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FENCHEL SUBDIFFERENTIAL OPERATORS: A CHARACTERIZATION WITHOUT CYCLIC MONOTONICITY

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Abstract. Fenchel subdifferential operators of lower semicontinuous proper convex functions on real Banach spaces are classically characterized as those operators that are maximally cyclically monotone or, equivalently, maximally monotone and cyclically monotone. This paper presents an alternative characterization, which does not involve cyclic monotonicity. In the case of subdifferential operators of sublinear functions, the new characterization substantially simplifies. Dually, the new characterization of normal cone operators is very simple, too.

Keywords. Fenchel subdifferential; Monotone operator; Normal cone, Sublinear function; Subdifferential operators.

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1. Introduction

The Fenchel subdifferential is arguably the most fundamental notion in convex analysis. The Fenchel subdifferential operator of a functional $f : X \to \mathbb{R} \cup \{+\infty\}$ defined on a real Banach space X is

$$\partial f : X \rightrightarrows X^*$$
$$\partial f(x) := \{x^* \in X^* : f(y) \geq f(x) + \langle y - x, x^* \rangle \ \forall y \in X\}.$$

Here and in the sequel, X^* is the dual space of X and $\langle \cdot, \cdot \rangle : X \times X^* \to \mathbb{R}$ denotes the duality product, that is, $\langle x, x^* \rangle$ means the value of the continuous linear functional $x^* \in X^*$ at $x \in X$. As is well known and easy to prove, ∂f is cyclically monotone. Recall that a set-valued operator $T : X \rightrightarrows X^*$ is said to be cyclically monotone if

$$\sum_{i=0}^{k} \langle x_i - x_{i+1}, x_i^* \rangle \geq 0 \qquad \text{for every } (x_i, x_i^*) \in T \quad (i = 0, 1, \ldots, k), \tag{1.1}$$

with $k \geq 1$ arbitrary and $x_{k+1} := x_0$.

Here and throughout the whole paper, operators are identified with their graphs, so that $(x, x^*) \in T$ means $x^* \in T(x)$. Every cyclically monotone operator is monotone since monotonicity corresponds to the case when $k = 1$ in (1.1). Cyclic monotonicity is closely connected to subdifferential operators as Rockafellar [1, Theorem 1] proved that, given $T : X \rightrightarrows X^*$, in order that

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there exist a proper convex functional $f : X \to \mathbb{R} \cup \{+\infty\}$ such that $T \subseteq \partial f$, it is necessary and sufficient that T is cyclically monotone. Consequently, if T is maximally (cyclically) monotone, which means that T is (cyclically) monotone and not properly contained in any other (cyclically) monotone operator, then $T = \partial f$. On the other hand, subdifferential operators of lower semicontinuous (l.s.c., in brief) proper convex functionals are maximally monotone [2, Theorem A]. Therefore, one concludes that $T : X \rightrightarrows X^*$ is the subdifferential operator of some l.s.c. proper convex functional if and only if it is maximally (cyclically) monotone [2, Theorem B]. An extension of this characterization to suitably defined subdifferentials of convex operators was obtained by Kusraev [3].

The aim of this paper is to obtain an alternative characterization of subdifferential operators not involving cyclic monotonicity. This is achieved in Theorem 3.1. However, as one may expect, the new characterization is not as simple and elegant as the one in [2, Theorem B]. It still involves maximal monotonicity, but the somewhat complicated conditions i) - iii) of Proposition 3.1, which replace cyclic monotonicity, make the new characterization less attractive than the classical one. By sharp contrast, in the case of subdifferential operators of sublinear functionals, the new characterization, which does not involve cyclic monotonicity either, is extremely simple and has a very easy proof. Furthermore, since normal cones of closed convex sets are the subdifferentials of their indicator functionals and the latter functionals are the conjugates of the corresponding support functionals, which characterize sublinear functionals, one easily obtains a simple characterization of normal cone operators (Theorem 2.1), because subdifferentials of mutually conjugate functionals are inverse to each other.

The rest of this paper is structured as follows. Section 2 contains characterizations of normal cone operators and subdifferential operators of l.s.c. proper sublinear functionals, and Section 3 characterizes subdifferential operators of general l.s.c. proper convex functionals.

