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CONE ENLARGEMENTS AND APPLICATIONS TO VECTOR OPTIMIZATION

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Abstract. We study four types of enlargements for cones in normed vector spaces. We identify some commune features and mutual inclusions that these enlargements enjoy under different classical properties of cones: normality, well-basedness and so on. The effect of such conic enlargements on the behavior of the Gerstewitz (Tammer) scalarizing functional is shortly presented. Then we prove that, in the virtue of their inclusions, all these enlargements are involved in the study of the properness of several types of solutions in a variety of vector optimization problems.

Keywords. Approximate efficiency; Cone enlargements; Vector optimization.

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1. Introduction

Certain types of conic enlargements are well known to be very useful in a wide range of applications from stability issues in variational analysis (see [1, 2] and the references therein) to results concerning the density of proper solutions into the set of strong solutions in vector optimization (Arrow-Barankin-Blackwell type theorems; see, e.g., [3]). More recently, some conic enlargement were introduced in order to deal with some concepts of directional solutions (see [4]) and a study of several possible enlargements was done in [5] within the scope of presenting their applications to cone separation results and to vector optimization problems with variable ordering structure in the setting designed in [6].

In this paper, we continue some ideas of investigation opened by the work [5]. Namely, we reconsider four types of enlargements for a given cone in a normed vector space and we explore them from several perspectives. In this vein, we provide more calculus rules for their inclusions and we put into light the exact constants of enlargement in all inclusions we prove. Moreover, we have a new look on these construction from the point of view of some usual properties of cones (the property to have a bounded base, to be normal, etc.). Actually, the normality property seems to be a good tool for obtaining interesting inclusions for the two of these constructions that are not convex, in general. A second part of the paper concerns the way these enlargements interacts with some important tools in vector optimization, namely scalarizing functionals and approximate solutions. Therefore, we are interested in knowing how the famous Gerstewitz (Tammer) functional is related to the cone enlargements under the study and in understanding

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under which topological conditions some directional approximate minima for sets and solutions for a class of problems with variable ordering structure can be seen as proper solutions (that is, solutions with respect to larger ordering cones and cones of directions). We consider that several facts presented here are actually generators of further results on similar ideas and in this sense we see these enlargements as the basis of some potential further applications in the theory of vector optimization.

The paper is organized as follows. Firstly, in Section 2, we identify some properties that the enlargements we consider share and this allows us to see them from the point of view of angle property of cones and the property to allow plastering. In particular, we concentrate on the inclusion of some conic neighborhoods and we make precise the radii of these generalized neighborhoods. Moreover, we show their mutual inclusions and, in contrast to [5], we provide exact inclusion constants in all situations. Then, in Section 3, we present some further inclusions using the trace of two (the most general and less demanding ones) of these enlargements on the unit sphere and this is motivated by the purpose to compensate the lack of convexity. These inclusions as well as the pointedness of these enlargements are studied under normality assumption of the underlying cone. The last two sections are dedicated to some potential application of the enlargements we study to vector optimization problems. We begin with the description in Section 4 of the behavior of Gerstewitz (Tammer) functional in relation to cone enlargements. After that, in Section 5, we present the way one (actually, all) of these enlargements is (are) involved in various situations where a directional solution of a given vector optimization problem can be seen as a proper one in the commonly understood sense (see [7]).

Throughout this paper, X denotes a normed vector space. We denote by $B(x,r)$, $D(x,r)$, and $S(x,r)$ the open ball, the closed ball, and the sphere of center $x \in X$ and radius $r > 0$; for $x = 0$ and $r = 1$, we will write B_X, D_X, and S_X, respectively. For a set $A \subset X$, we denote by $\operatorname{int} A$, $\operatorname{cl} A$ its topological interior and closure. The cone generated by A is designated by $\operatorname{cone} A$, and the distance from a point $x \in X$ to a nonempty set $A \subset X$ is $d(x,A) := \inf\{\|x-a\| \mid a \in A\}$. We denote by w the weak topology on X.

2. Cone Enlargements and General Properties

Let X be a normed vector space and $C \subset X$ be a closed cone. One says that C is based if there exists a convex set B such that $0 \notin \operatorname{cl} B$ and $C = \operatorname{cone} B$. Obviously, a based cone is convex. If B is also bounded, one says that C is well-based and in this case, since C is supposed to be closed, one can take B to be closed as well (see [7, Definition 2.2.14] and the subsequent comments). Moreover, it is well known that a cone which has a base is pointed.

We consider and compare four kinds of conic enlargements for C. Firstly, we note the obvious fact that $C = \operatorname{cone}(C \cap S_X)$ and we denote by S_C the set $C \cap S_X$. The first two enlargements are defined next.

Definition 2.1. Let $\varepsilon > 0$ and $C \subset X$ be a closed pointed cone. Define the following enlargements of the cone C:

(i) the first type enlargement is

$$C^{(1)\varepsilon} = \operatorname{cone}(\{x \in S_X \mid d(x,C) \le \varepsilon\}) = \{u \in X \mid d(u,C) \le \varepsilon \|u\|\};$$

(ii) the second type enlargement is

$$C^{(2)\varepsilon} = \operatorname{cone}(\{x \in S_X \mid d(x,S_C) \le \varepsilon\});$$

Remark 2.1. Notice that no convexity is involved in these definitions since C can be a nonconvex cone. For instance, $C^{(1)\varepsilon}$ was used in [1] under the name of "conic $\varepsilon-$neighborhood" in the context of some continuity properties of cone-values multifunctions, while $C^{(2)\varepsilon}$ was introduced in [4] in order to deal with directional efficiency in vector optimization.

In relation with these enlargements, we use the notation:

$$S_C^{(1)\varepsilon} = S_X \cap C^{(1)\varepsilon} = \{x \in S_X \mid d(x,C) \leq \varepsilon\},$$
$$S_C^{(2)\varepsilon} = S_X \cap C^{(2)\varepsilon} = \{x \in S_X \mid d(x,S_C) \leq \varepsilon\}.$$

Notice that the above sets are closed subsets of S_X, and since $C^{(*)\varepsilon} = \operatorname{cone} S_C^{(*)\varepsilon}$ for $* \in \{1,2\}$, these cones are closed.

The conic enlargement for based cones mostly used in literature was given by the Henig cone dilating procedure (see [7, Lemma 3.2.51]).

Definition 2.2. Let $\varepsilon > 0$ and $C \subset X$ be a closed convex based cone with the base B. The third type enlargement is $C^{(3)\varepsilon} = \operatorname{cone}(\{x \in X \mid d(x,B) \leq \varepsilon\})$.

Remark 2.2. $C^{(3)\varepsilon}$ is a closed convex cone for all $\varepsilon \in (0, d(0,B))$.

Remark 2.3. All three types of enlargements can be the whole space for ε large enough ($\varepsilon \geq 1$ for $C^{(1)\varepsilon}$, $\varepsilon \geq 2$ for $C^{(2)\varepsilon}$, $\varepsilon > d(0,B)$ for $C^{(3)\varepsilon}$) so we are interested to work with small values of ε in order to avoid this situation.

