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INERTIAL VISCOSITY WITH ALTERNATIVE REGULARIZATION FOR CERTAIN OPTIMIZATION AND FIXED POINT PROBLEMS

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Abstract. In this paper, we study the problem of finding a common solution of split equilibrium, variational inclusion, and fixed point problems in real Hilbert spaces. We propose two inertial viscosity algorithms to solve the problem and obtain two strong convergence theorems. The assumption of the upper semi-continuity of the bifunction in the split equilibrium problem is dispensed. We apply our results to the common solution of variational inequality, convex minimization, and fixed point problems. Finally, we give a numerical experiment to illustrate the performance of our algorithms and compare them with existing algorithms.

Keywords. Alternative regularization; Inertial viscosity method; Nonexpansive mapping; Split equilibrium problem, Variational inclusion problem.

1. Introduction

Throughout this paper, unless otherwise stated, H_1 and H_2 are assumed to be real Hilbert spaces with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$, C and Q are assumed to be nonempty, closed, and convex subsets of H_1 and H_2, respectively, and $A : H_1 \to H_2$ is assumed to be a bounded linear operator.

Let $F_1 : C \times C \to \mathbb{R}$ be a bifunction. The *Equilibrium Problem* (shortly, EP) in the sense of Blum and Oettli [1] is to find $x^* \in C$ such that $F_1(x^*,x) \geq 0$, $\forall x \in C$. The set of solutions of the EP is denoted by $EP(F_1,C)$. The EP attracts considerable research efforts and serves as a unified framework for many real problems; see, e.g., [2, 3, 4, 5, 6, 7] and the references therein.

Let $F_2 : Q \times Q \to \mathbb{R}$ be a bifunction. Recently, Kazmi and Rizvi [8] introduced the following *Split Equilibrium Problem*, (shortly, SEP): Find $x^* \in C$ such that

$$F_1(x^*,x) \geq 0,\ \forall\, x \in C \text{ and such that } y^* = Ax^* \text{ solves } F_2(y^*,y) \geq 0,\ \forall\, y \in Q. \tag{1.1}$$

The SEP consists of a pair of EPs such that the set of solutions of one is the image of the other under a bounded linear operator. We denote the solution set of SEP (1.1) by $SEP(F_1,F_2)$.

Let $B : H_1 \to H_1$ be an operator and $D : H_1 \to 2^{H_1}$ be a set-valued operator. The *Variational Inclusion Problem* is defined as find $x^* \in H_1$ such that

$$0 \in Bx^* + Dx^*. \tag{1.2}$$

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It is known that many practical problems arising in areas, such as image recovery, signal processing, and linear inverse problems can be modeled mathematically in the variational inclusion problem. For more details on the variational inclusion problem and existing solution methods, we refer to [9, 10, 11, 12, 13] and the references therein.

In [14], Cholamjiak et al. studied the problem of finding a common element of the set of solutions of SEP (1.1) and variational inclusion problem (1.2) in real Hilbert spaces. They proposed the following algorithm: For $x_0, x_1 \in H_1$, let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated as follows:

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n = \alpha_n y_n + (1-\alpha_n)T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)y_n, \\ x_{n+1} = \beta_n z_n + (1-\beta_n)(I + s_n D)^{-1}(I - s_n B)z_n, n \geq 1. \end{cases} \tag{1.3}$$

Noted that the step size γ in Algorithm (1.3) depends on a prior knowledge of the operator norm, which is difficult or impossible to calculate. In addition, the algorithm only guaranteed the weak convergence. In order to obtain strong convergence, they proposed the following modified inertial shrinking projection algorithm

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n = \alpha_n y_n + (1-\alpha_n)T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)y_n, \\ w_n = \beta_n z_n + (1-\beta_n)(I + s_n D)^{-1}(I - s_n B)z_n, \\ C_{n+1} = \{z \in C_n : \|w_n - z\|^2 \leq \|x_n - z\|^2 + 2\theta_n^2\|x_n - x_{n+1}\|^2 - 2\theta_n\langle x_n - z, x_{n-1} - x_n\rangle\}, \\ x_{n+1} = P_{C_{n+1}}x_1, n \geq 1. \end{cases} \tag{1.4}$$

The θ_n in Algorithms (1.3) and (1.4) is an inertial extrapolation factor. The term $\theta_n(x_n - x_{n-1})$ is called the inertial term. The idea of inertial extrapolation can be traced to Polyak [15], who proposed it as a discrete version of a second order time dynamical system to speed up convergence rate of smooth convex minimization problems. The main idea of this method is to make use of two previous iterates in order to update the next iterate, which results in speeding up the algorithm's convergence. Due to its tendency to accelerate convergence, inertial-type algorithms have attracted enormous attention of authors; see, e.g., [5, 16, 17, 18, 19]. Note that the implementation of Algorithm (1.4) requires the computation of the closed convex subset C_{n+1} and then the projection of the initial point x_1 onto C_{n+1} per iteration. A major drawback is that the structure of C_{n+1} may be very complicated in general, which makes it difficult to calculate the projection, if not impossible (see [20]). Attouch [21] in 1996 introduced the viscosity approximation method. This was further developed by Moudafi [22] and was used to find the fixed points of nonexpansive mappings. Motivated by Moudafi [22] and Xu [23], Yang and He [24] proposed a general alternative regularization method for approximating the fixed point of nonexpansive mappings in Hilbert spaces.

Inspired by the works of Cholamjiak et al. [14], Yang and He [24], and Luo et al. [25], in this paper, we propose two inertial viscosity type algorithms for finding the common element of the set of solutions of split equilibrium problem (1.1), variational inclusion problem 1.2, and fixed point problems of nonexpansive mappings in real Hilbert spaces. We prove two strong convergence theorems of the sequences generated by the algorithms without imposing upper semi-continuity on the equilibrium bifunction as in some existing works on SEP (see for example [14, 26, 27]). We also present some consequences of our strong convergence theorems and give an application. Finally, we show the usability and efficiency of our algorithms by giving a numerical example and compare with an existing result. The organization of the remaining part of this paper is as follows. In Section 2, we collect some useful definitions, notations, and lemmas. In Section 3, we present our algorithms and prove two strong convergence theorems.Some corollaries and an application are also given. In Section 4, we compare the performance of our algorithms with Algorithm 1.3 via a numerical example. Section 5 ends this paper.

2. PRELIMINARIES

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, C be a nonempty, closed, and convex subset of H, and let $I: H \to H$ be the identity mapping on H. We denote by '$x_n \rightharpoonup x$' and '$x_n \to x$', the weak and the strong convergence of $\{x_n\}$ to a point $x \in H$, respectively. An element $x \in H$ is called a fixed point of a mapping $T : H \to H$ if $x = Tx$. We denote the fixed point set of T by $\mathrm{Fix}(T)$. We recall the following definitions:

Definition 2.1. An operator $T : H \to H$ is said to be:

(i) Lipschitzian if there exists a constant $\beta > 0$ such that $\|Tx - Ty\| \le \beta\|x - y\|$, $\forall x, y \in H$. If $\beta \in [0, 1)$, then T is a contraction;

(ii) firmly nonexpansive if $\|Tx - Ty\|^2 + \|(I - T)x - (I - T)y\|^2 \le \|x - y\|^2$, $\forall x, y \in H$, or equivalently, $\|Tx - Ty\|^2 \le \langle x - y, Tx - Ty\rangle$, $\forall x, y \in H$;

(iii) nonexpansive if $\|Tx - Ty\| \le \|x - y\|$, $\forall x, y \in H$;

(iv) β-strongly monotone if there exists $\beta > 0$ such that $\langle Tx - Ty, x - y\rangle \ge \beta\|x - y\|^2$;

(v) β-inverse strongly monotone if there exists $\beta > 0$ such that $\langle Tx - Ty, x - y\rangle \ge \beta\|Tx - Ty\|^2$.

It is obvious that every firmly nonexpansive mapping is nonexpansive, and the set of fixed points of nonexpansive mappings is closed and convex. Let $h : C \to C$ be a nonlinear operator. The Variational Inequality Problem (shortly, VIP) is to find

$$x^* \in C \text{ such that } \langle h(x^*), x - x^*\rangle \ge 0\ , \forall\ x \in C.$$

Lemma 2.1. [28] *Let H be a real Hilbert space. Suppose that $h : H \to H$ is κ-Lipschitzian and β-strongly monotone over a closed convex subset $C \subset H$. Then, $\langle h(u^*), v - u^*\rangle \ge 0$, $\forall v \in C$ has its unique solution $u^* \in C$.*

Lemma 2.2. [29] *(Demiclosedness Principle) Suppose that $T : H \to H$ is a nonexpansive mapping. Let $\{x_n\}$ be a vector sequence in H and let p be a vector in H. If $x_n \rightharpoonup p$ and $x_n - Tx_n \to 0$, then $p = Tp$.*

Lemma 2.3. [30] *Let H be a real Hilbert space. Then the following assertions hold:*

(i) $\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$ for all $x, y \in H$;

(ii) for all $x_i \in H$ and $\alpha_i \in [0, 1]$ $(i = 1, 2, \ldots, n)$ such that $\sum_{i=1}^n \alpha_i = 1$, the following equality holds: $\|\sum_{i=1}^n \alpha_i x_i\|^2 = \sum_{i=1}^n \alpha_i\|x_i\|^2 - \sum_{1\le i<j\le n} \alpha_i\alpha_j\|x_i - x_j\|^2$.

