J. Appl. Numer. Optim. 4 (2022), No. 3, pp. 381-391
Available online at http://jano.biemdas.com
https://doi.org/10.23952/jano.4.2022.3.05

PRIMAL CRITERIA OF METRIC SUBREGULARITY FOR GENERALIZED EQUATIONS

JIE PENG[1], ZHOU WEI[2,*]

[1]*Comprehensive Experimental Center, Hebei University, Baoding 071002, China*
[2]*Hebei Key Laboratory of Machine Learning and Computational Intelligence,*
College of Mathematics and Information Science, Hebei University, Baoding 071002, China

Abstract. This paper is devoted to the study on metric subregularity of a generalized equation defined by a closed set-valued mapping and a closed constraint subset. In terms of Bouligand's contingent cones and tangent derivatives, several primal characterizations and criteria of metric subregularity for the generalized equation are presented with some mild assumptions. This work extends primal results on metric subregularity of the generalized equation from the convex case to the non-convex one.

Keywords. Contingent cone; Generalized equation; Metric subregularity; Pseudoconvexity; Tangent derivative.

1. Introduction

Many optimization problems appearing in variational analysis and mathematical programming can be modelled as finding a solution to a generalized equation. This generalized equation mathematically is defined as follows

$$\bar{y} \in F(x), \tag{1.1}$$

where $F : X \rightrightarrows Y$ is a set-valued mapping between Banach spaces X and Y, and $\bar{y}$ is a given point in Y.

It is well known that a key concept, when studying the behavior of the solution set to generalized equation (1.1), is the metric regularity. Recall that F is said to be metrically regular at $\bar{x} \in F^{-1}(\bar{y})$ if there exists $\tau \in (0, +\infty)$ such that

$$d(x, F^{-1}(y)) \leq \tau d(y, F(x)) \text{ for all } (x,y) \text{ close to } (\bar{x}, \bar{y}). \tag{1.2}$$

The study of this concept can be traced back to the Robinson-Ursescu theorem, Lyusternik-Graves theorem, or even Banach open mapping theorem. Readers are invited to consult [1, 2, 3, 4, 5, 6, 7] and the references therein for many theoretical results on metric regularity and its various applications.

A weaker property of the metric regularity is named as the metric subregularity which means that inequality (1.2) holds only for fixed $\bar{y}$. Recall that F is said to be metrically subregular at

*Corresponding author.
E-mail addresses: pengjie@hbu.edu.cn (J. Peng), weizhou@hbu.edu.cn (Z. Wei).
Received April 22, 2022; Accepted June 9, 2022.

$\bar{x} \in F^{-1}(\bar{y})$ if there exists $\tau \in (0, +\infty)$ such that

$$d(x, F^{-1}(\bar{y})) \leq \tau d(\bar{y}, F(x)) \text{ for all } x \text{ close to } \bar{x}. \tag{1.3}$$

This concept can be used to estimate the distance of a candidate x to the solution set of generalized equation (1.1). When the solution set $F^{-1}(\bar{y})$ reduces to the singleton locally, (1.3) becomes to the strong metric subregularity of (1.1), that is, F is said to be strongly metrically subregular at $\bar{x} \in F^{-1}(\bar{y})$ if there exists $\tau \in (0, +\infty)$ such that

$$\|x - \bar{x}\| \leq \tau d(\bar{y}, F(x)) \text{ for all } x \text{ close to } \bar{x}.$$

It is known from [8] that the metric subregularity of generalized equation (1.1) is equivalent to calmness of the inverse set-valued mapping (F^{-1}). This property is proved to closely relate with error bounds, linear regularity, and basic constraint qualification (BCQ) in optimization and have a huge range of applications in areas of variational analysis and mathematical programming like optimality conditions, variational inequalities, subdifferential theory, the sensitivity analysis of generalized equations and convergence analysis of algorithms for solving equations or inclusions. For these reasons, the concept of the metric subregularity has been extensively studied by many authors (see, e.g., [9, 10, 11, 12, 13, 14, 15, 16, 17] and the references therein).

