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SUBDIFFERENTIAL CALCULUS FOR ORDERED MULTIFUNCTIONS WITH APPLICATIONS TO SET-VALUED OPTIMIZATION

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Dedicated to Christiane Tammer, in friendship and great esteem

Abstract. This paper addresses the study of subdifferentials for set-valued mappings/multifunctions, which take values in ordered spaces. First we obtain the main calculus (sum and chain) rules for such subdifferentials. Then the developed subdifferential calculus is applied to establishing existence theorems for the so-called relative Pareto minimizers in general problems of set-valued optimization with constraints of various types.

Keywords. Coderivatives; Existence of optimal solutions; Relative Pareto minimizers; Subdifferential Palais-Smale condition; Variational analysis.

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1. INTRODUCTION

The paper concerns the areas of vector and set-valued optimization, where the achievements of Professor Christiane Tammer is difficult to overstate. In particular, she introduced and investigated a number of fundamental concepts in these areas that have been largely developed and applied by many mathematicians, economists as well as scientists and practitioners in other disciplines over the words. Professor Tammer wrote numerous very influential papers and monographs among which we mention the monumental one [1] and the most recent book [2], where the reader can find comprehensive bibliographies.

Our research here focuses on developing a *dual-space approach* to optimization problems related to variational analysis of nonsmooth problems via normal cones to sets, coderivatives of set-valued mappings, and subdifferentials of extended-real-valued functions in dual spaces that may not be generated by tangential approximations. The foundation of this approach with applications to scalar and vector (single-objective) optimization problems can be found in [3] and the references therein. A characteristic feature of the dual-space approach, mainly based on *extremal principles* of variational analysis, is *full calculus* available for the aforementioned dual-space generalized differential constructions in general finite-dimensional and infinite-dimensional nonconvex settings. The developed calculus induces broad applications of

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such constructions to various problems of optimization, equilibria, control, stability, applied sciences, etc.; see, e.g., [1, 2, 3, 4, 5] for more information.

Recall that *subdifferentials* of variational analysis are usually associated with extended-real-valued functions $\varphi\colon X \to \bar{\mathbb{R}} := (-\infty,\infty]$ by using the natural order of real numbers. The subdifferential notions for multifunctions (set-valued and, in particular, vector-valued mappings) with values in *ordered* spaces were introduced in [6, 7] and then were developed and applied in other publications; see, e.g., [4]. However, adequate calculus rules have not been developed for such subdifferentials. This paper aims at developing the major sum and chain rules for the subdifferentials of *ordered multifunctions* with their applications to the existence of optimal solutions and deriving necessary optimality conditions in various classes of problems in constrained set-valued optimization.

The solution concepts for set-valued optimization problems investigated in this paper under various types of constraints are unified in [4, 7] under the name of *relative Pareto minimizers* in the general framework of unconstrained set-valued optimization. Along with the conventional notions of Pareto and weak Pareto optimality (efficiency and weak efficiency, respectively), the relative Pareto minimizers include other Pareto-type notions for problems with *nonsolid* ordering cones in locally convex spaces. Existence theorems for relative Pareto minimizers for *unconstrained* problems were expressed in [4, 7] via subdifferentials of ordered cost multifunctions. The subdifferential calculus rules for ordered multifunctions obtained in this paper allow us to extend such results to several large classes of *constrained* set-valued optimization problems with *geometric, functional, operator*, and *equilibrium constraints*. The obtained results are illustrated by various examples.

The rest of the paper is organized as follows. Section 2 contains some *preliminaries* from variational analysis broadly used in the paper. In Section 3, we present the definitions of the two *subdifferentials* (*basic and singular*) of ordered multifunctions studied below and then derive general *sum rules* for both subdifferentials under the *singular subdifferential qualification condition*. Section 4 contains the corresponding subdifferential *chain rules* for compositions of ordered set-valued mappings.

The subsequent sections of the paper address applications of the obtained calculus rules to constrained problems of set-valued optimization by mainly concentrating here on deriving *existence theorems* for relative Pareto minimizers. In Section 5, we formulate the three notions of Pareto-type *optimal solutions* to such problems and discuss their important properties. Section 6 is devoted to establishing efficient conditions of the *subdifferential Palais-Smale* type, which ensure the existence of global Pareto-type minimizers in problems with *geometric constraints*. Section 7 contains existence results for set-valued optimization problems with *functional constraints* of equality and inequality types. In Section 8, we establish the existence of optimal solutions for set-valued problems under general constraints of the *operator* and *equilibrium* types. The concluding Section 9 summarizes the major results of the paper and discusses some *open questions* of our future research.

Throughout the paper, we use standard notation of variational analysis, set-valued analysis, and generalized differentiation; see, e.g., the books [1, 4, 5]. Recall that $\mathbb{N} := \{1,2,\dots\}$ and that $\mathbb{B}$ stands to the closed unit ball in the space in question.

2. PRELIMINARIES FROM VARIATIONAL ANALYSIS

For simplicity, we confine ourselves in this paper to the *finite-dimensional* settings, although most of the presented constructions and obtained results can be extended to infinite-dimensional spaces; cf. the books [3, 4] and their references for further discussions. The finite-dimensional notions and facts presented in this section can be founds, e.g., in [4, 5] with different proofs of the major results therein.

Employing the dual-space geometric approach from [3, 4], we start with recalling the notions of generalized normal to sets. Given $\Omega \subset \mathbb{R}^n$ and $x \in \Omega$, the (Fréchet) *regular normal cone* to $x \in \Omega$ is given by

$$\hat{N}(x;\Omega) := \Big\{ x^* \in \mathbb{R}^n \;\Big|\; \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u-x\rangle}{\|u-x\|} \le 0 \Big\}, \tag{2.1}$$

where the symbol '$u \xrightarrow{\Omega} x$' indicates that $u \to x$ with $u \in \Omega$. We let $\hat{N}(x;\Omega) := \emptyset$ if $x \notin \Omega$. The (limiting, Mordukhovich) *normal cone* to Ω at $\bar{x} \in \Omega$ is defined by

$$N(\bar{x};\Omega) := \big\{ x^* \in \mathbb{R}^n \;\big|\; \exists x_k \to \bar{x},\ x_k^* \to x^* \ \text{ with } \ x_k^* \in \hat{N}(x_k;\Omega) \ \text{ as } \ k \in \mathbb{N} \big\}. \tag{2.2}$$

Both sets (2.1) and (2.2) reduce to the normal cone of convex analysis if Ω is convex. While in the general case, (2.1) is always a convex set, the one in (2.2) is often nonconvex; e.g., for $\Omega_1 := \mathrm{epi}\,(-|x|)$ and $\Omega_2 := \mathrm{gph}\,(|x|)$ at $(0,0) \in \mathbb{R}^2$. Nevertheless, the normal cone (2.2) and the associated constructions for functions and multifunctions enjoy comprehensive *calculus rules* due to *variational/extremal principles* of variational analysis. In contrast, available calculus for (2.1) is quite limited. Note also that we often have the triviality $\hat{N}(\bar{x};\Omega) = \{0\}$ at boundary points $\bar{x} \in \mathrm{bd}\,\Omega$ of closed sets (as, e.g., for the set Ω_1 at $\bar{x} = (0,0)$ above), while $N(\bar{x};\Omega) \ne \{0\}$ *if and only if* $\bar{x} \in \mathrm{bd}\,\Omega$. In what follows, the normal cone (2.2) and the associated coderivative and subdifferential notions defined below are our *basic* constructions. Note that the nonconvexity of $N(\bar{x};\Omega)$ tells us that this basic normal cone cannot be generated in duality by any primal-space tangential approximation of Ω, since the duality/polarity operation yields convexity. The convex closure of (2.2) reduces to the (Clarke) *convexified normal cone* to Ω at $\bar{x}$. The latter may be *very large*, especially for graphical sets as, e.g., for the set Ω_2 above, the graph of the simplest nonsmooth convex function $|x|$, where the convexified normal cone at $(0,0)$ is the whole plane $\mathbb{R}^2$.

Given further a set-valued mapping $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, its *domain* and *graph* are defined in the standard way by, respectively.

$$\mathrm{dom}\,F := \big\{ x \in \mathbb{R}^n \;\big|\; F(x) \ne \emptyset \big\} \ \text{ and } \ \mathrm{gph}\,F := \big\{ (x,z) \in \mathbb{R}^n \times \mathbb{R}^m \;\big|\; z \in F(x) \big\}.$$

Then the *coderivative* of F at $(\bar{x},\bar{z}) \in \mathrm{gph}\,F$ is introduced by

$$D^*F(\bar{x},\bar{z})(z^*) := \big\{ x^* \in \mathbb{R}^n \;\big|\; (x^*,-z^*) \in N\big((\bar{x},\bar{z});\mathrm{gph}\,F\big\}, \quad z^* \in \mathbb{R}^m, \tag{2.3}$$

via the normal cone (2.2) of the graph at $(\bar{x},\bar{z})$, while graphical sets are the most troublesome with respect to convexification. If $F(\bar{x})$ is a singleton at $\bar{x}$, we skip $\bar{z} = F(\bar{x})$ from the coderivative notation (2.3). If $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is single-valued and continuously differentiable around $\bar{x} \in \mathrm{dom}\,F$, then we have the representation

$$D^*F(\bar{x})(z^*) = \big\{ \nabla F(\bar{x})^* z^* \big\} \ \text{ for all } \ z^* \in \mathbb{R}^m$$

via the adjoint/transpose Jacobian matrix $\nabla F(\bar{x})$. This tells us that the coderivative is a natural extension of the *adjoint derivative* operator to nonsmooth and set-valued mappings. In general, the coderivative (2.3) is a positive homogeneous multifunction with possibly nonconvex values, and thus it *cannot be dual* to any tangential graphical derivative of F.

One of the major advantages of the coderivative (2.3) is the possibility to completely characterize in its terms some fundamental *well-posedness* properties of set-valued mappings. The first property of this type concerns *robust Lipschitzian stability* of multifunctions with respect to small perturbations of the initial point. Recall that a multifunction $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ has the (Aubin) *Lipschitz-like* property around $(\bar{x},\bar{z}) \in \operatorname{gph} F$ with modulus $\ell > 0$ if there exist neighborhoods U of $\bar{x}$ and V of $\bar{z}$ such that

$$F(x) \cap V \subset F(u) + \ell \|x-u\| \mathbb{B} \text{ for all } x,u \in U. \tag{2.4}$$

This clearly extends the standard local Lipschitz continuity of single-valued mappings while being a graphical localization of the classical (Hausdorff) local Lipschitzian property of set-valued mappings F around $\bar{x} \in \operatorname{dom} F$, which corresponds to the case of $V = \mathbb{R}^m$ in (2.4). The *exact Lipschitzian bound* of F around $(\bar{x},\bar{z})$, denoted by $\operatorname{lip} F(\bar{x},\bar{z}) \geq 0$, is the infimum of all ℓ for which (2.4) holds with some neighborhoods U and V.

It has been well recognized in variational analysis that the Lipschitz-like property of F with some modulus ℓ in (2.4) is *equivalent* to the metric regularity of the inverse mapping $G := F^{-1}$ around $(\bar{z},\bar{x}) \in \operatorname{gph} G = \operatorname{gph} F^{-1}$. Recall that an arbitrary multifunction $G\colon \mathbb{R}^d \rightrightarrows \mathbb{R}^s$ is *metrically regular* around $(\bar{p},\bar{q}) \in \operatorname{gph} G$ with modulus $\mu > 0$ if there exist neighborhoods P of $\bar{p}$ and Q of $\bar{q}$ such that

$$\operatorname{dist}\big(p; G^{-1}(q)\big) \leq \mu \operatorname{dist}\big(q; G(p)\big) \text{ for all } p \in P \text{ and } q \in Q, \tag{2.5}$$

where 'dist' indicates the distance between a point and a set. The *exact regularity bound* of G around $(\bar{p},\bar{q})$, denoted by $\operatorname{reg} G(\bar{p},\bar{q})$, is the supremum of all μ for which the distance estimate (2.5) holds with some neighborhoods P and Q. In addition to the aforementioned equivalence between the Lipschitz-like and metric regularity properties, recall the following relationship between the exact bounds:

$$\operatorname{lip} F(\bar{x},\bar{z}) = \operatorname{reg} F^{-1}(\bar{z},\bar{x}).$$

We mention that the metric regularity property of G around $(\bar{p},\bar{q})$ is equivalent to yet another fundamental property of this multifunction, known as the *covering* (or *linear openness*) of G around the same point but with the exact covering bound *reciprocal* to $\operatorname{reg} G(\bar{p},\bar{q})$.

Using the coderivative (2.3) allows us to establish *complete characterizations* of all the above properties of closed-graph multifunctions with precise formulas for computing the corresponding exact bounds; see [8]. In particular, a closed-graph multifunction $F\colon \mathbb{R}^n \rightrightarrows R^m$ is Lipschitz-like around $(\bar{x},\bar{z})$ *if and only if*

$$D^*F(\bar{x},\bar{z})(0) = \{0\} \tag{2.6}$$

with the exact Lipschitzian bounds computed by

$$\operatorname{lip} F(\bar{x},\bar{z}) = \|D^*F(\bar{x},\bar{z})\| := \sup\big\{\|x^*\| \,\big|\, x^* \in D^*F(\bar{x},\bar{z})(z^*),\ \|z^*\| \leq 1\big\}. \tag{2.7}$$

The coderivative characterization of the Lipschitz-like property in (2.6) together with the exact bound formula (2.7) is labeled as the *Mordukhovich criterion* in [5].