Let $D : H \to 2^H$ be a set-valued operator. The graph of D denoted by $gr(D)$ is defined by $gr(D) = \{(x, u) \in H \times H : u \in Dx\}$. D is called a non-trivial operator if $gr(D) \ne \emptyset$. D is called a monotone operator if $\forall\ (x, u), (y, v) \in gr(D)$, $\langle x - y, u - v\rangle \ge 0$. D is said to be a maximal monotone operator if the graph of D is not a proper subset of the graph of any other monotone operator, or equivalently if its graph cannot be enlarged without destroying monotonicity. For a maximal monotone operator D, the resolvent of parameter $\lambda > 0$ is defined by

$J_\lambda^D := (I+\lambda D)^{-1} : H \to dom(D)$. It is known that $D^{-1}(0) = \text{Fix}(J_\lambda^D)$ for all $\lambda > 0$ and J_λ^D is firmly nonexpansive. Also, let $B : H \to H$ be an operator and defined the operator $J_\lambda^{D,B}$ by

$$J_\lambda^{D,B} := (I+\lambda D)^{-1}(I-\lambda B) = J_\lambda^D(I-\lambda B).$$

It is known that $x \in \text{Fix}(J_\lambda^{D,B})$ if and only if $x \in (B+D)^{-1}(0)$. In addition, $J_\lambda^{D,B}$ is nonexpansive; see, e.g., [31].

We make the following assumptions on the bifunction $F : C \times C \to \mathbb{R}$:

Assumption 2.1. Let C be a nonempty, closed, and convex subset of a Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the following conditions:

(A1) $F(x,x) = 0$, for all $x \in C$;
(A2) $F(x,y) + F(y,x) \le 0$, for all $x, y \in C$;
(A3) for each $x, y, z \in C$, $\limsup\limits_{t\to 0} F(tz+(1-t)x, y) \le F(x,y)$;
(A4) for each $x \in C$, $F(x,\cdot)$ is convex and lower semicontinuous.

Lemma 2.4. [32] *Let C be nonempty, closed, and convex subset of a Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 2.1. For $r > 0$ and $x \in H$, define a mapping $T_r^F : H \to C$ by $T_r^F(x) = \{z \in C : F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \ge 0, \forall y \in C\}$ for all $x \in H$. Then,*

(1) *For each $x \in H$, $T_r^F \ne \emptyset$;*
(2) *T_r^F is single-valued;*
(3) *T_r^F is firmly nonexpansive;*
(4) *$Fix(T_r^F) = EP(F,C)$ is closed and convex.*

Lemma 2.5. [33] *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying the following relation: $s_{n+1} \le (1-t_n)s_n + t_n\rho_n$, $n \ge n_0$, where $\{t_n\} \subset (0,1)$ and $\{\rho_n\} \subset \mathbb{R}$ satisfying the following conditions: $\lim_{n\to\infty} t_n = 0$, $\sum_{n=1}^{\infty} t_n = \infty$, and $\limsup_{n\to\infty} \rho_n \le 0$. Then $s_n \to 0$ as $n \to \infty$.*

Lemma 2.6. [34] *Let $\{\Gamma_n\}$ be a real number sequence that never gets monotonically decreasing from a certain $n_0 \in \mathbb{N}$, in the sense that there exists a subsequence $\{\Gamma_{n_j}\}_{j\ge 0}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_j} < \Gamma_{n_j+1}$ for all $j \ge 0$. Also consider the sequence of integers $\{\tau(n)\}_{n\ge n_0}$ defined by $\tau(n) := \max\{k \le n \mid \Gamma_k < \Gamma_{k+1}\}$. Then $\{\tau(n)\}_{n\ge n_0}$ is a non-decreasing sequence verifying $\lim\limits_{n\to\infty} \tau(n) = \infty$, and, for all $n \ge n_0$, the following two estimates hold: $\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}$, $\Gamma_n \le \Gamma_{\tau(n)+1}$.*

3. Main Results

In this section, we prove two strong convergence theorems in real Hilbert spaces. We also provide some consequences of our results and an application.

3.1. **Algorithms and their convergence analysis.**

We first state some notations and assumptions that are needed in the sequel.

Assumption 3.1. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\tau_n\}$, $\{\lambda_n\}$, $\{\sigma_n\}$, $\{\mu_n\}$, $\{\kappa_n\}$, $\{s_n\}$, and $\{r_n\}$ satisfy the following assumptions:

(B1) $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ such that
  i) $0 < \liminf\limits_{n\to\infty} \alpha_n \le \limsup\limits_{n\to\infty} \alpha_n < 1$;

ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(B2) $\{\lambda_n\}, \{\sigma_n\}, \{\mu_n\} \subset (0,1)$ such that $0 < a \leq \sigma_n, \mu_n \leq b < 1$, $\lambda_n + \sigma_n + \mu_n = 1$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, and $\lim_{n\to\infty} \lambda_n = 0$;

(B3) For $\theta, \varepsilon > 0$, $\{\delta_n\}$ is a positive sequence satisfying $0 \leq \delta_n < \theta, \varepsilon$, and $\lim_{n\to\infty} \frac{\delta_n}{\lambda_n} = 0$;

(B4) $\{r_n\} \subset (0,\infty)$ such that $\liminf_{n\to\infty} r_n > 0$;

(B5) $0 < a \leq \kappa_n, \tau_n \leq b < 1$, $a, b \in \mathbb{R}$, and γ is any nonnegative real number;

(B6) $\{s_n\} \subset (0, 2\alpha)$ such that $0 < \liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n < 2\alpha$.

Algorithm 3.1. Parallel inertial viscosity-type algorithm.

Initialization: Let $x_0, x_1 \in H_1$ be arbitrary.

Iterative steps: Calculate x_{n+1} as follows:

Step 1. Given the iterates x_{n-1} and x_n $(n \geq 1)$, choose θ_n and ε_n such that $0 \leq \theta_n \leq \theta_n^*$ and $0 \leq \varepsilon_n \leq \varepsilon_n^*$, respectively, where

$$\theta_n^* = \begin{cases} \min\left\{\theta, \frac{\delta_n}{\|x_n - x_{n-1}\|}\right\} & \text{if } x_n \neq x_{n-1}, \\ \theta & \text{if otherwise,} \end{cases} \qquad \varepsilon_n^* = \begin{cases} \min\left\{\varepsilon, \frac{\delta_n}{\|x_n - x_{n-1}\|}\right\} & \text{if } x_n \neq x_{n-1}, \\ \varepsilon & \text{if otherwise.} \end{cases}$$

Step 2. Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}),\ u_n = x_n + \varepsilon_n(x_n - x_{n-1}).$$

Step 3. Compute

$$z_n = \alpha_n w_n + (1-\alpha_n) T_{r_n}^{F_1}(I - \gamma_n A^*(I - T_{r_n}^{F_2})A) w_n,\ y_n = \beta_n u_n + (1-\beta_n) J_{s_n}^{D,B} u_n,$$

where

$$\gamma_n = \begin{cases} \frac{\tau_n \|(I - T_{r_n}^{F_2})Aw_n\|^2}{\|A^*(I - T_{r_n}^{F_2})Aw_n\|^2} & \text{if } Aw_n \neq T_{r_n}^{F_2} Aw_n, \\ \gamma & \text{if otherwise.} \end{cases}$$

Step 4: Compute

$$x_{n+1} = S_n(\lambda_n f(x_n) + \sigma_n z_n + \mu_n y_n),$$

where

$$S_n = (1-\kappa_n)I + \kappa_n S.$$

Set $n := n+1$ and go to **Step 1**.

Remark 3.1. From Step 1, it can be deduced that $\lim_{n\to\infty} \theta_n \|x_n - x_{n-1}\| = 0$ and $\lim_{n\to\infty} \frac{\theta_n}{\lambda_n} \|x_n - x_{n-1}\| = 0$. Similarly, $\lim_{n\to\infty} \varepsilon_n \|x_n - x_{n-1}\| = 0$ and $\lim_{n\to\infty} \frac{\varepsilon_n}{\lambda_n} \|x_n - x_{n-1}\| = 0$.