When dealing with metric subregularity (or calmness), a large literature was devoted to providing dual characterizations and criteria in terms of subdifferentials, normal cones, or coderivatives. Henrion et al. [18, 19, 20] derived calmness criteria of the constraint set mapping in terms of Clarke's subdifferential. Zheng and Ng [21] proved dual characterizations of the metric subregularity for a convex generalized equation in terms of normal cones and coderivatives. Subsequently, they [22] considered the metric subregularity for nonconvex generalized equation and gave its dual sufficient criteria. Gfrerer [23] developed first-order and second-order characterizations of the metric subregularity for the constraint set mapping. In [24], Huang, He, and Wei provided necessary and/or sufficient dual conditions of the metric subregularity for the non-convex generalized equation. It is noted that a pretty natural idea is to study the metric subregularity in terms of various primal derivative-like objects, such as directional derivatives, contingent cones, or slopes. Several criteria for metric subregularity of generalized equations were presented in [1, 9, 10, 11, 25, 26, 27, 28, 29, 30, 31] dependent on the primal-type estimate. Based on the works from [9, 30, 31] that are to give primal characterizations of metric subregularity for the convex generalized equation, a natural issue is to extend primal results on metric subregularity by dropping the convexity assumption. Inspired by this issue, our goal in the paper is to discuss primal criteria of metric subregularity for the generalized equation (not necessarily convex). Our work is to provide primal necessary and/or sufficient conditions of metric subregularity in terms of contingent cones and tangent derivatives with mild assumptions.

The paper is organized as follows. In Section 2, we give some preliminaries used in this paper. Our notation is basically standard and conventional in the area of variational analysis. Section 3 is devoted to the main results on metric subregularity for a generalized equation. Several primal criteria for metric subregularity are established in terms of contingent cones and tangent derivatives. The accurate estimate to the metric subregularity modulus is also given therein.

2. Preliminaries

Let X be a Banach space (or Euclidean space). Let B_X denote the closed unit ball of X. For $\bar{x} \in X$ and $\delta > 0$, let $B(\bar{x}, \delta)$ denote the open and ball with center $\bar{x}$ and radius δ. For a subset Ω of X, we denote by $\mathrm{cl}(\Omega)$ the closure of Ω.

Let A be a closed subset in X and $\bar{x} \in A$. We denote by

$$T(A, \bar{x}) := \operatorname*{Limsup}_{t \to 0^+} \frac{A - \bar{x}}{t}$$

the *Bouligand contingent cone* of A at $\bar{x}$. Thus $v \in T(A, \bar{x})$ if and only if there exist a sequence $\{v_n\}$ in X converging to v and a sequence $\{t_n\}$ in $(0, +\infty)$ decreasing to 0 such that $\bar{x} + t_n v_n \in A$ for all $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of all natural numbers. Recall (see [32]) that A is said to be *star-shaped* at $\bar{x}$ if

$$\lambda \bar{x} + (1 - \lambda)x \in A \ \text{ for all } x \in A \text{ and } \lambda \in [0, 1].$$

Recall from [32] that A is said to be *pseudoconvex* at $\bar{x}$ if $A - \bar{x} \subset T(A, \bar{x})$ and A is said to be a pseudoconvex subset if A is pseudoconvex at all points in A. It is from [32, Lemma 4.2.8] that if A is star-shaped at $\bar{x}$, then A is pseudoconvex at $\bar{x}$.

Let $M : Y \rightrightarrows X$ be a multifunction between Banach spaces Y and X. We denote by $\mathrm{dom}(M) := \{y \in Y : M(y) \neq \emptyset\}$ the *domain* of M. The *graph* of M is defined by

$$\mathrm{gph}(M) := \big\{(y, x) \in Y \times X : x \in M(y)\big\}.$$

As usual, M is said to be closed if $\mathrm{gph}(M)$ is a closed subset of $Y \times X$. Let $(\bar{y}, \bar{x}) \in \mathrm{gph}(M)$. The *Bouligand tangent derivative* $DM(\bar{y}, \bar{x}) : Y \rightrightarrows X$ of M at $(\bar{y}, \bar{x})$ is defined as

$$DM(\bar{y}, \bar{x})(v) := \big\{u \in X : (v, u) \in T(\mathrm{gph}(M), (\bar{y}, \bar{x}))\big\} \ \text{ for all } v \in Y.$$

We close this section with the following lemma cited from [33, Theorem 4.1].

Lemma 2.1. *Let Ω be a nonempty closed subset of X and $\gamma \in (0, 1)$. Then for any $x \notin \Omega$ there exists $z \in \Omega$ such that*

$$\gamma \|x - z\| < \min\{d(x, \Omega), d(x - z, T(\Omega, z))\}.$$

3. Main Results

In this section, we mainly study the metric subregularity for a generalized equation with constraints and aim to provide several primal criteria of metric subregularity in terms of contingent cones and tangent derivatives.

Throughout the rest of this paper, we suppose that $F : X \rightrightarrows Y$ is a closed set-valued mapping and A is a closed subset of X.

