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A UNIFIED APPROACH TO BISHOP-PHELPS AND SCALARIZING FUNCTIONALS

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Abstract. In this paper, functionals representing a negative convex cone as the solution set of an inequality are investigated. This general class of representing functionals includes various scalarizing functionals and Bishop-Phelps functionals. This unified approach extends some initial results to variable order structures, and additional properties of the representing functionals are given. The topics sublinearity, subdifferential, zeros, and separation are also treated in the context of representing functionals. Finally, the theory is applied to problems of vector and set optimization.

Keywords. Bishop-Phelps cones; Scalarizing functionals; Set optimization; Vector optimization.

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1. Introduction

Bishop-Phelps cones have a rich mathematical struture and therefore play an important role in optimization theory; see, e.g., [1, 2, 3, 4, 5]. On the other hand, scalarizing functionals (compare [6, 7, 8, 9, 10] among others, and the book [11] and the references therein) are often used to investigate minimal elements in vector optimization. In the past, both scientific fields have developed in parallel and often without interaction. As a result, there are various papers with similar optimization results but completely different approaches. This paper aims to provide a general unified approach for the treatment of vector and set optimization problems, which incorporates aspects of Bishop-Phelps theory and scalarizing functionals.

Bishop-Phelps cones are defined by an inequality with a so-called Bishop-Phelps functional. Under certain assumptions, it is well-known for scalarizing functionals that the negative order cone can be characterized by an inequality given by the scalarizing functional. The form of definition of Bishop-Phelps cones and a property of scalarizing functionals are covered by the unified approach of this paper. To be more concrete, we consider a negative convex cone defined by the solutions of an inequality given by a functional representing this negative convex cone. Special cases of this representing functional are Bishop-Phelps functionals and scalarizing functionals. In this way, different branches in optimization are brought together.

When working only with Bishop-Phelps or scalarizing functionals, it is obvious that the results are special results based on these special functionals. However, the unified approach of this

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paper leads to general results under certain assumptions about the functional representing the negative convex cone, and these results are generally easier to obtain. Furthermore, since these representing functionals are generally not unique, one can choose such a functional that satisfies appropriate assumptions. For instance, representing functionals can be chosen as smooth functionals in special cases, although Bishop-Phelps and scalarizing functionals are often nonsmooth. In this case, the theory of representing functionals is not only unifying but has its own merits.

The theory outlined in this paper has various applications in economics, finance, and optimization. We restrict ourselves to applications in optimization like vector optimization and set optimization. Since representing functionals can be defined in fixed and variable order structures, this theory is so general that vector and set optimization problems with standard order structure and those with variable order structure can be investigated.

The idea of representing functionals occured already in [12, 13] and first properties were specified there. However, the name of these functionals is used for the first time in the present paper.

This paper is organized as follows: Preliminaries are summarized in Section 2. Representing functionals are presented in Section 3. Topics are definition, basic properties, sublinearity, subdifferential, zeros, extensions to variable order structures, and separation results. The last two sections are devoted to applications to vector and set optimization with fixed and variable order structure.

2. Preliminaries

Throughout this paper, we consider a real linear space Y and a convex cone $C \subset Y$. The convex cone induces a *preorder* being reflexive, transitive, and compatible with the linear structure of Y. Recall that the cone C is said to be *pointed* iff $(-C) \cap C = \{0_Y\}$. A pointed convex cone induces a *partial order*, which is reflexive, transitive, antisymmetric, and compatible with the linear structure of Y. The dual space of Y is denoted by Y' whereas Y^* is used for its topological variant. For a nonempty subset A of Y cone(A) denotes the cone generated by the set A, and conv(A) is an abbreviation for the convex hull of the set A. The *algebraic interior* (or *core*) of a nonempty subset $A \subset Y$ is defined as

$$\operatorname{core}(A) := \{\bar{y} \in A \mid \forall\, y \in Y\ \exists\, \bar{\lambda} > 0 :\ \bar{y} + \lambda y \in A\ \forall\, \lambda \in [0, \bar{\lambda}]\}.$$

For two nonempty sets $A, B \subset Y$, we use the abbreviation

$$A \pm B := \{a \pm b \in Y \mid a \in A \text{ and } b \in B\}$$

for the sum and the difference, respectively.

In 1962, Bishop and Phelps [1] introduced a class of order cones with a rich mathematical structure (see also [2]). Nowadays, many authors use a slightly different definition than the original one. We recall the most commonly used version.

Definition 2.1. Let $(Y, \|\cdot\|_Y)$ be a real normed space, and let some continuous linear functional $\ell \in Y^*$ be arbitrarily chosen. The convex cone

$$C(\ell) := \{y \in Y \mid \ell(y) \geq \|y\|_Y\}$$

is called a *Bishop-Phelps cone*.

For detailed information about this important class of cones, we refer to [3, 4, 5].

Remark 2.1. It is obvious under the assumptions of Definition 2.1 that

$$-C(\ell) = \{y \in Y \mid -\ell(y) \geq \|y\|_Y\} = \{y \in Y \mid \ell(y) + \|y\|_Y \leq 0\}.$$

In this paper, the functional $\varphi_\ell : Y \to \mathbb{R}$ with

$$\varphi_\ell(y) := \ell(y) + \|y\|_Y \text{ for all } y \in Y$$

is called a *Bishop-Phelps functional*. Hence, we conclude $-C(\ell) = \{y \in Y \mid \varphi_\ell(y) \leq 0\}$. The Bishop-Phelps functional φ_ℓ is a special normlinear functional recently studied by Zaffaroni [14].

In vector optimization, there are various scalarization approaches, which work with a so-called scalarizing functional. We restrict ourselves to a version given by Tammer (formerly Gerstewitz) [8] in 1983.

Definition 2.2. Let A be a nonempty subset of a real linear space Y, and let $k \in Y \backslash \{0_Y\}$ be arbitrarily given. The functional $\xi_{A,k} : Y \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ with

$$\xi_{A,k}(y) := \inf\{t \in \mathbb{R} \mid y \in tk + A\} \text{ for all } y \in Y$$

is called the *Gerstewitz scalarizing functional*.

For properties of this functional, we refer to [15] among others.

Remark 2.2. Let Y be a real topological linear space, and let C be a closed convex cone in Y with $C \neq Y$ and $C \neq \{0_Y\}$. In Definition 2.2, we choose $A := -C$ and $k \in C \backslash (-C)$. With [15, Theorem 5.2.3], we then obtain the equality $-C = \{y \in Y \mid \xi_{-C,k}(y) \leq 0\}$.

As a consequence of Remark 2.1 and Remark 2.2, we obtain under appropriate assumptions that a negative convex cone equals the set of solutions of an inequality given by a Bishop-Phelps or a scalarizing functional. This result opens the door for a unified approach for the representation of a negative convex cone via solutions of a general inequality.

We conclude this section with some bibliographical notes. Under the assumptions of Definition 2.2 the functional $\xi_{A,k}$ is also said to be *translation invariant along k* (compare [11] and the references therein). In mathematical finance, this type of functionals plays an important role in the theory of risk measures ([16]), and a solution set of an inequality given by a risk measure is also called *acceptance set* (see [16, Definition 2.3]).

The Hiriart-Urruty scalarizing functional (see [6, 7] and later [9]) also called *signed distance functional* ([17, p. 313], [10, p. 439]) or *oriented distance functional* ([15, p. 215]) can also be used for the representation of the negative convex cone as solution set of an inequality ([9, Proposition 3.2, (4)]). The unified approach of the present paper also covers this type of functional. The relationships between several scalarizing functionals were studied in [18].

3. $-C$ REPRESENTING FUNCTIONALS

In this section, we introduce so-called $-C$ representing functionals where C is a convex cone in a real linear space. These functionals extend Bishop-Phelps and scalarizing functionals and have already been utilized in [12, 13] without using this special name. We investigate basic properties of these functionals together with sublinearity, dual cone, zeros, and separation results. Moreover, we extend this notion to variable order structures.

3.1. **Definition.**

For our investigations, we use the following standard assumption.

Assumption 3.1. Let Y be a real linear space and let C be a convex cone in Y.

Under this assumption we describe the convex cone $-C$ using a special functional ψ defined as follows.