3. Subdifferential Sum Rules for Ordered Multifunctions

First we introduce here the two subdifferential notions for ordered multifunctions that are studied and applied in this and subsequent sections.

Given an *ordering cone* Θ as a closed and convex subcone of $\mathbb{R}^m$, define in its term the *preference relation* $\preceq$ on $\mathbb{R}^m$ by

$$z_1 \preceq z_2 \Longleftrightarrow z_2 - z_1 \in \Theta$$

and then associate with $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and Θ the *generalized epigraph*

$$\operatorname{epi}_\Theta F = \big\{(x,z) \in \mathbb{R}^n \times \mathbb{R}^m \;\big|\; z \in F(x) + \Theta\big\}. \tag{3.1}$$

Considering the corresponding Θ-*epigraphical multifunction* $\mathscr{E}_{F,\Theta}\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ defined by

$$\mathscr{E}_{F,\Theta}(x) := \big\{z \in \mathbb{R}^m \;\big|\; z \in F(x) + \Theta\big\}, \tag{3.2}$$

we easily check that $\operatorname{gph}\mathscr{E}_{F,\Theta} = \operatorname{epi}_\Theta F$. The two subdifferential constructions for F of our interest in this paper are given next.

Definition 3.1 (subdifferentials of ordered multifunctions). *Let $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with Z ordered by a closed convex cone Θ. Then:*

(i) *The* BASIC SUBDIFFERENTIAL *of F at $(\bar{x},\bar{z}) \in \operatorname{epi}_\Theta F$ is*

$$\partial_\Theta F(\bar{x},\bar{z}) := \big\{x^* \in X^* \;\big|\; x^* \in D^*\mathscr{E}_{F,\Theta}(\bar{x},\bar{z})(z^*),\ -z^* \in N(0;\Theta),\ \|z^*\| = 1\big\}. \tag{3.3}$$

(ii) *The* SINGULAR SUBDIFFERENTIAL *of F at $(\bar{x},\bar{z}) \in \operatorname{epi}_\Theta F$ is*

$$\partial^\infty_\Theta F(\bar{x},\bar{z}) := D^*\mathscr{E}_{F,\Theta}(\bar{x},\bar{z})(0). \tag{3.4}$$

Observe that the subdifferentials in Definition 3.1(i,ii) expand to the case of ordered multifunctions the corresponding notions of the basic/general and singular/horizon subdifferentials for extended-real-valued functions $\varphi\colon \mathbb{R}^n \to \bar{\mathbb{R}} := (-\infty,\infty]$ as developed in [3, 4, 5]. Indeed, letting $\Theta := \mathbb{R}_+$, the set of nonnegative numbers, we have

$$\operatorname{epi}_\Theta \varphi = \big\{(x,z) \in \mathbb{R}^n \times \mathbb{R} \;\big|\; z \in \varphi(x) + \Theta\big\} = \big\{(x,z) \in \mathbb{R}^{n+1} \;\big|\; z \geq \varphi(x)\big\} = \operatorname{epi}\varphi.$$

Fix $\bar{x}$ with $\varphi(\bar{x}) < \infty$. The basic subdifferential (3.3) for φ at $(\bar{x},\bar{z})$ with $\bar{z} := \varphi(\bar{x})$ is

$$\partial_\Theta \varphi(\bar{x},\bar{z}) = \big\{x^* \in \mathbb{R}^n \;\big|\; x^* \in D^*\mathscr{E}_{\varphi,\Theta}(\bar{x},\bar{z})(z^*),\ -z^* \in N(0,\Theta),\ \|z^*\| = 1\big\}.$$

Since $N(0;\mathbb{R}_+) = \mathbb{R}_-$, it follows that

$$\begin{aligned}
\partial_\Theta \varphi(\bar{x},\bar{z}) &= \big\{x^* \in \mathbb{R}^n \;\big|\; x^* \in D^*\mathscr{E}_{\varphi,\Theta}(\bar{x},\bar{z})(z^*),\ -z^* \leq 0,\ \|z^*\| = 1\big\}\\
&= \big\{x^* \in \mathbb{R}^n \;\big|\; x^* \in D^*\mathscr{E}_{\varphi,\Theta}(\bar{x},\bar{z})(1)\big\}\ \big(\text{by } z^* = 1\big)\\
&= \big\{x^* \in \mathbb{R}^n \;\big|\; (x^*,-1) \in N\big((\bar{x},\bar{z});\operatorname{gph}\mathscr{E}_{\varphi,\Theta}\big)\big\}\\
&= \big\{x^* \in \mathbb{R}^n \;\big|\; (x^*,-1) \in N\big((\bar{x},\bar{z});\operatorname{epi}\varphi\big)\big\}\ \big(\text{by } \operatorname{gph}\mathscr{E}_{\varphi,\Theta} = \operatorname{epi}_\Theta\varphi = \operatorname{epi}\varphi\big)\\
&= \partial\varphi(\bar{x}).
\end{aligned}$$

Similarly we check that $\partial^\infty_\Theta \varphi(\bar{x},\bar{z}) = \partial^\infty \varphi(\bar{x})$ according to the [4, Definition 1.18].

We say that a mapping $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with values ordered by a closed convex cone $\Theta \subset \mathbb{R}^m$ is Θ-*Lipschitz-like* around $(\bar{x},\bar{z}) \in \operatorname{epi}_\Theta F$ if the Θ-epigraphical mapping (3.2) is Lipschitz-like around this point. It follows from the coderivative criterion (2.6) that

$$\partial^\infty_\Theta F(\bar{x},\bar{z}) = \{0\} \tag{3.5}$$

for the singular subdifferential (3.4) of F at $(\bar{x},\bar{z})$ provided that F is Θ-Lipschitz-like and closed-graph around this point.

To establish next *subdifferential sum rules* for both basic and singular subdifferentials of ordered multifunctions, we start with formulating for the reader's convenience appropriate results for coderivatives of general set-valued mappings. To this end, recall that an arbitrary multifunction $M\colon \mathbb{R}^d \rightrightarrows \mathbb{R}^s$ is said to be *inner semicontinuous* at $(\bar{p},\bar{q})$ if for every sequence $p_k \to \bar{p}$ there exists a sequence $q_k \in M(x_k)$ that converges to $\bar{q}$ as $k\to\infty$. This extends the continuity notions for mappings and holds, in particular, when M is Lipschitz-like around $(\bar{p},\bar{q})$. We also recall that M is *locally bounded* around $\bar{p}\in \operatorname{dom} M$ if there exists a bounded set $O\subset\mathbb{R}^s$ such that $M(p)\subset O$ whenever p is sufficiently close to $\bar{p}$.

Given two arbitrary multifunctions $F_1,F_2:\mathbb{R}^n\rightrightarrows\mathbb{R}^m$, consider the auxiliary mapping $S:\mathbb{R}^n\times\mathbb{R}^m\rightrightarrows\mathbb{R}^{2m}$ defined by

$$S(x,z):=\big\{(z_1,z_2)\in\mathbb{R}^m\times\mathbb{R}^m\;\big|\;z_1\in F_1(x),\;z_2\in F_2(x),\;z=z_1+z_2\big\}. \tag{3.6}$$

The first lemma is taken from [4, Theorem 3.9].

Lemma 3.2 (**sum rules for coderivatives of multifunctions**). *Let $F_i:\mathbb{R}^n\rightrightarrows\mathbb{R}^m$ for $i=1,2$ be closed-graph multifunctions, and let $(\bar{x},\bar{z})\in\operatorname{gph}(F_1+F_2)$. The following two independent assertions are satisfied:*

(i) *Fix $(\bar{z}_1,\bar{z}_2)\in S(\bar{x},\bar{z})$ from (3.6). Assume that this mapping is inner semicontinuous at $(\bar{x},\bar{z},\bar{z}_1,\bar{z}_2)$ and that the qualification condition*

$$D^*F_1(\bar{x},\bar{z}_1)(0)\cap\big(-D^*F_2(\bar{x},\bar{z}_2)(0)\big)=\{0\} \tag{3.7}$$

is fulfilled. Then for all $z^*\in\mathbb{R}^m$ we have

$$D^*(F_1+F_2)(\bar{x},\bar{z})(z^*)\subset D^*F_1(\bar{x},\bar{z}_1)(z^*)+D^*F_2(\bar{x},\bar{z}_2)(z^*). \tag{3.8}$$

(ii) *Suppose that the mapping S in (3.6) is locally bounded around $(\bar{x},\bar{z})$ and that the qualification condition (3.7) holds for every pair $(\bar{z}_1,\bar{z}_2)\in S(\bar{x},\bar{z})$. Then for all $z^*\in\mathbb{R}^m$, we have the inclusion*

$$D^*(F_1+F_2)(\bar{x},\bar{z})(z^*)\subset\bigcup_{(\bar{z}_1,\bar{z}_2)\in S(\bar{x},\bar{z})}\big[D^*F_1(\bar{x},\bar{z}_1)(z^*)+D^*F_2(\bar{x},\bar{z}_2)(z^*)\big].$$

The next lemma helps us go from coderivatives to subdifferentials of ordered mappings.

Lemma 3.3 (**epigraphical multifunctions for sums of mappings**). *Let $F_i:\mathbb{R}^n\rightrightarrows\mathbb{R}^m$ for $i=1,2$, let Θ be a closed and convex ordering cone on $\mathbb{R}^m$, and let the Θ-epigraphical multifunction $\mathscr{E}_{F,\Theta}$ be taken from (3.2). Then for any $x\in\mathbb{R}^n$ we have*

$$\mathscr{E}_{F_1+F_2,\Theta}(x)=\mathscr{E}_{F_1,\Theta}(x)+\mathscr{E}_{F_2,\Theta}(x).$$

Proof. Pick $x\in\mathbb{R}^n$ and $z\in\mathscr{E}_{F_1+F_2,\Theta}(x)$. Then we get by definition (3.2) that $z\in(F_1+F_2)(x)+\Theta$. This ensures the existence of vectors $z_1\in F_1(x)+\Theta$ and $z_2\in F_2(x)+\Theta$ such that $z=z_1+z_2$. Therefore, $\mathscr{E}_{F_1+F_2,\Theta}(x)\subset\mathscr{E}_{F_1,\Theta}+\mathscr{E}_{F_2,\Theta}(x)$.

To verify the opposite inclusion, fix $x\in\mathbb{R}^n$ and take $z_1\in\mathscr{E}_{F_1,\Theta}(x)$ and $z_2\in\mathscr{E}_{F_2,\Theta}(x)$. This implies by definition (3.2) that $z_1\in F_1(x)+\Theta$ and $z_2\in F_2(x)+\Theta$, which tells us that $z_1+z_2\in\mathscr{E}_{F_1+F_2,\Theta}(x)$ and thus completes the proof. $\square$

To proceed further, define the ordering counterpart of the mapping S in (3.6) by

$$S_{\mathscr{E}}(x,z):=\big\{(z_1,z_2)\in\mathbb{R}^m\times\mathbb{R}^m\;\big|\;z_1\in\mathscr{E}_{F_1,\Theta}(x),\;z_2\in\mathscr{E}_{F_2,\Theta}(x),\;z=z_1+z_2\big\}. \tag{3.9}$$

Now we are ready to establish the subdifferential sum rules for both basic and singular subdifferentials of ordered set-valued mappings defined in (3.3) and (3.4), respectively.

Theorem 3.4 (subdifferential sum rules for ordered multifunctions). *Let $F_1,F_2:\mathbb{R}^n\rightrightarrows\mathbb{R}^m$ be set-valued mappings with the image space $\mathbb{R}^m$ ordered by a closed convex cone Θ and such that the generalized epigraphical set in (3.1) is closed for both F_1 and F_2 around the corresponding points. Given $(\bar{x},\bar{z})\in\operatorname{epi}_\Theta(F_1+F_2)$, the following hold:*

(i) *Fix $(\bar{z}_1,\bar{z}_2)\in S_{\mathscr{E}}(\bar{x},\bar{z})$ from (3.9) and suppose that this mapping is inner semicontinuous at $(\bar{x},\bar{z},\bar{z}_1,\bar{z}_2)$ and that the singular subdifferential qualification condition*

$$\partial^\infty_\Theta F_1(\bar{x},\bar{z}_1)\cap\big(-\partial^\infty_\Theta F_2(\bar{x},\bar{z}_2)\big)=\{0\} \tag{3.10}$$

is satisfied. Then for we have the inclusions

$$\partial_\Theta(F_1+F_2)(\bar{x},\bar{z})\subset\partial_\Theta F_1(\bar{x},\bar{z}_1)+\partial_\Theta F_2(\bar{x},\bar{z}_2), \tag{3.11}$$

$$\partial^\infty_\Theta(F_1+F_2)(\bar{x},\bar{z})\subset\partial^\infty_\Theta F_1(\bar{x},\bar{z}_1)+\partial^\infty_\Theta F_2(\bar{x},\bar{z}_2). \tag{3.12}$$

(ii) *Suppose that the mapping $S_{\mathscr{E}}$ is locally bounded around $(\bar{x},\bar{z})$ and that the qualification condition (3.8) holds for every pair $(\bar{z}_1,\bar{z}_2)\in S_{\mathscr{E}}(\bar{x},\bar{z})$. Then we have*

$$\partial_\Theta(F_1+F_2)(\bar{x},\bar{z})\subset\bigcup_{(\bar{z}_1,\bar{z}_2)\in S_{\mathscr{E}}(\bar{x},\bar{z})}\big[\partial_\Theta F_1(\bar{x},\bar{z}_1)+\partial_\Theta F_2(\bar{x},\bar{z}_2)\big], \tag{3.13}$$

$$\partial^\infty_\Theta(F_1+F_2)(\bar{x},\bar{z})\subset\bigcup_{(\bar{z}_1,\bar{z}_2)\in S_{\mathscr{E}}(\bar{x},\bar{z})}\big[\partial^\infty_\Theta F_1(\bar{x},\bar{z}_1)+\partial^\infty_\Theta F_2(\bar{x},\bar{z}_2)\big]. \tag{3.14}$$