To define the fourth type of enlargement, one needs to recall the following ancillary result (see [7, Section 2.2]).

Lemma 2.1. *Let $C \subset X$ be a closed and convex cone. Then*
(i) C is based if and only if there is $x^* \in X^*$ such that $x^*(x) > 0$ for all $x \in C \setminus \{0\}$. In this case $C \cap \{x \in X \mid x^*(x) = 1\}$ is a base for C;
(ii) C is well-based if and only if there are $x^* \in X^*$ and $\alpha > 0$ such that $x^*(x) \geq \alpha \|x\|$ for all $x \in C$. In this case $C \cap \{x \in X \mid x^*(x) = 1\}$ is a bounded base for C.

Now we can define a specific enlargement for well-based cones.

Definition 2.3. Let $C \subset X$ be a closed, convex, well-based cone and $\varepsilon > 0$. Let $x^* \in X^*$ be the functional from Lemma 2.1 (ii) and $A := \{u \in X \mid x^*(u) = 1\}$. The fourth type enlargement is:

$$C^{(4)\varepsilon} = \operatorname{cone}(\{x \in A \mid d(x, C \cap A) \leq \varepsilon\}).$$

Remark 2.4. Since A and $C \cap A$ are convex and closed, the set $B_\varepsilon := \{x \in A \mid d(x, C \cap A) \leq \varepsilon\}$ enjoys the same properties. Moreover, $0 \notin B_\varepsilon$ since $0 \notin A$, while the boundedness of $C \cap A$ implies the boundedness of B_ε. Therefore, $C^{(4)\varepsilon}$ is always a proper, closed, convex, well-based cone.

Remark 2.5. While in every normed vector spaces one can easily construct well-based cones, and on finite dimensional spaces every closed convex pointed cone is well-based, among the classical infinite dimensional normed vector spaces only for ℓ^1 and $L^1(\Omega)$ the natural ordering cones are well-based.

These enlargements share some similar features. For instance, for $* \in \{1,2,3,4\}$ and $\varepsilon > 0$, $C \setminus \{0\} \subset \operatorname{int} C^{(*)\varepsilon}$, but in fact, these inclusions can be significantly improved in the sense that these enlargements, except $C^{(3)\varepsilon}$, enjoy a property that is used in literature in relation to cones that "allow plastering" (see [8]). In this vein, we have the following result.

Proposition 2.1. *For all $* \in \{1,2,4\}$ and $\varepsilon > 0$, there is $\rho > 0$ such that*

$$D(x, \rho \|x\|) \subset C^{(*)\varepsilon}, \ \forall x \in C. \tag{2.1}$$

(Here we consider $D(0,0) = \{0\}$.) More precisely,

$$\rho = \begin{cases} (1+\varepsilon)\varepsilon^{-1}, \ \text{if } * = 1, \\ 2^{-1}\varepsilon, \ \text{if } * = 2, \\ \frac{\varepsilon\alpha^2}{\|x^*\|(2+\alpha\varepsilon)+2\varepsilon\alpha^2}, \ \text{if } * = 4, \end{cases}$$

with α from Lemma 2.1 (ii) (and, implicitly, from Definition 2.3).

Proof. Firstly, we show the assertion for $* = 1$. Indeed, let $x \in C \setminus \{0\}$ and take $u \in D(x,(1+\varepsilon)^{-1}\varepsilon\|x\|)$. It is obvious that

$$\|u\| \geq \|x\| - (1+\varepsilon)^{-1}\varepsilon\|x\| = (1+\varepsilon)^{-1}\|x\|.$$

On the other hand,

$$d(u,C) \leq \|u-x\| \leq (1+\varepsilon)^{-1}\varepsilon\|x\| \leq \varepsilon\|u\|.$$

This shows that $u \in C^{(1)\varepsilon}$.

Now we show the conclusion for $* = 2$. Let $x \in S_C$ and take $u \in D(x, 2^{-1}\varepsilon)$. Then there is $v \in D\left(0, 2^{-1}\varepsilon\right)$ such that $u = x + v$. Consequently,

$$\left\| x - \frac{x+v}{\|x+v\|} \right\| = \left\| x+v-v - \frac{x+v}{\|x+v\|} \right\| \leq \|v\| + \|x+v\| \left| 1 - \frac{1}{\|x+v\|} \right|$$
$$\leq \|v\| + |\|x+v\| - 1| \leq 2^{-1}\varepsilon + |\|x+v\| - 1|.$$

But $1 - 2^{-1}\varepsilon \leq \|x+v\| \leq 1 + 2^{-1}\varepsilon$, whence $|\|x+v\| - 1| \leq 2^{-1}\varepsilon$, so

$$\left\| x - \frac{u}{\|u\|} \right\| \leq \varepsilon.$$

We conclude that $u \in C^{(2)\varepsilon}$. Now, for $x \in C \setminus \{0\}$, one has $\|x\|^{-1} x \in S_C$, whence

$$D\left(\frac{x}{\|x\|}, 2^{-1}\varepsilon\right) \subset C^{(2)\varepsilon},$$

which means that $D\left(x, 2^{-1}\varepsilon\|x\|\right) \subset C^{(2)\varepsilon}$. Finally, we consider the case $* = 4$. Take first

$$\theta \in \left(0, \frac{\varepsilon\alpha^2}{\|x^*\|(2+\varepsilon\alpha)}\right) \text{ and } \gamma = \frac{\theta}{1+2\theta}.$$

Furthermore, consider $\zeta > 0$ such that

$$\theta + \zeta < \frac{\varepsilon\alpha^2}{\|x^*\|(2+\varepsilon\alpha)}.$$

We firstly prove that for $x \in C \setminus \{0\}$, $D(x, \gamma\|x\|) \subset C^{(4)\varepsilon}$. Consider $u \in D(x, \gamma\|x\|)$. Therefore,

$$d(u,C) \leq \|u-x\| \leq \gamma\|x\|.$$

But $\|u\| \geq \|x\| - \|u-x\| \geq \|x\| - \gamma\|x\|$ whence

$$d(u,C) \leq \frac{\gamma}{1-\gamma}\|u\|.$$

Denote

$$\nu := \frac{\gamma}{1-\gamma} = \frac{\theta}{1+\theta}.$$

Clearly, for all $\delta > 0$, there is $c_\delta \in C$ such that $\|u-c_\delta\| < \nu\|u\| + \delta$. Notice that, for small δ, $c_\delta \neq 0$ since $\nu < 1$ whence $x^*(c_\delta) \geq \alpha\|c_\delta\| > 0$. We obtain

$$\|u-c_\delta\| < \nu\|u-c_\delta\| + \nu\|c_\delta\| + \delta,$$

which yields

$$\|u-c_\delta\| < \frac{\nu}{1-\nu}\|c_\delta\| + \frac{\delta}{1-\nu} = \theta\|c_\delta\| + \frac{\delta}{1-\nu}.$$