Let $\bar{y} \in Y$ be given. We consider the following generalized equation with constraint (GEC for short):

$$\bar{y} \in F(x) \ \text{ subject to } \ x \in A. \tag{3.1}$$

We denote by $S := F^{-1}(\bar{y}) \cap A$ the solution set of (3.1). Recall that GEC (3.1) is said to be *metrically subregular* at $\bar{x} \in S$ if there exist $\tau, \delta \in (0, +\infty)$ such that

$$d(x, S) \leq \tau(d(\bar{y}, F(x)) + d(x, A)) \ \ \forall x \in B(\bar{x}, \delta). \tag{3.2}$$

We denote by $\text{subreg}_A F(\bar{x},\bar{y})$ the *modulus of metric subregularity* at $\bar{x}$ for GEC (3.1) which is defined by

$$\text{subreg}_A F(\bar{x},\bar{y}) := \inf\{\tau > 0 : \exists\, \delta > 0 \text{ s.t. (3.2) holds}\},$$

where we use the convention that the infimum over the empty set is $+\infty$. Thus GEC (3.1) is metrically subregular at $\bar{x}$ if and only if $\text{subreg}_A F(\bar{x},\bar{y}) < +\infty$.

The following theorem is to give primal equivalent conditions of metric subregularity for GEC (3.1) in terms of contingent cones and tangent derivatives with mild assumptions.

Theorem 3.1. *Let $\bar{x} \in S$. Suppose that there exists a neighborhood U of $\bar{x}$ such that* gph(F) *is pseudoconvex at $(u,\bar{y})$ for all $u \in F^{-1}(\bar{y}) \cap U$ and A is star-shaped such that S is pseudoconvex. Then GEC (3.1) is metrically subregular at $\bar{x}$ if and only if there exist $\delta, \eta > 0$ such that*

$$\left[DF^{-1}(\bar{y},x)(\eta_1 B_Y) \cap (T(A,x) + \eta_2 B_X) \subseteq T(S,x) + B_X \middle| \begin{array}{l} \forall \eta_1,\eta_2 \in [0,+\infty) \\ with\ \eta_1+\eta_2 < \eta \end{array} \right] \tag{3.3}$$

holds for all $x \in S \cap B(\bar{x},\delta)$.

Proof. The necessity part. Since GEC (3.1) is metrically subregular at $\bar{x}$, then there exist $\tau, \delta > 0$ such that (3.2) holds. Let $\eta \in (0, \frac{1}{\tau})$ and $x \in S \cap B(\bar{x},\delta)$. Take any $\eta_1, \eta_2 \in [0,+\infty)$ such that $\eta_1+\eta_2 < \eta$ and any $u \in DF^{-1}(\bar{y},x)(\eta_1 B_Y) \cap (T(A,x)+\eta_2 B_X)$. Thus there exist $z \in B_Y, b \in B_X$, $t_n \to 0^+$, $(v_n,u_n) \to (\eta_1 z, u)$, $s_k \to 0^+$, and $w_k \to u + \eta_2 b$ such that

$$x + t_n u_n \in F^{-1}(\bar{y} + t_n v_n) \ \ \forall n \ \text{ and } \ x + s_k w_k \in A \ \ \forall k. \tag{3.4}$$

Then there exists $\{n_k\} \subseteq \mathbb{N}$ such that $t_{n_k} < s_k$ for all k. Since A is star-shaped, it follows from (3.4) that

$$x + t_{n_k} w_k = (1 - \frac{t_{n_k}}{s_k})x + \frac{t_{n_k}}{s_k}(x + s_k w_k) \in A.$$

This and (3.1) imply that, for any k sufficiently large,

$$\begin{aligned} d(x+t_{n_k}u_{n_k}, S) &\leq \tau(d(\bar{y}, F(x+t_{n_k}u_{n_k})) + d(x+t_{n_k}u_{n_k}, A)) \\ &\leq \tau t_{n_k}(\|v_{n_k}\| + \|u_{n_k} - w_k\|). \end{aligned}$$

Consequently,

$$d(u_{n_k}, T(S,x)) \leq d\Big(u_n, \frac{S-x}{t_{n_k}}\Big) \leq \tau(\|v_{n_k}\| + \|u_{n_k} - w_k\|),$$

as S is pseudoconvex. Letting $k \to \infty$, one has

$$d(u, T(S,x)) \leq \tau(\|\eta_1 z\| + \|u - (u+\eta_2 b)\|) \leq \tau(\eta_1+\eta_2).$$

Thus

$$u \in \text{cl}(T(S,x) + \tau(\eta_1+\eta_2)B_X) \subseteq T(S,x) + B_X$$

thanks to $\eta_1 + \eta_2 < \eta < \frac{1}{\tau}$. This means that (3.3) holds.