Definition 3.1. ([12, Remark 3.3], [13, Assumption 2.1]) Let Assumption 3.1 be satisfied. A functional $\psi : Y \to \mathbb{R}$ is called $-C$ *representing* iff

$$-C = \{y \in Y \mid \psi(y) \leq 0\}. \tag{3.1}$$

It is obvious from Remarks 2.1 and 2.2 that the Bishop-Phelps functional φ_ℓ and the Gerstewitz scalarizing functional $\xi_{-C,k}$ are $-C$ representing functionals. In a recent paper by Ha [19], equality (3.1) is also called the cone representation property (P4), where ψ is named an abstract scalarizing function.

Example 3.1. As a simple illustrative example we consider the special case $Y := \mathbb{R}^2$ and $C := \mathbb{R}^2_+$. Then we obtain

$$-C = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1, y_2 \leq 0\} = \{(y_1, y_2) \in \mathbb{R}^2 \mid \max\{y_1, y_2\} \leq 0\},$$

i.e. the function $\psi_1 : \mathbb{R}^2 \to \mathbb{R}$ with

$$\psi_1(y) = \max\{y_1, y_2\} \text{ for all } y = (y_1, y_2) \in \mathbb{R}^2 \tag{3.2}$$

is $-C$ representing. Figure 1 illustrates the function ψ_1. It is evident that function ψ_1 is contin-

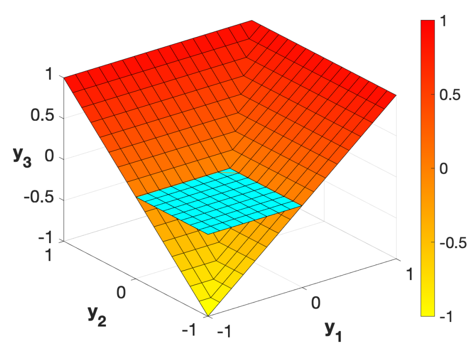

FIGURE 1. Illustration of the nonsmooth function $\psi_1 : \mathbb{R}^2 \to \mathbb{R}$ given by (3.2) together with a part of the cone $-\mathbb{R}^2_+$.

uous but nonsmooth. A continuously differentiable $-C$ representing function can be obtained if we choose $\psi_2 : \mathbb{R}^2 \to \mathbb{R}$ with

$$\psi_2(y) = \big(\max\{y_1, 0\}\big)^2 + \big(\max\{y_2, 0\}\big)^2 \text{ for all } y = (y_1, y_2) \in \mathbb{R}^2. \tag{3.3}$$

The function ψ_2 is illustrated in Figure 2. From numerical optimization, it is well-known that ψ_2 is a penalty function with respect to $-\mathbb{R}^2_+$.

Various other $-C$ representing functionals are known from the specialist literature for different spaces and cones; see [12, Example 3.4].

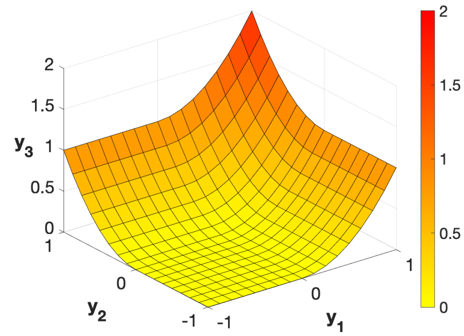

FIGURE 2. Illustration of the smooth function $\psi_2 : \mathbb{R}^2 \to \mathbb{R}$ given by (3.3).

Remark 3.1.

(a) It is evident that a $-C$ representing functional ψ also gives a representation of the convex cone C, i.e. $C = \{y \in Y \mid \psi(-y) \le 0\}$.
(b) Example 3.1 shows that a functional ψ is not uniquely defined by the set equality (3.1).
(c) For every $y^1, y^2 \in Y$ the equivalence

$$\boxed{y^1 \le_C y^2 \iff \psi(y^1 - y^2) \le 0}$$

holds where $\le_C$ denotes the preorder in Y induced by the convex cone C. Hence, a preorder inequality can be written as a standard inequality in $\mathbb{R}$. This simplifies the investigation of preorder inequalities.

3.2. **Basic properties.**

Proposition 3.1. *Let Assumption 3.1 be satisfied, and let ψ be a $-C$ representing functional. Then*

(a) $C \neq \{0_Y\} \iff \exists\, y \in Y \backslash \{0_Y\} : \psi(y) \le 0,$
(b) C *pointed* $\iff \big(y \in Y,\ \max\{\psi(-y), \psi(y)\} \le 0 \Rightarrow y = 0_Y\big).$

Proof.

(a) Since $C \neq \{0_Y\} \Leftrightarrow -C \neq \{0_Y\}$, the assertion follows with equation (3.1).
(b) Since the convex cone C is pointed iff $C \cap (-C) = \{0_Y\}$, the assertion follows with equation (3.1). □

Proposition 3.2. *Let Assumption 3.1 be satisfied, and let ψ be a $-C$ representing functional. Then*

(a) $\forall\, y \in Y : \psi(y) \le 0 \Rightarrow \psi(\lambda y) \le 0$ *for all* $\lambda \ge 0$,
(b) $\forall\, y^1, y^2 \in Y : \max\{\psi(y^1), \psi(y^2)\} \le 0 \Rightarrow \psi(y^1 + y^2) \le 0,$
(c) $\psi(0_Y) \le 0,$
(d) If, in addition, Y is a real topological linear space, ψ is continuous at 0_Y and $C \neq Y$, then we have $\psi(0_Y) = 0$.

Proof. The assertions (a) and (b) follow from the conditions of the convex cone $-C$, i.e. $y \in -C \Rightarrow \lambda y \in -C$ for all $\lambda \geq 0$ and $y^1, y^2 \in -C \Rightarrow y^1 + y^2 \in -C$. Part (c) is a simple consequence of $0_Y \in -C$. In Part (d) notice that there is some $y \notin -C$ implying $\psi(\lambda y) > 0$ for all $\lambda > 0$. Consequently, we obtain $\psi(0_Y) \geq 0$ and the assertion follows from Part (c). $\square$

Proposition 3.3. *Let Assumption 3.1 be satisfied, and let ψ be a $-C$ representing functional. Then*

(a) $\{\bar{y}\} - C = \{y \in Y \mid \psi(y - \bar{y}) \leq 0\}$ *for all* $\bar{y} \in Y$*,*
(b) $\{\bar{y}\} + C = \{y \in Y \mid \psi(\bar{y} - y) \leq 0\}$ *for all* $\bar{y} \in Y$*,*
(c) $S - C = \{y \in Y \mid \exists\, s \in S : \psi(y - s) \leq 0\}$ *for all nonempty sets* $S \subset Y$*,*
(d) $S + C = \{y \in Y \mid \exists\, s \in S : \psi(s - y) \leq 0\}$ *for all nonempty sets* $S \subset Y$*.*

Proof. We only show the assertions (a) and (c) because the other two ones can be proven analogously.

(a) $\{\bar{y}\} - C = \{y \in Y \mid y - \bar{y} \in -C\} = \{y \in Y \mid \psi(y - \bar{y}) \leq 0\}$ for all $\bar{y} \in Y$,
(c) $S - C = \{y \in Y \mid \exists\, s \in S : y \in \{s\} - C\} = \{y \in Y \mid \exists\, s \in S : \psi(y - s) \leq 0\}$ for all nonempty sets $S \subset Y$. $\square$

3.3. **Sublinearity.**

In this subsection we investigate $-C$ representing sublinear functionals. Recall that in a real linear space Y a functional $\eta : Y \to \mathbb{R}$ is called *sublinear* iff

(a) $\eta(\lambda y) = \lambda \eta(y)$ for all $y \in Y$ and $\lambda \geq 0$
(b) $\eta(y^1 + y^2) \leq \eta(y^1) + \eta(y^2)$ for all $y^1, y^2 \in Y$.

Proposition 3.4. *Let Y be a real linear space. For every sublinear functional $\psi : Y \to \mathbb{R}$, the set $S := \{y \in Y \mid \psi(y) \leq 0\}$ is a convex cone.*

Proof. Let $\psi : Y \to \mathbb{R}$ be a sublinear functional. For arbitrarily chosen $y \in S$ and $\lambda \geq 0$ we have $\psi(\lambda y) = \lambda \psi(y) \leq 0$, i.e. $\lambda y \in S$. Moreover, for all $y^1, y^2 \in S$ we obtain $\psi(y^1 + y^2) \leq \psi(y^1) + \psi(y^2) \leq 0$, which means $y^1 + y^2 \in S$. Consequently, the set S is a convex cone. $\square$

But not every functional $\psi : Y \to \mathbb{R}$ (where Y denotes a real linear space) with the property that the sub-level set $\{y \in Y \mid \psi(y) \leq 0\}$ is a convex cone, is sublinear (see ψ_2 given by (3.3)).