Theorem 3.2. *Let H_1, H_2 be real Hilbert spaces, and let $C \subset H_1$, $Q \subset H_2$ be nonempty, closed, and convex subsets. Let $S : H_1 \to H_1$ be a nonexpansive mapping, and let $f : H_1 \to H_1$ be a contraction mapping with contraction coefficient $\nu \in [0, \frac{1}{\sqrt{2}})$. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^*, and let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumption 2.1. Let $B : H_1 \to H_1$ be an α-inverse strongly monotone operator, and let $D : H_1 \to 2^{H_1}$ be a maximal monotone operator. Denote $\Omega := Fix(S) \cap (B+D)^{-1}(0) \cap SEP(F_1, F_2)$, and assume $\Omega \neq \emptyset$. Let $\{x_n\}$ be generated by Algorithm 3.1 and Assumption 3.1* *hold. Then, $\{x_n\}$ converges strongly to $x^* \in \Omega$, where x^* is the unique solution to VIP (3.1): Find $p \in \Omega$ such that*

$$\langle (I-f)p, x-p\rangle \geq 0 \;\forall\, x \in \Omega. \tag{3.1}$$

Proof. We divide this proof into three steps as follows.

Step 1: The sequence $\{x_n\}$ is bounded.

We first show that $\mathrm{Fix}(S) = \mathrm{Fix}(S_n)$ for each $n \in \mathbb{N}$. Let $\bar{x} \in Fix(S)$. Then $S_n\bar{x} = (1-\kappa_n)\bar{x} + \kappa_n S\bar{x} = \bar{x}$. Also S_n is nonexpansive. Indeed, $\|S_nx - S_ny\| \leq (1-\kappa_n)\|x-y\| + \kappa_n\|Sx-Sy\| \leq \|x-y\|$. Let $p \in \Omega$. Then $p \in \mathrm{Fix}(S) = \mathrm{Fix}(S_n)$, $T_{r_n}^{F_1}p = p$, $T_{r_n}^{F_2}Ap = Ap$, and $p \in \mathrm{Fix}(J_{s_n}^{D,B})$. Since $T_{r_n}^{F_2}$ is nonexpansive, we find from Lemma 2.4 and Lemma 2.3(i) that

$$\begin{aligned}
2\langle Ap - Aw_n, (I-T_{r_n}^{F_2})Aw_n\rangle &= \|Ap - T_{r_n}^{F_2}Aw_n\|^2 - \|Ap - Aw_n\|^2 - \|(I-T_{r_n}^{F_2})Aw_n\|^2\\
&\leq \|Ap - Aw_n\|^2 - \|Ap - Aw_n\|^2 - \|(I-T_{r_n}^{F_2})Aw_n\|^2\\
&= -\|(I-T_{r_n}^{F_2})Aw_n\|^2.
\end{aligned}$$

which together with Lemma 2.3(i), and the fact that $T_{r_n}^{F_1}$ is nonexpansive obtains that

$$\begin{aligned}
&\|T_{r_n}^{F_1}(I-\gamma_nA^*(I-T_{r_n}^{F_2})A)w_n - T_{r_n}^{F_1}p\|^2\\
&\leq \|(I-\gamma_nA^*(I-T_{r_n}^{F_2})A)w_n - p\|^2\\
&= \|w_n-p\|^2 + 2\gamma_n\langle Ap - Aw_n, (I-T_{r_n}^{F_2})Aw_n\rangle + \gamma_n^2\|A^*(I-T_{r_n}^{F_2})Aw_n\|^2\\
&= \|w_n-p\|^2 - \gamma_n\|(I-T_{r_n}^{F_2})Aw_n\|^2 + \gamma_n^2\|A^*(I-T_{r_n}^{F_2})Aw_n\|^2\\
&= \|w_n-p\|^2 - \frac{\tau_n(1-\tau_n)\|(I-T_{r_n}^{F_2})Aw_n\|^4}{\|A^*(I-T_{r_n}^{F_2})Aw_n\|^2}\\
&\leq \|w_n-p\|^2.
\end{aligned}$$

From Lemma 2.3(ii), we have

$$\begin{aligned}
\|z_n-p\|^2 &= \|\alpha_nw_n + (1-\alpha_n)T_{r_n}^{F_1}(I-\gamma_nA^*(I-T_{r_n}^{F_2})A)w_n - p\|^2\\
&= \alpha_n\|w_n-p\|^2 + (1-\alpha_n)\|T_{r_n}^{F_1}(I-\gamma_nA^*(I-T_{r_n}^{F_2})A)w_n - p\|^2\\
&\quad -\alpha_n(1-\alpha_n)\|T_{r_n}^{F_1}(I-\gamma_nA^*(I-T_{r_n}^{F_2})A)w_n - w_n\|^2\\
&\leq \|w_n-p\|^2 - \frac{(1-\alpha_n)\tau_n(1-\tau_n)\|(I-T_{r_n}^{F_2})Aw_n\|^4}{\|A^*(I-T_{r_n}^{F_2})Aw_n\|^2}\\
&\quad -\alpha_n(1-\alpha_n)\|T_{r_n}^{F_1}(I-\gamma_nA^*(I-T_{r_n}^{F_2})A)w_n - w_n\|^2 \qquad (3.2)\\
&\leq \|w_n-p\|^2. \qquad (3.3)
\end{aligned}$$

Furthermore, since $J_{s_n}^{D}$ is nonexpansive, Lemma 2.3 (ii) gives

$$\begin{aligned}
&\|y_n-p\|^2\\
&= \beta_n\|u_n-p\|^2 + (1-\beta_n)\|J_{s_n}^{D,B}u_n - p\|^2 - \beta_n(1-\beta_n)\|J_{s_n}^{D,B}u_n - u_n\|^2\\
&\leq \beta_n\|u_n-p\|^2 + (1-\beta_n)(\|u_n-p\|^2 - s_n(2\alpha - s_n)\|Bu_n - Bp\|^2) - \beta_n(1-\beta_n)\|J_{s_n}^{D,B}u_n - u_n\|^2\\
&\leq \|u_n-p\|^2 - \beta_n(1-\beta_n)\|J_{s_n}^{D,B}u_n - u_n\|^2 \qquad (3.4)\\
&\leq \|u_n-p\|^2. \qquad (3.5)
\end{aligned}$$

As $w_n = x_n + \theta_n(x_n - x_{n-1})$, we have $\|w_n - p\| \le \|x_n - p\| + \theta_n\|x_n - x_{n-1}\| \le \|x_n - p\| + \lambda_n M_1$, where $M_1 = \sup_{n\in\mathbb{N}}\left\{\frac{\theta_n}{\lambda_n}\|x_n - x_{n-1}\|\right\}$. Similarly, $\|u_n - p\| \le \|x_n - p\| + \lambda_n M_2$, where $M_2 = \sup_{n\in\mathbb{N}}\left\{\frac{\varepsilon_n}{\lambda_n}\|x_n - x_{n-1}\|\right\}$. Note that the existence of M_1 and M_2 follows from Condition (B2) and Remark 3.1. Let $M^* := \max\{M_1, M_2, \|f(p) - p\|\}$. Then, from (3.3) and (3.5), we obtain

$$\begin{aligned}
\|x_{n+1} - p\| &\le \lambda_n\|f(x_n) - p\| + \sigma_n\|z_n - p\| + \mu_n\|y_n - p\| \\
&\le \lambda_n\nu\|x_n - p\| + \lambda_n\|f(p) - p\| + \sigma_n\|z_n - p\| + \mu_n\|y_n - p\| \\
&\le (1 - \lambda_n + \lambda_n\nu)\|x_n - p\| + \sigma_n\lambda_n M_1 + \mu_n\lambda_n M_2 + \lambda_n\|f(p) - p\| \\
&\le (1 - \lambda_n(1-\nu))\|x_n - p\| + \lambda_n(1-\nu)\frac{2M^*}{1-\nu} \\
&\le \max\left\{\|x_n - p\|, \frac{2M^*}{1-\nu}\right\} \\
&\ \ \vdots \\
&\le \max\left\{\|x_0 - p\|, \frac{2M^*}{1-\nu}\right\}.
\end{aligned}$$

This from (3.6) that $\{\|x_n - p\|\}$ is bounded. Consequently, $\{x_n\}, \{w_n\}, \{u_n\}, \{z_n\}$, and $\{y_n\}$ are bounded.