The notation and terminology used in the paper is mostly standard, but it is explained here for the reader's convenience. The zero elements in X and X^* are denoted 0_X and 0_{X^*}, respectively. The projection of $X \times X^*$ onto X^* is

$$\Pi_{X^*} : X \times X^* \to X^*$$
$$\Pi_{X^*}(x, x^*) := x^*.$$

The bidual space of X is the dual X^{**} of X^*. The restriction of a functional $g : X^{**} \to \mathbb{R} \cup \{+\infty\}$ to X (canonically identified with a subset of X^{**}) is denoted $g_{|X}$. The domain and the range of an operator $T : X \rightrightarrows X^*$ are $dom\, T := \{x \in X : T(x) \neq \emptyset\}$ and $range\, T := \cup_{x \in X} T(x)$, respectively. The inverse operator of T is

$$T^{-1} : X^* \rightrightarrows X$$
$$T^{-1}(x^*) := \{x \in X : x^* \in T(x)\}.$$

The closure and the convex hull of a subset C of a real Banach space X are denoted $cl\, C$ and $conv\, C$, repectively. Its barrier cone, its recession cone, and its indicator functional are

$$barr(C) := \left\{ x^* \in X^* : \sup_{x \in C} \langle x, x^* \rangle < +\infty \right\},$$

$$0^+(C) := \{d \in X : C + \mathbb{R}_+ d = C\},$$

and

$$\delta_C : X \to \mathbb{R} \cup \{+\infty\}$$
$$\delta_C(x) := \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{if } x \notin C, \end{cases}$$

respectively. The normal cone operator to C is $N_C := \partial \delta_C$. If $C \neq \emptyset$, its support functional is

$$\sigma_C : X \to \mathbb{R} \cup \{+\infty\}$$
$$\sigma_C(x^*) := \sup_{x \in C} \langle x, x^* \rangle .$$

In the case when X is the dual of another real Banach space Y, the support functional σ_C is defined on the bidual Y^{**} since $X^* = Y^{**}$ in such a case. In the same way, in such a situation, $\partial \sigma_C$ is a set-valued operator from X^* into X^{**}. The epigraph of a functional $f : X \to \mathbb{R} \cup \{+\infty\}$ is the set

$$epi\ f := \{(x, \alpha) \in X \times \mathbb{R} : f(x) \leq \alpha\} .$$

A functional $s : X \to \mathbb{R} \cup \{+\infty\}$ is said to be sublinear if it is convex and positively homogeneous, the latter property meaning that for $x \in s^{-1}(\mathbb{R})$ and $\lambda \geq 0$ one has $s(\lambda x) = \lambda s(x)$. Clearly, if s is proper, then $s(0) = 0$.

The classical reference on convexity in finite dimension is Rockafellar's book [4]. Convexity in Banach spaces has been the subject of many excellent monographs, including [5, 6], and the very recent [8]; the latter two books also considered functionals defined on locally convex real topological vector spaces. Concerning monotonicity and its close relationship with convexity, the interested reader may consult, for instance, [7, 9] and, for operators defined on Hilbert spaces, the more recent [10].

2. Normal Cone Operators of Closed Convex Sets and Subdifferentials of Sublinear Functionals

This section contains new and simple characterizations of normal cone operators of closed convex sets and subdifferential operators of l.s.c. sublinear functionals. The first result gives a simple sufficient condition for a monotone operator to be contained in the normal cone operator of some closed convex set.

Proposition 2.1. *If $T : X \rightrightarrows X^*$ is monotone and $0_{X^*} \in \cap_{x \in dom\ T} T(x)$. Then*

$$T \subseteq N_{cl\ conv\ dom\ T}. \tag{2.1}$$

Proof. Let $(x, x^*) \in T$. For every $y \in dom\ T$, we have $0_{X^*} \in T(y)$. Hence, by the monotonicity of T, we have $\langle y - x, x^* \rangle \leq 0$. Thus,

$$dom\ T \subseteq \{y \in X : \langle y - x, x^* \rangle \leq 0\} . \tag{2.2}$$

Since the right hand side in (2.2) is a closed convex set, it immediately follows that

$$cl\ conv\ dom\ T \subseteq \{y \in X : \langle y - x, x^* \rangle \leq 0\} ,$$

which, in view of $x \in dom\ T \subseteq cl\ conv\ dom\ T$, implies that $x^* \in N_{cl\ conv\ dom\ T}(x)$, that is, $(x, x^*) \in N_{cl\ conv\ dom\ T}$. This proves (2.1). □