The sufficiency part. We next prove that

$$d(x,S) \leq \tau(d(\bar{y},F(x)) + d(x,A)) \ \ \forall x \in B(\bar{x}, \frac{\delta}{2}). \tag{3.5}$$

It is clear that (3.5) holds for any $x \in B(\bar{x}, \frac{\delta}{2}) \cap S$. Let $x \in B(\bar{x}, \frac{\delta}{2}) \backslash S$. Then $d(x,S) \leq \|x - \bar{x}\| < \frac{\delta}{2}$. Take any $\gamma \in (\frac{2d(x,S)}{\delta}, 1)$. By applying Lemma 2.1, there exists $z \in S$ such that

$$\gamma \|x - z\| \leq d(x - z, T(S,z)). \tag{3.6}$$

Since S is pseudoconvex, then $S - z \subseteq T(S,z)$ and it follows that

$$\gamma \|x - z\| \leq d(x - z, T(S,z)) \leq d(x,S).$$

This and the choice of γ imply that $\|x - z\| \leq \frac{d(x,S)}{\gamma} < \frac{\delta}{2}$. Thus

$$\|z - \bar{x}\| \leq \|z - x\| + \|x - \bar{x}\| < \frac{\delta}{2} + \frac{\delta}{2} = \delta. \tag{3.7}$$

Choose sequences $\{y_n\} \subseteq F(x)$ and $\{u_n\} \subseteq A$ such that $\|\bar{y} - y_n\| \to d(\bar{y}, F(x))$ and $\|x - u_n\| \to d(x,A)$. For each $n \in \mathbb{N}$, let

$$r_n := \frac{1}{\tau(\|\bar{y} - y_n\| + \|x - u_n\|)} > 0$$

and take $\varepsilon_n > 0$ sufficiently small such that

$$r_n(\|\bar{y} - y_n\| + (1 + \varepsilon_n)\|x - u_n\|) < \eta. \tag{3.8}$$

Note that $\|r_n(x - z) - r_n(u_n - z)\| = r_n \|x - u_n\|$, and so

$$d(r_n(x - z), T(A,z)) \leq \|r_n(x - z) - r_n(u_n - z)\| = r_n \|x - u_n\|.$$

This implies that

$$r_n(x - z) \in \mathrm{cl}(T(A,z) + r_n \|x - u_n\| B_X) \subseteq T(A,z) + (1 + \varepsilon_n) r_n \|x - u_n\| B_X. \tag{3.9}$$

Since $\mathrm{gph}(F)$ is pseudoconvex at $(z, \bar{y})$, it follows that

$$r_n(x - z) \in D(F^{-1})(\bar{y}, z)(r_n(y_n - \bar{y})) \subseteq DF^{-1}(\bar{y}, z)(r_n \|y_n - \bar{y}\| B_Y).$$

By virtue of (3.2), (3.7), (3.8), and (3.9), one has $r_n(x - z) \in T(S,z) + B_X$ and consequently

$$d(x - z, T(S,z)) \leq \frac{1}{r_n} = \tau(\|\bar{y} - y_n\| + \|x - u_n\|).$$

Letting $n \to \infty$, one has $d(x - z, T(S,z)) \leq \tau(d(\bar{y}, F(x)) + d(x,A))$. This and (3.6) mean that

$$\gamma d(x,S) \leq \tau(d(\bar{y}, F(x)) + d(x,A)).$$

Letting $\gamma \to 1^-$, one has $d(x,S) \leq \tau(d(\bar{y}, F(x)) + d(x,A))$. Hence (3.5) holds. The proof is complete. $\square$

As a consequence of Theorem 3.1, the following theorem provides an accurate estimate on the modulus $\mathrm{subreg}_A F(\bar{x}, \bar{y})$.

Theorem 3.2. *Let $\bar{x} \in S$. Suppose that there exists a neighborhood U of $\bar{x}$ such that $\mathrm{gph}(F)$ is pseudoconvex at $(u, \bar{y})$ for all $u \in F^{-1}(\bar{y}) \cap U$ and A is star-shaped such that S is pseudoconvex. Then*

$$\frac{1}{\mathrm{subreg}_A F(\bar{x}, \bar{y})} = \eta_A(F, \bar{x}, \bar{y}), \tag{3.10}$$

where

$$\eta_A(F, \bar{x}, \bar{y}) := \sup\{\eta > 0 : \exists\, \delta > 0 \text{ s.t. (3.3) holds for any } x \in S \cap B(\bar{x}, \delta)\}.$$

Proof. We first consider the case that $\text{subreg}_A F(\bar{x},\bar{y}) < +\infty$.