Proof. First we verify the basic subdifferential inclusion (3.11) in (i). Picking any $x^*\in\partial_\Theta(F_1+F_2)(\bar{x},\bar{z})$ and using (3.3) give us $-z^*\in N(0;\Theta)$ such that $\|z^*\|=1$ and $x^*\in D^*\mathscr{E}_{F_1+F_2,\Theta}(\bar{x},\bar{z})(z^*)$. By Lemma 3.3 we get $x^*\in D^*(\mathscr{E}_{F_1,\Theta}+\mathscr{E}_{F_2,\Theta})(\bar{x},\bar{z})(z^*)$. It follows from the singular subdifferential qualification condition (3.10) that

$$D^*\mathscr{E}_{F_1,\Theta}(\bar{x},\bar{z}_1)(0)\cap\big(-D^*\mathscr{E}_{F_2,\Theta}(\bar{x},\bar{z}_2)(0)\big)=\{0\}.$$

This together with the assumed inner semicontinuity of $S_{\mathscr{E}}$ at $(\bar{z}_1,\bar{z}_2)$ allows us to apply the coderivative sum rule from Lemma 3.2 to the epigraphical sum $\mathscr{E}_{F_1,\Theta}+\mathscr{E}_{F_2,\Theta}$. The latter brings us therefore to the inclusion

$$x^*\in D^*\mathscr{E}_{F_1,\Theta}(\bar{x},\bar{z}_1)(z^*)+D^*\mathscr{E}_{F_2,\Theta}(\bar{x},\bar{z}_2)(z^*).$$

Then employing definition (3.3) and the choice of z^* tells us that

$$x^*\in\partial_\Theta F_1(\bar{x},\bar{z}_1)+\partial_\Theta F_2(\bar{x},\bar{z}_2),$$

which readily verifies (3.11). The proof of the singular subdifferential sum rule (3.12) is similar by using (3.4) and putting $z^*=0$ in the arguments above.

To furnish now the proof of (ii), we proceed with (3.13) while observing that the proof of (3.14) is similar. Pick any $x^*\in\partial_\Theta(F_1+F_2)(\bar{x},\bar{z})$ and then find by definition (3.3) a vector $z^*\in\mathbb{R}^m$ such that $-z^*\in N(0;\Theta)$ with $\|z^*\|=1$ and $x^*\in D^*\mathscr{E}_{F_1+F_2,\Theta}(\bar{x},\bar{z})(z^*)$. Since $S_{\mathscr{E}}$ is locally bounded around $(\bar{x},\bar{z})$ and the epigraphical multifunctions $\mathscr{E}_{F_i,\Theta}$, $i=1,2$, have closed graphs, it follows from Theorem 3.2(ii) that

$$\begin{aligned} D^*\mathscr{E}_{F_1+F_2,\Theta}(\bar{x},\bar{z})(z^*) &= D^*(\mathscr{E}_{F_1,\Theta}+\mathscr{E}_{F_2,\Theta})(\bar{x},\bar{z})(z^*) \\ &\subset \bigcup_{(\bar{z}_1,\bar{z}_2)\in S_{\mathscr{E}}(\bar{x},\bar{z})} \big[D^*\mathscr{E}_{F_1,\Theta}(\bar{x},\bar{z}_1)(z^*)+D^*\mathscr{E}_{F_2,\Theta}(\bar{x},\bar{z}_2)(z^*)\big], \end{aligned}$$

which justifies (3.13) and completes the proof of the theorem. □

Observe that the singular subdifferential qualification condition (3.10) serves both basic and singular subdifferential rules in Theorem 3.4. Moreover, the coderivative criterion (2.6) applied to ordered multifunctions allows us to ensure that this qualification condition fulfillment holds *automatically* under the Θ-Lipschitz-like property of either F_1 at $(\bar{x},\bar{z}_1)$ or F_2 at $(\bar{x},\bar{z}_2)$ in Theorem 3.4. For simplicity, we formulate the corresponding consequence only for assertion (i) of the above theorem.

Corollary 3.5 (subdifferential sum rules for Θ-Lipschitz-like mappings). *Consider the general setting of Theorem* 3.4(i) *without imposing the qualification condition* (3.10) *while assuming that either F_1 is Θ-Lipschitz-like around $(\bar{x},\bar{z}_1)$, or F_2 is Θ-Lipschitz-like around $(\bar{x},\bar{z}_2)$. Then both subdifferential sum rules in* (3.11) *and* (3.12) *are satisfied.*

Proof. This follows directly from the fulfillment of the singular subdifferential condition (3.5) for Θ-Lipschitz-like ordered multifunctions. □

Let us illustrate the results of Theorem 3.4 by the following simple example.

Example 3.6 (illustration of the subdifferential sum rules). Consider the two set-valued mappings $F_i:\mathbb{R}\rightrightarrows\mathbb{R}^2$, for $i=1,2$ given by

$$F_1(x) := \begin{cases} [0,\infty), & x\le 0 \\ x, & x>0 \end{cases}$$

$$F_2(x) := \begin{cases} -x, & x< 0 \\ [0,\infty), & x\ge 0 \end{cases}$$

with the ordering cone $\Theta := \{x\in\mathbb{R} \mid x\ge 0\}$. Then the sum of these two mappings is

$$F(x) := (F_1+F_2)(x) = \begin{cases} [-x,\infty), & x<0 \\ [x,\infty), & x\ge 0 \end{cases}$$

which is depicted as the epigraph of $|x|$. It is easy to compute the basic subdifferentials by

$$\partial_\Theta F_1(x,y) = \begin{cases} [0,1], & (x,y)=(0,0) \\ \{0\}, & x<0,\ y=0 \\ \{1\}, & x>0 \end{cases}$$

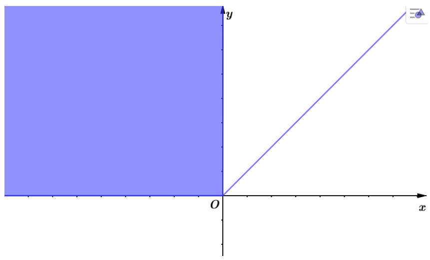

FIGURE 1. Graph of F_1

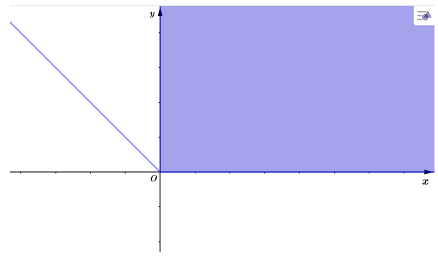

FIGURE 2. Graph of F_2

and

$$\partial_\Theta F_2(x,y) = \begin{cases} [-1,0], & (x,y)=(0,0) \\ \{-1\}, & x<0 \\ \{0\}, & x>0,\ y=0 \end{cases}.$$

We can similarly compute the singular subdifferentials of the above mappings at $(0,0)$ by

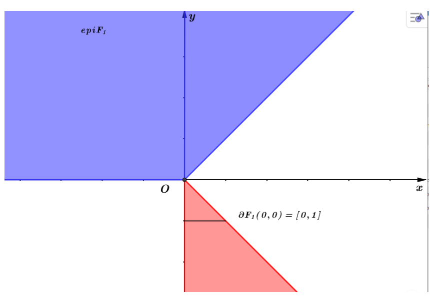

FIGURE 3. Subdifferential of F_1

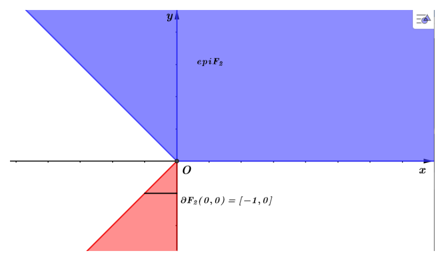

FIGURE 4. Subdifferential of F_2

$$\partial^\infty_\Theta F_1(0,0) = \partial^\infty_\Theta F_2(0,0) = \{0\}.$$

This tells us that the qualification condition (3.10) holds. Moreover, the mapping $S_{\mathscr{E}}$ defined in (3.9) is inner semicontinuous at $(0,0,0,0) \in \operatorname{gph}(F_1+F_2)$. Therefore, employing the sum rule (3.11) in Theorem 3.4 at $(0,0)$ gives us the evaluation

$$\partial_\Theta F(0,0) \subset \partial_\Theta F_1(0,0) + \partial_\Theta F_2(0,0) = [-1,1].$$

We derive from Theorem 3.4 its counterpart for finitely many multifunctions.

Corollary 3.7 (subdifferential sum rules for finitely many multifunctions). *Let $F_i : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, $i=1,\ldots,k$, be set-valued mappings with the image space $\mathbb{R}^m$ ordered by a closed convex cone Θ and such that each F_i is locally epiclosed around the corresponding points. Let $\mathscr{F}_k := F_1+\ldots+F_k$ and take $(\bar{x},\bar{z}) \in \operatorname{epi}_\Theta \mathscr{F}_k$. Define*

$$S_{\mathscr{E}}(\bar{x},\bar{z}) := \left\{ (\bar{z}_1,\ldots,\bar{z}_k) \in \mathbb{R}^{mk} \;\middle|\; \bar{z}_i \in \mathscr{E}_{F_i,\Theta}(\bar{z}),\ i=1,\ldots,m,\ \bar{z}=\bar{z}_1+\ldots+\bar{z}_k \right\}. \tag{3.15}$$

Fix $(\bar{z}_1,\ldots,\bar{z}_k) \in S_{\mathscr{E}}(\bar{x},\bar{z})$ and suppose that this mapping is inner semicontinuous at $(\bar{x},\bar{z},\bar{z}_1,\ldots,\bar{z}_k)$ and being such that

$$x_1^*+\ldots+x_k^*=0,\ x_i^* \in \partial^\infty_\Theta F_i(\bar{x},\bar{z}_i) \Longrightarrow x_i^*=0 \ \text{for}\ i=1,\ldots,k. \tag{3.16}$$

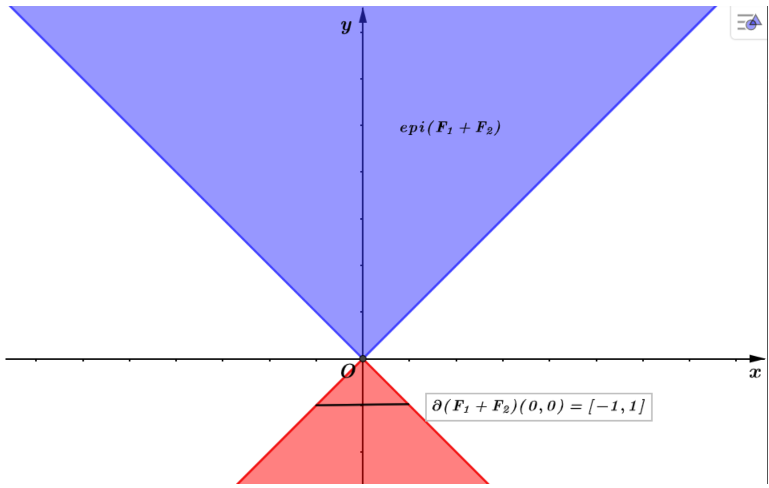

FIGURE 5. Subdifferential of F_1+F_2

Then we have the subdifferential inclusions

$$\partial_\Theta \mathscr{F}_k(\bar{x},\bar{z}) \subset \partial_\Theta F_1(\bar{x},\bar{z}_1)+\ldots+\partial_\Theta F_k(\bar{x},\bar{z}_k), \tag{3.17}$$

$$\partial^\infty_\Theta \mathscr{F}_k(\bar{x},\bar{z}) \subset \partial^\infty_\Theta F_1(\bar{x},\bar{z}_1)+\ldots+\partial^\infty_\Theta F_k(\bar{x},\bar{z}_k). \tag{3.18}$$

Proof. Proceeding by induction, observe that by Lemma 3.3 we have the representation

$$\mathscr{E}_{\mathscr{F}_k,\Theta}(\bar{x}) = \mathscr{E}_{F_1,\Theta}(\bar{x})+\ldots+\mathscr{E}_{F_k,\Theta}(\bar{x}).$$

When $k=2$, inclusions (3.17) and (3.18) are proved in Theorem 3.4. Assume now that these inclusions (3.17) are satisfied for $k-1$, i.e.,

$$\partial_\Theta \mathscr{F}_k(\bar{x},\bar{z}_1+\ldots+\bar{z}_{k-1}) \subset \partial_\Theta F_1(\bar{x},\bar{z}_1)+\ldots+\partial_\Theta F_{k-1}(\bar{x},\bar{z}_{k-1}),$$

$$\partial^\infty_\Theta \mathscr{F}_k(\bar{x},\bar{z}_1+\ldots+\bar{z}_{k-1}) \subset \partial^\infty_\Theta F_1(\bar{x},\bar{z}_1)+\ldots+\partial^\infty_\Theta F_{k-1}(\bar{x},\bar{z}_{k-1}). \tag{3.19}$$

We are going to verify the fulfillment of the following ones:

$$\partial_\Theta \mathscr{F}_k(\bar{x},\bar{z}_1+\ldots+\bar{z}_k) \subset \partial_\Theta \mathscr{F}_{k-1}(\bar{x},\bar{z}_1+\ldots+\bar{z}_{k-1})+\partial_\Theta F_k(\bar{x},\bar{z}_k), \tag{3.20}$$

$$\partial^\infty_\Theta \mathscr{F}_k(\bar{x},\bar{z}_1+\ldots+\bar{z}_k) \subset \partial^\infty_\Theta \mathscr{F}_{k-1}(\bar{x},\bar{z}_1+\ldots+\bar{z}_{k-1})+\partial^\infty_\Theta F_k(\bar{x},\bar{z}_k). \tag{3.21}$$

Applying the subdifferential sum rule in Theorem 3.4(i) to $\mathscr{F}_{k-1}$ and F, we need to check the qualification condition (3.10) and the inner semicontinuity of the corresponding mapping $S_\mathscr{E}$ in (3.15) for these two multifunctions. To proceed, take any $x^* \in \partial^\infty_\Theta \mathscr{F}_k(\bar{x},\bar{z}_1+\ldots+\bar{z}_{k-1})$ with $-x^* \in \partial^\infty_\Theta F(\bar{x},\bar{z}_k)$. The former inclusion ensures by (3.19) that

$$x^* \in \partial^\infty_\Theta F_1(\bar{x},\bar{z}_1)+\ldots+\partial^\infty_\Theta F_{k-1}(\bar{x},\bar{z}_{k-1}).$$

Thus there exist $x_i^* \in \partial^\infty_\Theta F_i(\bar{x},\bar{z}_i)$ for $i=1,\ldots,k$ such that $x^*=x_1^*+\ldots+x_{k-1}^*$. This yields $x_1^*+\ldots+x_{k-1}^*+(-x^*)=0$, and therefore tells us by (3.16) that $x^*=0$, which verifies (3.10) for the multifunctions $\mathscr{F}_{k-1}$ and F_k.