Step 2: Let $M_3 = \sup\limits_{n\in\mathbb{N}}\{\theta_n\|x_n - x_{n-1}\|, \|x_n - p\|\}$, $M_4 = \sup\limits_{n\in\mathbb{N}}\{\varepsilon_n\|x_n - x_{n-1}\|, \|x_n - p\|\}$, and $b_n = \frac{\sigma_n}{\sigma_n+\mu_n}z_n + \frac{\mu_n}{\sigma_n+\mu_n}y_n$. Then, the following inequality holds:

$$\|x_{n+1} - p\|^2 \le (1-\rho_n)\|x_n - p\|^2 + \rho_n\xi_n, \tag{3.6}$$

where $\rho_n = \lambda_n(1 - (1+\lambda_n)\nu^2)$ and

$$\xi_n = \frac{3M_3\frac{\theta_n\sigma_n\|x_n - x_{n-1}\|}{\lambda_n} + 3M_4\frac{\varepsilon_n\mu_n\|x_n - x_{n-1}\|}{\lambda_n} + 2\lambda_n\|f(p) - p\|^2 + 2(1-\lambda_n)\langle f(p) - p, b_n - p\rangle}{1 - (1+\lambda_n)\nu^2}. \tag{3.7}$$

Indeed, let $b_n = \tilde{\sigma}_n z_n + \tilde{\mu}_n y_n$, where $\tilde{\sigma}_n = \frac{\sigma_n}{\sigma_n+\mu_n}$ and $\tilde{\mu}_n = \frac{\mu_n}{\sigma_n+\mu_n}$. From Lemma 2.3(ii), we have

$$\|b_n - p\|^2 = \tilde{\sigma}_n\|z_n - p\|^2 + \tilde{\mu}_n\|y_n - p\|^2 - \tilde{\sigma}_n\tilde{\mu}_n\|z_n - y_n\|^2. \tag{3.8}$$

Observe that

$$\|f(x_n) - p\|^2 \le (\|f(x_n) - f(p)\| + \|f(p) - p\|)^2 \le 2\nu^2\|x_n - p\|^2 + 2\|f(p) - p\|^2$$

and

$$\begin{aligned}
\langle f(x_n) - p, b_n - p\rangle &\le \|f(x_n) - f(p)\|\|b_n - p\| + \langle f(p) - p, b_n - p\rangle \\
&\le \nu\|x_n - p\|\|b_n - p\| + \langle f(p) - p, b_n - p\rangle \\
&\le \frac{1}{2}(\nu^2\|x_n - p\|^2 + \|b_n - p\|^2) + \langle f(p) - p, b_n - p\rangle.
\end{aligned}$$

We also have

$$\begin{aligned} x_{n+1} &= S_n\big(\lambda_n f(x_n)+(1-\lambda_n)(\frac{\sigma_n}{1-\lambda_n}z_n+\frac{\mu_n}{1-\lambda_n}y_n)\big) \\ &= S_n\big(\lambda_n f(x_n)+(1-\lambda_n)(\frac{\sigma_n}{\sigma_n+\mu_n}z_n+\frac{\mu_n}{\sigma_n+\mu_n}y_n)\big) \\ &= S_n(\lambda_n f(x_n)+(1-\lambda_n)b_n). \end{aligned} \tag{3.9}$$

From (3.8), (3.9), Lemma 2.3(i), and the fact that S_n is nonexpansive, we have

$$\begin{aligned} \|x_{n+1}-p\|^2 &\leq \|\lambda_n f(x_n)+(1-\lambda_n)b_n-p\|^2 \\ &= \lambda_n^2\|f(x_n)-p\|^2+(1-\lambda_n)^2\|b_n-p\|^2+2\lambda_n(1-\lambda_n)\langle f(x_n)-p,b_n-p\rangle \\ &\leq (\lambda_n\nu^2+\lambda_n^2\nu^2)\|x_n-p\|^2+(1-\lambda_n)\|b_n-p\|^2 \\ &\quad +\lambda_n\Big(2\lambda_n\|f(p)-p\|^2+2(1-\lambda_n)\langle f(p)-p,b_n-p\rangle\Big) \\ &\leq (\lambda_n\nu^2+\lambda_n^2\nu^2)\|x_n-p\|^2+\sigma_n\|z_n-p\|^2+\mu_n\|y_n-p\|^2-\frac{\sigma_n\mu_n}{\sigma_n+\mu_n}\|z_n-y_n\|^2 \\ &\quad +\lambda_n\Big(2\lambda_n\|f(p)-p\|^2+2(1-\lambda_n)\langle f(p)-p,b_n-p\rangle\Big). \end{aligned} \tag{3.10}$$

By Lemma 2.3(i), we obtain

$$\begin{aligned} \|w_n-p\|^2 &\leq \|x_n-p\|^2+\theta_n^2\|x_n-x_{n-1}\|^2+2\theta_n\|x_n-p\|\|x_n-x_{n-1}\| \\ &\leq \|x_n-p\|^2+3M_3\theta_n\|x_n-x_{n-1}\|. \end{aligned} \tag{3.11}$$

Similarly,

$$\|u_n-p\|^2 \leq \|x_n-p\|^2+3M_4\varepsilon_n\|x_n-x_{n-1}\|. \tag{3.12}$$

Substituting (3.3), (3.5), (3.11), and (3.12) in (3.10), we have

$$\begin{aligned} \|x_{n+1}-p\|^2 &\leq (\lambda_n\nu^2+\lambda_n^2\nu^2)\|x_n-p\|^2+\sigma_n(\|x_n-p\|^2+3M_3\theta_n\|x_n-x_{n-1}\|) \\ &\quad +\mu_n(\|x_n-p\|^2+3M_4\varepsilon_n\|x_n-x_{n-1}\|)-\frac{\sigma_n\mu_n}{\sigma_n+\mu_n}\|z_n-y_n\|^2 \\ &\quad +\lambda_n\Big(2\lambda_n\|f(p)-p\|^2+2(1-\lambda_n)\langle f(p)-p,b_n-p\rangle\Big) \\ &\leq (1-\lambda_n+\lambda_n\nu^2+\lambda_n^2\nu^2)\|x_n-p\|^2 \\ &\quad +\lambda_n\Big(3M_3\frac{\theta_n}{\lambda_n}\sigma_n\|x_n-x_{n-1}\|+3M_4\frac{\varepsilon_n}{\lambda_n}\mu_n\|x_n-x_{n-1}\|\Big) \\ &\quad +\lambda_n\Big(2\lambda_n\|f(p)-p\|^2+2(1-\lambda_n)\langle f(p)-p,b_n-p\rangle\Big) \\ &\leq (1-\rho_n)\|x_n-p\|^2+\rho_n\xi_n. \end{aligned}$$

Step 3: The sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to $x^*\in\Omega$, where x^* is the unique solution of the following VIP: Find $p\in\Omega$ such that

$$\langle (I-f)p,x-p\rangle \geq 0 \;\forall\, x\in\Omega.$$

Let $d_n = \lambda_n f(x_n) + \sigma_n z_n + \mu_n y_n$. Then by Lemma 2.3(ii) and the fact that S is nonexpansive, we have

$$\begin{aligned} \|x_{n+1}-p\|^2 &= (1-\kappa_n)\|d_n-p\|^2+\kappa_n\|Sd_n-Sp\|^2-\kappa_n(1-\kappa_n)\|Sd_n-d_n\|^2 \\ &\leq \lambda_n\|f(x_n)-p\|^2+\sigma_n\|z_n-p\|^2+\mu_n\|y_n-p\|^2-\lambda_n\sigma_n\|f(x_n)-z_n\|^2 \\ &\quad -\lambda_n\mu_n\|f(x_n)-y_n\|^2-\sigma_n\mu_n\|z_n-y_n\|^2-\kappa_n(1-\kappa_n)\|Sd_n-d_n\|^2. \end{aligned} \tag{3.13}$$

Substituting (3.2), (3.4), (3.11), and (3.12) into (3.13), we have

$$\begin{aligned} &\|x_{n+1}-p\|^2 \\ &\leq \lambda_n\|f(x_n)-p\|^2+\sigma_n\left(\|w_n-p\|^2-\frac{(1-\alpha_n)\tau_n(1-\tau_n)\|(I-T_{r_n}^{F_2})Aw_n\|^4}{\|A^*(I-T_{r_n}^{F_2})Aw_n\|^2}\right) \\ &\quad -\sigma_n\alpha_n(1-\alpha_n)\|T_{r_n}^{F_1}(I-\lambda_nA^*(I-T_{r_n}^{F_2})A)w_n-w_n\|^2 \\ &\quad +\mu_n(\|u_n-p\|^2-\beta_n(1-\beta_n)\|J_{s_n}^{D,B}u_n-u_n\|^2)-\lambda_n\sigma_n\|f(x_n)-z_n\|^2 \\ &\quad -\lambda_n\mu_n\|f(x_n)-y_n\|^2-\sigma_n\mu_n\|z_n-y_n\|^2-\kappa_n(1-\kappa_n)\|Sd_n-d_n\|^2 \\ &\leq (1-\lambda_n)\|x_n-p\|^2+\lambda_n\left(\|f(x_n)-p\|^2+3M_3\frac{\sigma_n\theta_n\|x_n-x_{n-1}\|}{\lambda_n}+3M_4\frac{\mu_n\varepsilon_n\|x_n-x_{n-1}\|}{\lambda_n}\right) \\ &\quad -\sigma_n\frac{(1-\alpha_n)\tau_n(1-\tau_n)\|(I-T_{r_n}^{F_2})Aw_n\|^4}{\|A^*(I-T_{r_n}^{F_2})Aw_n\|^2}-\sigma_n\alpha_n(1-\alpha_n)\|T_{r_n}^{F_1}(I-\lambda_nA^*(I-T_{r_n}^{F_2})A)w_n-w_n\|^2 \\ &\quad -\mu_n\beta_n(1-\beta_n)\|J_{s_n}^{D,B}u_n-u_n\|^2-\kappa_n(1-\kappa_n)\|Sd_n-d_n\|^2. \end{aligned} \tag{3.14}$$

We further divide this step into two cases.