Let $\tau > \text{subreg}_A F(\bar{x},\bar{y})$. Then there exists $\delta > 0$ such that (3.2) holds. Using the proof of the necessity part in Theorem 3.1, one has $\eta_A(F,\bar{x},\bar{y}) \geq \eta$ for all $\eta \in (0,\frac{1}{\tau})$. By letting $\eta \uparrow \frac{1}{\tau}$ and then $\tau \downarrow \text{subreg}_A F(\bar{x},\bar{y})$, one has

$$\eta_A(F,\bar{x},\bar{y}) \geq \frac{1}{\text{subreg}_A F(\bar{x},\bar{y})} > 0. \tag{3.11}$$

Let $\eta \in (0,\eta_A(F,\bar{x},\bar{y}))$. Then there exists $\delta > 0$ such that (3.3) holds for all $x \in S \cap B(\bar{x},\delta)$ and all $\eta_1,\eta_2 \geq 0$ with $\eta_1+\eta_2 < \eta$. By the proof of the sufficiency part in Theorem 3.1, one has that

$$\frac{1}{\text{subreg}_A F(\bar{x},\bar{y})} \geq \frac{1}{\tau}$$

holds for all $\tau > \frac{1}{\eta}$. By letting $\tau \downarrow \frac{1}{\eta}$ and then $\eta \uparrow \eta_A(F,\bar{x},\bar{y})$, one has

$$\frac{1}{\text{subreg}_A F(\bar{x},\bar{y})} \geq \eta_A(F,\bar{x},\bar{y}).$$

This and (3.11) imply that (3.10) holds.

We next consider the case that $\text{subreg}_A F(\bar{x},\bar{y}) = +\infty$. We claim that $\eta_A(F,\bar{x},\bar{y}) = 0$. Otherwise, one can verify that

$$\text{subreg}_A F(\bar{x},\bar{y}) \leq \frac{1}{\eta_A(F,\bar{x},\bar{y})} < +\infty,$$

which is a contradiction. The proof is complete. □

It is noted that inclusion (3.3) is a key tool to characterize metric subregularity of GEC (3.1). The following proposition provides an equivalent condition of inclusion (3.3).

Proposition 3.1. *Let $x \in S$ and $\eta > 0$. Then (3.3) holds if and only if there exists $\tau > 0$ such that*

$$d(h,T(S,x)) \leq \tau(d(0,DF(x,\bar{y})(h)) + d(h,T(A,x))), \quad \forall h \in X. \tag{3.12}$$

More precisely, for $\bar{x} \in S$, one has the following accurate quantitative equality:

$$\eta_A(F,\bar{x},\bar{y}) = \frac{1}{\tau_A(F,\bar{x},\bar{y})} \tag{3.13}$$

where $\tau_A(F,\bar{x},\bar{y})$ is defined by

$$\tau_A(F,\bar{x},\bar{y}) := \inf\{\tau > 0 : \exists\, \delta > 0 \text{ s.t. (3.12) holds for any } x \in S \cap B(\bar{x},\delta)\}.$$

Proof. To complete the proof, we only need to prove (3.13).

Let $\eta \in (0,\eta_A(F,\bar{x},\bar{y}))$. Then there exists $\delta > 0$ such that (3.3) holds for all $x \in S \cap B(\bar{x},\delta)$. Let $\tau > \frac{1}{\eta}$ and $x \in S \cap B(\bar{x},\delta)$. Take any $h \in X$. Choose sequences $\{v_n\} \subseteq DF(x,\bar{y})(h)$ and $\{u_n\} \subseteq T(A,x)$ such that

$$\|v_n\| \to d(0,DF(x,\bar{y})(h)) \ \text{ and } \ \|h-u_n\| \to d(h,T(A,x)).$$

For each $n \in \mathbb{N}$, let

$$r_n := \frac{1}{\tau(\|v_n\|+\|h-u_n\|)} > 0.$$

Noting that $h = u_n + \|h-u_n\| \cdot \frac{h-u_n}{\|h-u_n\|}$ and $v_n \in DF(x,\bar{y})(h)$, it follows that

$$\begin{aligned} r_n h &\in DF^{-1}(\bar{y},x)(r_n v_n) \cap \big(T(A,x) + r_n\|h-u_n\| \cdot \frac{h-u_n}{\|h-u_n\|}\big) \\ &\subseteq DF^{-1}(\bar{y},x)(r_n\|v_n\|B_Y) \cap (T(A,x) + r_n\|h-u_n\|B_X). \end{aligned}$$