It remains to check the inner semicontinuity of $S_\mathscr{E}$ from (3.15) with respect to $\mathscr{F}_{k-1}$ and F. We are going to show that the mapping

$$M_\mathscr{E}(\bar{x},\bar{z}) := \left\{(u,v)\;\middle|\; u\in\mathscr{E}_{\mathscr{F}_{k-1}}(x)\;,\; v\in\mathscr{E}_{F_k}(x)\;,\; \bar{z}=u+v\right\}$$

is inner semicontinuous at $(\bar{x},\bar{z},\bar{u},\bar{v})$, where $\bar{u} := \bar{z}_1 + \ldots + \bar{z}_{k-1}$ and $\bar{v} := \bar{z}_k$. Indeed, take $\{(x^j, z^j)\} \subset \operatorname{dom} M_{\mathscr{E}}(\bar{x},\bar{z})$ such that $(x^j, z^j) \to (\bar{x},\bar{z})$ as $j \to \infty$. Since $S_{\mathscr{E}}$ is inner semicontinuous at $(\bar{x},\bar{z},\bar{z}_1,\ldots,\bar{z}_k)$, there exists $(z_1^j,\ldots,z_k^j)$ such that

$$z_i^j \in \mathscr{E}_{F_i}(\bar{x}),\ i = 1,\ldots,k,\ z^j = z_1^j + \ldots + z_k^j \ \text{ and } \ (z_1^j,\ldots,z_k^j) \to (\bar{z}_1,\ldots,\bar{z}_k) \ \text{ as } \ j \to \infty.$$

Letting $u^j := z_1^j + \ldots + z_{k-1}^j$ and $v^j := z_k^j$ gives us that $u^j \in \mathscr{E}_{\mathscr{F}_{k-1}}(\bar{x})$, $v^j \in \mathscr{E}_{F_k}(\bar{x})$, and $u^j + v^j = z_1^j + \ldots + z_k^j = z^j$. Hence $(u^j, v^j) \in M_{\mathscr{E}}(x^j, z^j)$ and $(u^j, v^j) \to (\bar{u},\bar{v})$, which justifies the inner semicontinuity of $M_{\mathscr{E}}$ at $(\bar{x},\bar{z},\bar{u},\bar{v})$.

Finally, employing the subdifferential sum rules for $\mathscr{F}_{k-1}$ and F_k gives us (3.20) and (3.21), we complete the proof of the corollary. $\square$

4. Subdifferential Chain Rules for Ordered Mappings

This section is devoted to deriving general *subdifferential chain rules* for compositions of ordered multifunctions. Given arbitrary set-valued mappings $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^q$ and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, their *composition* $(F \circ G) : \mathbb{R}^n \rightrightarrows \mathbb{R}^q$ is defined by

$$(F \circ G)(x) := \bigcup_{y \in G(x)} F(y) = \big\{ z \in \mathbb{R}^q \,\big|\, \exists y \in G(x) \ \text{ with } \ z \in F(y) \big\}, \quad x \in \mathbb{R}^n. \tag{4.1}$$

Our first lemma here presents the *coderivative chain rules* taken from [4, Theorem 3.11]. Recall that the *kernel* of a multifunction $M \colon \mathbb{R}^d \rightrightarrows \mathbb{R}^s$ is

$$\ker M := \big\{ p \in \mathbb{R}^d \,\big|\, 0 \in M(p) \big\}.$$

Lemma 4.1 (**chain rules for coderivatives**). *Given $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^q$ and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, pick $\bar{z} \in (F \circ G)(\bar{x})$ and consider the auxiliary set-valued mapping:*

$$K(x,z) := G(x) \cap F^{-1}(z) = \big\{ y \in G(x) \,\big|\, z \in F(y) \big\} \tag{4.2}$$

for all $(x,z) \in \mathbb{R}^n \times \mathbb{R}^q$. The following assertions hold:

(i) *Fixing $\bar{y} \in K(\bar{x},\bar{z})$ from (4.2), suppose that this mapping is inner semicontinuous at $(\bar{x},\bar{z},\bar{y})$ and that the qualification condition*

$$D^*F(\bar{y},\bar{z})(0) \cap \ker D^*G(\bar{x},\bar{y}) = \{0\} \tag{4.3}$$

is satisfied. Then, for all $z^* \in \mathbb{R}^q$, we have the inclusion

$$D^*(F \circ G)(\bar{x},\bar{z})(z^*) \subset D^*G(\bar{x},\bar{y}) \circ D^*F(\bar{y},\bar{z})(z^*).$$

(ii) *Suppose that the multifunction K from (4.2) is locally bounded around $(\bar{x},\bar{z})$ and that the qualification condition (4.3) holds for every $\bar{y} \in K(\bar{x},\bar{z})$. Then for all vectors $z^* \in \mathbb{R}^q$ we have the inclusion*

$$D^*(F \circ G)(\bar{x},\bar{z})(z^*) \subset \bigcup_{\bar{y} \in K(\bar{x},\bar{z})} \big[D^*G(\bar{x},\bar{y}) \circ D^*F(\bar{y},\bar{z})(z^*) \big].$$

The next lemma provides a certain *chain rule* for epigraphical multifunctions (3.2) that plays an important role in deriving subdifferential chain rules for ordered mappings.

Lemma 4.2 (**epigraphical multifunctions for compositions of ordered mappings).** *Let $G:\mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and $F:\mathbb{R}^m \rightrightarrows \mathbb{R}^q$ be set-valued mappings with the space $\mathbb{R}^m$ ordered by a closed convex cone Θ_1 and the space $\mathbb{R}^q$ ordered by a closed convex cone Θ_2. Then for all $x\in\mathbb{R}^n$ we have the inclusion*

$$\mathscr{E}_{F\circ G,\Theta_2}(x)\subset(\mathscr{E}_{F,\Theta_2}\circ\mathscr{E}_{G,\Theta_1})(x). \tag{4.4}$$

Moreover, this inclusion holds as an identity if the mapping F is CONE MONOTONE *with respect to (Θ_1,Θ_2), i.e., the implication*

$$\big[y_1\in y_2+\Theta_1\big]\Longrightarrow\big[F(y_1)\subset F(y_2)+\Theta_2\big] \tag{4.5}$$

is fulfilled for all vectors $y_1,y_2\in\mathbb{R}^m$.

Proof. To verify the inclusion in (4.4), take any $x\in\mathbb{R}^n$ and $z\in\mathscr{E}_{F\circ G,\Theta_2}(x)$. Then we get that $z\in(F\circ G)(x)+\Theta_2$, i.e., there exists $y\in G(x)$ such that $z\in\mathscr{E}_{F,\Theta_2}(y)$. Choosing $0\in\Theta_1$ gives us $y=y+0\in G(x)+\Theta_1$, which ensures that $y\in\mathscr{E}_{G,\Theta_1}(x)$. Hence we conclude that $z\in(\mathscr{E}_{F,\Theta_2}\circ\mathscr{E}_{G,\Theta_1})(x)$, which justifies the claimed inclusion (4.4).

Assuming further the cone monotonicity property (4.5), pick any $z\in(\mathscr{E}_{F,\Theta_2}\circ\mathscr{E}_{G,\Theta_1})(x)$ and find $y\in\mathscr{E}_{G,\Theta_1}(x)$ such that $z\in\mathscr{E}_{F,\Theta_2}(y)$. This yields $y\in G(x)+\Theta_1$ and $z\in F(y)+\Theta_2$. It follows from the first inclusion that there exists $y_0\in G(x)$ with $y\in y_0+\Theta_1$, which ensures by the imposed cone monotonicity (4.5) property that $F(y)\in F(y_0)+\Theta_2$. Therefore, $z\in F(y_0)+\Theta_2$, i.e., $z\in\mathscr{E}_{F\circ G,\Theta_2}(x)$ as claimed. □

To derive now the subdifferential chain rules, we need an ordered counterpart of the mapping K from (4.2), which is defined by

$$K_{\mathscr{E}}(x,z):=\mathscr{E}_{G,\Theta_1}(x)\cap(\mathscr{E}_{F,\Theta_2})^{-1}(z)=\big\{y\in\mathscr{E}_{G,\Theta_1}(x)\;\big|\;z\in\mathscr{E}_{F,\Theta_2}(y)\big\}. \tag{4.6}$$

Here is the main result of this section.

Theorem 4.3 (**subdifferential chain rules for ordered multifunctions).** *Consider the composition $F\circ G$ in (4.1) of multifunctions $G\colon\mathbb{R}^n\rightrightarrows\mathbb{R}^m$ and $F\colon\mathbb{R}^m\rightrightarrows\mathbb{R}^q$ ordered by closed convex cones $\Theta_1\subset\mathbb{R}^m$ and $\Theta_2\subset\mathbb{R}^q$, respectively, such that the generalized epigraphs $\operatorname{epi}_{\Theta_1}G$ and $\operatorname{epi}_{\Theta_2}F$ are closed around the corresponding points in question. Assume that the mapping F satisfies the cone monotonicity property with respect to $(\Theta_1,\,\Theta_2)$. Then the following subdifferential chain rules for both basic and singular subdifferentials hold:*

(i) *Fix $\bar y\in K_{\mathscr{E}}(\bar x,\bar z)$ from (4.6) and suppose that this mapping is inner semicontinuous at $(\bar x,\bar z,\bar y)$ and that the qualification condition*

$$\partial^\infty_{\Theta_2}F(\bar y,\bar z)\cap\ker D^*\mathscr{E}_{G,\Theta_1}(\bar x,\bar y)=\{0\} \tag{4.7}$$

is satisfied. Then for both subdifferentials (3.3) and (3.4) we have the inclusions

$$\partial_{\Theta_2}(F\circ G)(\bar x,\bar z)\subset D^*\mathscr{E}_{G,\Theta_1}(\bar x,\bar y)\big(\partial_{\Theta_2}F(\bar y,\bar z)\big), \tag{4.8}$$

$$\partial^\infty_{\Theta_2}(F\circ G)(\bar x,\bar z)\subset D^*\mathscr{E}_{G,\Theta_1}(\bar x,\bar y)\big(\partial^\infty_{\Theta_2}F(\bar y,\bar z)\big), \tag{4.9}$$

where $D^*\mathscr{E}_{G,\Theta_1}(\bar x,\bar y)\big(\partial_{\Theta_2}F(\bar y,\bar z)\big):=\big\{D^*\mathscr{E}_{G,\Theta_1}(\bar x,\bar y)(x^*)\;\big|\;x^*\in\partial_{\Theta_2}F(\bar y,\bar z)\big\}$ and $D^*\mathscr{E}_{G,\Theta_1}(\bar x,\bar y)\big(\partial^\infty_{\Theta_2}F(\bar y,\bar z)\big):=\big\{D^*\mathscr{E}_{G,\Theta_1}(\bar x,\bar y)(x^*)\;\big|\;x^*\in\partial^\infty_{\Theta_2}F(\bar y,\bar z)\big\}$.

