x^1) + (1 - \lambda)r(x^2))\bar{\mathbb{B}} = \lambda r(x^1)\bar{\mathbb{B}} + (1 - \lambda)r(x^2)\bar{\mathbb{B}},$$

respectively. Hence, we obtain

$$\begin{aligned}
 y &\in \lambda \{g(x^1)\} + (1 - \lambda)\{g(x^2)\} + \lambda r(x^1)\bar{\mathbb{B}} + (1 - \lambda)r(x^2)\bar{\mathbb{B}} - C \\
 &= \lambda (\{g(x^1)\} + r(x^1)\bar{\mathbb{B}}) + (1 - \lambda)(\{g(x^2)\} + r(x^2)\bar{\mathbb{B}}) - C \\
 &= \lambda F_3(x^1) + (1 - \lambda)F_3(x^2) - C
 \end{aligned}$$

and thus  $F_3(\lambda x^1 + (1 - \lambda)x^2) \subseteq \lambda F_3(x^1) + (1 - \lambda)F_3(x^2) - C$ .

(iv) Let  $x^1, x^2 \in \mathbb{R}^n$  and  $\lambda \in [0, 1]$ . It holds

$$\begin{aligned}
 F_4(\lambda x^1 + (1 - \lambda)x^2) &= \{L(\lambda x^1 + (1 - \lambda)x^2)\} + A \\
 &= \lambda \{L(x^1)\} + (1 - \lambda)\{L(x^2)\} + A \\
 &\subseteq \lambda \{L(x^1)\} + (1 - \lambda)\{L(x^2)\} + \lambda A + (1 - \lambda)A \\
 &= \lambda (\{L(x^1)\} + A) + (1 - \lambda)(\{L(x^2)\} + A) \\
 &= \lambda F_4(x^1) + (1 - \lambda)F_4(x^2) \\
 &= \lambda F_4(x^1) + (1 - \lambda)F_4(x^2) - \{0\} \\
 &\subseteq \lambda F_4(x^1) + (1 - \lambda)F_4(x^2) - C.
 \end{aligned}$$

(v) As an immediate consequence of the definition of a  $u$ -type  $C$ -convex map, the sum of two  $u$ -type  $C$ -convex maps is again  $u$ -type  $C$ -convex. As  $g$  is a  $C$ -convex function, the set-valued map  $x \mapsto \{g(x)\}$  is  $u$ -type  $C$ -convex and, as a consequence,  $F_5$  is  $u$ -type  $C$ -convex. □

**Remark 3.5.** The assumption  $P \in \mathcal{C}_u(\mathbb{R}^n, \mathbb{R}^m, C)$  of Theorem 3.6 (v) is, as the map  $P$  is cone-valued, equivalent to

$$P(\lambda x^1 + (1 - \lambda)x^2) \subseteq \lambda P(x^1) + (1 - \lambda)P(x^2) - C = P(x^1) + P(x^2) - C$$

for all  $x^1, x^2 \in \mathbb{R}^n$  and all  $\lambda \in [0, 1]$ . This condition is fulfilled in case one of the following statements holds:

- For all  $x \in \mathbb{R}^n$  it holds  $P(x) \subseteq -C$ .
- $C - C = \mathbb{R}^m$  and for all  $x \in \mathbb{R}^n$  it holds  $C \subseteq P(x)$ .

To see the first statement note that  $0 \in P(x)$  for all  $x \in \mathbb{R}^n$ . For the second note that the assumption already implies  $P(x^1) + P(x^2) - C = \mathbb{R}^m$ .

We conclude this section by investigating the convexity of  $\varphi_{\min}^{F_j, \ell}$ ,  $j \in [6]$  and  $\varphi_{\max}^{F_j, \ell}$ ,  $j \in [5]$  in the following theorem.