Then

$$\begin{aligned}\frac{3}{2}(1+\nu)\|u\| \geq (1+\nu)\|u\| + \delta \geq \|c_\delta\| &\geq \|u\| - \|u-c_\delta\| \\ &\geq \|u\|(1-\nu) - \delta > 2^{-1}\|u\|(1-\nu)\end{aligned}$$

for small δ. Observe now that

$$\begin{aligned}x^*(u) &= x^*(c_\delta) + x^*(u-c_\delta) \geq \alpha\|c_\delta\| - \|x^*\|\|u-c_\delta\| \\ &\geq \alpha\|c_\delta\| - \theta\|c_\delta\|\|x^*\| - \frac{\delta}{1-\nu}\|x^*\|. \\ &= \|c_\delta\|(\alpha - \theta\|x^*\|) - \frac{\delta}{1-\nu}\|x^*\| \\ &> 2^{-1}\|u\|(1-\nu)\cdot\frac{2\alpha}{2+\varepsilon\alpha} - \frac{\delta}{1-\nu}\|x^*\| > 0,\end{aligned}$$

for all small δ. Therefore,

$$\frac{u}{x^*(u)} \in A \text{ and } \frac{c_\delta}{x^*(c_\delta)} \in C \cap A.$$

We aim to show that, for small δ,

$$\left\|\frac{u}{x^*(u)} - \frac{c_\delta}{x^*(c_\delta)}\right\| \leq \varepsilon.$$

By denoting $y := u - c_\delta$, this becomes

$$\left\|\frac{y+c_\delta}{x^*(y)+x^*(c_\delta)} - \frac{c_\delta}{x^*(c_\delta)}\right\| \leq \varepsilon,$$

that is,

$$\|x^*(c_\delta)y - x^*(y)c_\delta\| \leq \varepsilon(x^*(y)+x^*(c_\delta))x^*(c_\delta). \tag{2.2}$$

Now, the left-hand side of (2.2) satisfies

$$\begin{aligned}\left\|x^*(c_\delta)y-x^*(y)c_\delta\right\| &\le 2\|x^*\|\|y\|\|c_\delta\| \\ &< 2\left(\theta\|c_\delta\|+\frac{\delta}{1-\nu}\right)\|x^*\|\|c_\delta\| \\ &= 2\theta\|x^*\|\|c_\delta\|^2+2\frac{\delta}{1-\nu}\|x^*\|\|c_\delta\| \\ &\le 2\|x^*\|\left(\frac{\varepsilon\alpha^2}{\|x^*\|(2+\varepsilon\alpha)}-\zeta\right)\|c_\delta\|^2+2\frac{\delta}{1-\nu}\|x^*\|\|c_\delta\| \\ &= 2\frac{\varepsilon\alpha^2}{2+\varepsilon\alpha}\|c_\delta\|^2-2\zeta\|x^*\|\|c_\delta\|^2+2\frac{\delta}{1-\nu}\|x^*\|\|c_\delta\|.\end{aligned}$$

The boundedness of the norm of c_δ and this inequality ensure that there is $\xi>0$ such that

$$\left\|x^*(c_\delta)y-x^*(y)c_\delta\right\|<2\frac{\varepsilon\alpha^2}{2+\varepsilon\alpha}\|c_\delta\|^2-\xi,$$

again, for all small δ. The right-hand side of (2.2) satisfies

$$\begin{aligned}\varepsilon\left(x^*(y)+x^*(c_\delta)\right)x^*(c_\delta) &\ge \varepsilon\alpha\|c_\delta\|\left(\alpha\|c_\delta\|-\|x^*\|\|y\|\right) \\ &\ge \varepsilon\alpha\|c_\delta\|\left(\alpha\|c_\delta\|-\|x^*\|\left(\theta\|c_\delta\|+\frac{\delta}{1-\nu}\right)\right) \\ &= \varepsilon\alpha\|c_\delta\|^2\left(\alpha-\frac{\varepsilon\alpha^2}{\|x^*\|(2+\varepsilon\alpha)}\|x^*\|\right)-\varepsilon\alpha\|c_\delta\|\|x^*\|\frac{\delta}{1-\nu} \\ &= \frac{2\varepsilon\alpha^2}{2+\varepsilon\alpha}\|c_\delta\|^2-\varepsilon\alpha\|c_\delta\|\|x^*\|\frac{\delta}{1-\nu}.\end{aligned}$$

These inequalities yield that that for small δ, (2.2) holds. Therefore, for all $x\in C\setminus\{0\}$,

$$D\left(x,\frac{\theta}{1+2\theta}\|x\|\right)\subset C^{(4)\varepsilon}$$

for all

$$\theta\in\left(0,\frac{\varepsilon\alpha^2}{\|x^*\|(2+\varepsilon\alpha)}\right).$$

Since $C^{(4)\varepsilon}$ is closed, by making $\theta\to\left(\|x^*\|(2+\varepsilon\alpha)\right)^{-1}\varepsilon\alpha^2$, we obtain the conclusion. □

Remark 2.6. Consider now the case $*=3$. If C is based but not well-based, then $C^{(3)\varepsilon}$ does not have property (2.1) since this cone is always convex. In such a case, (2.1) implies that the cone is well-based (see [8]). However, if B is bounded, then there is $M>0$ such that for all $b\in B$, $\|b\|\le M$, and we can take $\rho=M^{-1}\varepsilon$. Indeed, for $x\in C\setminus\{0\}$, pick $u\in D(x,\rho\|x\|)$. Then there is $b\in B$ and $\alpha>0$ such that $x=\alpha b$. Clearly, $\|x\|\le\alpha M$. Then

$$d\left(\alpha^{-1}u,B\right)\le\left\|\alpha^{-1}u-\alpha^{-1}x\right\|=\alpha^{-1}\|u-x\|\le\alpha^{-1}M^{-1}\varepsilon\|x\|\le\varepsilon,$$

so $\alpha^{-1}u\in C^{(3)\varepsilon}$ and this proves the assertion.

Let $x^*\in X^*$ and $\alpha\in(0,1)$. Define $C_{x^*,\alpha}=\{x\in X\mid x^*(x)\ge\alpha\|x\|\|x^*\|\}$. The interesting situation is $x^*\neq 0$ and it is obvious that in this case $C_{x^*,\alpha}$ is a closed convex pointed cone with nonempty interior. Actually, $\operatorname{int}C_{x^*,\alpha}=\{x\in X\mid x^*(x)>\alpha\|x\|\|x^*\|\}$. A well-known result (see

[7, p. 2]) says that a cone K is contained into a cone of this type (in which case K is said to have the angle property) iff it is well-based. For the relationships between the properties of cones we use, see also [9].