(ii) *If $K_{\mathscr{E}}$ in (4.6) is locally bounded around $(\bar{x},\bar{z})$ and the qualification condition (4.7) holds for every $\bar{y}\in K_{\mathscr{E}}(\bar{x},\bar{z})$, then we have*

$$\partial_{\Theta_2}(F\circ G)(\bar{x},\bar{z})\subset\bigcup_{\bar{y}\in K_{\mathscr{E}}(\bar{x},\bar{z})}\big[D^*\mathscr{E}_{G,\Theta_1}(\bar{x},\bar{y})\big(\partial_{\Theta_2}F(\bar{y},\bar{z})\big)\big],$$

$$\partial^{\infty}_{\Theta_2}(F\circ G)(\bar{x},\bar{z})\subset\bigcup_{\bar{y}\in K_{\mathscr{E}}(\bar{x},\bar{z})}\big[D^*\mathscr{E}_{G,\Theta_1}(\bar{x},\bar{y})\big(\partial^{\infty}_{\Theta_2}F(\bar{y},\bar{z})\big)\big].$$

Proof. It suffices to verify the basic subdifferential chain rule in (4.8), while observing that the singular subdifferential one in (4.9) as well as those in (ii) can be justified similarly; cf. the proof of the subdifferential sum rules in Theorem 3.4.

To proceed with the verification of (4.8), fix $\bar{y}\in K_{\mathscr{E}}(\bar{x},\bar{z})$ and take any $x^*\in\partial_{\Theta_2}(F\circ G)(\bar{x},\bar{z})$. Then Definition 3.3 tells us that there exists $-v^*\in N(0;\Theta_2)$ such that $\|v^*\|=1$ and $x^*\in D^*\mathscr{E}_{F\circ G,\Theta_2}(\bar{x},\bar{y})(v^*)$. Employing Lemma 4.2, where we have the identity in (4.4) thanks to the imposed cone monotonicity property of F, yields

$$x^*\in D^*(\mathscr{E}_{F,\Theta_2}\circ\mathscr{E}_{G,\Theta_1})(\bar{x},\bar{y})(v^*).$$

Observe that the qualification condition (4.7) can be equivalently written as

$$D^*\mathscr{E}_{F,\Theta_2}(\bar{y},\bar{z})(0)\cap\ker D^*\mathscr{E}_{G,\Theta_1}(\bar{x},\bar{y})=\{0\},$$

and thus we can apply Theorem 4.1 to the composition $\mathscr{E}_{F,\Theta_2}\circ\mathscr{E}_{G,\Theta_1}$. This tells us that

$$x^*\in D^*\mathscr{E}_{F\circ G,\Theta_2}(\bar{x},\bar{z})(v^*)\subset D^*\mathscr{E}_{G,\Theta_1}(\bar{x},\bar{y})\circ D^*\mathscr{E}_{F,\Theta_2}(\bar{y},\bar{z})(v^*),$$

which therefore justifies (4.8) and completes the proof of the theorem. □

Next let us present the following efficient consequence of Theorem 4.3, where the subdifferential qualification condition (4.7) is guaranteed by the fulfillment of the major well-posedness properties of ordered multifunctions due to their coderivative characterizations. We say a set-valued mapping $M\colon\mathbb{R}^d\rightrightarrows\mathbb{R}^s$ ordered by a closed convex cone $\Theta\subset\mathbb{R}^s$ is Θ-*metrically regular* around a point $(\bar{p},\bar{q})\in\operatorname{epi}_{\Theta}M$ if the Θ-epigraphical multifunction (3.2) is metrically regular (2.5) around this point. For simplicity, we confine ourselves to formulating just the consequence of assertion (i) in Theorem 4.3.

Corollary 4.4 (subdifferential chain rules for well-posed ordered mappings). *In the setting of Theorem 4.3*(i)*, suppose that all the assumptions but (4.7) are satisfied. Assume in addition that either the outer mapping $F:\mathbb{R}^m\rightrightarrows\mathbb{R}^q$ is Θ_2-Lipschitz-like around $(\bar{y},\bar{z})$, or the inner mapping $G\colon\mathbb{R}^n\rightrightarrows\mathbb{R}^m$is Θ_1-metrically regular around $(\bar{x},\bar{y})$. Then both subdifferential chain rules (4.8) and (4.9) are fulfilled.*

Proof. If F is Θ_2-Lipschitz-like around $(\bar{y},\bar{z})$, then the qualification condition (4.7) holds due to singular subdifferential criterion (3.5) for this Lipschitz-like property. In the alternative case where the inner mapping G is Θ_1-metrically regular around $(\bar{x},\bar{y})$, we deduce from the equivalence between this property and the Θ_1-Lipschitz-like property of the inverse G^{-1} around $(\bar{y},\bar{x})$ due to the discussions in Section 2 that

$$D^*\mathscr{E}^{-1}_{G,\Theta_1}(\bar{y},\bar{x})(0)=\{0\}$$

by the coderivative criterion (2.6). It is easy to check that the latter condition reduces to

$$\ker D^*\mathscr{E}_{G,\Theta_1}(\bar{x},\bar{y})=\{0\},$$

by definition (3.4) of the singular subdifferential. This verifies (4.7) and thus completes the proof of the corollary. □

5. Solution Notions and Underlying Properties in Problems of Set-Valued Optimization

The rest of the paper is devoted to applications of the subdifferential calculus developed above for both basic and singular subdifferentials of ordered set-valued mappings to the study of appropriate solution notions in problems of set-valued optimization. Our major attention is paid to the existence of optimal solutions. As before, we confine ourselves for simplicity to problems in finite-dimensional spaces and discuss below their possible extensions to infinite dimensions.

First we recall the definitions of *Pareto-type minimizers* in set-valued optimization following the scheme of [4, 7]. Besides the notions of *Pareto/efficient* and *weak Pareto/weak efficient* solutions to set-valued problems, we study in what follows the notion of *relative Pareto optimality* induced by relative interiors of ordering cones.

Definition 5.1 (Pareto-type minimizers in set-valued optimization). *Let $F:\mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a multifunction with values ordered by a nonempty, closed, and convex cone $\Theta \subset \mathbb{R}^m$, and let $(\bar{x},\bar{z}) \in \operatorname{epi}_\Theta F$. We say that:*

(i) *$(\bar{x},\bar{z})$ is a* Pareto minimizer *of F if*

$$(\bar{z}-\Theta)\cap F(\mathbb{R}^n)=\{\bar{z}\}.$$

(ii) *$(\bar{x},\bar{z})$ is a* weak Pareto minimizer *of F if $\operatorname{int}\Theta \neq \emptyset$ and*

$$(\bar{z}-\operatorname{int}\Theta)\cap F(\mathbb{R}^n)=\emptyset.$$

(iii) *$(\bar{x},\bar{z})$ is a* relative Pareto minimizer *of F if*

$$(\bar{z}-\operatorname{ri}\Theta)\cap F(\mathbb{R}^n)=\emptyset.$$

Note the mapping F in Definition 5.1 may have empty values at some points. This implies that we can incorporate the constraint $x\in\Omega$ by putting $F(x):=\emptyset$ for $x\in\Omega$. However, to proceed efficiently in this direction for by using subdifferentials of F requires *subdifferential calculus*, which is done in the subsequent sections.

Appealing to the developments in [4, 7] for general set-valued mappings, we now recall some properties of ordered mappings $F:\mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with the ordering cone $\Theta\subset\mathbb{R}^m$. Such a mapping F is *epiclosed* of its generalizes epigraph (3.1) is closed in $\mathbb{R}^n\times\mathbb{R}^m$. We say that F is $\Theta-$*quasibounded from below* if there exists a ball $B\subset\mathbb{R}^m$ such that $F(\mathbb{R}^n)\subset B+\Theta$. A set Ω is $\Theta-$quasibounded from below if the constant mapping $F(x)\equiv\Omega$ enjoys this property.

Given an arbitrary set $A\subset\mathbb{R}^m$, define the collection of the *Pareto minimal points* of A with respect to the ordering cone Θ on $\mathbb{R}^m$ by

$$\operatorname{Min}_\Theta A:=\big\{\bar{z}\in A\;\big|\;\bar{z}-z\notin\Theta\ \text{ whenever }\ z\in A,\ z\neq\bar{z}\big\}.$$

We say that a Θ-ordered multifunction F satisfies the *strong limiting monotonicity property* at $\bar{x}\in\operatorname{dom}F$ if for any sequence $(x_k,z_k)\subset\operatorname{gph}F$ with $x_k\to\bar{x}$ as $k\to\infty$ it holds

$$\big[z_k\preceq v_k,\ v_{k+1}\preceq v_k\big]\Longrightarrow\big[\exists\,\bar{z}\in\operatorname{Min}F(\bar{x})\ \mathit{with}\ \bar{z}\preceq v_k\big] \tag{5.1}$$

with $\bar{z} \preceq \bar{v}$ if $v_k \to \bar{v}$ as $k \to \infty$. The reader can find more about this and related properties in [4, 7]; see, in particular, [4, Proposition 9.8 and Exercise 9.39] about relationships between the strong limiting monotonicity and *domination* properties. On the other hand, condition (5.1) may fail for simple set-valued mappings as, e.g., for $F\colon \mathbb{R} \rightrightarrows \mathbb{R}$ defined by

$$F(x) := \begin{cases} \{0\}, & x \neq 0 \\ (-1,0], & x = 0 \end{cases}$$

with $\Theta := \{x \in \mathbb{R} \mid x \geq 0\}$, where (5.1) fails at $\bar{x} := 0$ since $\operatorname{Min} F(0) = \emptyset$.

The next condition, which was first introduced and applied in [6] in general Banach spaces, involves–in contrast to the previous ones–the *basic subdifferential* (3.3) of ordered multifunctions. This condition can be viewed as a nonsmooth and set-valued extension of the classical *Palais-Smale condition* that postulates a relationship between convergence properties of derivative and function values.

Definition 5.2. *Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a Θ-valued multifunction. We say that the* SUBDIFFERENTIAL PALAIS-SMALE CONDITION *holds for F if any sequence $\{x_k\} \subset \mathbb{R}^n$ with*

$$\exists z_k \in F(x_k),\ x_k^* \in \partial_\Theta F(x_k,z_k),\ x_k^* \to 0 \ \text{ as } \ k \to \infty \tag{5.2}$$

contains a convergent subsequence, provided that $\{z_k\}$ is Θ-quasibounded from below.

The classical Palais-Smale condition for a smooth function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ corresponds to the subdifferential one from Definition 5.2 with $\partial_\Theta \varphi(x_k) = \{\nabla\varphi(x_k)\}$ in (5.2).

6. EXISTENCE OF OPTIMAL SOLUTIONS TO SET-VALUED OPTIMIZATION PROBLEMS WITH GEOMETRIC CONSTRAINTS

This section addresses the *existence* of Pareto-type minimizers in constrained problems of set-valued optimization. We begin with presenting an adaptation and specification in the finite-dimensional setting under consideration of the main existence theorem from [7] (see also [4, Theorem 9.15]) ensuring the existence of the so-called *intrinsic relative minimizers* for ordered multifunctions between Asplund spaces without explicit constraints. Then we derive from this result and the obtained *subdifferential calculus rules* the existence theorems for set-valued optimization problems with *explicit geometric constraints*.

First we present the aforementioned theorem in the general unconstrained setting and illustrate it by a numerical example.

Theorem 6.1 (existence of relative Pareto minimizers without constraints). *Let $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a Θ-ordered multifunction that is epiclosed, Θ-quasibounded from below, and satisfies the strong limiting monotonicity condition (5.1) on* $\operatorname{dom} F$*, whether the ordering cone Θ is closed and convex while not a subspace of $\mathbb{R}^m$. If in addition the subdifferential Palais-Smale condition holds, then there exists a relative Pareto minimizer of F.*

Proof. Since $\operatorname{ri}\Theta \neq \emptyset$ in finite dimensions, the relative and intrinsic relative minimizers of F agree. Furthermore, the assumed version of the Palais-Smale condition (5.2) yields the one imposed in [4, Theorem 9.15], and thus the result follows. □

The next existence result is an immediate consequence of Theorem 6.1.

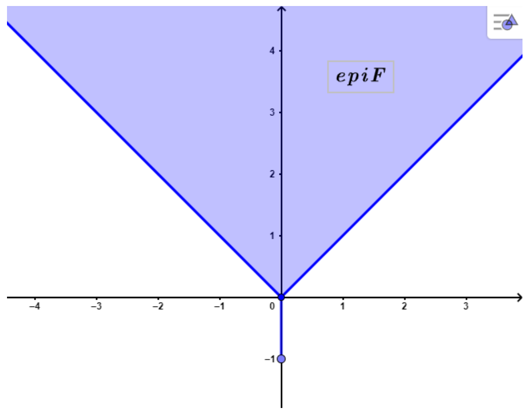

FIGURE 6. Epigraph of F

Corollary 6.2 (existence of weak Pareto minimizers). *If in addition to the assumptions of Theorem 6.1 we have* $\operatorname{int}\Theta \neq \emptyset$*, then there is a weak Pareto minimizer of* F.

Proof. It follows from Theorem 6.1 due to $\emptyset \neq \operatorname{int}\Theta = \operatorname{ri}\Theta$. □

Let us illustrate the assumptions of Theorem 6.1 by the following simple example.

Example 6.3 (illustration of the unconstrained existence theorem). Consider the set-valued mapping $F\colon \mathbb{R} \rightrightarrows \mathbb{R}$ defined by

$$F(x) := \begin{cases} |x|, & x \neq 0 \\ [-1,0], & x = 0 \end{cases} \tag{6.1}$$

with the ordering cone $\Theta := \{x \in \mathbb{R} \mid x \geq 0\}$. F is obviously epiclosed and Θ-quasibounded from below. It follows also that for any $x \in \operatorname{dom} F$, F satisfies the strong limiting monotonicity condition (5.1) at x. Indeed, at $x = 0$, $\operatorname{Min} F(0) = \{-1\}$, which is smaller than v_k whenever $(x_k, z_k) \in \operatorname{gph} F$, $x_k \to 0$, $z_k \leq v_k$, and $v_k \downarrow 0$ as $k \to \infty$. Whenever $x \neq 0$, $F(x)$ is single-valued, and so it has the strong limiting monotonicity property.