**Theorem 3.7.** *Let  $\ell \in C^*$ . Then the following statements hold:*

- (i) *If  $g$  is  $C$ -convex, then  $\varphi_{\min}^{F^1, \ell}$  is convex.*
- (ii) *If  $g^1$  is  $C$ -convex, then  $\varphi_{\min}^{F^2, \ell}$  is convex.*
- (iii) *If  $g$  is  $C$ -convex and  $r$  is concave, then  $\varphi_{\min}^{F^3, \ell}$  is convex.*
- (iv) *If  $A$  is compact and  $A + C$  is convex, then  $\varphi_{\min}^{F^4, \ell}$  is convex.*
- (v) *If  $g$  is  $C$ -convex,  $P \in \mathcal{C}_\ell(\mathbb{R}^n, \mathbb{R}^m, C)$  and  $P(x) + C$  is closed for all  $x \in \mathbb{R}^n$ , then  $\varphi_{\min}^{F^5, \ell}$  is convex.*
- (vi) *If Assumption 3.1 is fulfilled and  $h$  is  $(D, K)$ -quasiconvex-like on  $\text{dom}(F_6) \times \mathbb{R}^m$  for  $D := \{0_{\mathbb{R}^n}\} \times C$  and  $K = \mathbb{R}_+^k$ , then  $\varphi_{\min}^{F_6, \ell}$  is convex.*

Furthermore, it holds:

- (I) *If  $g$  is  $C$ -convex, then  $\varphi_{\max}^{F^1, \ell}$  is convex.*
- (II) *If  $g^2$  is  $C$ -convex, then  $\varphi_{\max}^{F^2, \ell}$  is convex.*
- (III) *If  $g$  is  $C$ -convex and  $r$  is convex, then  $\varphi_{\max}^{F^3, \ell}$  is convex.*
- (IV) *If  $A$  is compact and  $A - C$  is convex, then  $\varphi_{\max}^{F^4, \ell}$  is convex.*
- (V) *If  $g$  is  $C$ -convex,  $P \in \mathcal{C}_u(\mathbb{R}^n, \mathbb{R}^m, C)$  and  $P(x) - C$  is closed for all  $x \in \mathbb{R}^n$ , then  $\varphi_{\max}^{F^5, \ell}$  is convex.*

*Proof.* According to our assumptions  $\text{dom}(F_j)$  is convex for all  $j \in [6]$ . Together with Lemma 3.1, the set  $F_j(x)$  is compact for all  $j \in [6] \setminus \{5\}$  and all  $x \in \mathbb{R}^n$ .

For the proof of statements (i) to (vi) we observe that  $F_j(x) + C$  is closed for all  $j \in [6] \setminus \{5\}$  and all  $x \in \mathbb{R}^n$ , since  $C$  is closed. By assumption we obtain that  $F_5(x) + C$  is closed for all  $x \in \mathbb{R}^n$ . Due to Theorem 3.5 we have  $F_j \in \mathcal{C}_\ell(\mathbb{R}^n, \mathbb{R}^m, C)$  for all  $j \in [6]$ . This implies, by [22, Lemma 2.2], the convexity of  $F_j(x) + C$  for all  $j \in [6]$  and all  $x \in \mathbb{R}^n$ . Applying Proposition 3.3 (i) concludes the proofs of (i) to (vi).

For the proof of statements (I) to (V) we analogously obtain that  $F_j(x) - C$  is closed for all  $j \in [5]$  and all  $x \in \mathbb{R}^n$ . According to Lemma 3.3 (i), we have that the set  $F_j(x)$  is convex for all  $j \in [5] \setminus \{4\}$  and all  $x \in \mathbb{R}^n$ . Therefore,  $F_j(x) - C$  is convex for all  $j \in [5] \setminus \{4\}$  and all  $x \in \mathbb{R}^n$ . Additionally, we have  $F_4(x) - C$  is convex for all  $x \in \mathbb{R}^n$  since  $A - C$  is convex by assumption.

Furthermore, by Theorem 3.6 we have  $F_j \in \mathcal{C}_u(\mathbb{R}^n, \mathbb{R}^m, C)$  for all  $j \in [5]$ . Applying Proposition 3.3 (ii) concludes the proofs of (I) to (V). □

#### 4. CONCLUSION

In this paper we have studied set-valued maps of certain structures. We have given an overview over sufficient conditions that guarantee properties like (Lipschitz-)continuity and convexity for these set-valued maps. Furthermore, we have investigated the respective relationships to (Lipschitz-)continuity and convexity of their extremal value functions.

#### Acknowledgments

The work of the first and the third author is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - Project-ID 392195690.

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