Proposition 2.2. *With the above notation, take* $\beta \in (\alpha, 1)$. *Then, for all* $x \in C_{x^*,\beta} \setminus \{0\}$,

$$D(x, (\beta - \alpha) \|x\|) \subset C_{x^*,\alpha}.$$

In particular,

$$C_{x^*,\beta} \setminus \{0\} \subset \operatorname{int} C_{x^*,\alpha}.$$

Proof. Indeed, for all $x \in C_{x^*,\beta} \setminus \{0\}$ and $u \in D_X$,

$$\begin{aligned} x^*(x + (\beta - \alpha)\|x\| u) &= x^*(x) + (\beta - \alpha)\|x\| x^*(u) \\ &\geq \beta \|x\| \|x^*\| + (\beta - \alpha)\|x\| (-\|x^*\| \|u\|) \\ &= \beta \|x\| \|x^*\| - (\beta - \alpha)\|x\| \|x^*\| \\ &= \alpha \|x\| \|x^*\|, \end{aligned}$$

and the conclusion follows. □

Remark 2.7. Let C be a closed convex pointed cone, $\rho > 0$ and define a generalized conic neighborhood of C (with radius ρ) by $C_\rho := \bigcup_{x \in C} D(x, \rho \|x\|)$. This is a cone. We have shown that in several situations there is a ρ such that C_ρ is a subset of an enlargement of C. If $x^* \in (C_\rho)^+$, then $x^*(x) \geq \rho \|x\| \|x^*\|$ for all $x \in C$, that is, $C \subset C_{x^*,\rho}$. Consequently, if $(C_\rho)^+ \neq \{0\}$ the cone C is well-based. We conclude that if C is not well-based the duals of both $C^{(1)\varepsilon}$ and $C^{(2)\varepsilon}$ are $\{0\}$.

In fact, if $C^{(1)\varepsilon}$ or $C^{(2)\varepsilon}$ is convex proper cone then, by a standard separation result, their duals are not trivial whence there is $\rho > 0$ such that $(C_\rho)^+ \neq \{0\}$. Consequently, C is well-based. Then, if C is not well-based, the enlargements $C^{(1)\varepsilon}$ and $C^{(2)\varepsilon}$ are convex only if they coincide with the whole space.

On the other hand, if C_ρ is convex, then again C is well-based (see [10, p. 222]). The converse is not true, in general. If C is well-based then one can find ρ and a convex cone K such that $C_\rho \subset K$.

In [5], some inclusions between these types of enlargements were given, but in some of them no concrete constant were pointed out. In the next result we provide such constants, improving in this way [5, Proposition 2.7].

Proposition 2.3. *Let* $C \subset X$ *be a closed convex cone, and let* $\varepsilon \in (0,1)$. *Then*

(i) $C^{(2)\varepsilon} \subset C^{(1)\varepsilon}$ *and* $C^{(1)\theta} \subset C^{(2)\varepsilon}$, *where*

$$\theta = \frac{\sqrt{\varepsilon^2 + 12\varepsilon + 4} - \varepsilon - 2}{4};$$

(ii) if C *is has a base* B, *then* $C^{(3)\theta} \subset C^{(1)\varepsilon}$ *for all*

$$\theta \in \left(0, \frac{\varepsilon \cdot d(0,B)}{1+\varepsilon}\right),$$

and if C *is well-based with boundedness constant* M, *then* $C^{(1)\theta} \subset C^{(3)\varepsilon}$, *where*

$$\theta = \frac{\varepsilon}{\varepsilon + M};$$

(iii) if C is well-based, then $C^{(4)\theta} \subset C^{(1)\varepsilon}$, where

$$\theta = \frac{\varepsilon}{\|x^*\|},$$

and $C^{(1)\theta} \subset C^{(4)\varepsilon}$ for all

$$\theta \in \left(0, \frac{\varepsilon\alpha^2}{\varepsilon\alpha^2 + \|x^*\| (2+\varepsilon\alpha)}\right).$$

Proof. (i) The first part is obvious. For the second part, let $y \in S_C^{(1)\theta}$. This means that $\|y\| = 1$ and for all $\delta > 0$ there is $x \in C$ such that $\|y - x\| < \theta + \delta$. Clearly,

$$1 + (\theta + \delta) \geq \|x\| \geq 1 - (\theta + \delta).$$

In particular $x \neq 0$ for small δ. Therefore,

$$d(y, S_C) \leq \left\| y - \frac{x}{\|x\|} \right\| = \frac{1}{\|x\|} \| \|x\| y - x\| < \frac{\| \|x\| y - \|x\| x + \|x\| x - x\|}{1 - (\theta + \delta)}$$
$$\leq \frac{\|x\| \|y - x\| + \|x\| \cdot |\|x\| - 1|}{1 - (\theta + \delta)} < \frac{2 \cdot (1 + \theta + \delta)(\theta + \delta)}{1 - (\theta + \delta)}.$$

Passing to the limit as $\delta \to 0$, we deduce that

$$d(y, S_C) \leq \frac{2\theta(1+\theta)}{1-\theta} = \varepsilon,$$

and this is enough to conclude the proof of this item.

(ii) As seen before (first part of Proposition 2.1), for all $x \in C$, $D(x, (1+\varepsilon)^{-1}\varepsilon \|x\|) \subset C^{(1)\varepsilon}$. Since $0 \notin \mathrm{cl} B$, $\|b\| \geq d(0, B) > 0$ for every $b \in B$. Then, from the above inclusion, for every $b \in B$ one has $D(b, (1+\varepsilon)^{-1}\varepsilon \cdot d(0, B)) \subset C^{(1)\varepsilon}$. This means that if $x \in X$ satisfies $d(x, B) < (1+\varepsilon)^{-1}\varepsilon \cdot d(0, B)$, then $x \in C^{(1)\varepsilon}$, so we get the announced the inclusion.

For the second part, take $x \in S_C^{(1)\theta}$. Then, for all $\delta > 0$ there are $\alpha > 0$ and $b \in B$ such that $\|x - \alpha b\| < \theta + \delta$. This shows that

$$1 - \alpha M \leq 1 - \alpha \|b\| = \|x\| - \|\alpha b\| \leq \|x - \alpha b\| < \theta + \delta,$$

that is,

$$\alpha > \frac{1 - \theta - \delta}{M}.$$

Therefore, for all $\delta > 0$ small enough,

$$d\left(\frac{x}{\alpha}, B\right) \leq \left\| \frac{x}{\alpha} - b \right\| < \frac{\theta + \delta}{\alpha} < \frac{M(\theta + \delta)}{1 - \theta - \delta},$$

whence

$$d\left(\frac{x}{\alpha}, B\right) \leq \frac{M\theta}{1 - \theta} = \varepsilon,$$

and the conclusion follows.

(iii) For this part, see the proof of [5, Proposition 2.7] □

3. Properties Under Normality Condition

In this section, we use the notion of normal cone (see [7, Definition 2.1.21] for the definition) by means of some of its characterization (see [7, Theorem 2.2.10]) in order to further explore the concepts studied before. Rather than the formal definition we present some characterizations of this concept.