Let us now check that the subdifferential Palais-Smale condition (5.2) holds for (6.1). It follows from the definition of F that $\operatorname{gph}\mathscr{E}_{F,\Theta} = \operatorname{epi}_{\Theta} F = (\{0\} \times [-1,\infty)) \cup \big\{(x,y) \mid y \geq |x|,\ x \in \mathbb{R}\big\}$; see Figure 6. This in turn gives us the following calculations of the normal cone to $\operatorname{gph}\mathscr{E}_{F,\Theta}$ at any point $(\bar{x},\bar{z}) \in \mathbb{R}^2$:

- If $(\bar{x},\bar{z}) = (0,0)$, then
$$N\big((\bar{x},\bar{z}); \operatorname{gph}\mathscr{E}_{F,\Theta}\big) = \Big(\big\{(x,x) \;\big|\; x < 0\big\}\Big) \cup \Big(\big\{(x,-x) \;\big|\; x > 0\big\}\Big) \cup \big(\mathbb{R} \times \{0\}\big).$$
- If $(\bar{x},\bar{z}) = (0,-1)$, then
$$N\big((\bar{x},\bar{z}); \operatorname{gph}\mathscr{E}_{F,\Theta}\big) = \mathbb{R} \times (-\infty, 0].$$
- If $\bar{x} = 0$, $\bar{z} \in (-1,0)$, then
$$N\big((\bar{x},\bar{z}); \operatorname{gph}\mathscr{E}_{F,\Theta}\big) = \mathbb{R} \times \{0\}.$$
- If $\bar{x} > 0$, then
$$N\big((\bar{x},\bar{z}); \operatorname{gph}\mathscr{E}_{F,\Theta}\big) = \{(x,-x) \mid x > 0\}.$$

- If $\bar{x} < 0$, then

$$N\big((\bar{x},\bar{z});\mathrm{gph}\,\mathscr{E}_{F,\Theta}\big) = \{(x,x) \mid x < 0\}.$$

This tells us by definition (3.3) that the basic subdifferential of F at any point $(\bar{x},\bar{z}) \in \mathrm{epi}_{\Theta}F$ is computed by

$$\begin{aligned}
\partial_{\Theta}F(\bar{x},\bar{z}) \quad &:= \big\{x^* \in \mathbb{R} \,\big|\, x^* \in D^*\mathscr{E}_{F,\Theta}(\bar{x},\bar{z})(z^*),\ -z^* \in N(0;\Theta),\ \|z^*\| = 1\big\} \\
&= \big\{x^* \in \mathbb{R} \,\big|\, x^* \in D^*\mathscr{E}_{F,\Theta}(\bar{x},\bar{z})(1)\big\} \\
&= \begin{cases} \{-1,1\}, & (\bar{x},\bar{z}) = (0,0) \\ \mathbb{R}, & \bar{x} = 0, \bar{z} = -1 \\ \emptyset, & \bar{x} = 0, \bar{z} \in (-1,0) \\ \{1\}, & \bar{x} > 0 \\ \{-1\}, & \bar{x} > 0 \end{cases}
\end{aligned}$$

which ensures the fulfillment of the subdifferential Palais-Smale condition from in (5.2) for the mapping F. Indeed, taking any $\{x_k\} \subset \mathbb{R}$, consider $z_k \in F(x_k)$. If $x_k^* \in \partial_{\Theta}F(x_k,z_k)$ and $x_k^* \to 0$. Then for k large enough, (x_k,z_k) has to be $(0,-1)$, which obviously means that $\{x_k\}$ has a convergent subsequence. Therefore, all the assumptions of Theorem 6.1 are satisfied, and we have the existence of a relative Pareto minimizer of this mapping, which also is a weak Pareto minimizer of F by Corollary 6.2 since $\mathrm{int}\,\Theta \neq \emptyset$.

To pass from Theorem 6.1 in the unconstrained format to existence results for set-valued optimization problems with explicit constraints of different types, we are going to use in what follows calculus rules for the basic subdifferential (3.3) in the subdifferential Palais-Smale condition from Definition 5.2. Proceeding in this way, we address in this subsection set-valued optimization problems with *geometric constraints* given by:

$$\text{minimize } F(x) \quad \text{subject to} \quad x \in \Omega, \tag{6.2}$$

where $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is a Θ-ordered mapping, and where Ω is a nonempty subset of $\mathbb{R}^n$. Observe that problem (6.2) can be equivalently rewritten in the unconstrained form

$$\text{minimize } F_{\Omega}(x) := F(x) + \Delta(x;\Omega),\ x \in X, \tag{6.3}$$

via the *indicator mapping* of Ω defined by

$$\Delta(x;\Omega) := \begin{cases} 0 & \text{if } x \in \Omega, \\ \emptyset & \text{if } x \notin \Omega. \end{cases} \tag{6.4}$$

The following result employs the sum rule for subdifferentials of ordered set-valued mappings in Theorem 3.4 to derive the existence condition for problem (6.2).

Theorem 6.4 (existence of relative Pareto minimizers under geometric constraints). *Let $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a Θ-ordered multifunction satisfying all the assumptions of Theorem 6.1 but the subdifferential Palais-Smale condition. Supposed in addition that the constraint set $\Omega \subset \mathbb{R}^n$ in (6.2) is closed and that the following qualification condition*

$$\partial^{\infty}_{\Theta}F(x,z) \cap \big(-N(x;\Omega)\big) = \{0\} \tag{6.5}$$

holds on $\mathrm{epi}_{\Theta}F$. *If furthermore any sequence $\{x_k\} \subset \Omega$ such that*

$$\textit{there are } z_k \in F(x_k) \textit{ and } x_k^* \in \partial_{\Theta}F(x_k,z_k) + N(x_k;\Omega) \textit{ with } \|x_k^*\| \to 0 \tag{6.6}$$

contains a convergent subsequence given that $\{z_k\}$ is Θ-quasibounded from below, then problem (6.2) *admits a relative Pareto minimizer.*

Proof. We need to show that the assumptions imposed in the theorem ensure that the summation multifunction F_Ω defined in (6.3) satisfies all the assumptions of Theorem 6.1. First check that F_Ω is epiclosed. Indeed, for any $(x,z) \in \mathrm{gph}\, F_\Omega$ we have that

$$\mathrm{epi}_\Theta F_\Omega = \left\{(x,z) \in \mathbb{R}^n \times \mathbb{R}^m \;\middle|\; z \in F_\Omega(x) + \Theta\right\} = \left\{(x,z) \in \Omega \times \mathbb{R}^m \;\middle|\; z \in F(x) + \Theta\right\}$$

by the definition of $\Delta(x;\Omega)$ in (6.4). Take any $\{(x_k,z_k)\} \subset \mathrm{epi}_\Theta F_\Omega$ with $(x_k,z_k) \to (\bar{x},\bar{z})$ as $k \to \infty$. This tells us that $z_k \in F(x_k) + \Delta(x_k,\Omega) + \Theta$, and so $z_k \in F(x_k) + \Theta$ with $x_k \in \Omega$. The last inclusion yields $\{(x_k,z_k)\} \subset \mathrm{epi}_\Theta F$, and hence $(\bar{x},\bar{z}) \in \mathrm{epi}_\Theta F$ by the epiclosedness of F, which ensures that $(\bar{x},\bar{z}) \in \mathrm{epi}_\Theta F_\Omega$ and thus verifies that the summation multifunction F_Ω is epiclosed.

Next we check that F_Ω is Θ-quasibounded from below. Since F has this property, there exists a ball $B \subset \mathbb{R}^m$ such that $F(\mathbb{R}^n) \subset B + \Theta$. For any $x \in \Omega$ we get that $F_\Omega(x) = F(x) + \Delta(x;\Omega) = F(x)$, and hence F_Ω is Θ-quasibounded from below.

Let us now verify that F_Ω enjoys the strong limiting monotonicity property on its domain. Indeed, pick any $(\bar{x},\bar{z}) \in \mathrm{dom}\, F_\Omega = \mathrm{dom}\, F \cap \Omega$ and consider a sequence $\{(x_k,z_k)\} \subset \mathrm{gph}\, F_\Omega$ with $x_k \to \bar{x}$ and $z_k \preceq v_k, v_{k+1} \preceq v_k$. The first inclusion implies that $z_k \in F_\Omega(x_k) + \Delta(x_k;\Omega) = F(x_k)$. The strong limiting monotonicity property of F tells us that there exists $\bar{z} \in \mathrm{Min}\, F(\bar{x})$ with $\bar{z} \preceq \bar{x}$, and therefore $\bar{z} \in \mathrm{Min}\, F_\Omega(\bar{x})$, which verifies the strong limiting monotonicity property of the multifunction F_Ω.

Finally, we are going to prove that the imposed assumption (6.6) in terms of the initial data of the constrained problem (6.2) guarantees the fulfillment of the subdifferential Palais-Smale condition from in Definition 5.2 for the multifunction F_Ω. To furnish this requires applying the subdifferential sum rule from Theorem 3.4 to the summation mapping (6.3). Observe first that for any $(x,z) \in \mathrm{gph}\, F$, the corresponding mapping $S_{\mathscr{E}}$ in (3.9) with respect to F and Δ is inner semicontinuous at $(x,z,z,0) \in \mathrm{gph}\, S_{\mathscr{E}}$. Indeed, note that

$$S_{\mathscr{E}}(x,z) = \left\{(z_1,z_2) \in \mathbb{R}^m \times \mathbb{R}^m \;\middle|\; z_1 \in \mathscr{E}_{F,\Theta}(x),\; z_2 \in \mathscr{E}_{\Delta,\Theta}(x),\; z = z_1 + z_2\right\}.$$

Since $z \in F(x) \subset F(x) + \Theta$ and $0 \in \Delta(x,\Omega) + \Theta$ for $x \in \Omega$, it follows that $(z,0) \in S_{\mathscr{E}}(x,z)$, i.e., $(x,z,z,0) \in \mathrm{gph}\, S_{\mathscr{E}}$. To show the inner semicontinuity of $S_{\mathscr{E}}$ at $(x,z,z,0)$, take any $(x_k,z_k) \to (x,z)$ from the set $\mathrm{dom}\, S_{\mathscr{E}}$ and choose $z_{1k} := z_k$ and $z_{2k} := 0$ for all $k \in \mathbb{N}$. Then $(z_{1k},z_{2k}) \in S_{\mathscr{E}}(x_k,z_k)$ and $(z_{1k},z_{2k}) \to (z,0)$ as $k \to \infty$. which ensures that $S_{\mathscr{E}}$ is inner semicontinuous at $(x,z,z,0)$.

Next we compute the basic and singular subdifferentials of the indicator mapping $\Delta(x;\Omega)$ from (6.4). We claim that

$$\partial_\Theta \Delta(x;\Omega) = \partial^\infty_\Theta \Delta(x;\Omega) = N(x;\Omega). \tag{6.7}$$

Let us check (6.7) for the case of $\partial_\Theta\Delta(x;\Omega)$ while observing that the singular subdifferential case can be treated similarly. It follows from the definitions that

$$\begin{aligned}
&x^* \in \partial_\Theta\Delta(x;\Omega)\\
&\Longleftrightarrow \begin{cases} x^* \in D^*\mathscr{E}_{\Delta,\Theta}(x,0)(z^*)\\ -z^* \in N(0;\Theta)\\ \|z^*\| = 1\end{cases}\\
&\Longleftrightarrow \begin{cases} (x^*,-z^*) \in N\big((x,0);\operatorname{epi}_\Theta\Delta\big),\\ -z^* \in N(0;\Theta)\\ \|z^*\| = 1\end{cases}\\
&\Longleftrightarrow \begin{cases} (x^*,-z^*) \in N\big((x,0);\Omega\times\Theta\big)\\ -z^* \in N(0;\Theta),\\ \|z^*\| = 1\end{cases}\\
&\Longleftrightarrow \begin{cases} (x^*,-z^*) \in N(x;\Omega)\times N(0;\Theta)\\ -z^* \in N(0;\Theta)\\ \|z^*\| = 1\end{cases}\\
&\Longleftrightarrow \begin{cases} x^* \in N(x;\Omega)\\ -z^* \in N(0;\Theta)\\ \|z^*\| = 1\end{cases}\\
&\Longleftrightarrow x^* \in N(x;\Omega)
\end{aligned}$$

which verifies (6.7). This tells us that the imposed assumption (6.5) ensures the fulfillment of the qualification condition (3.10) in Theorem 3.4 for the mappings F and $\Delta(\cdot;\Omega)$. Employing the basic subdifferential sum rule (3.8), we arrive at

$$\partial_\Theta F_\Omega(x,z) = \partial_\Theta(F+\Delta)(x,z) \subset \partial_\Theta F(x,z) + \partial_\Theta\Delta(x;\Theta) = \partial_\Theta F(x,z) + N(x;\Omega). \tag{6.8}$$

Now we are ready to verify the fulfillment of the Palais-Smale condition for F_Ω. Take any sequence $\{x_k\} \subset \mathbb{R}^n$ with

$$\exists z_k \in F_\Omega(x_k),\ x_k^* \in \partial_\Theta F_\Omega(x_k,z_k),\ ||x_k^*|| \to 0 \ \text{ as } \ k \to \infty,$$

where $\{z_k\}$ is Θ-quasibounded from below. Then it follows from (6.8) that

$$\exists z_k \in F_\Omega(x_k),\ x_k^* \in \partial_\Theta F(x_k,z_k) + N(x_k;\Omega),\ ||x_k^*|| \to 0 \ \text{ as } \ k \to \infty,$$

which shows that the relationships in (6.6) are satisfied. Thus the imposed assumption (6.5) implies that $\{x_k\}$ contains a convergent subsequence. Applying finally Theorem 6.1 to the mapping F_Ω verifies the existence of a relative Pareto minimizer in problem (6.3) and thus in the original constrained problem (6.2). □

It follows from Corollary 6.2 and Theorem 6.4 that problem (6.2) admits a *weak Pareto* minimizer provided that, in addition of the assumptions of Theorem 6.4, we have $\operatorname{int}\Theta \neq \emptyset$.