Proposition 3.5. *Let Assumption 3.1 be satisfied, and let ψ be a $-C$ representing sublinear functional. Then there exists some linear functional $\ell \in Y'$ so that $-C$ is contained in the halfspace $\{y \in Y \mid \ell(y) \leq 0\}$.*

Proof. Since ψ is a sublinear functional, by the Hahn-Banach theorem there is some linear functional $\ell \in Y'$ with $\psi(y) \geq \ell(y)$ for all $y \in Y$. For every $y \in -C$ we then obtain $0 \geq \psi(y) \geq \ell(y)$, which proves the assertion. $\square$

Proposition 3.6. *Let Assumption 3.1 be satisfied, and let ψ be a $-C$ representing sublinear functional. Then*

(a) $\{y \in Y \mid \psi(y) < 0\} \subset \mathrm{core}(-C)$*,*
(b) $\mathrm{core}(-C) \subset \{y \in Y \mid \psi(y) < 0\}$*, if the latter set is nonempty.*

Proof.

(a) Let $y \in Y$ with $\psi(y) < 0$ be arbitrarily chosen. For any $h \in Y$ there is some sufficiently small $\bar{\lambda} > 0$ with

$$\psi(y+\lambda h) \leq \underbrace{\psi(y)}_{<0} + \lambda\, \psi(h) < 0 \text{ for all } \lambda \in [0, \bar{\lambda}].$$

This implies $y \in \operatorname{core}(-C)$.

(b) Let $y \in \operatorname{core}(-C)$ be arbitrarily chosen. By assumption there exists some $\bar{y} \in Y$ with $\psi(\bar{y}) < 0$. Then there is some $\lambda > 0$ with $y - \lambda \bar{y} \in -C$. Consequently, we obtain

$$\psi(y) = \psi(y - \lambda \bar{y} + \lambda \bar{y}) \leq \underbrace{\psi(y - \lambda \bar{y})}_{\leq 0} + \psi(\lambda \bar{y}) \leq \lambda\, \psi(\bar{y}) < 0,$$

which has to be shown. □

Under the assumptions of Proposition 3.6, it then follows

$$\operatorname{core}(-C) = \{y \in Y \mid \psi(y) < 0\},$$

if the latter set is nonempty. Hence, we have a very simple characterization of the core of $-C$.

Proposition 3.7. *Let Assumption 3.1 be satisfied, and let ψ be a $-C$ representing sublinear functional. If there exists some $\bar{y} \in Y$ with $\psi(\bar{y}) < 0$, then the convex cone C is reproducing, i.e. $Y = C - C$.*

Proof. Since the inclusion $C - C \subset Y$ is evident, we only prove $Y \subset C - C$. By Proposition 3.6, (a) we obtain $\bar{y} \in \operatorname{core}(-C)$. Then for every $y \in Y$ there is some $\lambda > 0$ with $\bar{y} + \lambda y \in -C$. And we conclude $y \in \big\{ -\frac{1}{\lambda}\bar{y} \big\} - \frac{1}{\lambda} C \subset C - C$. Hence, the convex cone C is reproducing. □

In the case of the Bishop-Phelps functional φ_ℓ (see Remark 2.1), the existence of some $\bar{y} \in Y$ with $\varphi_\ell(\bar{y}) < 0$ can be simply ensured.

Proposition 3.8. *Let $(Y, \|\cdot\|_Y)$ be a real normed space, and let $C(\ell) := \{y \in Y \mid \ell(y) \geq \|y\|_Y\}$ be a Bishop-Phelps cone for some $\ell \in Y^*$ with the $-C(\ell)$ representing functional $\varphi_\ell := \ell + \|\cdot\|_Y$. If $\|\ell\|_{Y^*} > 1$, then there exists some $\bar{y} \in Y \backslash \{0_Y\}$ with $\varphi_\ell(\bar{y}) < 0$.*

Proof. Since $1 < \|\ell\|_{Y^*} = \sup_{y \neq 0_Y} \frac{|\ell(y)|}{\|y\|_Y}$, there is some $\bar{y} \neq 0_Y$ with $-\ell(\bar{y}) = |\ell(\bar{y})| > \|\bar{y}\|_Y$. This implies $\varphi_\ell(\bar{y}) = \ell(\bar{y}) + \|\bar{y}\|_Y < 0$, which has to be shown. □

3.4. **Subdifferential.**

Under appropriate assumptions a relation between the subdifferential of the $-C$ representing functional at the origin and the dual cone can be given.

Theorem 3.1. *Let C be a convex cone of a real normed space $(Y, \|\cdot\|_Y)$, and let ψ be a $-C$ representing functional, which is convex with $\psi(0_Y) = 0$ and has a nonempty subdifferential $\partial \psi(0_Y)$. Then it holds*

$$\operatorname{cl}\big[\operatorname{cone}\big(\partial \psi(0_Y)\big)\big] \subset C^*.$$

Proof. Let $\bar{\ell} \in \text{cl}\left[\text{cone}\left(\partial\psi(0_Y)\right)\right]$ be arbitrarily chosen. Then there exist sequences $(\lambda_n)_{n\in\mathbb{N}}$ and $(\ell_n)_{n\in\mathbb{N}}$ with $\lambda_n \geq 0$ and $\ell_n \in \partial\psi(0_Y)$ for all $n \in \mathbb{N}$ so that $\bar{\ell} = \lim_{n\to\infty} \lambda_n \ell_n$. By the definition of the subdifferential we have for every $n \in \mathbb{N}$

$$\psi(y) \geq \underbrace{\psi(0_Y)}_{=0} + \ell_n(y) = \ell_n(y) \text{ for all } y \in Y.$$

This implies for every $n \in \mathbb{N}$

$$-\ell_n(c) = \ell_n(-c) \leq \psi(-c) \leq 0 \text{ for all } c \in C.$$

Hence, we obtain

$$\bar{\ell}(c) = \lim_{n\to\infty} \underbrace{\lambda_n}_{\geq 0} \underbrace{\ell_n(c)}_{\geq 0} \geq 0 \text{ for all } c \in C,$$

i.e., $\bar{\ell} \in C^*$. □

We now discuss a very simple example.

Example 3.2. For the cone $C := \mathbb{R}^2_+$ in $\mathbb{R}^2$, it is well-known that C coincides with its dual cone C^*, i.e., C is self-dual. For the $-C$ representing functional $\psi : \mathbb{R}^2 \to \mathbb{R}$ with

$$\psi(y) = \max\{y_1, y_2\} \text{ for all } y = (y_1, y_2) \in \mathbb{R}^2$$

we obtain the subdifferential

$$\begin{aligned} \partial\psi(0_{\mathbb{R}^2}) &= \{\ell \in \mathbb{R}^2 \mid \psi(y) \geq \ell^T y \; \forall \, y \in \mathbb{R}^2\} \\ &= \{(\ell_1, \ell_2) \in \mathbb{R}^2 \mid \max\{y_1, y_2\} \geq \ell_1 y_1 + \ell_2 y_2 \; \forall \, (y_1, y_2) \in \mathbb{R}^2\}. \end{aligned}$$

By a simple calculation, one can see that the inequality

$$\max\{y_1, y_2\} \geq \ell_1 y_1 + \ell_2 y_2 \text{ for all } (y_1, y_2) \in \mathbb{R}^2$$

is equivalent to the condition $(\ell_1, \ell_2) \in \text{conv}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$, i.e.

$$\text{cone}\left(\partial\psi(0_{\mathbb{R}^2})\right) = \text{cone}\left(\text{conv}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}\right) = \mathbb{R}^2_+ = C^*.$$

This example shows that the inclusion in the assertion of Theorem 3.1 can also be a set equation.

Finally, we turn our attention to Bishop-Phelps cones, which are very simple to treat.