Since $r_n(\|v_n\| + \|h-u_n\|) = \frac{1}{\tau} < \eta$, by (3.3), one has $r_n h \in T(S,x) + B_X$ and consequently $d(r_n h, T(S,x)) \leq 1$. This implies that

$$d(h, T(S,x)) \leq \frac{1}{r_n} = \tau(\|v_n\| + \|h-u_n\|),$$

and consequently $\tau_A(F,\bar{x},\bar{y}) \leq \tau$ by letting $n \to \infty$. Thus

$$\frac{1}{\tau_A(F,\bar{x},\bar{y})} \geq \eta_A(F,\bar{x},\bar{y})$$

by letting $\tau \downarrow \frac{1}{\eta}$ and then $\eta \uparrow \eta_A(F,\bar{x},\bar{y})$.

Let $\tau > \tau_A(F,\bar{x},\bar{y})$. Then there exists $\delta > 0$ such that (3.13) holds for any $x \in S \cap B(\bar{x},\delta)$. Let $\eta \in (0, \frac{1}{\tau})$ and $x \in S \cap B(\bar{x},\delta)$. Take any $\eta_1, \eta_2 \in [0,+\infty)$ with $\eta_1 + \eta_2 < \eta$ and $u \in DF^{-1}(\bar{y},x)(\eta_1 B_Y) \cap (T(A,x) + \eta_2 B_X)$. Then there exist $(v,b) \in B_Y \times B_X$ such that

$$u \in DF^{-1}(\bar{y},x)(\eta_1 v) \text{ and } u - \eta_2 b \in T(A,x).$$

By virtue of (3.13), one has

$$\begin{aligned} d(u, T(S,x)) &\leq \tau(d(0, DF(x,\bar{y})(u)) + d(u, T(A,x))) \\ &\leq \tau(\eta_1\|v\| + \eta_2\|b\|) \\ &\leq \tau(\eta_1 + \eta_2). \end{aligned}$$

Thus

$$u \in \overline{T(S,x) + \tau(\eta_1+\eta_2)B_X} \subseteq T(S,x) + B_X$$

as $\eta_1 + \eta_2 < \eta < \frac{1}{\tau}$. This means that $\eta_A(F,\bar{x},\bar{y}) \geq \eta$ and consequently

$$\eta_A(F,\bar{x},\bar{y}) \geq \frac{1}{\tau_A(F,\bar{x},\bar{y})}$$

by taking limits as $\eta \uparrow \frac{1}{\tau}$ and then $\tau \downarrow \tau_A(F,\bar{x},\bar{y})$. Hence (3.13) holds. The proof is complete. □

The following theorem, as a main result in this paper, is immediate from Theorem 3.2 and Proposition 3.1.

Theorem 3.3. *Let $\bar{x} \in S$. Suppose that there exists a neighborhood U of $\bar{x}$ such that* gph(F) *is pseudoconvex at $(u,\bar{y})$ for all $u \in F^{-1}(\bar{y}) \cap U$ and A is star-shaped such that S is pseudoconvex. Then*

$$\frac{1}{\mathrm{subreg}_A F(\bar{x},\bar{y})} = \eta_A(F,\bar{x},\bar{y}) = \frac{1}{\tau_A(F,\bar{x},\bar{y})}.$$

Remark 3.1. For the case that gph(F) and A are convex, it is easy to verify that all assumptions in Theorem 3.3 are satisfied automatically and then Theorem 3.3 reduces to [24, Theorem 3.1]. This means that primal results on metric subregularity of the generalized equation can be extended from the convex case to the non-convex one with the help of pseudoconvexity.

For the special case, we have the following sharper primal result on metric subregularity, that is, the validity of (3.3) only at $\bar{x}$ can ensure the metric subregularity of GEC (3.1).