Proposition 3.1. *A convex cone C in a normed vector space X is normal iff there exists $\alpha > 0$ such that $\alpha \|x\| \leq \|y\|$ whenever $x, y \in C$, $y - x \in C$ iff there exists $\beta > 0$ such that $\|x + y\| \geq \beta$ whenever $x, y \in S_C$.*

In infinite dimensional vector spaces, this property of normality is in general weaker than the property of the cone to be well-based (see [7, p. 3]). For instance, if one considers the normed space $(C([0,1]), \|\cdot\|_\infty)$ and $C = \{x \in C([0,1]) \mid x(t) \geq 0,\ \forall t \in [0,1]\}$, then C is normal but it is not well-based.

We readily get the following assertion.

Proposition 3.2. *If C is normal, then C_ρ is pointed for $\rho \in (0, 2^{-1}\alpha)$, where α is the constant from the equivalences in Proposition 3.1.*

Proof. Take $u, -u \in C_\rho$. Then there are $x, y \in C$ such that

$$\|u - x\| \leq \rho \|x\| \text{ and } \|-u - y\| \leq \rho \|y\|.$$

So,

$$\frac{\alpha}{2}(\|x\| + \|y\|) \leq \|x + y\| \leq \|u - x\| + \|-u - y\| \leq \rho(\|x\| + \|y\|),$$

which is possible iff $x = y = 0$, that is, $u = 0$. □

Proposition 3.3. *If C is normal, then*

(i) $C^{(1)\varepsilon}$ is pointed for all $\varepsilon \in \left(0, (2+\alpha)^{-1}\alpha\right)$;

(ii) $C^{(2)\varepsilon}$ is pointed for all $\varepsilon \in (0, 2^{-1}\beta)$;

(where α and β are the constants from the equivalences in Proposition 3.1).

Proof. (i) Take $\varepsilon \in \left(0, (2+\alpha)^{-1}\alpha\right)$ and suppose, by way of contradiction, that $C^{(1)\varepsilon}$ is not pointed. Then for all $\delta > 0$, one can find $x \in S_X$, $u, v \in C$ such that $\|x - u\| < \varepsilon + \delta$ and $\|-x - v\| < \varepsilon + \delta$. In particular, $\|u\| > 1 - (\varepsilon + \delta)$ and $\|v\| > 1 - (\varepsilon + \delta)$. Since

$$\|u + v\| \geq \frac{\alpha}{2}(\|u\| + \|v\|) > \alpha(1 - (\varepsilon + \delta))$$

and

$$\|u + v\| = \|u - x + (x + v)\| < 2(\varepsilon + \delta),$$

we have $\alpha(1 - (\varepsilon + \delta)) < 2(\varepsilon + \delta)$ for all δ, that is, $\alpha(1 - \varepsilon) \leq 2\varepsilon$, which contradicts the choice of ε.

(ii) Take $\varepsilon \in (0, 2^{-1}\beta)$. The argument is similar with that of (i). Suppose that $C^{(2)\varepsilon}$ is not pointed. Then, for all $\delta > 0$, one can find $x \in S_X$, $u, v \in S_C$ such that

$$\|x - u\| < \varepsilon + \delta \text{ and } \|-x - v\| < \varepsilon + \delta.$$

We obtain that

$$\beta \leq \|u + v\| = \|u - x + (x + v)\| < 2(\varepsilon + \delta),$$

and we reach a contradiction. □

As we mentioned, the first two types of enlargement are not necessarily convex so a inclusion of the form $C^{(*)\varepsilon}+C^{(*)\varepsilon}\subset C^{(*)\varepsilon}$ does not necessarily hold for $*\in\{1,2\}$. Instead, we provide some inclusions for $S_C^{(*)\varepsilon}+S_C^{(*)\varepsilon}$.

Proposition 3.4. *Suppose that C is normal and let α,β be the constants from the equivalences in Proposition 3.1.*

(i) Let $\varepsilon<2^{-1}\beta$. Then $S_C^{(2)\varepsilon}+S_C^{(2)\varepsilon}\subset C^{(1)\theta}$, where

$$\theta=\frac{2\varepsilon}{\beta-2\varepsilon}.$$

(ii) Let $\varepsilon<(2+\alpha)^{-1}\alpha$. Then $S_C^{(1)\varepsilon}+S_C^{(1)\varepsilon}\subset C^{(1)\theta}$, where

$$\theta=\frac{2\varepsilon}{\alpha-\varepsilon(2+\alpha)}.$$

(iii) Let $\varepsilon<2^{-1}\beta$. Then $S_C^{(2)\varepsilon}+S_C^{(2)\varepsilon}\subset C^{(2)\theta}$, where

$$\theta=\frac{8\varepsilon}{\beta(\beta-2\varepsilon)}.$$

(iv) Let $\varepsilon<(2+\alpha)^{-1}\alpha$. Then $S_C^{(1)\varepsilon}+S_C^{(1)\varepsilon}\subset C^{(2)\theta}$, where

$$\theta=\frac{8\varepsilon(1+\varepsilon)}{\alpha^2(1-\varepsilon)^2-2\alpha\varepsilon(1-\varepsilon)}.$$

Proof. (i) Let us take $y_1,y_2\in S_C^{(2)\varepsilon}$. Then for all $\delta>0$ there are $x_1,x_2\in S_C$ such that $\|y_1-x_1\|<\varepsilon+\delta$ and $\|y_2-x_2\|<\varepsilon+\delta$. Then

$$\begin{aligned} d\left(\frac{y_1+y_2}{\|y_1+y_2\|},C\right)&=\frac{1}{\|y_1+y_2\|}d(y_1+y_2,C)\\ &\le\frac{1}{\|y_1+y_2\|}\|y_1+y_2-(x_1+x_2)\|\le\frac{2(\varepsilon+\delta)}{\|y_1+y_2\|}. \end{aligned}$$

But

$$\|y_1+y_2\|\ge\|x_1+x_2\|-\|y_1+y_2-(x_1+x_2)\|\ge\beta-2(\varepsilon+\delta)>0$$

for all small δ. Therefore, for all small enough δ,

$$d\left(\frac{y_1+y_2}{\|y_1+y_2\|},C\right)\le\frac{2(\varepsilon+\delta)}{\beta-2(\varepsilon+\delta)}.$$

Making $\delta\to0$, we have

$$d\left(\frac{y_1+y_2}{\|y_1+y_2\|},C\right)\le\frac{2\varepsilon}{\beta-2\varepsilon},$$

and the conclusion follows.