<u>**Case I**</u>: Assume there exists some $n_0 \in \mathbb{N}$ such that $\{\|x_n-p\|^2\}$ is monotonically non-increasing for $n > n_0$. Since it is bounded, it implies that it is convergent. Consequently, we have $\|x_n-p\|^2-\|x_{n+1}-p\|^2 \to 0$ as $n \to \infty$. From (3.14), we obtain

$$\begin{aligned} &\sigma_n\frac{(1-\alpha_n)\tau_n(1-\tau_n)\|(I-T_{r_n}^{F_2})Aw_n\|^4}{\|A^*(I-T_{r_n}^{F_2})Aw_n\|^2}+\sigma_n\alpha_n(1-\alpha_n)\|T_{r_n}^{F_1}(I-\gamma_nA^*(I-T_{r_n}^{F_2})A)w_n-w_n\|^2 \\ &\quad +\mu_n\beta_n(1-\beta_n)\|J_{s_n}^{D,B}u_n-u_n\|^2+\kappa_n(1-\kappa_n)\|Sd_n-d_n\|^2 \\ &\leq \lambda_n\left(\|f(x_n)-p\|^2+3M_3\frac{\sigma_n\theta_n\|x_n-x_{n-1}\|}{\lambda_n}+3M_4\frac{\mu_n\varepsilon_n\|x_n-x_{n-1}\|}{\lambda_n}\right)(1-\lambda_n)\|x_n-p\|^2 \\ &\quad -\|x_{n+1}-p\|^2. \end{aligned} \tag{3.15}$$

Taking the limit of (3.15) and using Assumption 3.1, we obtain

$$\|T_{r_n}^{F_1}(I-\gamma_nA^*(I-T_{r_n}^{F_2})A)w_n-w_n\| \to 0 \text{ as } n \to \infty, \tag{3.16}$$

$$\|J_{s_n}^{D,B}u_n-u_n\| \to 0 \text{ as } n \to \infty, \tag{3.17}$$

$$\|Sd_n-d_n\| \to 0 \text{ as } n \to \infty, \tag{3.18}$$

and

$$\frac{\|(I-T_{r_n}^{F_2})Aw_n\|^2}{\|A^*(I-T_{r_n}^{F_2})Aw_n\|} \to 0 \text{ as } n \to \infty. \tag{3.19}$$

It then follows from (3.19) that

$$\|(I-T_{r_n}^{F_2})Aw_n\| \leq \|A^*\| \frac{\|(I-T_{r_n}^{F_2})Aw_n\|^2}{\|A^*(I-T_{r_n}^{F_2})Aw_n\|} \to 0 \text{ as } n \to \infty. \tag{3.20}$$

It also follows from Algorithm 3.1 and Remark 3.1 that

$$\|w_n - x_n\| \to 0 \text{ as } n \to \infty, \text{ and } \|u_n - x_n\| \to 0 \text{ as } n \to \infty. \tag{3.21}$$

In addition, using (3.16) and (3.17), respectively, we obtain

$$\|z_n - w_n\| = (1-\alpha_n)\|T_{r_n}^{F_1}(I-\gamma_n A^*(I-T_{r_n}^{F_2})A)w_n - w_n\| \to 0 \text{ as } n \to \infty, \tag{3.22}$$

and

$$\|y_n - u_n\| = (1-\beta_n)\|J_{s_n}^{D,B}u_n - u_n\| \to 0 \text{ as } n \to \infty. \tag{3.23}$$

From (3.21), (3.22), and (3.23), we obtain $\lim_{n\to\infty}\|y_n - x_n\| = \lim_{n\to\infty}\|z_n - x_n\| = 0$. It follows that $\|d_n - x_n\| \leq \lambda_n\|f(x_n) - x_n\| + \sigma_n\|z_n - x_n\| + \mu_n\|y_n - x_n\| \to 0$. In view of (3.18), we have

$$\begin{aligned} \|x_{n+1} - Sx_n\| &= \|(1-\kappa_n)(d_n - Sd_n) + Sd_n - Sx_n\| \\ &\leq (1-\kappa_n)\|d_n - Sd_n\| + \|Sd_n - Sx_n\| \\ &\leq (1-\kappa_n)\|d_n - Sd_n\| + \|d_n - x_n\| \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.24}$$

It is easy to see that

$$\|x_n - Sx_n\| \to 0 \text{ as } n \to \infty. \tag{3.25}$$

Hence, combining (3.24) and (3.25), we arrive at $\|x_{n+1} - x_n\| \leq \|x_{n+1} - Sx_n\| + \|Sx_n - x_n\| \to 0$ as $n \to \infty$.

Next, we prove that $\limsup_{n\to\infty} \xi_n \leq 0$. To achieve this, we choose a subsequence $\{x_{n_m}\}$ of $\{x_n\}$ such that

$$\lim_{m\to\infty}\langle (I-f)p, b_{n_m} - p\rangle = \limsup_{n\to\infty}\langle (I-f)p, b_n - p\rangle.$$

Since $\{x_{n_m}\}$ is bounded, then there exists a subsequence $\{x_{n_{m_k}}\}$ of $\{x_{n_m}\}$ such that $x_{n_{m_k}} \rightharpoonup \bar{x} \in H_1$ as $k \to \infty$. Without any loss of generality, we may assume that $x_{n_m} \rightharpoonup \bar{x} \in H_1$ as $m \to \infty$. Thus we can conclude from (3.21) that $w_{n_m} \rightharpoonup \bar{x}$ and $u_{n_m} \rightharpoonup \bar{x}$, as $m \to \infty$. Since A is a bounded linear operator, it implies that $Aw_{n_m} \rightharpoonup A\bar{x}$. Since S is nonexpansive, it follows then from (3.25) and Lemma 2.2 that $\bar{x} \in \text{Fix}(S)$. Similarly, by (3.17) and the fact that $J_{s_n}^{D,B}$ is nonexpansive, we have that $\bar{x} \in \text{Fix}(J_{s_{n_m}}^{D,B})$ and hence $\bar{x} \in (B+D)^{-1}(0)$. Replacing n with n_m in (3.20), we have $\lim_{m\to\infty}\|(I-T_{r_{n_m}}^{F_2})Aw_{n_m}\| = 0$. Since $T_{r_{n_m}}^{F_2}$ is nonexpansive, then $(I-T_{r_{n_m}}^{F_2})$ is demiclosed at 0. From Lemma 2.2, we have that $A\bar{x} \in \text{Fix}(T_{r_{n_m}}^{F_2})$, that is, $A\bar{x} \in EP(F_2, Q)$ by Lemma 2.4. Furthermore, letting $c_{n_m} = (I-\gamma_{n_m}A^*(I-T_{r_{n_m}}^{F_2})A)w_{n_m}$, we conclude from (3.20) that

$$\|c_{n_m} - w_{n_m}\| \leq \gamma_{n_m}\|A^*\|\|(I-T_{r_{n_m}}^{F_2})Aw_{n_m}\| \to 0$$

as $m \to \infty$. It then follows that $c_{n_m} \rightharpoonup \bar{x}$. Now using (3.16), we have

$$\|T_{r_{n_m}}^{F_1}c_{n_m} - c_{n_m}\| \leq \|T_{r_{n_m}}^{F_1}c_{n_m} - w_{n_m}\| + \|w_{n_m} - c_{n_m}\| \to 0$$

as $m \to \infty$. Since $T_{r_{n_m}}^{F_1}$ is nonexpansive and $(I-T_{r_{n_m}}^{F_1})$ is demiclosed at 0, we conclude from Lemma 2.2 that $\bar{x} \in \text{Fix}(T_{r_{n_m}}^{F_1})$, that is, $\bar{x} \in EP(F_1, C)$ by Lemma 2.4. So, we obtain that $F_1(\bar{x}, y) \geq 0$, $\forall y \in C$ and $F_2(A\bar{x}, q) \geq 0$ $\forall q \in Q$. Thus $\bar{x} \in \Omega := \text{Fix}(S) \cap (B+D)^{-1}(0) \cap SEP(F_1, F_2)$. It can be deduced from Lemma 2.4 that Ω is a closed and convex subset of H_1. In addition, since

f is a ν-contraction mapping, it holds that $I-f$ is $(1+\nu_1)$-Lipschitzian and $(1-\nu_1)$-strongly monotone; see [2]. It implies from Lemma 2.1 that the $VI(f,\Omega)$ (3.1) has a unique solution, say, $x^* \in \Omega$, i.e., $\langle (I-f)x^*, x-x^*\rangle \geq 0$ for all $x \in \Omega$. Indeed, since $\bar{x} \in \Omega$ and $x_{n_m} \rightharpoonup \bar{x}$, we obtain

$$\begin{aligned}
\limsup_{n\to\infty}\langle (I-f)x^*, b_n - x^*\rangle &= \lim_{m\to\infty}\langle (I-f)x^*, b_{n_m} - x^*\rangle \\
&= \lim_{m\to\infty}\langle (I-f)x^*, \tilde{\sigma_{n_m}} z_{n_m} + \tilde{\mu_{n_m}} y_{n_m} - x^*\rangle \\
&= \langle (I-f)x^*, \bar{x} - x^*\rangle \\
&\geq 0.
\end{aligned}$$

Replacing p with x^* in (3.7), and using Remark 3.1 and the fact that $\lim_{n\to\infty}\lambda_n = 0$, we obtain that $\limsup_{n\to\infty}\xi_n \leq 0$. By the definition of $\rho_n = \lambda_n(1-(1+\lambda_n)\nu^2)$ in Step 2 and the condition on λ_n, it holds that $\lim_{n\to\infty}\rho_n = 0$. In addition, since $\rho_n = \lambda_n(1-(1+\lambda_n)\nu^2) > (1-2\nu^2)\lambda_n$, we deduce that $\sum_{n=1}^{\infty}\rho_n = \infty$. By replacing p with x^* in (3.6) and invoking Lemma 2.5, we derive $\lim_{n\to\infty}\|x_n - x^*\| = 0$.