Example 3.3. Let C be a Bishop-Phelps cone in a real normed space $(Y, \|\cdot\|_Y)$ given by some continuous linear functional $\ell \in Y^*$. As in Remark 2.1, we consider the $-C$ representing functional $\varphi_\ell : Y \to \mathbb{R}$ with $\varphi_\ell = \ell + \|\cdot\|_Y$. The functional φ_ℓ is convex with $\varphi_\ell(0_Y) = 0$. By [20, Example 3.24, (b)], we obtain the subdifferential

$$\partial\varphi_\ell(0_Y) = \ell + \partial\|0_Y\|_Y = \ell + B(0_Y, 1) = B(\ell, 1),$$

where $B(\ell, 1)$ denotes the closed ball with center ℓ and radius 1. With Theorem 3.1, we then obtain

$$\text{cl}\left[\text{cone}\left(B(\ell, 1)\right)\right] \subset C^*.$$

This result is included in [3, Thm. 2.13].

Investigations of the subdifferential of the signed distance functional were already given in [7, p. 69] (compare also [17, Proposition 21.11]). In [15, Subsection 5.2.5], the investigations of the subdifferential were carried out for the Gerstewitz scalarizing functional.

3.5. **Zeros.**

It is obvious from the definition of a $-C$ representing functional ψ that the zeros of ψ belong to the negative convex cone of the real linear space. For the functional ψ_1 in the illustrative Example 3.1, we know that the set of zeros equals the boundary of $-C$ (see Figure 1). But it is also known from Bishop-Phelps theory that in special spaces these zeros equal the whole negative convex cone $-C$ (for a comprehensive investigation see [5]).

Example 3.4. As in Example 3.1, let the convex cone $C := \mathbb{R}^2_+$ in $Y := \mathbb{R}^2$ be given. But now we consider the $-C$ representing functional $\psi : \mathbb{R}^2 \to \mathbb{R}$ with

$$\psi(y) = \max\{y_1, y_2, 0\} \text{ for all } y = (y_1, y_2) \in \mathbb{R}^2 \tag{3.4}$$

(compare Fig. 3). It is evident that $-\mathbb{R}^2_+ = \{y \in \mathbb{R}^2 \mid \psi(y) = 0\}$, i.e., the convex cone $-C$ equals

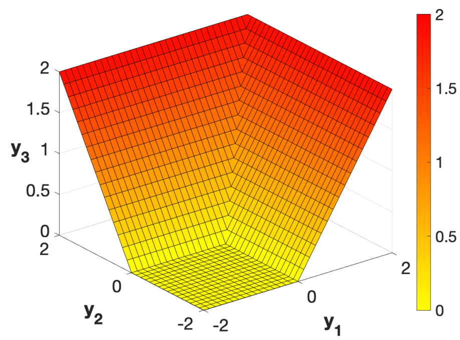

FIGURE 3. Illustration of the function $\psi : \mathbb{R}^2 \to \mathbb{R}$ given by (3.4).

the set of all zeros of the functional ψ. The continuously differentiable function ψ_2 given by (3.3) also has this property (compare Figure 2). In general, standard scalarizing functionals are not smooth.

Proposition 3.9. *Let C be a convex cone of a real normed space $(Y, \|\cdot\|_Y)$, and let ψ be a $-C$ representing convex functional with the property that for $y \in Y$ we have $y \in -C \iff \psi(y) = 0$. Then it holds $0_{Y^*} \in \partial\psi(0_Y)$.*

Proof. For every $y \in Y$, we have

$$\psi(y) \geq 0 = \underbrace{\psi(0_Y)}_{=0} + \underbrace{0_{Y^*}(y)}_{=0}$$

implying $0_{Y^*} \in \partial\psi(0_Y)$. □

Finally, we give an additional relation between the dual cone and the subdifferential.

Theorem 3.2. *Let C be a convex cone of a real normed space $(Y, \|\cdot\|_Y)$, and let ψ be a $-C$ representing convex functional with the property that, for $y \in Y$, it holds $y \in -C \Leftrightarrow \psi(y) = 0$. Let $\psi'(0_Y)(\cdot)$ denote the directional derivative of ψ at 0_Y. Under the additional assumption*

$$\forall\, \ell \in C^* \;\exists\, \alpha_\ell > 0 :\; \alpha_\ell \ell(y) \leq \psi'(0_Y)(y) \text{ for all } y \in Y \tag{3.5}$$

it then holds

$$C^* \subset \text{cone}\left(\partial \psi(0_Y)\right).$$

Proof. Let $\bar{\ell} \in C^*$ be arbitrarily chosen. By assumption (3.5), there exists some $\alpha_{\bar{\ell}} > 0$ so that

$$\alpha_{\bar{\ell}} \bar{\ell}(y) \leq \psi'(0_Y)(y) \text{ for all } y \in Y.$$

With [20, Lemma 3.25], we then obtain $\alpha_{\bar{\ell}} \bar{\ell} \in \partial \psi(0_Y)$ implying $\bar{\ell} \in \frac{1}{\alpha_{\bar{\ell}}} \partial \psi(0_Y)$. Consequently, we have $\bar{\ell} \in \text{cone}\left(\partial \psi(0_Y)\right)$. $\square$

Figure 4 illustrates assumption (3.5) of Theorem 3.2. This assumption is stronger than the

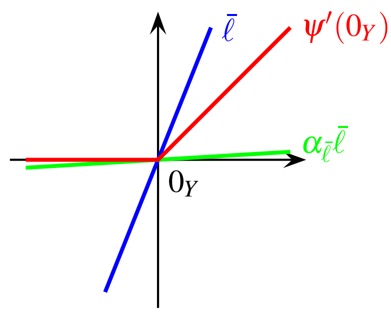

FIGURE 4. Illustration of assumption (3.5).

condition $0_{Y^*} \in \partial \psi(0_Y)$ shown in Proposition 3.9. Although the equivalence $y \in -C \Leftrightarrow \psi(y) = 0$ is not explicitly used in the proof of Theorem 3.2, it is a reasonable condition for assumption (3.5). Notice that Theorems 3.1 and 3.2 present nearly converse inclusions; there are close relations between the dual cone and the subdifferential of ψ at the origin under appropriate assumptions.

3.6. **Extension to variable order structures.**

The theory of $-C$ representing functionals can also be extended to variable order structures in a real linear space. In this general case we assume the following.

Assumption 3.2. Let Y be a real linear space, and let $\mathscr{K} : Y \rightrightarrows Y$ denote a set-valued map so that the images $\mathscr{K}(y)$ are convex cones for all $y \in Y$.

Definition 3.2. Under Assumption 3.2, a functional $\psi : Y \times Y \to \mathbb{R}$ is called a $-\mathscr{K}$ *representing functional* iff

$$-\mathscr{K}(\bar{y}) = \{y \in Y \mid \psi(y, \bar{y}) \leq 0\} \text{ for all } \bar{y} \in Y.$$

Example 3.5. Let $(Y, \|\cdot\|_Y)$ be a real normed space, and let a map $L : Y \to Y^*$ be given. Then we investigate the set-valued map $\mathscr{K} : Y \rightrightarrows Y$ with

$$\mathscr{K}(\bar{y}) = \{y \in Y \mid L(\bar{y})(y) \geq \|y\|_Y\} \text{ for all } \bar{y} \in Y$$

being a Bishop-Phelps cone for every $\bar{y} \in Y$. In Figure 5, different Bishop-Phelps cones are illustrated, which may define the variable order structure in the real normed space $(Y, \|\cdot\|_Y)$. The functional $\psi : Y \times Y \to \mathbb{R}$ given by

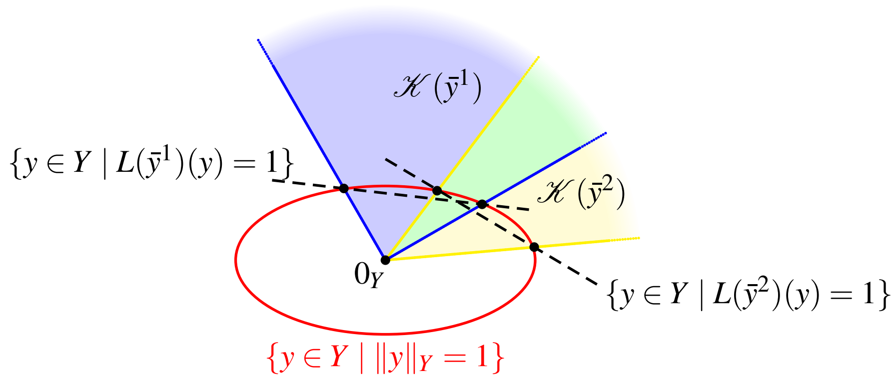

FIGURE 5. Illustration of two overlapping Bishop-Phelps cones $\mathscr{K}(\bar{y}^1)$ and $\mathscr{K}(\bar{y}^2)$.

$$\psi(y, \bar{y}) = L(\bar{y})(y) + \|y\|_Y \text{ for all } y, \bar{y} \in Y$$

is a $-\mathscr{K}$ representing functional.