Theorem 3.4. *Let $\bar{x} \in S$. Suppose that* $\mathrm{gph}(F)$ *is pseudoconvex at $(\bar{x},\bar{y})$, and A is star-shaped such that S is pseudoconvex and that there exists a closed pseudoconvex cone K and a neighborhood V of $\bar{x}$ such that $S \cap V = (\bar{x}+K) \cap V$. Then*

$$\frac{1}{\mathrm{subreg}_A F(\bar{x},\bar{y})} = \sup\{\eta > 0 : (3.3) \text{ holds only with } x = \bar{x}\}. \tag{3.14}$$

Proof. We denote

$$\alpha := \sup\{\eta > 0 : (3.3) \text{ holds only with } x = \bar{x}\}.$$

By using the proof of Theorem 3.2, one has

$$\frac{1}{\mathrm{subreg}_A F(\bar{x},\bar{y})} \leq \eta_A(F,\bar{x},\bar{y}) \leq \alpha.$$

Let $\eta \in (0,\alpha)$ and $\tau > \frac{1}{\eta}$. Take $\delta > 0$ such that $B(\bar{x},2\delta) \subseteq V$. Then

$$S \cap B(\bar{x},2\delta) = (\bar{x}+K) \cap B(\bar{x},2\delta).$$

Since K is a pseudoconvex cone, one can verify that

$$T(S,\bar{x}) = K. \tag{3.15}$$

Let $u \in B(\bar{x},\delta) \backslash S$. Take any $\gamma \in (\frac{d(u,S)}{\delta},1)$. By Lemma 2.1, there exists $z \in S$ such that

$$\gamma\|u-z\| \leq d(u-z,T(S,z)). \tag{3.16}$$

Note that $S-z \subseteq T(S,z)$ by the convexity of S and so $\|u-z\| \leq \frac{d(u-z,T(S,z))}{\gamma} < \delta$. Then

$$\|z-\bar{x}\| \leq \|z-u\| + \|u-\bar{x}\| < \delta+\delta = 2\delta.$$

Noting that K is a convex cone, it follows from (3.15) that

$$T(S,z) = T(\bar{x}+K,z) = T(K,z-\bar{x}) \supseteq K - z + \bar{x} = T(S,\bar{x}) - z + \bar{x}.$$

This and (3.16) imply that

$$\gamma d(u,S) \leq \gamma\|u-z\| \leq d(u-\bar{x},T(S,\bar{x})). \tag{3.17}$$

Similar to the proof of $\frac{1}{\mathrm{subreg}_A F(\bar{x},\bar{y})} \geq \eta_A(F,\bar{x},\bar{y})$ in Theorem 3.2, by using (3.17), one can prove that $d(u,S) \leq \tau(d(\bar{y},F(u)) + d(u,A))$. This means that $\mathrm{subreg}_A F(\bar{x},\bar{y}) \leq \tau$. By taking limits as $\tau \downarrow \frac{1}{\eta}$ and $\eta \uparrow \alpha$, one has that

$$\frac{1}{\mathrm{subreg}_A F(\bar{x},\bar{y})} \geq \alpha.$$

Hence (3.14) holds. The proof is complete. □

Next, we consider strong metric subregularity for GEC (3.1). Recall that GEC (3.1) is said to be strongly metrically subregular at $\bar{x} \in S$ if there exist $\tau, \delta \in (0,+\infty)$ such that

$$\|x-\bar{x}\| \leq \tau(d(\bar{y},F(x)) + d(x,A)) \quad \forall x \in B(\bar{x},\delta). \tag{3.18}$$

We denote by $\mathrm{ssubreg}_A F(\bar{x},\bar{y})$ the modulus of strong metric subregularity at $\bar{x}$ for GEC (3.1); that is,

$$\mathrm{ssubreg}_A F(\bar{x},\bar{y}) := \inf\{\tau > 0 : \exists \delta > 0 \text{ s.t. } (3.18) \text{ holds}\}. \tag{3.19}$$

Thus, GEC (3.1) is strongly metrically subregular at $\bar{x} \Longleftrightarrow \operatorname{ssubreg}_A F(\bar{x},\bar{y}) < +\infty$.

It is easy to verify that GEC (3.1) is strongly metrically subregular at $\bar{x} \in S$ if and only if (3.1) is metrically subregular at $\bar{x}$ and $S = \{\bar{x}\}$.

To present primal characterizations of strong metric subregularity for GEC (3.1), we consider the following inclusion which is given by contingent cones and tangent derivatives.
For given $\bar{x} \in S$ and $\eta \in (0,+\infty)$, we consider the following inclusion:

$$DF^{-1}(\bar{y},\bar{x})(\eta_1 B_Y) \cap (T(A,\bar{x}) + \eta_2 B_X) \subseteq B_X \ \ \forall \eta_1,\eta_2 \geq 0 \text{ with } \eta_1 + \eta_2 < \eta. \tag{3.20}$$

Proposition 3.2. *Let $\bar{x} \in S$ such that* $\operatorname{gph}(F)$ *is pseudoconvex at $(\bar{x},\bar{y})$ and A is pseudoconvex at $\bar{x}$. Suppose that (3.20) holds with $\eta \in (0,+\infty)$. Then $S = \{\bar{x}\}$.*