(ii) Even the arguments are similar, there are some differences that give us the impetus to write all the details. So, let $y_1,y_2\in S_C^{(1)\varepsilon}$. Then, for all $\delta>0$, there are $x_1,x_2\in C$ such that $\|y_1-x_1\|<\varepsilon+\delta$ and $\|y_2-x_2\|<\varepsilon+\delta$. In particular,

$$\|x_1\|\ge\|y_1\|-\|y_1-x_1\|>1-(\varepsilon+\delta),$$

and similarly for $\|x_2\|$. One has, as before,

$$d\left(\frac{y_1+y_2}{\|y_1+y_2\|},C\right)\leq\frac{2(\varepsilon+\delta)}{\|y_1+y_2\|}.$$

Now

$$\begin{aligned}\|y_1+y_2\| &\geq \|x_1+x_2\|-\|y_1+y_2-(x_1+x_2)\|\geq 2^{-1}\alpha(\|x_1\|+\|x_2\|)-2(\varepsilon+\delta)\\ &> 2^{-1}\alpha(2-2(\varepsilon+\delta))-2(\varepsilon+\delta)=\alpha(1-(\varepsilon+\delta))-2(\varepsilon+\delta)>0\end{aligned}$$

for all small δ. Therefore, for δ small enough,

$$d\left(\frac{y_1+y_2}{\|y_1+y_2\|},C\right)\leq\frac{2(\varepsilon+\delta)}{\alpha(1-(\varepsilon+\delta))-2(\varepsilon+\delta)}.$$

Making $\delta\to 0$, we have

$$d\left(\frac{y_1+y_2}{\|y_1+y_2\|},C\right)\leq\frac{2\varepsilon}{\alpha-\varepsilon(2+\alpha)}.$$

Again, we obtain the conclusion.

(iii) To prove that, let us consider $y_1,y_2\in S_C^{(2)\varepsilon}$. As before, for all $\delta>0$ there are $x_1,x_2\in S_C$ such that

$$\|y_1-x_1\|<\varepsilon+\delta \text{ and } \|y_2-x_2\|<\varepsilon+\delta.$$

Then

$$d\left(\frac{y_1+y_2}{\|y_1+y_2\|},S_C\right)\leq\left\|\frac{y_1+y_2}{\|y_1+y_2\|}-\frac{x_1+x_2}{\|x_1+x_2\|}\right\|=\frac{\|\|x_1+x_2\|(y_1+y_2)-\|y_1+y_2\|(x_1+x_2)\|}{\|y_1+y_2\|\cdot\|x_1+x_2\|}.$$

On the one hand, $\|x_1+x_2\|\geq\beta$ and $\|y_1+y_2\|\geq\beta-2(\varepsilon+\delta)>0$ for small δ. On the other hand,

$$\begin{aligned}&\|\|x_1+x_2\|(y_1+y_2)-\|y_1+y_2\|(x_1+x_2)\|\\ &=\|\|x_1+x_2\|(y_1+y_2)-\|x_1+x_2\|(x_1+x_2)+\|x_1+x_2\|(x_1+x_2)-\|y_1+y_2\|(x_1+x_2)\|\\ &\leq\|x_1+x_2\|\|(y_1+y_2)-(x_1+x_2)\|+\|x_1+x_2\|\cdot|\|x_1+x_2\|-\|y_1+y_2\||\\ &\leq 2\cdot 2(\varepsilon+\delta)+2\cdot\|(y_1+y_2)-(x_1+x_2)\|\leq 4(\varepsilon+\delta)+2\cdot 2(\varepsilon+\delta)=8(\varepsilon+\delta).\end{aligned}$$

We conclude that, for small δ,

$$d\left(\frac{y_1+y_2}{\|y_1+y_2\|},S_C\right)\leq\frac{8(\varepsilon+\delta)}{\beta\cdot(\beta-2(\varepsilon+\delta))}.$$

Now we pass to the limit as $\delta\to 0$ and this concludes the proof of this item.

(iv) Let $y_1,y_2\in S_C^{(1)\varepsilon}$. Then, for all $\delta>0$, there are $x_1,x_2\in C$ such that $\|y_1-x_1\|<\varepsilon+\delta$ and $\|y_2-x_2\|<\varepsilon+\delta$. Then, again,

$$d\left(\frac{y_1+y_2}{\|y_1+y_2\|},S_C\right)\leq\frac{\|\|x_1+x_2\|(y_1+y_2)-\|y_1+y_2\|(x_1+x_2)\|}{\|y_1+y_2\|\cdot\|x_1+x_2\|}.$$

Using the same estimations steps, we have

$$\begin{aligned}&2(1+\varepsilon+\delta)\geq\|x_1+x_2\|\geq 2^{-1}\alpha(\|x_1\|+\|x_2\|)\geq 2^{-1}\alpha\cdot 2(1-(\varepsilon+\delta))=\alpha(1-(\varepsilon+\delta)),\\ &\|y_1+y_2\|\geq\alpha(1-(\varepsilon+\delta))-2(\varepsilon+\delta);\\ &\|\|x_1+x_2\|(y_1+y_2)-\|y_1+y_2\|(x_1+x_2)\|\leq 8(1+\varepsilon+\delta)\cdot(\varepsilon+\delta).\end{aligned}$$

So, for all small δ,

$$d\left(\frac{y_1+y_2}{\|y_1+y_2\|}, S_C\right) \leq \frac{8(1+\varepsilon+\delta)\cdot(\varepsilon+\delta)}{\alpha(1-(\varepsilon+\delta))\cdot(\alpha(1-(\varepsilon+\delta))-2(\varepsilon+\delta))}.$$

We conclude that

$$d\left(\frac{y_1+y_2}{\|y_1+y_2\|}, S_C\right) \leq \frac{8\varepsilon(1+\varepsilon)}{\alpha^2(1-\varepsilon)^2-2\alpha\varepsilon(1-\varepsilon)},$$

and the assertion follows. □

4. Gerstewitz (Tammer) Functional for Cone Enlargements

In this section, we study the behavior of the Gerstewitz (Tammer) functional in relation to cone enlargements.

Let $C \subset X$ a closed pointed cone $e \in C \setminus \{0\}$. In this case, the Gerstewitz (Tammer) scalarizing functional is $\varphi_{C,e} : X \to \overline{\mathbb{R}}$ defined by

$$\varphi_{C,e}(x) := \varphi(x) := \inf\{t \in \mathbb{R} \mid x \in te - C\}. \tag{4.1}$$

Here we use the convention $\inf \emptyset = +\infty$. Notice that in this setting φ is proper, i.e., does not take the value $-\infty$. Indeed, this would happen if it would exist $x \in X$ such that for all $t \in \mathbb{R}$ one has $-x + te \in C$. In particular, this implies

$$n\left(-\frac{1}{n}x+e\right) \in C \text{ and } n\left(-\frac{1}{n}x-e\right) \in C,\ \forall n \in \mathbb{N}\setminus\{0\}.$$

Since C is a closed cone, this yields $e, -e \in C$ and this is not possible in view of the pointedness of C.

We list here the properties of φ we would like to recall in this section (see [7, Theorem 2.3.1], [11]).