In [21], Eichfelder developed a comprehensive theory on variable order structures. We follow these lines and use two order structures.

Definition 3.3. ([21, p. 5]) Let Assumption 3.2 be satisfied. Then we define the order relations $\leq_1$ and $\leq_2$ by

$$y^1 \leq_1 y^2 \;:\Longleftrightarrow\; y^2 - y^1 \in \mathscr{K}(y^1)$$

and

$$y^1 \leq_2 y^2 \;:\Longleftrightarrow\; y^2 - y^1 \in \mathscr{K}(y^2)$$

for arbitrary $y^1, y^2 \in Y$.

Remark 3.2. Let Assumption 3.2 be satisfied, and let $\psi : Y \times Y \to \mathbb{R}$ be a $-\mathscr{K}$ representing functional. Then it is obvious from the definition of the order relations $\leq_1$ and $\leq_2$ that for every $y^1, y^2 \in Y$ the equivalences

$$\boxed{y^1 \leq_1 y^2 \iff \psi(y^1 - y^2, y^1) \leq 0}$$

and

$$\boxed{y^1 \leq_2 y^2 \iff \psi(y^1 - y^2, y^2) \leq 0}$$

hold. Also in the case of a variable order structure, a complicated order inequality can be written as an inequality in $\mathbb{R}$.

Proposition 3.10. *Let Assumption 3.2 be satisfied, and let $\psi : Y \times Y \to \mathbb{R}$ be a $-\mathscr{K}$ representing functional. Then*

(a) $\{\bar{y}\} - \mathscr{K}(\bar{y}) = \{y \in Y \mid \psi(y - \bar{y}, \bar{y}) \leq 0\}$ for all $\bar{y} \in Y$,
(b) $\{\bar{y}\} + \mathscr{K}(\bar{y}) = \{y \in Y \mid \psi(\bar{y} - y, \bar{y}) \leq 0\}$ for all $\bar{y} \in Y$,
(c) $S - \mathscr{K}(\bar{y}) = \{y \in Y \mid \exists\, s \in S : \psi(y - s, \bar{y}) \leq 0\}$ for all nonempty sets $S \subset Y$ and all $\bar{y} \in Y$,

(d) $S+\mathscr{K}(\bar{y})=\{y\in Y \mid \exists\, s\in S: \psi(s-y,\bar{y})\leq 0\}$ *for all nonempty sets* $S\subset Y$ *and all* $\bar{y}\in Y$.

Proof. For an arbitrary $\bar{y}\in Y$ we obtain

(a) $\{\bar{y}\}-\mathscr{K}(\bar{y})=\{y\in Y \mid y-\bar{y}\in -\mathscr{K}(\bar{y})\}=\{y\in Y \mid \psi(y-\bar{y},\bar{y})\leq 0\}$,
(b) $\{\bar{y}\}+\mathscr{K}(\bar{y})=\{y\in Y \mid \bar{y}-y\in -\mathscr{K}(\bar{y})\}=\{y\in Y \mid \psi(\bar{y}-y,\bar{y})\leq 0\}$.

And for an arbitrary nonempty set $S\subset Y$ we get

(c) $S-\mathscr{K}(\bar{y})=\{y\in Y \mid \exists\, s\in S: y-s\in -\mathscr{K}(\bar{y})\}=\{y\in Y \mid \exists\, s\in S: \psi(y-s,\bar{y})\leq 0\}$,
(d) $S+\mathscr{K}(\bar{y})=\{y\in Y \mid \exists\, s\in S: s-y\in -\mathscr{K}(\bar{y})\}=\{y\in Y \mid \exists\, s\in S: \psi(s-y,\bar{y})\leq 0\}$. □

3.7. **Separation results.**

By definition, $-C$ representing functionals are qualified for simple separation results.

Proposition 3.11. *Let Assumption 3.1 be satisfied, let ψ be a $-C$ representing functional, and let A be a nonempty subset of Y. Then*

(a) $A\cap C=\emptyset \iff \psi(-y)>0\ \forall\, y\in A$,
(b) for an arbitrary $\bar{y}\in A\cap C$:
$A\cap C=\{\bar{y}\} \iff \psi(-y)>0\ \forall\, y\in A, y\neq\bar{y}$.

Proof.

(a) The assertion is a direct consequence of the definition of the functional ψ because $A\cap C=\emptyset \Leftrightarrow -A\cap(-C)=\emptyset \Leftrightarrow \psi(y)>0\ \forall\, y\in -A$, which leads to the assertion.
(b) With the equivalence $A\cap C=\{\bar{y}\} \Leftrightarrow \big(A\backslash\{\bar{y}\}\big)\cap C=\emptyset$ the assertion follows from part (a). □

Proposition 3.12. *Let C_1 and C_2 be convex cones in a real linear space Y with $C_1,C_2\neq\{0_Y\}$ and let ψ_1 and ψ_2 be $-C_1$ and $-C_2$ representing functionals, respectively. Then*

$$\begin{aligned} C_1\cap C_2=\{0_Y\} &\iff \psi_2(-c_1)>0\ \forall\, c_1\in C_1\backslash\{0_Y\}\\ &\iff \psi_1(-c_2)>0\ \forall\, c_2\in C_2\backslash\{0_Y\}\\ &\iff \Big(y\neq 0_Y,\ \psi_1(-y)\leq 0\ \Rightarrow\ \psi_2(-y)>0\Big)\\ &\iff \Big(y\neq 0_Y,\ \psi_2(-y)\leq 0\ \Rightarrow\ \psi_1(-y)>0\Big). \end{aligned}$$

Proof. The first and second equivalences follow from Proposition 3.11, (b). The third equivalence can be deduced from the first one because $C_1=\{y\in Y \mid \psi_1(-y)\leq 0\}$ or $y\in C_1\backslash\{0_Y\} \Leftrightarrow \psi_1(-y)\leq 0$ for $y\neq 0_Y$. The last equivalence is a result of the second one. □

4. Application to Vector Optimization

Vector optimization is a typical application of $-C$ or $-\mathscr{K}$ representing functionals. Here we study general problems of vector optimization involving those with vector-valued objective map. We investigate vector optimization problems with a fixed order structure given by a pointed convex cone C and those with a variable order structure.

4.1. **Fixed order structure.**

We start our investigations with a fixed order cone given in a real linear space.

Assumption 4.1. Let Y be a real linear space, let C be a pointed convex cone in Y, and let T be a nonempty subset of Y.

Although there are various optimality notions in use, we restrict ourselves to the known notion of minimal elements of a set (for instance, compare [22]). Under Assumption 4.1 we ask for minimal elements of the set T. Recall that an element $\bar{y} \in T$ is called a *minimal* element of the set T iff

$$y \leq_C \bar{y},\ y \in T \implies y = \bar{y} \tag{4.1}$$

($\leq_C$ denotes the partial order induced by C). For the sake of simplicity we do not work with the definition of minimality for a non-pointed convex cone C, which is different and subsumes the aforementioned notion.

Now, we present a characterization of minimal elements.

Theorem 4.1. *Let Assumption 4.1 be satisfied, and let ψ be a $-C$ representing functional with $\psi(0_Y) = 0$. Then*

$$\bar{y} \text{ minimal element of } T \iff \bar{y} \text{ unique minimal solution of } \min_{y \in T} \psi(y - \bar{y}).$$

Proof. By definition, $\bar{y}$ is a minimal element of the set T iff the implication (4.1) holds. Since for any $y \in Y$ the inequality $y \leq_C \bar{y}$ means $y - \bar{y} \in -C$ or $\psi(y - \bar{y}) \leq 0$, the implication (4.1) is equivalent to

$$\begin{aligned} & \{y \in T \mid \psi(y - \bar{y}) \leq 0\} = \{\bar{y}\} \\ \iff \quad & \psi(y - \bar{y}) > 0 \text{ for all } y \in T \setminus \{\bar{y}\}. \end{aligned} \tag{4.2}$$

With the assumption $\psi(0_Y) = 0$, condition (4.2) is equivalent to

$$\begin{aligned} & \psi(y - \bar{y}) > \psi(\bar{y} - \bar{y}) \text{ for all } y \in T \setminus \{\bar{y}\} \\ \iff \quad & \bar{y} \text{ is a unique minimal solution of the problem } \min_{y \in T} \psi(y - \bar{y}), \end{aligned}$$

which has to be shown. □

This theorem extends the result [9, Theorem 4.3], which was shown for the signed distance functional (compare also [17, Proposition 21.10]).