Using the fundamental intersection rule for the normal cone (2.2) leads us to the existence theorem for set-valued optimization problems with many geometric constraints.

Corollary 6.5 (existence of optimal solutions under many geometric constraints). *Consider the set-valued optimization problem:*

$$\text{minimize } F(x) \quad \text{subject to} \quad x \in \Omega_i \quad \text{for} \quad i = 1, \ldots, s, \tag{6.9}$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is a Θ-ordered multifunction satisfying all the assumptions of Theorem 6.4 together with the ordering cone Θ, and where $\Omega_1, \ldots, \Omega_s$ are closed subsets of $\mathbb{R}^n$ such that the relationships

$$v_i \in N(x; \Omega_i) \text{ for } i = 1, \ldots, s \text{ and } v_1 + \ldots + v_s = 0 \tag{6.10}$$

hold only when $v_1 = \ldots = v_s = 0$ whenever $x \in \bigcap_{i=1}^s \Omega_i$. Suppose further that

$$\partial^\infty_\Theta F(x, z) \cap \Big(-\sum_{i=1}^s N(x; \Omega_i) \Big) = \{0\}$$

for all $(x, z) \in \operatorname{epi}_\Theta F$ with $x \in \bigcap_{i=1}^s \Omega_i$, and that every sequence $\{x_k\} \subset \bigcap_{i=1}^s \Omega_i$ such that

$$\text{there are } z_k \in F(x_k) \text{ and } x_k^* \in \partial_\Theta F(x_k, z_k) + \sum_{i=1}^s N(x; \Omega_i) \text{ with } \|x_k^*\| \to 0$$

contains a convergent subsequence when $\{z_k\}$ is Θ-quasibounded from below. Then problem (6.9) admits a relative Pareto minimizer. If furthermore $\operatorname{int}\Theta \neq \emptyset$, then there exists a weak Pareto minimizer of (6.9).

Proof. Consider the set

$$\Omega := \bigcap_{i=1}^s \Omega_i,$$

and observe first that Ω is closed since all the sets Ω_i are closed for $i = 1, \ldots, s$. Applying the normal cone intersection rule to Ω, which holds under the qualification condition in (6.10) by [4, Corollary 2.17], we have that

$$N(x; \Omega) \subset N(x; \Omega_1) + \ldots + N(x; \Omega_s)$$

for any $(x, z) \in \operatorname{gph} F$. Substituting the latter into the corresponding conditions of Theorem 6.4 gives us the existence of relative Pareto minimizers in (6.9). If in addition $\operatorname{int}\Theta \neq \emptyset$, we get the existence of weak Pareto minimizers as in Corollary 6.2. □

7. Existence of Solutions to Set-Valued Optimization Problems with Functional Constraints

In this section, we consider a constrained problem of set-valued optimization of type (6.2), where the constraint set Ω is described by finitely many *equalities* and *inequalities* via *Lipschitz continuous* functions as

$$\Omega := \big\{ x \in X \;\big|\; \varphi_i(x) \le 0,\ i = 1, \ldots, m;\ \varphi_i(x) = 0,\ i = m+1, \ldots, m+r \big\}. \tag{7.1}$$

Using appropriate calculus rules leads us to deriving efficient conditions for the existence of optimal solutions to this problem expressed in terms of the basic subdifferentials of the Θ-ordered mapping F and of the functions φ_i in (7.1).

Theorem 7.1 (existence of solutions under functional constraints). *Consider the set-valued optimization problem (6.2), where Ω is given by (7.1). Suppose that F and Θ satisfy the assumptions of Theorem 6.4, and that the functions φ_i, $i=1,\ldots,m+r$, are locally Lipschitz continuous on* $\operatorname{dom}F$*. In addition, impose following conditions:*

(i) *For any $x\in\operatorname{dom}F\cap\Omega$, the inclusion*

$$0\in\sum_{i=1}^{m}\lambda_i\partial\varphi_i(x)+\sum_{i=m+1}^{m+r}|\lambda_i|\big(\partial\varphi_i(x)\cup\partial(-\varphi_i)(x)\big) \tag{7.2}$$

with $\lambda_i\ge 0$ and $\lambda_i\varphi_i(x)=0$ for $i=1,\ldots m$ implies that $\lambda_i=0$ for all $i=1,\ldots,m+r$.

(ii) *For any $(x,z)\in\operatorname{epi}_\Theta F$ with $x\in\Omega$, the inclusions $x^*\in\partial^\infty_\Theta F(x,z)$ and*

$$-x^*\in\sum_{i=1}^{m}\lambda_i\partial\varphi_i(x)+\sum_{i=m+1}^{m+r}|\lambda_i|\big(\partial\varphi_i(x)\cup\partial(-\varphi_i)(x)\big)$$

with $\lambda_i\ge 0$ and $\lambda_i\varphi_i(x)=0$ for $i=1,\ldots m$ imply that $x^*=0$.

(iii) *Every sequence $\{x_k\}\subset\Omega$ such that*

$$\exists z_k\in F(x_k)\ \text{and}\ x_k^*\in\partial_\Theta F(x_k,z_k)+\sum_{i=1}^{m}\lambda_{ik}\partial\varphi_i(x_k)+\sum_{i=m+1}^{m+r}|\lambda_{ik}|\big(\partial\varphi_i(x_k)\cup\partial(-\varphi_i)(x_k)\big),$$

where $\|x_k^*\|\to 0$ as $k\to\infty$ and $\lambda_{ik}\ge 0$ with $\lambda_{ik}\varphi_i(x_k)=0$ for $i=1,\ldots m$ and $k\in\mathbb{N}$, contains a convergent subsequence provided that $\{z_k\}$ is $\Theta-$quasibounded from below.

Then the problem of minimizing F under the constraints in (7.1) admits a relative Pareto minimizer. If $\operatorname{int}\Theta\neq\emptyset$*, then there exists a weak Pareto minimizer of this problem.*

Proof. To verify the existence of a relative Pareto minimizer in the problem under consideration, we check that the explicit assumptions imposed in this theorem in terms of φ_i ensure the fulfillment of the assumptions in Theorem 6.4 for the set Ω given in (7.1).

First we show that the qualification condition (6.5) of Theorem 6.4 is satisfied in the current setting. To furnish this, let $g:=(\varphi_1,\ldots,\varphi_{m+r})$ and define the set

$$S:=\big\{(\alpha_1,\ldots,\alpha_{m+r})\;\big|\;\alpha_i\le 0\ \text{ for }\ i=1,\ldots,m;\ \alpha_i=0\ \text{ for }\ i=m+1,\ldots,m+r\big\}.$$

Observe that the set Ω from (7.1) is represented as the inverse image $\Omega=g^{-1}(S)$. Indeed, $x\in g^{-1}(S)$ means that $g(x)\in S$, which yields the inverse image description of Ω by the constructions of g and S. Using the indicator mapping (6.4) of the set S, we claim that

$$\begin{aligned} N\big(x;g^{-1}(S)\big)\ &=D^*\big(\Delta_S\circ g\big)(x,0)\subset D^*g(x)\circ N\big(g(x);S\big)\\ &=\bigcup\big\{D^*g(x)(\lambda)\;\big|\;\lambda\in N\big(g(x);S\big)\big\}. \end{aligned} \tag{7.3}$$

This claim is verified by using the upper estimate of basic normals to inverse images given in [4, Corollary 3.13]. To apply this result in the setting of the theorem, we need to check that the qualification condition

$$N\big(g(x);S\big)\cap\ker D^*g(x)=\{0\},\quad x\in\operatorname{dom}F\cap\Omega, \tag{7.4}$$

which is required for using [4, Corollary 3.13] in our setting, follows from the qualification condition (7.2) imposed in the theorem. Indeed, picking $\lambda=(\lambda_1,\ldots,\lambda_{m+r})\in N(g(x);S)\cap$

$\ker D^*g(x,z)$ yields $0 \in D^*g(x)(\lambda)$. Since the functions φ_i, $i=1,\ldots,m+r$, are locally Lipschitzian, we deduce from the scalarization formula of [4, Theorem 1.32] that

$$D^*g(x)(\lambda) = \partial\langle\lambda,g\rangle(x) = \partial\left(\sum_{i=1}^{m+r}\lambda_i\varphi_i\right)(x).$$

Furthermore, the subdifferential sum rule from [4, Corollary 2.21] tells us that

$$\partial\left(\sum_{i=1}^{m+r}\lambda_i\varphi_i\right)(x) \subset \sum_{i=1}^{m+r}\partial\left(\lambda_i\varphi_i\right)(x).$$

On the other hand, it is easy to see that

$$N\big(g(x);S\big) = \big\{(\lambda_1,\ldots,\lambda_{m+r}) \in \mathbb{R}^{m+r}\;\big|\;\lambda_i \geq 0,\;\lambda_i\varphi_i(x)=0\;\text{ for }\;i=1,\ldots,m\big\},$$

and that for all $\lambda_i \geq 0$ with $\lambda_i\varphi_i(x)=0$ as $i=1,\ldots,m$ and for all $\lambda_i \in \mathbb{R}$ as $i=m+1,\ldots,m+r$ we have the equalities

$$\begin{aligned}\sum_{i=1}^{m+r}\partial\big(\lambda_i\varphi_i\big)(x) &= \sum_{i=1}^{m}\partial\big(\lambda_i\varphi_i\big)(x) + \sum_{i=m+1}^{m+r}\partial\big(\lambda_i\varphi_i\big)(x)\\ &= \sum_{i=1}^{m}\lambda_i\partial\varphi_i(x) + \sum_{i=m+1}^{m+r}|\lambda_i|\big(\partial\varphi_i(x)\cup\partial(-\varphi_i)(x)\big).\end{aligned}$$

Unifying the above tells us that the imposed qualification condition (7.2) ensures the fulfillment of (7.4) required in[4, Corollary 3.13], which justifies the inclusion in (7.3).

To proceed further, let us show that the assumption imposed in (ii) guarantees the fulfillment of the qualification condition (6.5) of Theorem 6.4 for the constraint set Ω from (7.1). Pick $x^* \in \partial^\infty_\Theta F(x,z)\cap\big(-N(x;\Omega)\big)$ for $(x,z) \in \operatorname{epi}_\Theta F$ with $x \in \Omega$. Employing (7.3) and the above representations gives us the inclusion

$$N(x;\Omega) \subset \left\{\;\sum_{i=1}^{m}\lambda_i\partial\varphi_i(x) + \sum_{i=m+1}^{m+r}|\lambda_i|\big(\partial\varphi_i(x)\cup\partial(-\varphi_i)(x)\big)\;\middle|\;\lambda_i \geq 0,\;\lambda_i\varphi_i(x)=0\;\text{ as }\;i=1,\ldots,m\right\}. \tag{7.5}$$

It follows from (ii) that $x^*=0$, and hence the qualification condition (6.5) is satisfied.

It remains to check that assumption (iii) of this theorem ensures the fulfillment of condition (6.6) in Theorem 6.4. But this follows directly from the upper estimate (7.5) of the normal cone to the set Ω in (7.1). This completes the proof of the theorem. $\square$

To conclude this section, observe that the term

$$\sum_{i=m+1}^{m+r}|\lambda_i|\big(\partial\varphi_i(x)\cup\partial(-\varphi_i)(x)\big)$$

in (7.2) and everywhere below can be replaced by its upper estimate

$$\sum_{i=m+1}^{m+r}\lambda_i\partial^0\varphi_i(x)$$

with $\lambda_i \in \mathbb{R}$ via the *symmetric subdifferential* of each function $\varphi := \varphi_i$ defined by

$$\partial^0\varphi(x) := \partial\varphi(x)\cup\big(-\partial(-\varphi)(x)\big).$$

It easily follows from the *plus-minus symmetry* property of the latter construction meaning that $\partial^0(-\varphi)(\bar{x}) = -\partial^0\varphi(x)$, which does not hold for the basic subdifferential.

8. Existence of Solutions to Problems with Operator and Equilibrium Constraints

This section is devoted to deriving efficient conditions of the existence of relative Pareto minimizers for two large classes of constrained problems of set-valued optimization. The first class deals with the so-called *operator constraints* described by

$$G(x)\cap P \neq \emptyset, \tag{8.1}$$

via a set-valued mapping $G\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^q$ and a set $P\subset\mathbb{R}^q$.

The next theorem establishes the existence of relative Pareto minimizers in problems of this type in terms of their initial data.

Theorem 8.1 (existence of solutions under operator constraints). *Given a Θ-ordered multifunction $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, consider the set-valued optimization problem:*

$$\textit{minimize } \; F(x) \; \textit{ subject to } \; G(x)\cap P\neq\emptyset, \tag{8.2}$$

where F and Θ satisfies the corresponding assumptions of Theorem 6.4, and where the sets gph G *and P are closed. Suppose in addition that:*

(i) *The qualification condition*

$$N(y;P)\cap \ker D^*G(x,y)=\{0\}$$

holds for all $y\in G(x)\cap P$ with $x\in \operatorname{dom} F$.