Corollary 2.1. *If $T : X \rightrightarrows X^*$ is monotone and range $T = T(0_X)$, then*

$$T \subseteq \partial \left(\sigma_{cl\ conv\ T(0_X)}\right)_{|X}. \tag{2.3}$$

Proof. The monotonicity of T is equivalent to that of T^{-1}, and the assumption *range* $T = T(0_X)$ is equivalent to the inclusion $0_X \in \bigcap_{x^* \in dom\ T^{-1}} T^{-1}(x^*)$. Hence, using that

$$dom\ T^{-1} = range\ T = T(0_X),$$

Proposition 2.1, applied to T^{-1} (regarded as a set-valued operator into X^{**}), yields

$$T^{-1} \subseteq N_{cl\ conv\ T(0_X)} = \partial \delta_{cl\ conv\ T(0_X)}.$$

Therefore, for every $(x, x^*) \in T$, we have $x \in T^{-1}(x^*) \subseteq \partial \delta_{cl\ conv\ T(0_X)}(x^*)$. Thus, since $x \in X$, we obtain

$$x^* \in \left(\partial \delta_{cl\ conv\ T(0_X)}\right)^{-1}(x) \subseteq \partial \left(\sigma_{cl\ conv\ T(0_X)}\right)_{|X}(x).$$

This proves (2.3). □

Corollary 2.1 is to be compared to [11, Theorem 1], which establishes that a correspondence $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ (interpreted as assigning to each price vector p a set of possible production plans $T(p)$) is consistent with profit maximization behavior, that is, there exists a convex closed production set $Y \subseteq \mathbb{R}^n$ such that for every price vector $p \in \mathbb{R}^n$ each supply decision $z \in T(p)$ maximizes the scalar product $p \cdot y$ (i.e., the profit of producing y under the given prices) subject to $y \in Y$, if it satisfies the law of supply (i.e., it is monotone) and is positively homogeneous of degree 0 (i.e., $T(\lambda p) = T(p)$ for every $p \in \mathbb{R}^n$ and $\lambda > 0$). Corollary 2.1 is simpler, as it does not require the homogeneity condition; in its place, it has the assumption *range* $T = T(0_X)$, which would be an immediate consequence of positive homogeneity of degree 0 if imposing the mild extra hypothesis of T^{-1} being closed-valued.

Theorem 2.1. *Let $T : X \rightrightarrows X^*$. There exists a nonempty closed convex set $C \subseteq X$ such that $T = N_C$ if and only if T is maximally monotone and $0_{X^*} \in \cap_{x \in dom\ T} T(x)$.*

Proof. The "only if" statement is immediate. The "if statement" follows from Proposition 2.1, since $N_{cl\ conv\ dom\ T}$ is monotone. □

The following theorem is related to [11, Theorem 2] in a similar way as Corollary 2.1 is related to [11, Theorem 1],

Theorem 2.2. *Let $T : X \rightrightarrows X^*$. There exists an l.s.c. proper sublinear functional $s : X \to \mathbb{R} \cup \{+\infty\}$ such that $T = \partial s$ if and only if T is maximally monotone and range $T = T(0_X)$.*

Proof. The "only if" statement is immediate. The "if statement" follows from Corollary 2.1, since $\partial \left(\sigma_{cl\ conv\ T(0_X)}\right)_{|X}$ is monotone. □

3. General Subdifferential Operators

To a given operator $A : X \times \mathbb{R} \rightrightarrows X^* \times \mathbb{R}$, we associate another operator $A_X : X \rightrightarrows X^*$, defined by

$$A_X(x) := \Pi_{X^*}\left(\left(\bigcup_{\lambda\in\mathbb{R}} A(x,\lambda)\right) \cap (X^* \times \{-1\})\right).$$

This section begins with two simple lemmas.

Lemma 3.1. *Let $C \subseteq X \times \mathbb{R}$. There exists an l.s.c. proper convex functional $f : X \to \mathbb{R}\cup\{+\infty\}$ such that*

$$C = epi\ f, \tag{3.1}$$

if and only if the following conditions hold:
i) C is nonempty, convex and closed,
ii) $(barr(C)) \cap (X^* \times \{-1\}) \neq \emptyset$,
iii) $(0_X, 1) \in 0^+(C)$.

Proof. Only if. Conditions i) and iii) are immediate. Condition ii) follows from the fact that $(x^*, -1) \in barr(C)$ for every continuous affine minorant $\langle \cdot, x^* \rangle + b$ of f.