Theorem 4.1 is a so-called scalarization result because minimal elements of a set are characterized as unique minimal solutions of an optimization problem with a scalar-valued objective functional. But these scalar optimization problems have the drawback that the unique minimal solution is simultaneously a parameter in the objective functional. The next theorem avoids this disadvantage to some extend.

Theorem 4.2. *Let Assumption 4.1 be satisfied, let ψ be a $-C$ representing functional, and let some $\hat{y} \in Y$ with $T \subset \{\hat{y}\} + C \setminus \{0_Y\}$ be arbitrarily chosen. $\bar{y}$ is a minimal element of the set T if and only if there is some $a \in C \setminus \{0_Y\}$ so that $(\bar{y}, 1)$ is a unique minimal solution of the* *optimization problem*

$$\begin{array}{c} \min\ \lambda \\ \text{subject to the constraint} \\ \psi(y-\hat{y}-\lambda a) \leq 0 \\ (y,\lambda) \in T \times \mathbb{R}. \end{array} \tag{4.3}$$

Proof. Since ψ is a $-C$ representing functional, the optimization problem (4.3) can also be rewritten as

$$\min_{\substack{y \in (\{\hat{y}+\lambda a\}-C)\cap T \\ \lambda \in \mathbb{R}}} \lambda. \tag{4.4}$$

Next, we begin with the actual proof.

(a) Let there exist some $a \in C\backslash\{0_Y\}$ so that $(\bar{y},1)$ is a unique minimal solution of the optimization problem (4.3) being equivalent to problem (4.4). Assume that $\bar{y}$ is not a minimal element of the set T. Then there exists some $y \neq \bar{y}$ with $y \in (\{\bar{y}\}-C)\cap T$. Because of the constraint $\bar{y} \in \{\hat{y}+a\}-C$ we get $\{\bar{y}\}-C \subset \{\hat{y}+a\}-C-C = \{\hat{y}+a\}-C$. Then we conclude $y \in (\{\hat{y}+a\}-C)\cap T$. Consequently, $(y,1)$ is also a minimal solution of problem (4.4) — a contradiction to the assumption that $(\bar{y},1)$ uniquely solves problem (4.3). Hence, $\bar{y}$ is a minimal element of the set T.

(b) Now, let $\bar{y}$ be a minimal element of the set T, i.e. $(\{\bar{y}\}-C)\cap T = \{\bar{y}\}$. With $a := \bar{y}-\hat{y} \in C\backslash\{0_Y\}$ we then have $(\{\hat{y}+1\cdot a\}-C)\cap T = \{\bar{y}\}$. Consequently, $(\bar{y},1)$ is the only feasible point at objective level 1 of problem (4.4). If there would be a feasible pair (y,λ) with $\lambda < 1$, we would obtain

$$\begin{array}{rcl} y & \in & \big(\{\hat{y}+\lambda a\}-C\big)\cap T \\ & = & \big(\{\hat{y}+a\}-C-\{\underbrace{(1-\lambda)}_{>0}\underbrace{a}_{\in C\backslash\{0_Y\}}\}\big)\cap T \\ & \subset & \big(\{\hat{y}+a\}-C\backslash\{0_Y\}\big)\cap T \\ & = & \big(\{\bar{y}\}-C\backslash\{0_Y\}\big)\cap T \\ & = & \emptyset \end{array}$$

(notice that $-C-C\backslash\{0_Y\} = -C\backslash\{0_Y\}$ because the convex cone C is assumed to be pointed). So, the minimal value of problem (4.4) equals 1, and $(\bar{y},1)$ is a unique minimal solution of the optimization problem (4.3). □

The scalarized problem (4.3) is a simple optimization problem, which can be used for the determination of minimal elements of a non-convex set T with a certain lower bound $\hat{y}$.

Example 4.1. In a real normed space $(Y,\|\cdot\|_Y)$ we consider the Bishop-Phelps cone $C(\ell) := \{y \in Y \mid \ell(y) \geq \|y\|_Y\}$ for some given continuous linear functional $\ell \in Y^*$. Moreover, we choose the $-C(\ell)$ representing functional $\varphi_\ell : Y \to \mathbb{R}$ with

$$\varphi_\ell(y) = \ell(y) + \|y\|_Y \text{ for all } y \in Y.$$

Then the scalar optimization problem (4.3) can be written as

$$\begin{gathered} \min\ \lambda \\ \text{subject to the constraint} \\ \ell(y-\hat{y}-\lambda a)+\|y-\hat{y}-\lambda a\|_Y \leq 0 \\ (y,\lambda)\in T\times\mathbb{R} \end{gathered}$$

with an inequality constraint given by a convex functional.

4.2. **Variable order structure.**

Now we investigate vector optimization problems with variable order structure under the following assumption.

Assumption 4.2. Let Y be a real linear space, let $\mathscr{K}: Y \rightrightarrows Y$ denote a set-valued map so that the images $\mathscr{K}(y)$ are convex cones for all $y\in Y$, and let T be a nonempty subset of Y.

In general, the known order relations $\leq_1$ and $\leq_2$ given in Definition 3.3 are used in vector optimization with variable order structure. Based on these order relations two optimality notions are in use (see [21, Definition 2.7]).

Definition 4.1. Let Assumption 4.2 be satisfied.

(a) An element $\bar{y}\in T$ is called a *nondominated* element of the set T iff there is no $y\in T$ with $y\leq_1\bar{y}$ and $y\neq\bar{y}$.
(b) An element $\bar{y}\in T$ is called a *minimal* element of the set T iff there is no $y\in T$ with $y\leq_2\bar{y}$ and $y\neq\bar{y}$.

Next, we present a characterization of the two optimality notions.

Theorem 4.3. *Let Assumption 4.2 be satisfied, and let ψ be a $-\mathscr{K}$ representing functional. Then*

(a) $\bar{y}$ nondominated element of T $\iff$ $\psi(y-\bar{y},y)>0$ for all $y\in T\backslash\{\bar{y}\}$,
(b) $\bar{y}$ minimal element of T $\iff$ $\psi(y-\bar{y},\bar{y})>0$ for all $y\in T\backslash\{\bar{y}\}$.

Proof.

(a) By Definition 4.1, (a) and Remark 3.2 we have

$$\begin{aligned} &\bar{y} \text{ is a nondominated element of the set } T \\ \iff\ & \nexists\, y\in T: y\leq_1\bar{y} \text{ and } y\neq\bar{y} \\ \iff\ & \nexists\, y\in T\backslash\{\bar{y}\}: \psi(y-\bar{y},y)\leq 0 \\ \iff\ & \psi(y-\bar{y},y)>0 \text{ for all } y\in T\backslash\{\bar{y}\}, \end{aligned}$$

and the assertion is shown.

(b) Following the lines of the proof of part (a), we obtain

$$\begin{aligned} &\bar{y} \text{ is a minimal element of the set } T \\ \iff\ & \nexists\, y\in T: y\leq_2\bar{y} \text{ and } y\neq\bar{y} \\ \iff\ & \nexists\, y\in T\backslash\{\bar{y}\}: \psi(y-\bar{y},\bar{y})\leq 0 \\ \iff\ & \psi(y-\bar{y},\bar{y})>0 \text{ for all } y\in T\backslash\{\bar{y}\}, \end{aligned}$$

which completes the proof.

□

Under an additional strong assumption, the following corollary presents a scalarization result extending Theorem 4.1 to the case of variable order structures.