(ii) *If $x^*\in\partial^\infty_\Theta F(x,z)$ is such that*

$$-x^*\in\bigcup\big\{D^*G(x,y)(y^*)\;\big|\;y^*\in N(y;P),\;y\in G(x)\cap P\big\}$$

with some $y\in G(x)\cap P$ and $(x,z)\in\operatorname{epi}_\Theta F$, then $x^*=0$.

(iii) *Every sequence $\{x_k\}\subset G^{-1}(P)$ for which*

$$\textit{there are } \; z_k\in F(x_k) \; \textit{ and } \; x_k^*\in\partial_\Theta F(x_k,z_k)+D^*G(x_k,y_k)(y_k^*),$$

where $y_k\in G(x_k)\cap P$, $y_k^*\in N(y_k;P)$, and $\|x_k^*\|\to 0$ as $k\to\infty$, contains a convergent subsequence provided that $\{z_k\}$ is $\Theta-$quasibounded from below.

Then problem (8.2) admits a relative Pareto minimizer. If furthermore int $\Theta\neq\emptyset$*, then there exists a weak Pareto minimizers of* (8.2)

Proof. Denote $\Omega := G^{-1}(P)$ and observe that taking $x\in G^{-1}(P)$ means that there exists $y\in G(x)\cap P$. Check that the imposed conditions in (i)–(iii) ensure that all the assumptions of Theorem 6.4 with this set Ω are satisfied.

First we verify that the qualification condition (6.5) in Theorem 6.4 holds under the fulfillment of the assumption in (i). Let $\Omega_1 := \operatorname{gph} G$ and $\Omega_2 := \mathbb{R}^n\times P$, and then note that

$$\Omega_1\cap\Omega_2 = G^{-1}(P)\times P. \tag{8.3}$$

Indeed, it is straightforward to check that

$$(x,y)\in\Omega_1\cap\Omega_2 \Longleftrightarrow y\in G(x) \;\text{ and }\; y\in P \Longleftrightarrow (x,y)\in G^{-1}(P)\times P.$$

We now claim the fulfillment of the inclusion

$$N(x;\Omega)\subset\bigcup\big\{D^*G(x,y)(y^*)\;\big|\;y^*\in N(y;P),\ y\in G(x)\cap P\big\}. \tag{8.4}$$

To verify (8.4), we take $x^*\in N(x;\Omega)$ and obtain that

$$(x^*,0)\in N(x;G^{-1}(P))\times N(y;P)=N((x,y);G^{-1}(P)\times P).$$

By representation (8.3), this yields

$$(x^*,0)\in N\big((x,y);\Omega_1\cap\Omega_2\big). \tag{8.5}$$

To employ in (8.5) the normal cone intersection rule, we check the qualification condition

$$N\big((x,y);\Omega_1\big)\cap\big[-N\big((x,y);\Omega_2\big)\big]=\{0\}. \tag{8.6}$$

Indeed, pick any $(x^*,y^*)\in N((x,y);\Omega_2)=N(x;\mathbb{R}^n)\times N(y;P)$ such that $(-x^*,-y^*)\in N((x,y);\Omega_1)$. The first inclusion implies that $x^*=0$ and $y^*\in N(y;P)$ while the second one gives us $(0,-y^*)\in N((x,y);\operatorname{gph}G)$, which means that $y^*\in\ker D^*G(x,y)$. It follows from the imposed assumption (i) that $y^*=0$, and thus the qualification condition (8.6) is justified. Now we are in a position to apply the basic normal cone intersection rule from [4, Theorem 2.16] to the set intersection in (8.5). This yields $(x^*,0)\in N\big((x,y);\operatorname{gph}G\big)+N\big((x,y);\mathbb{R}^n\times P\big)$. The last inclusion implies that $(x^*,0)\in N((x,y);\operatorname{gph}G)+\{0\}\times N(y;S)$, and hence

$$(x^*,0)=(x^*,-y^*)+(0,y^*)\ \text{ for some }\ y^*\in N(y;P)\ \text{ and }\ (x^*,-y^*)\in N\big((x,y);\operatorname{gph}G\big).$$

The latter inclusion implies that $x^*\in D^*G(x,y)(y^*)$, which therefore justifies the claim in (8.4). Taking further $x^*\in\partial_\Theta F(x,y)\cap(-N(x;\Omega))$ and using(8.4) tell us by assumption (ii) that $x^*=0$. Thus we arrive at the qualification condition (6.5) of Theorem 6.4.

To complete the proof of this theorem, it remains to show that the conditions imposed in (iii) yields the fulfillment of assumption (6.6) in Theorem 6.4 for the operator constraint set $\Omega=G^{-1}(P)$ from (8.1). Indeed, take $\{x_k\}\subset G^{-1}(P)$ such that there are $z_k\in F(x_k)$ and $x_k^*\in\partial_\Theta F(x_k,z_k)+N(x_k;\Omega)$ as $k\in\mathbb{N}$ with $\|x_k^*\|\to 0$ as $k\to\infty$ and $\{z_k\}$ is $\Theta-$quasibounded from below. Using the normal cone inclusion (8.4) gives us $y_k\in G(x_k)\cap P$ and $y_k^*\in N(y_k;P)$ such that $x_k^*\in\partial_\Theta F(x_k,z_k)+D^*G(x_k,y_k)(y_k^*)$. Now the imposed assumption (iii) guarantees that the sequence $\{x_k\}$ contains a convergent subsequence, i.e., (6.6) holds. Combining all the above verifies the existence of a relative Pareto minimizer in problem (8.2) and also the existence of a weak Pareto minimizer in this problem provided that $\operatorname{int}\Theta\neq\emptyset$. □

The final result of this section addresses set-valued optimization problems with the so-called *equilibrium constraints* given by

$$0\in G(x)+Q(x), \tag{8.7}$$

where both G and Q are generally set-valued mappings from $\mathbb{R}^n$ into $\mathbb{R}^q$, while of different natures. While G may often be a single-valued mapping, Q is always a multifunction, which is often described by the normal cone to a set, or a subgradient mapping for an extended-real-valued functions. In particular, model (8.7) is convenient to study *vector variational inequalities* and other variational systems modeling various *equilibria*; see, e.g., [3, Section 5.3] for more discussions, references, and commentaries.

Here is the existence theorem for set-valued optimization problems with constraints of type (8.7) expressed in terms of the initial problem data.

Theorem 8.2 (**existence of solutions under equilibrium constraints).** *Given a Θ-ordered multifunction $F:\mathbb{R}^n \rightrightarrows \mathbb{R}^m$, consider the set-valued optimization problem with equilibrium constraints defined by:*

$$\text{minimize } F(x) \text{ subject to } 0 \in G(x)+Q(x) \tag{8.8}$$

where F and Θ satisfies the corresponding assumptions of Theorem 6.4, and where both multifunctions G and Q are closed-graph. Impose in addition the following conditions:

(i) *The only vector $x^* \in \mathbb{R}^n$ satisfying the inclusions $x^* \in \partial^\infty_\Theta F(x,z)$ and*

$$-x^* \in D^*G(x,y)(y^*)+D^*Q(x,-y)(y^*),$$

where $(x,z)\in \operatorname{epi}_\Theta F$, $(x,y)\in \operatorname{gph} G$ and $(x,-y)\in \operatorname{gph} Q$, is $x^*=0$.

(ii) *If for such (x,y) we have that $x_1^* \in D^*G(x,y)(y^*)$, $x_2^* \in D^*Q(x,-y)(y^*)$, and $x_1^*+x_2^*=0$, then $y^*=0$ and $x_1^*=x_2^*=0$.*

(iii) *Every sequence $\{x_k\}\subset \mathbb{R}^n$ for which there are $z_k \in F(x_k), y_k \in G(x_k)\cap(-Q(x_k))$,*

$$x_k^* \in \partial_\Theta F(x_k,z_k)+D^*G(x_k,y_k)(y_k^*)+D^*Q(x_k,-y_k)(y_k^*),$$

and $\|x_k^*\|\to 0$ as $k\to\infty$, contains a convergent subsequence provided that $\{z_k\}$ is Θ-quasibounded from below.

Then problem (8.8) admits a relative Pareto minimizer. If $\operatorname{int}\Theta \neq \emptyset$, there exists a weak Pareto minimizer of problem (8.8).

Proof. It is similar to the proof of Theorem 8.1 by using the sets $\Omega_1 := \operatorname{gph} G$ and $\Omega_2 := \operatorname{gph}(-Q)$ and applying to them the normal cone intersection rule. $\square$

We finish this section with two important remarks. The first remark discusses the usage of subdifferential calculus in deriving necessary optimality conditions for local Pareto-type minimizers in constrained set-valued optimization.

Remark 8.3 (**optimality conditions for constrained set-valued problems).** The developed and utilized results of subdifferential calculus, which are used above to establish the existence of optimal solutions in constrained problems of set-valued optimization, can be similarly applied to deriving *subdifferential necessary optimality conditions* for local counterparts of all the three kinds of *local* Pareto-type minimizers given in Definition 5.1. To proceed in this way, we start from the subdifferential necessary condition $0\in\partial_\Theta F(\bar{x},\bar{z})$ for minimizing unconstrained Θ-ordered multifunctions F as in [4, 7] and then utilize calculus rules similarly to the device in Sections 6–8 for the existence theorems.

The second remark concerns extensions of the obtained results to constrained set-valued optimization problems in infinite-dimensional spaces.

Remark 8.4 (**infinite-dimensional extensions).** As mentioned above, in this paper we confine ourselves for simplicity to the finite-dimensional framework, since the employed machinery of variational analysis and generalized differentiation does not require any additional assumptions in this setting. However, basic variational tools have been well understood and developed in infinite dimensions; mainly in *Asplund spaces*, i.e., Banach spaces where every separable subspace has a separable dual (as, e.g., in any reflexive space); see a comprehensive account in [3].

A major ingredient for infinite-dimensional developments consists of imposing the so-called *sequential normal compactness* (SNC) conditions, which are automatic in finite dimension while admit a rich calculus in Banach/Asplund spaces. Using these tools together with variational principles, some existence theorems and necessary optimality conditions for vector and set-valued optimization problems have been established in [3, 4, 7], while the set-valued results in [4, 7] mostly concern unconstrained problems with no subdifferential calculus behind. Nevertheless, the approach of this paper married to the SNC calculus developed in [4] makes it possible to extend the obtained results to to constrained set-valued optimization problems in Asplund spaces.

Note to this end that the case of infinite-dimensional ordered spaces offers more varieties of *relative Pareto minimizers* due to several possibilities of relaxing the restrictive assumption on nonempty interiors of convex ordering cones. We refer the reader to [4, 7], and also to [1, 9], for comprehensive discussions and various results in this direction.

9. Conclusions

This paper develops major calculus rules for basic and singular subdifferentials of ordered multifunctions and then applies them to obtain existence theorems for relative Pareto minimizers in constrained problems of set-valued optimization in finite-dimensional spaces. Besides extensions of the obtained results to the existence of optimal solutions and necessary optimality conditions for problems in infinite-dimensional spaces as discussed in Remark 8.4 and Remark 8.3, respectively, we intend to address the following important issues, unsolved even in finite dimensions, in our future research:

(i) An attractive question arises about the existence of (usual) *Pareto minimizers* from Definition 5.1(i) in the unconstrained framework of set-valued optimization. We plan to attack this question by using tools of variational analysis and generalized differentiation.

(ii) In [10], existence results for the usual *Pareto efficient* solutions (i.e., Pareto minimizers) and *Geoffrion-properly efficient* solutions are obtained for problems of *vector optimization* with Lipschitzian cost functions in finite-dimensional spaces. The approach developed in [10] is based on tools of variational analysis and generalized differentiation married to those in *semi-algebraic geometry*. Our intention in the future research is to extend the methods and results of [10] to problems of *set-valued optimization*.

References

[1] A. A. Khan, C. Tammer, C. Zălinescu, Set-Valued Optimization. An Introduction with Applications, Springer, Berlin. 2015.

[2] C. Tammer, P. Weidner, Scalarization and Separation by Translation Invariant Functions, Springer, Cham, 2020.

[3] B. S. Mordukhovich, Variational Analysis and Generalized Differentiation, I: Basic Theory, II: Applications, Springer, Berlin, 2006.

[4] B. S. Mordukhovich, Variational Analysis and Applications, Cham, Switzerland, 2018.

[5] R. T. Rockafellar, R. J-B. Wets, Variational Analysis, Springer, Berlin, 1998.

[6] T. Q. Bao, B. S. Mordukhovich, Variational principles for set-valued mappings with applications to multiobjective optimization, Control Cybern. 36 (2007), 531-562.

[7] T. Q. Bao, B. S. Mordukhovich, Relative Pareto minimizers for multiobjective problems: existence and optimality conditions, Math. Program. 122 (2010), 301–347.

[8] B. S. Mordukhovich, Complete characterizations of covering, metric regularity, and Lipschitzian properties of multifunctions, Trans. Amer. Math. Soc. 340 (1993), 1-35.

[9] B. S. Mordukhovich, N. M. Nam, Convex Analysis and Beyond, I: Basic Theory, Springer, Cham, Switzerland, 2022.

[10] D.S. Kim, B. S. Mordukhovich, T.-S. Pham, N. V. Tuyen, Existence of efficient and properly efficient solutions to problems of constrained vector optimization, Math. Program. 190 (2021), 259-283.