Theorem 4.1. *The functional φ in (4.1) has the following properties:*

(i) φ is lower semicontinuous and positively homogeneous;

(ii) $\operatorname{dom}\varphi = \mathbb{R}e - C$;

(iii) for every $\lambda \in \mathbb{R}$ and $x \in X$, $\{u \in X \mid \varphi(u) \leq \lambda\} = \lambda e - C$, and $\varphi(x+\lambda e) = \varphi(x) + \lambda$;

(iv) for all $x \in \operatorname{dom}\varphi$, the inf in (4.1) is attained, that is, $x \in \varphi(x)e - C$;

(v) if C is convex, then φ is convex and for all $\bar{x} \in \operatorname{dom}\varphi$ the Fenchel subdifferential of φ at $\bar{x}$ is $\partial\varphi(\bar{x}) = \{u^* \in C^+ \mid u^*(e) = 1, u^*(\bar{x}) = \varphi(\bar{x})\}$;

(vi) if, additionally, $\operatorname{int}C \neq \emptyset$ and $e \in \operatorname{int}C$, then φ is also $d(e, \operatorname{bd}(K))^{-1}$–Lipschitz.

Proposition 4.1. *Let $\varepsilon > 0$, $C \subset X$ a convex closed pointed cone, and $e \in S_C$. Consider one of the enlargements $C^{(*)\varepsilon}$ of C for which there is $\rho > 0$ such that $C^\rho \subset C^{(*)\varepsilon}$ and $C^{(*)\varepsilon}$ is pointed. Denote $\varphi_{C,e}$ by φ and $\varphi_{C^{(*)\varepsilon},e}$ by φ_ε. Then:*

(i) for all $x \in \mathbb{R}e$, $\varphi(x) = \varphi_\varepsilon(x) = \alpha$, where $x = \alpha e$;

(ii) for all $x \in (X \setminus \mathbb{R}e) \cap (\mathbb{R}e - C)$, $\varphi_\varepsilon(x) \leq \varphi(x) - \rho\,\|\varphi(x)e - x\|$.

Proof. Since $C \subset C^{(*)\varepsilon}$, it is clear that $\varphi_\varepsilon(x) \leq \varphi(x)$ for all $x \in X$.

(i) Suppose that there is $\alpha \in \mathbb{R}$ such that $x = \alpha e$. Then $\varphi(x) = \alpha$ and if there exists $\beta > 0$ such that $(\varphi(x) - \beta)e - x \in C^{(*)\varepsilon}$. We obtain $-\beta e \in C^{(*)\varepsilon}$, and this contradicts the pointedness of $C^{(*)\varepsilon}$. Then $\varphi_\varepsilon(x) = \alpha$.

(ii) Take $x \in (X \setminus \mathbb{R}e) \cap (\mathbb{R}e - C)$. Then $\varphi(x) \in \mathbb{R}$ and $\varphi(x)e - x \in C \setminus \{0\}$. Since, $C^{\rho} \subset C^{(*)\varepsilon}$, we deduce that $D(\varphi(x)e - x, \rho\|\varphi(x)e - x\|) \subset C^{(*)\varepsilon}$. In particular,

$$\varphi(x)e - x - \rho\|\varphi(x)e - x\| e \in C^{(*)\varepsilon},$$

which implies that $(\varphi(x) - \rho\|\varphi(x)e - x\|)e - x \in C^{(*)\varepsilon}$. We deduce that

$$\varphi_{\varepsilon}(x) \le \varphi(x) - \rho\|\varphi(x)e - x\|,$$

whence the conclusion. □

The inequality in the second item of Proposition 4.1 can be strict even if φ takes only finite values as the next example demonstrates.

Example 4.1. Take $X := \mathbb{R}^2$, $C = \mathbb{R}^2_+$,

$$e = \frac{(1,2)}{\sqrt{5}},\ x = \frac{(2,1)}{\sqrt{5}},\ \rho = \frac{1}{\sqrt{5}},\ \varepsilon = \frac{\sqrt{2\sqrt{5}-4}}{\sqrt{5}},$$

in which case $C^{\rho} = C^{(2)\varepsilon}$ if one considers the Euclidean norm. Observe that:

$$\varphi(x) = 2,\ \varphi(x) - \rho\|\varphi(x)e - x\| = \frac{7}{5},\ \varphi_{\varepsilon}(x) = \frac{1}{4}.$$

5. Application to Optimization Problems

In this section, we consider some types of generalized vector optimization problems under the vectorial approach. We aim at presenting topological conditions under which a certain type of efficiency can be seen with respect to an (all) enlargement(s) studied above. This perspective is widely motivated by the concept of proper efficiency that is very useful in the standard vector optimization problems (see [7, Section 3.2.6], [12, Chapter 10] and [13]).

We start with a problem concerning directional minima of sets. Consider a nonempty set $L \subset S_X$ and take K as a pointed closed cone in X. The following notion of directional efficiency is introduced and briefly studied in [14].

Definition 5.1. Let $M \subset X$ be a nonempty set. One says that $\bar{x} \in M$ is a local directional Pareto minimum point for M with respect to L and K if

$$(M \cap (\bar{x} + \operatorname{cone} L) - \bar{x}) \cap (-K) \subset \{0\}. \tag{5.1}$$

If $\operatorname{int} K \neq \emptyset$, one says that $\bar{x} \in M$ is a weak directional Pareto minimum for M with respect to L and K if

$$(M \cap (\bar{x} + \operatorname{cone} L) - \bar{x}) \cap (-\operatorname{int} K) = \emptyset. \tag{5.2}$$

It is simple to see that relation (5.1) is equivalent to

$$(M - \bar{x}) \cap \operatorname{cone} L \cap (-K) = \{0\},$$

while relation (5.2) actually means

$$(M - \bar{x}) \cap \operatorname{cone} L \cap (-\operatorname{int} K) = \emptyset.$$

Therefore, (5.1) is relevant only if $\operatorname{cone} L \cap -K \neq \{0\}$, while for (5.2) it is important to have $\operatorname{cone} L \cap -\operatorname{int} K \neq \emptyset$.

We have the following result which basically says that under certain conditions one can enlarge both the set of directions and the ordering cone from Definition 5.1.

Proposition 5.1. *Let $\overline{x} \in M$ be a local directional Pareto minimum point for M with respect to L and K. Suppose that X is reflexive,* $\operatorname{cone}(M-\overline{x})$, $\operatorname{cone} L$, *K are weakly sequentially closed and, moreover, K is well-based. Then there is $\varepsilon > 0$ such that $\overline{x}$ is a local directional Pareto minimum point for M with respect to $S^{(2)\varepsilon}_{\operatorname{cone} L}$ and $K^{(2)\varepsilon}$.*

Proof. We have to show that there is $\varepsilon > 0$ such that

$$(M-\overline{x}) \cap (\operatorname{cone} L)^{(2)\varepsilon} \cap \left(-K^{(2)\varepsilon}\right) = \{0\}.$$

Suppose, by way of contradiction, that for all $n \in \mathbb{N} \setminus \{0\}$, there exists

$$a_n \in (M-\overline{x}) \cap (\operatorname{cone} L)^{(2)n^{-1}} \cap \left(-K^{(2)n^{-1}}\right) \setminus \{0\}.$$

This means that for all such a_n one can find $\ell_n \in L$ and $k_n \in S_K$ such that

$$\left\| \frac{a_n}{\|a_n\|} - \ell_n \right\| < \frac{2}{n}, \ \left\| \frac{a_n}{\|a_n\|} + k_n \right\| < \frac{2}{n}, \ \forall n > 0.$$

Since the space is reflexive and $\operatorname{cone}(M-\overline{x})$ is weakly sequentially closed, the bounded sequence $\left(\|a_n\|^{-1} a_n\right)$ is weakly convergent on a subsequence to some $a \in \operatorname{cone}(M-\overline{x})$. The above relations show that a is also the weak limit of (ℓ_n) and (k_n). Because $\operatorname{cone} L$ and $-K$ are weakly sequentially closed, one deduces that $a \in \operatorname{cone} L \cap (-K)$.