Corollary 4.1. *Let the assumptions in Theorem 4.3 be satisfied. In addition, let $\psi(0_Y,\bar{y})=0$ be fulfilled for some $\bar{y}\in T$. Then we have*

(a) $\bar{y}$ nondominated element of T $\iff$ $\bar{y}$ unique minimal solution of $\min\limits_{y\in T}\psi(y-\bar{y},y)$,

(b) $\bar{y}$ minimal element of T $\iff$ $\bar{y}$ unique minimal solution of $\min\limits_{y\in T}\psi(y-\bar{y},\bar{y})$.

Proof.

(a) By Theorem 4.3, (a) and the assumption $\psi(0_Y,\bar{y})=0$, we have

$$\begin{aligned}&\bar{y} \text{ is a nondominated element of the set } T\\ \iff\ & \psi(y-\bar{y},y)>0=\psi(0_Y,\bar{y})=\psi(\bar{y}-\bar{y},\bar{y}) \text{ for all } y\in T\backslash\{\bar{y}\}\\ \iff\ & \bar{y} \text{ unique minimal solution of } \min_{y\in T}\psi(y-\bar{y},y),\end{aligned}$$

which has to be shown.

(b) In a similar way as in the proof of part (a) we obtain

$$\begin{aligned}&\bar{y} \text{ is a minimal element of the set } T\\ \iff\ & \psi(y-\bar{y},\bar{y})>0=\psi(\bar{y}-\bar{y},\bar{y}) \text{ for all } y\in T\backslash\{\bar{y}\}\\ \iff\ & \bar{y} \text{ unique minimal solution of } \min_{y\in T}\psi(y-\bar{y},\bar{y}),\end{aligned}$$

which completes the proof. □

5. Application to set optimization

In set optimization, one investigates optimization problems with a set-valued objective. As in the previous section, we distinguish between fixed and variable order structures.

5.1. **Fixed order structure.**

In a first step, one needs an order relation for the comparison of nonempty sets. There are different order relations, which can be used in practice (e.g. compare [23]). For simplicity, we restrict ourselves to the well-known set less order relation already introduced by Young [24] in 1931.

Definition 5.1. Let C be a convex cone in a real linear space Y. For nonempty subsets $A,B\subset Y$ the *set less* order relation $\preccurlyeq_s$ is defined by

$$A\preccurlyeq_s B \quad :\iff \quad B\subset A+C \text{ and } A\subset B-C.$$

These set inclusions can be simply characterized by a functional representing the negative cone.

Proposition 5.1. ([13, Prop. 2.1]) *Let Assumption 3.1 be satisfied, and let ψ be a $-C$ representing functional. Then for arbitrary nonempty sets $A,B\subset Y$ we have*

$$B\subset A+C\left\{\begin{array}{ll}\Longrightarrow & \sup_{b\in B}\inf_{a\in A}\psi(a-b)\le 0\\ \Longleftarrow & \sup_{b\in B}\min_{a\in A}\psi(a-b)\le 0,\ \textit{if this } \min \textit{ term exists}\end{array}\right.$$

and

$$A \subset B - C \left\{ \begin{array}{ll} \Longrightarrow & \sup_{a\in A}\inf_{b\in B}\psi(a-b) \leq 0 \\ \Longleftarrow & \sup_{a\in A}\min_{b\in B}\psi(a-b) \leq 0, \text{ \textit{if this} min \textit{term exists.}} \end{array} \right.$$

With Proposition 5.1, the set inclusions used in Definition 5.1 can be characterized by inequalities in $\mathbb{R}$, which are not simple to check because sup inf or sup min problems have to be solved. A rewriting of set inclusions as inequalities has been already given by Hernández and Rodríguez-Marín [25, Thm. 3.10, (iii)] in 2007 using an extension of the Gerstewitz scalarizing functional [8]. Later such a rewriting of set inclusions as inequalities was also given in [26, Thm. 3.3 and 3.8] for Gerstewitz scalarizing functionals and various order relations. For the signed distance functional similar investigations can be found in [27, Proposition 4.11], [28, Proposition 3.8], [29, Lemma 3.3], [30, Theorem 5.1] [31, Theorem 4.1] and [32, Proposition 3]. However, in Proposition 5.1 one does not need an explicit form of the scalarizing functional.

Next, we investigate problems of set optimization. We consider a family $\mathscr{F}$ of nonempty subsets of a real linear space Y and we ask for minimal sets of $\mathscr{F}$ where we use the well-known set less order relation.

Definition 5.2. Let Assumption 3.1 be satisfied, and let $\mathscr{F}$ be a family of nonempty subsets of Y. A set $\bar{A} \in \mathscr{F}$ is called a *minimal* set of $\mathscr{F}$ iff

$$A \preccurlyeq_s \bar{A},\ \ A \in \mathscr{F} \ \ \Longrightarrow\ \ \bar{A} \preccurlyeq_s A.$$

Next, we recall a characterization of minimal sets. Under Assumption 4.1 we use the abbreviation $\min T$ for the set of all minimal elements of T and $\max T$ for the set of all maximal elements of T.

Proposition 5.2. ([13, Prop. 3.1]) *Let C be a pointed convex cone in a real linear space Y, and let $\mathscr{F}$ be a family of nonempty subsets of Y, for which the set of minimal elements and the set of maximal elements are nonempty. For every $A \in \mathscr{F}$ let the set equalities*

$$A + C = (\min A) + C \text{ \textit{and} } A - C = (\max A) - C \tag{5.1}$$

be satisfied. For some $\bar{A} \in \mathscr{F}$ we then have

$$\bar{A} \text{ \textit{minimal set of} } \mathscr{F} \iff \min A = \min \bar{A} \text{ \textit{and} } \max A = \max \bar{A} \text{ \textit{for all} } A \in \mathscr{F} \text{ \textit{with} } A \preccurlyeq_s \bar{A}.$$

Note that condition (5.1) is a well-known assumption in vector optimization, which may also hold for certain non-convex sets (see [13, Figure 1]). For the formulation of a further optimality condition without using the set less order relation, we combine Proposition 5.2 with Proposition 5.1.

Corollary 5.1. *Let C be a pointed convex cone in a real linear space Y, and let ψ be a $-C$ representing functional. Let $\mathscr{F}$ be a family of nonempty subsets of Y, for which the set of minimal elements and the set of maximal elements are nonempty and the set equalities (5.1) are satisfied. Moreover, let the equalities*

$$\sup_{a\in A}\inf_{b\in B}\psi(a-b) = \max_{a\in A}\min_{b\in B}\psi(a-b) \text{ \textit{and} } \sup_{b\in B}\inf_{a\in A}\psi(a-b) = \max_{b\in B}\min_{a\in A}\psi(a-b)$$

be satisfied for all $A, B \in \mathscr{F}$. For some $\bar{A} \in \mathscr{F}$ we then have

$$\begin{aligned} \bar{A} \text{ \textit{minimal set of} } \mathscr{F} \iff\ & \min A = \min \bar{A} \text{ \textit{and} } \max A = \max \bar{A} \text{ \textit{for all} } A \in \mathscr{F} \text{ \textit{with}} \\ & \max\left\{ \max_{\bar{a}\in\bar{A}}\min_{a\in A}\psi(a-\bar{a}),\ \max_{a\in A}\min_{\bar{a}\in\bar{A}}\psi(a-\bar{a}) \right\} \leq 0. \end{aligned}$$

Proof. It follows from Proposition 5.1 that the set inequality $A \preccurlyeq_s \bar{A}$ (for $A, \bar{A} \in \mathscr{F}$) can be rewritten as

$$\max\left\{ \max_{\bar{a}\in\bar{A}} \min_{a\in A} \psi(a-\bar{a}),\ \max_{a\in A} \min_{\bar{a}\in\bar{A}} \psi(a-\bar{a}) \right\} \leq 0.$$

Hence, the assertion is an immediate consequence of Proposition 5.2. □

The characterization of a minimal set in the previous corollary shows that set optimization problems can be difficult optimization problems.

5.2. **Variable order structure.**

Set optimization problems with a variable order structure are even more complicated problems than those with fixed order structure. Nonetheless, $-\mathscr{K}$ representing functionals seem to be an appropriate tool for the investigation of these types of problems.

We begin with an extension of the set less order relation given in Definition 5.1 to the case of a variable order structure.