Case II: Assume $\{\|x_n - p\|\}$ is not monotonically decreasing. Define $\tau : \mathbb{N} \to \mathbb{N}$ by

$$\tau(n) := \max\{m \in \mathbb{N} | m \leq n, \|x_m - p\| \leq \|x_{m+1} - p\|\}$$

for all $n \geq n_0$ (for some n_0 large enough). Obviously, $\{\tau(n)\}$ is non-decreasing with $\lim_{n\to\infty}\tau(n) = \infty$ and

$$0 \leq \|x_{\tau(n)} - p\| \leq \|x_{\tau(n)+1} - p\|, \ \forall\, n \geq n_0. \tag{3.26}$$

Following the similar arguments as in Case I, we see that, as $n \to \infty$,

$$\|T^{F_1}_{r_{\tau(n)}} c_{\tau(n)} - c_{\tau(n)}\| \to 0, \ \|(I - T^{F_2}_{r_{\tau(n)}})Aw_{\tau(n)}\| \to 0, \ \|x_{\tau(n)} - Tx_{\tau(n)}\| \to 0,$$
$$\limsup_{n\to\infty}\langle (I-f)x^*, b_{\tau(n)} - x^*\rangle \geq 0, \text{ and } \lim_{n\to\infty}\xi_{\tau(n)}(x^*) \leq 0.$$

Using (3.6) and (3.26), and replacing p with x^*, we obtain

$$\begin{aligned}
0 &\leq (1-\rho_{\tau(n)})\|x_{\tau(n)} - x^*\| - \|x_{\tau(n)+1} - x^*\| + \rho_{\tau(n)}\xi_{\tau(n)}(x^*) \\
&\leq (1-\rho_{\tau(n)})\|x_{\tau(n)+1} - x^*\| - \|x_{\tau(n)+1} - x^*\| + \rho_{\tau(n)}\xi_{\tau(n)}(x^*) \\
&= -\rho_{\tau(n)}\|x_{\tau(n)+1} - x^*\| + \rho_{\tau(n)}\xi_{\tau(n)}(x^*).
\end{aligned}$$

which implies that $\|x_{\tau(n)+1} - x^*\| \leq \xi_{\tau(n)}(x^*)$. Thus $\|x_{\tau(n)+1} - x^*\| \to 0$ and consequently, $\|x_{\tau(n)} - x^*\| \to 0$. From Lemma 2.6, we have

$$0 \leq \|x_n - x^*\| \leq \max\{\|x_n - x^*\|, \|x_{\tau(n)} - x^*\|\} \leq \|x_{\tau(n)+1} - x^*\|. \tag{3.27}$$

Hence, we can infer from (3.27) that $\lim_{n\to\infty}\|x_n - x^*\| = 0$. Thus $\{x_n\}$ converges strongly to $x^* \in \Omega$. So in both cases, we have that $\{x_n\}$ converges strongly to $x^* \in \Omega$, where x^* is the unique solution to (3.1). □

We next present the following strong convergence theorem. The proof is similar to the proof of Theorem 3.2 and hence it is omitted.

Theorem 3.3. *Let H_1, H_2 be real Hilbert spaces, and let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty, closed, and convex subsets. Let $S : H_1 \to H_1$ be a nonexpansive mapping, and let $f : H_1 \to H_1$ be a contraction mapping with contraction coefficient $\nu \in [0, \frac{1}{\sqrt{2}})$. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^*, and let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be bifunctions* *satisfying Assumption 2.1. Let $B : H_1 \to H_1$ be an α-inverse strongly monotone operator, and let $D : H_1 \to 2^{H_1}$ be a maximal monotone operator. Denote $\Omega := Fix(S) \cap (B+D)^{-1}(0) \cap SEP(F_1, F_2)$ and assume $\Omega \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by Algorithm 3.2 and Assumption 3.1 hold. Then, $\{x_n\}$ converges strongly to $x^* \in \Omega$, where x^* is the unique solution to the following VIP (3.1): Find $p \in \Omega$ such that $\langle (I-f)p, x-p\rangle \geq 0$, $\forall x \in \Omega$.*

Algorithm 3.2. : Inertial viscosity-type algorithm.

Initialization: Let $x_0, x_1 \in H_1$ be arbitrary.
Iterative steps: Calculate x_{n+1} as follows:
Step 1. Given the iterates x_{n-1} and x_n $(n \geq 1)$, choose θ_n such that $0 \leq \theta_n \leq \theta_n^*$, where

$$\theta_n^* = \begin{cases} \min\{\theta, \frac{\delta_n}{\|x_n - x_{n-1}\|}\} & \text{if } x_n \neq x_{n-1}, \\ \theta & \text{if otherwise.} \end{cases}$$

Step 2. Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 3. Compute

$$z_n = \alpha_n w_n + (1-\alpha_n)T_{r_n}^{F_1}(I - \gamma_n A^*(I - T_{r_n}^{F_2})A)w_n,$$

where

$$\gamma_n = \begin{cases} \frac{\tau_n \|(I - T_{r_n}^{F_2})Aw_n\|^2}{\|A^*(I - T_{r_n}^{F_2})Aw_n\|^2} & \text{if } Aw_n \neq T_{r_n}^{F_2}Aw_n, \\ \gamma & \text{if otherwise.} \end{cases}$$

Step 4 Compute

$$y_n = \beta_n z_n + (1-\beta_n)J_{s_n}^{D,B} z_n.$$

Step 5: Compute

$$x_{n+1} = S_n(\lambda_n f(x_n) + (1-\lambda_n)y_n),$$

where

$$S_n = (1-\kappa_n)I + \kappa_n S.$$

Set $n := n+1$ and go to **Step 1**.

3.2. **Some subresults.**

Corollary 3.1. *Let H_1, H_2 be real Hilbert spaces, and let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty, closed, and convex subsets. Let $f : H_1 \to H_1$ be a contraction mapping with contraction coefficient $\nu \in [0, \frac{1}{\sqrt{2}})$. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^*, and let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumption 2.1. Let $B : H_1 \to H_1$ be an α-inverse strongly monotone operator, and let $D : H_1 \to 2^{H_1}$ be a maximal monotone operator. Denote $\Omega_2 := (B+D)^{-1}(0) \cap SEP(F_1, F_2)$ and assume $\Omega_2 \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by Algorithm 3.3 and Assumption 3.1 hold. Then, $\{x_n\}$ converges strongly to $x^* \in \Omega_2$, where x^* is the unique solution to the following VIP: Find $p \in \Omega_2$ such that $\langle (I-f)p, x-p\rangle \geq 0$ $\forall x \in \Omega_2$.*

Algorithm 3.3.

Initialization: Let $x_0, x_1 \in H_1$ be arbitrary.

Iterative steps: Calculate x_{n+1} as follows:

Step 1. Given the iterates x_{n-1} and x_n $(n \geq 1)$, choose θ_n and ε_n such that $0 \leq \theta_n \leq \theta_n^*$ and $0 \leq \varepsilon_n \leq \varepsilon_n^*$, respectively, where

$$\theta_n^* = \begin{cases} \min\{\theta, \frac{\delta_n}{\|x_n - x_{n-1}\|}\} & \text{if } x_n \neq x_{n-1}, \\ \theta & \text{if otherwise}, \end{cases} \quad \varepsilon_n^* = \begin{cases} \min\{\varepsilon, \frac{\delta_n}{\|x_n - x_{n-1}\|}\} & \text{if } x_n \neq x_{n-1}, \\ \varepsilon & \text{if otherwise}. \end{cases}$$

Step 2. Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}), \; u_n = x_n + \varepsilon_n(x_n - x_{n-1}).$$

Step 3. Compute

$$z_n = \alpha_n w_n + (1-\alpha_n) T_{r_n}^{F_1}(I - \gamma_n A^*(I - T_{r_n}^{F_2})A)w_n, \; y_n = \beta_n u_n + (1-\beta_n) J_{s_n}^{D,B} u_n,$$

where

$$\gamma_n = \begin{cases} \frac{\tau_n \|(I - T_{r_n}^{F_2})Aw_n\|^2}{\|A^*(I - T_{r_n}^{F_2})Aw_n\|^2} & \text{if } Aw_n \neq T_{r_n}^{F_2} Aw_n, \\ \gamma & \text{if otherwise}. \end{cases}$$

Step 4: Compute

$$x_{n+1} = \lambda_n f(x_n) + \sigma_n z_n + \mu_n y_n.$$

Set $n := n+1$ and go to **Step 1**.