Now, if $a = 0$, then $k_n \xrightarrow{w} 0$. But, since K is well-based it is supernormal (see [7, p. 37]) and one gets that $k_n \to 0$ which means that $\left(\|a_n\|^{-1} a_n\right) \to 0$. The last convergence is obviously not true, whence $a \neq 0$. This yields the contradiction $0 \neq a \in \operatorname{cone}(M-\overline{x}) \cap \operatorname{cone} L \cap (-K)$. □

Now we consider a vector optimization problem with variable ordering structure in the sense presented in [6]. Let us take $F, Q : X \rightrightarrows Y$ as multifunctions (set-valued maps), where F acts as the objective mapping, while Q is the ordering mapping. Recall that the domain of F is

$$\operatorname{Dom} F = \{x \in X \mid F(x) \neq \emptyset\}.$$

Definition 5.2. Let $\overline{x} \in A$, $c \in Y \setminus \{0\}$, $\varepsilon \geq 0$ and $L \subset S_X$ a closed set. The point $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ is a local $\varepsilon c-$directional nondominated point for F on A with respect to Q and L if there is a neighborhood U of $\overline{x}$ such that

$$(F(x) - \overline{y} + \varepsilon c) \cap (-Q(x)) = \emptyset, \ \forall x \in A \cap U \cap (\overline{x} + \operatorname{cone} L).$$

If $\operatorname{int} Q(x) \neq \emptyset$ for every $x \in A$ the point $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ is a local $\varepsilon c-$directional weak nondominated point for F on A with respect to Q and L if there is a neighborhood U of $\overline{x}$ such that

$$(F(x) - \overline{y} + \varepsilon c) \cap (-\operatorname{int} Q(x)) = \emptyset, \ \forall x \in A \cap U \cap (\overline{x} + \operatorname{cone} L).$$

Some stability issues for this type of solutions in a nondirectional setting were studied in [5]. Clearly, the first of the above notions can be rephrased as follows: $(\overline{x}, \overline{y})$ is a local $\varepsilon c-$directional nondominated point for F on A with respect to Q and L if there is $\gamma > 0$ such that

$$(F(x) - \overline{y} + \varepsilon c) \cap (-Q(x)) = \emptyset, \ \forall x \in A \cap (\overline{x} + [0, \gamma] L).$$

A similar assertion holds for the second concept.

Definition 5.3. Let X,Y be topological spaces and $F : X \rightrightarrows Y$ a set-valued map. One says that F is upper semicontinuous (usc, for short) at $\overline{x} \in \operatorname{Dom} F$ if for every open set V such that $F(\overline{x}) \subset V$ there is a neighborhood U of $\overline{x}$ such that $F(U) \subset V$. One says that F is upper semicontinuous on a set if it is so at every point of that set.

Lemma 5.1. *Let X,Y be topological spaces such that Y is normal and $F,G : X \rightrightarrows Y$ be set-valued maps. Let $\overline{x} \in X$ such that F,G are usc at $\overline{x}$, and $F(\overline{x}), G(\overline{x})$ are nonempty closed disjoint sets. Then there is a neighborhood U of $\overline{x}$ such that for all $x \in U$, $F(x) \cap G(x) = \emptyset$.*

Proof. Since Y is normal, the closed sets $F(\overline{x}), G(\overline{x})$ can be separated by open sets, which means that there are $V,W \subset Y$ open such that $F(\overline{x}) \subset V$, $G(\overline{x}) \subset W$ and $V \cap W = \emptyset$. By the fact that F and G are usc at $\overline{x}$, one can find a neighborhood U of $\overline{x}$ such that for all $x \in U$, $F(x) \subset V$ and $G(x) \subset W$. Now, the conclusion follows. □

Proposition 5.2. *Suppose that $(\overline{x},\overline{y})$ is a local $\varepsilon c-$directional nondominated point for F on A with respect to Q and L. Suppose that one of the following sets of hypotheses holds:*

(i) A is closed, L is compact or X is finite dimensional, F,Q have nonempty closed values and are $(\|\cdot\|, \|\cdot\|)-$usc on $\overline{x} + [0,\gamma]L$.

(ii) A is weakly closed, L is weakly sequentially compact, F,Q have nonempty closed values and are $(w, \|\cdot\|)-$usc on $\overline{x} + [0,\gamma]L$.

Then there is $\theta > 0$ such that $(\overline{x},\overline{y})$ is a local $\varepsilon c-$directional nondominated point for F on A with respect to Q and $S^{(1)\theta}_{\operatorname{cone} L}$

Proof. Suppose, by way of contradiction, that the conclusion is not true. This means that for every positive natural number n one can find $u_n \in L^{(1)n^{-1}}$, $\alpha_n \in [0,\gamma]$ such that $x_n := \overline{x} + \alpha_n u_n \in A$ and $y_n \in (F(x_n) - \overline{y} + \varepsilon c) \cap (-Q(x_n))$. By the definition of $L^{(1)\varepsilon}$, for every u_n, there is a $v_n \in L$ such that $\|u_n - v_n\| < 2 \cdot n^{-1}$.

Suppose that assumptions from (i) hold. This ensures that (v_n) converges (on a subsequence) to some $u \in L$, and we deduce that (u_n) converges to the same limit. Clearly, (α_n) converges (on a subsequence) to some $\alpha \in [0,\gamma]$, so $x_n \to x := \overline{x} + \alpha u \in A \cap (\overline{x} + [0,\gamma]L)$. Then $(F(x) - \overline{y} + \varepsilon c) \cap (-Q(x)) = \emptyset$. By Lemma 5.1, there is a neighborhood U of $\overline{x}$ such that for all $z \in U$, $(F(z) - \overline{y} + \varepsilon c) \cap (-Q(z)) = \emptyset$, which means that this applies to (x_n), eventually. We reach a contradiction, so the conclusion is true.

For the assumptions (ii) the reasoning is similar using the weak convergence on the input space X. □

Remark 5.1. Of course, in view of the mutual inclusions between the four types of cone enlargement we studied in Proposition 2.3, one can replace in several ways the type of enlargement considered in the above results.

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