Definition 5.3. Let Assumption 3.2 be satisfied. For nonempty subsets $A, B \subset Y$ the *set less variable* order relation $\preccurlyeq_{s,\mathscr{K}}$ is defined by

$$A \preccurlyeq_{s,\mathscr{K}} B \ :\Longleftrightarrow\ B \subset \bigcup_{a\in A} \big(\{a\} + \mathscr{K}(a)\big) \text{ and } A \subset \bigcup_{b\in B} \big(\{b\} - \mathscr{K}(b)\big).$$

This definition was first given in [33, Definition 2.4] together with a variety of other variable order relations for sets (for a subsequent paper see also [34]). For convenience, we restrict ourselves to the notion given in Definition 5.3.

Proposition 5.3. *Let Assumption 3.2 be satisfied, and let ψ be a $-\mathscr{K}$ representing functional. Then, for arbitrary nonempty sets $A, B \subset Y$,*

$$B \subset \bigcup_{a\in A} \big(\{a\} + \mathscr{K}(a)\big) \left\{ \begin{array}{ll} \Longrightarrow & \sup_{b\in B} \inf_{a\in A} \psi(a-b,a) \leq 0 \\ \Longleftarrow & \sup_{b\in B} \min_{a\in A} \psi(a-b,a) \leq 0, \text{ \textit{if this} min \textit{term exists}} \end{array} \right.$$

and

$$A \subset \bigcup_{b\in B} \big(\{b\} - \mathscr{K}(b)\big) \left\{ \begin{array}{ll} \Longrightarrow & \sup_{a\in A} \inf_{b\in B} \psi(a-b,b) \leq 0 \\ \Longleftarrow & \sup_{a\in A} \min_{b\in B} \psi(a-b,b) \leq 0, \text{ \textit{if this} min \textit{term exists.}} \end{array} \right.$$

Proof. For arbitrary nonempty sets $A, B \subset Y$, it follows from Proposition 3.10, (b)

$$\begin{array}{lll} B \subset \bigcup_{a\in A} \big(\{a\} + \mathscr{K}(a)\big) & \Longleftrightarrow & \forall\, b \in B\ \exists\, a \in A:\ b \in \{a\} + \mathscr{K}(a) \\ & & \left\{ \begin{array}{ll} \Longrightarrow & \sup_{b\in B} \inf_{a\in A} \psi(a-b,a) \leq 0 \\ \Longleftarrow & \sup_{b\in B} \min_{a\in A} \psi(a-b,a) \leq 0, \text{ if this min term exists.} \end{array} \right. \end{array}$$

For the proof of the second characterization, we write

$$A \subset \bigcup_{b\in B} \big(\{b\} - \mathscr{K}(b)\big) \Longleftrightarrow \forall\, a \in A\ \exists\, b \in B:\ a \in \{b\} - \mathscr{K}(b),$$

and then we apply Proposition 3.10, (a). This completes the proof. □

This proposition was first given in the proof of Theorem 4.1 in [35] and subsequently in [34, Theorems 3 and 6] for special scalarizing functionals.

Proposition 5.3 immediately implies the following characterization of a set less variable inequality.

Proposition 5.4. *Let Assumption 3.2 be satisfied, and let ψ be a $-\mathscr{K}$ representing functional. Then for arbitrary nonempty sets $A,B \subset Y$ we have*

$$A \preccurlyeq_{s,\mathscr{K}} B \iff \max\left\{ \sup_{b\in B}\min_{a\in A} \psi(a-b,a),\ \sup_{a\in A}\min_{b\in B} \psi(a-b,b) \right\} \leq 0,$$

if the arising min *terms exist.*

It is obvious that these min terms exist, if the real linear space Y is normed, the sets A and B are weakly compact and the functional ψ is weakly lower semicontinuous.

We now investigate set optimization problems with a variable order structure. Based on the set less variable order relation in a real linear space Y we consider a family $\mathscr{F}$ of nonempty subsets of Y and we investigate minimal sets of $\mathscr{F}$ in a similar way as in Definition 5.2.

Definition 5.4. Let Assumption 3.2 be satisfied, and let $\mathscr{F}$ be a family of nonempty subsets of Y. A set $\bar{A} \in \mathscr{F}$ is called a *minimal* set of $\mathscr{F}$ iff

$$A \preccurlyeq_{s,\mathscr{K}} \bar{A},\ A \in \mathscr{F} \implies \bar{A} \preccurlyeq_{s,\mathscr{K}} A.$$

Minimal sets can be characterized using the equivalence in Proposition 5.4.

Theorem 5.1. *Let Assumption 3.2 be satisfied, and let ψ be a $-\mathscr{K}$ representing functional. Then we have*

$$\begin{aligned} &\bar{A} \in \mathscr{F} \text{ is a minimal set of } \mathscr{F} \\ \iff\ & \forall\, A \in \mathscr{F}:\ \max\left\{ \sup_{a\in A}\min_{\bar{a}\in\bar{A}} \psi(\bar{a}-a,\bar{a}),\ \sup_{\bar{a}\in\bar{A}}\min_{a\in A} \psi(\bar{a}-a,a) \right\} \leq 0 \text{ or} \\ & \qquad\qquad \max\left\{ \sup_{\bar{a}\in\bar{A}}\min_{a\in A} \psi(a-\bar{a},a),\ \sup_{a\in A}\min_{\bar{a}\in\bar{A}} \psi(a-\bar{a},\bar{a}) \right\} > 0, \end{aligned} \tag{5.2}$$

if the arising min *terms exist.*

Proof.

(a) Let $\bar{A} \in \mathscr{F}$ be a minimal set of $\mathscr{F}$, and let $A \in \mathscr{F}$ be arbitrarily chosen. Then we consider two cases:

(i) If $A \preccurlyeq_{s,\mathscr{K}} \bar{A}$, we then obtain $\bar{A} \preccurlyeq_{s,\mathscr{K}} A$, i.e. with Proposition 5.4

$$\max\left\{ \sup_{a\in A}\min_{\bar{a}\in\bar{A}} \psi(\bar{a}-a,\bar{a}),\ \sup_{\bar{a}\in\bar{A}}\min_{a\in A} \psi(\bar{a}-a,a) \right\} \leq 0.$$

(ii) If $A \not\preccurlyeq_{s,\mathscr{K}} \bar{A}$, we conclude with Proposition 5.4

$$\max\left\{ \sup_{\bar{a}\in\bar{A}}\min_{a\in A} \psi(a-\bar{a},a),\ \sup_{a\in A}\min_{\bar{a}\in\bar{A}} \psi(a-\bar{a},\bar{a}) \right\} > 0.$$

This shows the first part of the assertion.

(b) Let $\bar{A} \in \mathscr{F}$ be not a minimal set of $\mathscr{F}$. Then there exists some $A \in \mathscr{F}$ with $A \preccurlyeq_{s,\mathscr{K}} \bar{A}$, i.e. with Proposition 5.4

$$\max\left\{\sup_{\bar{a}\in\bar{A}}\min_{a\in A}\psi(a-\bar{a},a),\ \sup_{a\in A}\min_{\bar{a}\in\bar{A}}\psi(a-\bar{a},\bar{a})\right\} \leq 0,$$

and $\bar{A} \not\preccurlyeq_{s,\mathscr{K}} A$, i.e. with Proposition 5.4

$$\max\left\{\sup_{a\in A}\min_{\bar{a}\in\bar{A}}\psi(\bar{a}-a,\bar{a}),\ \sup_{\bar{a}\in\bar{A}}\min_{a\in A}\psi(\bar{a}-a,a)\right\} > 0.$$

Hence, the condition (5.2) is not fulfilled. □

Theorem 5.1 is only a slight extension of [35, Theorem 4.1, (iii)] to the set less variable order relation, and its proof follows the lines of the proof of [35, Theorem 4.1, (iii)] (bibliographical note: In [35] the exact definition of the function g is not used in the proof of [35, Theorem 4.1, (iii)] so that this result is more general and can simply be rewritten for a $-\mathscr{K}$ representing functional).

Conclusion

The introduced class of $-C$ representing functionals subsumes Bishop-Phelps functionals and various scalarizing functionals. Together with the class of $-\mathscr{K}$ representing functionals it can be used for the investigation of fixed and variable order structures, respectively. The $-C$ representing functionals have the significant advantage that one is not restricted to an explicit function rule, since only a property of the negative convex cone is used. This approach is more than unifying because functionals, such as the penalty function in Example 3.1, can be chosen that have not previously been used for order structures. Representing functionals are defined in such a way that one has the right toolbox for studying vector and set optimization problems.

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