Proof. The result directly follows from Theorem 3.2 by making $S = I$, where I is the identity mapping on H_1. □

Corollary 3.2. *Let H_1, H_2 be real Hilbert spaces, and let $C \subseteq H_1$, $Q \subseteq H_2$ be nonempty, closed, and convex subsets. Let $S : H_1 \to H_1$ be a nonexpansive mapping, and let $f : H_1 \to H_1$ be a contraction mapping with contraction coefficient $\nu \in [0, \frac{1}{\sqrt{2}})$. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^*, and let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumption 2.1. Let $B : H_1 \to H_1$ be an α-inverse strongly monotone operator, and let $D : H_1 \to 2^{H_1}$ be a maximal monotone operator. Denote $\Omega := Fix(S) \cap (B+D)^{-1}(0) \cap SEP(F_1, F_2)$ and assume $\Omega \neq \emptyset$. Let $\{x_n\}$ be the seqeunce generated by Algorithm 3.4 and Assumption 3.1 hold. Then, $\{x_n\}$ converges strongly to $x^* \in \Omega$, where x^* is the unique solution to the following VIP: Find $p \in \Omega$ such that $\langle (I - f)p, x - p \rangle \geq 0 \; \forall x \in \Omega$.*

Algorithm 3.4.

Initialization: Let $x_0, x_1 \in H_1$ be arbitrary.

Iterative steps: Calculate x_{n+1} as follows:

Step 1. Given the iterates x_{n-1} and x_n $(n \geq 1)$, choose θ_n such that $0 \leq \theta_n \leq \theta_n^*$, where

$$\theta_n^* = \begin{cases} \min\{\theta, \frac{\delta_n}{\|x_n - x_{n-1}\|}\} & \text{if } x_n \neq x_{n-1}, \\ \theta & \text{if otherwise}, \end{cases}$$

Step 2. Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 3. Compute

$$z_n = \alpha_n w_n + (1-\alpha_n) T_{r_n}^{F_1}(I - \gamma_n A^*(I - T_{r_n}^{F_2})A)w_n, \; y_n = \beta_n w_n + (1-\beta_n) J_{s_n}^{D,B} w_n,$$

where

$$\gamma_n = \begin{cases} \frac{\tau_n\|(I-T_{r_n}^{F_2})Aw_n\|^2}{\|A^*(I-T_{r_n}^{F_2})Aw_n\|^2} & \text{if } Aw_n \neq T_{r_n}^{F_2}Aw_n, \\ \gamma & \text{if otherwise.} \end{cases}$$

Step 4: Compute

$$x_{n+1} = S_n(\lambda_n f(x_n) + \sigma_n z_n + \mu_n y_n),$$

where

$$S_n = (1-\kappa_n)I + \kappa_n S.$$

Set $n := n+1$ and go to **Step 1**.

Proof. The proof is similar to the proof of Theorem 3.2 by making $\theta = \varepsilon$ in Algorithm 3.1. □

Next, we give an application to split variational inequality, convex minimization, and fixed point problems.

3.3. **An application.** Let C be a nonempty, closed, and convex subset of a real Hilbert space H_1, and let $G_1 : H_1 \to H_1$ be a single-valued mapping. The *Variational Inequality Problem* (in short, VIP) associated with G_1 and C is to find $x^* \in C$ such that $\langle G_1x^*, y - x^*\rangle \geq 0$, $\forall y \in C$. We denote its the solution set by $VIP(G_1, C)$. The VIP has been studied by many authors and several methods have been proposed for finding its solution and related optimization problems; see, e.g., [5, 35, 36] and the references therein. Here, we assume G_1 is a monotone mapping, i.e., $\langle G_1y - G_1x, y - x\rangle \geq 0$, $\forall x, y \in H_1$. In addition, let Q be a nonempty, closed, and convex subset of a real Hilbert space H_2, and $G_2 : H_2 \to H_2$ a monotone mapping, and $A : H_1 \to H_2$ a bounded linear operator. We consider the *Split Variational Inequality Problem* (in short, SVIP) of finding

$$x^* \in C \text{ such that } \langle G_1x^*, y - x^*\rangle \geq 0 \; \forall\, y \in C \tag{3.28}$$

and

$$t = Ax^* \in Q \text{ such that } \langle G_2t, z - t\rangle \geq 0 \; \forall\, z \in Q. \tag{3.29}$$

We denote the solution set of SVIP (3.28)-(3.29) by $SVIP(G_1, G_2, C, Q)$.

Furthermore, let $h : H_1 \to \mathbb{R} \cup \{+\infty\}$ be a function that can be expressed as sum of two functions M, m, i.e., $h(x) \equiv m(x) + M(x)$, where $M : H_1 \to \mathbb{R} \cup \{+\infty\}$ is proper, convex, and lower semicontinuous, and $m : H_1 \to \mathbb{R}$ is convex and differentiable. We consider the problem of finding $x^* \in H_1$ such that

$$h(x^*) = m(x^*) + M(x^*) \leq m(x) + M(x) \equiv h(x), \; \forall x \in H_1. \tag{3.30}$$

It is observed that (3.30) serves as a model for several optimization problems in signal and image processing, such as image segmentation, image restoration, image compression and in painting, image deconvolution, non-negative matrix and tensor factorization (see e.g., [37, 38, 39]). We assume that m has a Lipschitz continuous gradient ∇m. Thus ∇m satisfies the following inequality (see [40]):

$$\langle \nabla m(x) - \nabla m(y), x - y\rangle \geq \frac{1}{L}\|\nabla m(x) - \nabla m(y)\|^2, \tag{3.31}$$

where L is the Lipschitz constant of the gradient of m. Therefore, we see from (3.31) that ∇m is $\frac{1}{L}$-inverse strongly monotone. Also, $\partial M : H_1 \to 2^{H_1}$ is maximal monotone. Note that (3.30) is equivalent to the following problem:

$$\text{find } x^* \in H_1 \text{ such that } 0 \in \nabla m(x^*) + \partial M(x^*).$$

Our aim in this subsection is to solve the following problem:

$$\text{find } x^* \in H_1 \text{ such that } x^* \in \mathrm{Fix}(S) \cap SVIP(G_1, G_2, C, Q) \cap \arg\min_{x \in H_1} h(x), \tag{3.32}$$

where $S : H_1 \to H_1$ is nonexpansive. By making $F_1(x,y) = \langle G_1 x, y-x\rangle$, $(x,y \in H_1)$ and $F_2(x,y) = \langle G_2 x, y-x\rangle$ $(x,y \in H_2)$, the SVIP (3.28) - (3.29) becomes the $SEP(F_1, F_2)$ and Assumption 2.1 is satisfied. Thus, by setting $F_1 = \langle G_1 x, y-x\rangle$, $F_2 = \langle G_2 x, y-x\rangle$, and $B = \nabla m$ and $D = \partial M$ in Theorem 3.2, we obtain the following result.

Theorem 3.4. *Let H_1, H_2 be real Hilbert spaces, and let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty, closed, and convex subsets. Let $S : H_1 \to H_1$ be a nonexpansive mapping, and let $f : H_1 \to H_1$ be a contraction mapping with contraction coefficient $\nu \in [0, \frac{1}{\sqrt{2}})$. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^*, and let $G_1 : C \to H$ and $G_2 : Q \to H$ be continuous monotone mappings. Let $m : H_1 \to \mathbb{R}$ be convex, differentiable with Lipschitz continuous gradient ∇m, $M : H_1 \to \mathbb{R} \cup \{+\infty\}$ be a proper convex and lower semicontinuous functions, and $J_s^{\partial M, \nabla m} = (I + s\partial M)^{-1}(I - s\nabla m)$ for $s > 0$. Denote $\Omega_3 := Fix(S) \cap SVIP(G_1, G_2, C, Q) \cap \arg\min_{x \in H_1} m(x) + M(x)$ and assume $\Omega_3 \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by Algorithm 3.5 and Assumption 3.1 hold. Then, $\{x_n\}$ converges strongly to $x^* \in \Omega_3$, where x^* is the unique solution to the following VIP: Find $p \in \Omega_3$ such that $\langle (I-f)p, x-p\rangle \geq 0$, $\forall x \in \Omega$.*

Algorithm 3.5. : Parallel inertial viscosity-type algorithm.

Initialization: Let $x_0, x_1 \in H_1$ be arbitrary.