If. Define $f : X \to \mathbb{R}\cup\{+\infty, -\infty\}$ by

$$f(x) := \inf\{\lambda \in \mathbb{R} : (x,\lambda) \in C\}.$$

From i) and ii), it easily follows that f is minorized by a continuous affine functional. Hence $f(x) > -\infty$ for every $x \in X$. It is also clear that $C \subseteq epi\ f$, which, since $C \neq \emptyset$, implies that f is proper. To see that the opposite inclusion also holds, let $(x,\lambda) \in epi\ f$. Then, for every $\mu > \lambda$, there exists $\lambda' < \mu$ such that $(x,\lambda') \in C$; hence, by iii), $(x,\mu) \in C$. Letting $\mu \to \lambda^+$, we obtain that $(x,\lambda) \in C$ since C is closed according to i). We have thus proved (3.1). □

Lemma 3.2. *Let $f : X \to \mathbb{R}\cup\{+\infty\}$ and $(x,\lambda) \in X \times \mathbb{R}$. If $(x^*, -1) \in N_{epi\ f}(x,\lambda)$, then $\lambda = f(x)$.*

Proof. If $(x^*, -1) \in N_{epi\ f}(x,\lambda)$, then $(x,\lambda) \in epi\ f$ since otherwise $N_{epi\ f}(x,\lambda)$ would be empty. It follows that $f(x) \leq \lambda < +\infty$, and hence $(x, f(x)) \in epi\ f$. This inclusion, together with $(x^*, -1) \in N_{epi\ f}(x,\lambda)$, yields

$$\langle (x, f(x)) - (x,\lambda), (x^*, -1) \rangle \leq 0,$$

which simply means that $\lambda \leq f(x)$, thus proving that $\lambda = f(x)$. □

The following corollary is an easy consequence of Lemma 3.2.

Corollary 3.1. *If $f : X \to \mathbb{R}\cup\{+\infty\}$ is convex and l.s.c., then $\left(N_{epi\ f}\right)_X = \partial f$.*

The next result gives sufficient conditions for the operator $A_X : X \rightrightarrows X^*$ induced by a monotone operator $A : X \times \mathbb{R} \rightrightarrows X^* \times \mathbb{R}$ to be included in the subdifferential operator of an l.s.c. convex functional.

Proposition 3.1. *If $A : X \times \mathbb{R} \rightrightarrows X^* \times \mathbb{R}$ is monotone and satisfies:*
i) $(barr(cl\ conv\ dom\ A)) \cap (X^* \times \{-1\}) \neq \emptyset$,
ii) $(0_X, 1) \in 0^+(cl\ conv\ dom\ A)$ and
iii) $(0_{X^*}, 0) \in \cap_{(x,\lambda)\in dom\ A} A(x,\lambda)$,

then the functional $f : X \to \mathbb{R} \cup \{+\infty\}$ *given by epi* $f = $ *cl conv dom* A *is well defined and satisfies*

$$A_X \subseteq \partial f. \tag{3.2}$$

Proof. By Lemma 3.1, f is indeed well defined. Since A is monotone, by iii), Proposition 2.1 and (3.1), we have

$$A \subseteq N_{epi\ f}. \tag{3.3}$$

Let $(x, x^*) \in A_X$. Then, by (3.3) and Lemma 3.2, we have

$$(x^*, -1) \in \bigcup_{\lambda \in \mathbb{R}} N_{epi\ f}(x, \lambda) = N_{epi\ f}(x, f(x)),$$

and hence $x^* \in \partial f(x)$. This proves (3.2). □

The following result is the main one in this paper. It characterizes subdifferential operators of general l.s.c. proper convex functionals within the class of maximally monotone operators. Unlike the classical characterization [2, Theorem B], the new one does not involve cyclic monotonicity.

Theorem 3.1. *Let* $T : X \rightrightarrows X^*$. *There exists an l.s.c. proper convex functional* $f : X \to \mathbb{R} \cup \{+\infty\}$ *such that* $T = \partial f$ *if and only if* T *is maximally monotone and there exists a monotone operator* $A : X \times \mathbb{R} \rightrightarrows X^* \times \mathbb{R}$ *satisfying conditions i) - iii) of Proposition 3.1 such that* $T = A_X$.

Proof. To prove the "only if" statement, take $A := N_{epi\ f}$. Since *dom* $N_{epi\ f} = $ *epi* f and *epi* f is convex and closed, conditions i) and ii) follow from Lemma 3.1, whereas iii) is immediate. Moreover, by Corollary 3.1, we have $\partial f = A_X$.

The "if statement" is an immediate consequence of Proposition 3.1 since ∂f is monotone. □

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