Iterative steps: Calculate x_{n+1} as follows:

Step 1. Given the iterates x_{n-1} and x_n $(n \geq 1)$, choose θ_n and ε_n such that $0 \leq \theta_n \leq \theta_n^*$ and $0 \leq \varepsilon_n \leq \varepsilon_n^*$, respectively, where

$$\theta_n^* = \begin{cases} \min\{\theta, \frac{\delta_n}{\|x_n - x_{n-1}\|}\} \text{ if } x_n \neq x_{n-1}, \\ \theta \quad \text{if otherwise,} \end{cases} \qquad \varepsilon_n^* = \begin{cases} \min\{\varepsilon, \frac{\delta_n}{\|x_n - x_{n-1}\|}\} \text{ if } x_n \neq x_{n-1}, \\ \varepsilon \quad \text{if otherwise.} \end{cases}$$

Step 2. Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}),\ u_n = x_n + \varepsilon_n(x_n - x_{n-1}).$$

Step 3. Compute

$$z_n = \alpha_n w_n + (1-\alpha_n) T_{r_n}^{F_1}(I - \gamma_n A^*(I - T_{r_n}^{F_2})A) w_n,\ y_n = \beta_n u_n + (1-\beta_n) J_{s_n}^{\partial M, \nabla m} u_n,$$

where

$$\gamma_n = \begin{cases} \frac{\tau_n \|(I - T_{r_n}^{F_2})Aw_n\|^2}{\|A^*(I - T_{r_n}^{F_2})Aw_n\|^2} \text{ if } Aw_n \neq T_{r_n}^{F_2} A w_n, \\ \gamma \quad \text{if otherwise.} \end{cases}$$

Step 4: Compute

$$x_{n+1} = S_n(\lambda_n f(x_n) + \sigma_n z_n + \mu_n y_n),$$

where

$$S_n = (1-\kappa_n) I + \kappa_n S.$$

Set $n := n+1$ and go to **Step 1**.

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4. NUMERICAL EXAMPLE

In this section, we give an example in an infinite dimensional space to illustrate the applicability and behaviour of our Algorithms 3.1 and 3.2 as well as compare them with a related method.

Example 4.1. Let $H_1 = H_2 = (\ell_2(\mathbb{R}), \|\cdot\|_2)$, where $\ell_2(\mathbb{R}) := \{x = (x_1, x_2, \ldots, x_n, \ldots), x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$, $\|x\|_2 = (\sum_{i=1}^{\infty} |x_i|)^{\frac{1}{2}}\ \forall\, x \in \ell_2(\mathbb{R})$, $C = Q = \{x \in \ell_2(\mathbb{R}) : \|x\|_2 \leq 10\}$, and $Ax = \frac{2}{5}x$. Let the bifunctions $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be defined by $F_1(x,y) = \langle G_1 x, y - x\rangle$, $\forall x, y \in C$ and $F_2(x,y) = \langle G_2 x, y - x\rangle$, $\forall x, y \in Q$, respectively, where $G_1 x = \frac{x}{2}$ and $G_2 x = 3x$. Also, let $B : H_1 \to H_1$ and $D : H_1 \to 2^{H_1}$ be defined by $Bx = 2x + (4, -2, 1, \cdots)$ and $Dx = \{4x\}$ $\forall x \in H_1$, respectively. Define the nonexpansive mapping $S : H_1 \to H_1$ by $Sx = x\ \forall x \in H_1$. It can be verified that F_1 and F_2 satisfy Assumption 2.1, B is $\frac{1}{2}$-inverse strongly monotone, D is maximal monotone, and $0 \in \mathrm{Fix}(S) \cap (B+D)^{-1}(0) \cap SEP(F_1, F_2)$. Moreover, by a simple calculation, we obtain, for $x \in H_1, r, \gamma > 0$,

$$T_r^{F_1}(I - \gamma A^*(I - T_r^{F_2})A)x = \frac{2(75rx + 25x - 12\gamma rx)}{25(r+2)(3r+1)},$$

and for $\lambda > 0$

$$\begin{aligned} J_\lambda^{D,B} x &= (I + \lambda D)^{-1}(I - \lambda B) = J_\lambda^D (I - \lambda B)x \\ &= \frac{(1-2\lambda)x}{1+4\lambda} - \frac{\lambda(4,-2,1,\cdots)}{1+4\lambda}. \end{aligned}$$

We choose $\alpha_n = \beta_n = \frac{n}{4n+1}$, $\lambda_n = \frac{1}{n+1}, \sigma_n = \mu_n = \frac{1-\lambda_n}{2}, \kappa_n = \frac{n}{3n+4}, r_n = \frac{n}{2n+1}, s_n = \frac{3n}{10n+3}$, $\theta = \varepsilon = 0.5$, $\delta_n = \frac{1}{(n+1)^2}, \gamma = 0.1$, and $\tau_n = \frac{n}{5n+1}$. We then make different choices of x_0, x_1 as follows:
Case Ia: $x_0 = (4, -1, \frac{1}{4}, \cdots), x_1 = (3, 1, \frac{1}{3}, \cdots)$;
Case Ib: $x_0 = (-3, \frac{3}{2}, -\frac{3}{4}, \cdots), x_1 = (4, -2, 1, \cdots)$;
Case Ic: $x_0 = (5, \frac{5}{2}, \frac{5}{4} \cdots), x_1 = (8, 2, \frac{1}{4}, \cdots)$;
Case Id: $x_0 = (4, 1, \frac{1}{4}, \cdots), x_1 = (-1, \frac{1}{2}, -\frac{1}{4}, \cdots)$.

Using MATLAB R2019(b), we compare the performance of our Algorithms 3.1 and 3.2 with Algorithm 1.3, which is Algorithm (3.1) of Cholamjiak et al. [14]. The stopping criterion used for our computation is $\|x_{n+1} - x_n\|_2 < 10^{-6}$. We plot the graphs of errors against the number of iterations in each case. The figures and numerical results are shown in Figure 1 and Table 1, respectively.

Remark 4.1. The numerical example shows that our algorithms is easy to compute. In addition, it reveals that our algorithms perform better in time of computation and number of iterations than Algorithm (3.1) of Cholamjiak et al. [14].

5. CONCLUSION

We studied the split equilibrium, variational inclusion, and fixed point problems in real Hilbert spaces. We proposed two inertial viscosity algorithms with self-adaptive step-size and proved strong convergence theorems without imposing upper semi-continuity condition on the

TABLE 1. Numerical results.

		Alg. 3.1	Alg. 3.2	Alg. (3.1) Chola [14]
Case Ia	CPU time (sec) No of Iter.	0.0056 14	0.0036 10	0.0074 16
Case Ib	CPU time (sec) No. of Iter.	0.0010 15	0.0012 10	0.0013 17
Case Ic	CPU time (sec) No of Iter.	0.0016 15	9.5560e-4 11	0.0020 17
Case Id	CPU time (sec) No of Iter.	0.0033 13	0.0013 10	0.0042 15

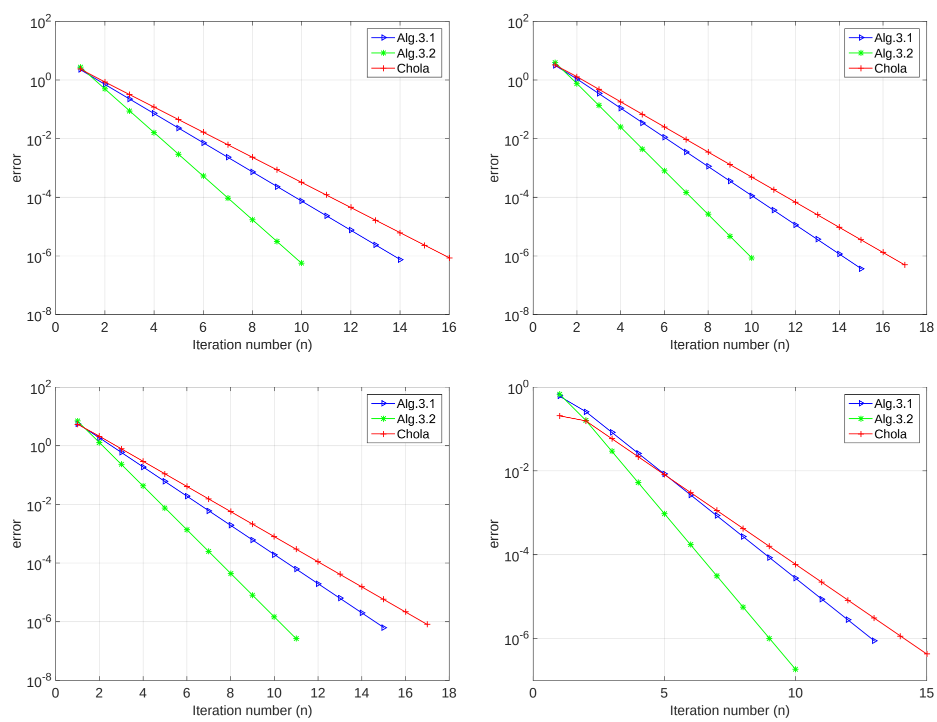

FIGURE 1. Top left: Case Ia; Top right: Case Ib; Bottom left: Case Ic; Bottom right: Case Id.

bifunctions. We applied our main results to split variational inequality, convex minimization, and fixed point problems. Finally, we illustrated our main results with a numerical example and compared with existing related result that can be computed. The numerical results indicate that our algorithms perform well.

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