Proof. Let $x \in S$. Then, for any $t > 0$,

$$t(x-\bar{x}) \in DF^{-1}(\bar{y},\bar{x})(0) \cap T(A,\bar{x})$$

as $\operatorname{gph}(F)$ is pseudoconvex at $(\bar{x},\bar{y})$ and A is pseudoconvex at $\bar{x}$. It follows from (3.20) that

$$t(x-\bar{x}) \in B_X \ \ \forall t > 0.$$

This implies that $x = \bar{x}$ and thus $S = \{\bar{x}\}$ holds. The proof is complete. □

The following theorem provides an accurate quantitative estimate on the modulus of (3.19) and also gives primal characterizations for strong metric subregularity of GEC (3.1). The proof can be obtained by Theorem 3.3 and Proposition 3.2.

Theorem 3.5. *Let $\bar{x} \in S$. Suppose that* $\operatorname{gph}(F)$ *is pseudoconvex at $(\bar{x},\bar{y})$ and A is star-shaped such that S is pseudoconvex. Then*

$$\frac{1}{\operatorname{ssubreg}_A F(\bar{x},\bar{y})} = \sup\{\eta > 0 : (3.20) \text{ holds}\}.$$

Proposition 3.3. *Let $\bar{x} \in S$. Suppose that* $\operatorname{gph}(F)$ *is pseudoconvex at $(\bar{x},\bar{y})$, A is star-shaped such that S is pseudoconvex and that $DF^{-1}(\bar{y},\bar{x})(B_Y) \cap (T(A,\bar{x}) + B_X) \cap B_X$ is relatively compact. Then* $\operatorname{ssubreg}_A F(\bar{x},\bar{y}) < +\infty$ *if and only if*

$$DF^{-1}(\bar{y},\bar{x})(0) \cap T(A,\bar{x}) = \{0\}. \tag{3.21}$$

Proof. The necessity part. By Theorem 3.5, there exists $\eta > 0$ such that (3.20) holds. Then

$$DF^{-1}(\bar{y},\bar{x})(0) \cap T(A,\bar{x}) \subseteq \varepsilon B_X, \ \ \forall \varepsilon > 0.$$

This means that (3.21) holds.

The sufficiency part. By using Theorem 3.5, we only need to prove that there exists $\eta > 0$ such that (3.20) holds.

Suppose on the contrary that, for any $n \in \mathbb{N}$, there exist $r_n, s_n \geq 0$ with $r_n + s_n < \frac{1}{n}$, $(y_n, b_n) \in B_Y \times B_X$, and $x_n \in X$ such that

$$x_n \in \big(DF^{-1}(\bar{y},\bar{x})(r_n y_n) \cap (T(A,\bar{x}) + s_n b_n)\big) \backslash B_X. \tag{3.22}$$

This implies that

$$\frac{x_n}{\|x_n\|} \in DF^{-1}(\bar{y},\bar{x})(B_Y) \cap (T(A,\bar{x}) + B_X).$$

Since $DF^{-1}F(\bar{y},\bar{x})(B_Y)\cap(T(A,\bar{x})+B_X)\cap B_X$ is relatively compact, without loss of generality, we can assume that $\frac{x_n}{\|x_n\|}\to x_0$ with $\|x_0\|=1$ (considering subsequence if necessary). By (3.22), one has

$$\frac{x_n}{\|x_n\|}\in DF^{-1}(\bar{y},\bar{x})(\frac{r_n}{\|x_n\|}y_n)\cap(T(A,\bar{x})+\frac{s_n}{\|x_n\|}b_n).$$

By taking the limit as $n\to\infty$, one has $x_0\in DF^{-1}(\bar{y},\bar{x})(0)\cap T(A,\bar{x})$, which contradicts (3.21) since $x_0\neq 0$. The proof is complete. □

The following proposition is immediate from Proposition 3.3.

Proposition 3.4. *Let $\bar{x}\in S$. Suppose that X is of finite dimension,* gph(F) *is pseudoconvex at $(\bar{x},\bar{y})$ and that A is star-shaped such that S is pseudoconvex. Then GEC (3.1) is strongly metrically regular at $\bar{x}$ if and only if $DF^{-1}(\bar{y},\bar{x})(0)\cap T(A,\bar{x})=\{0\}$.*

Acknowledgments

This paper was supported by the National Natural Science Foundations of China (Grant Nos. 11971422 and 12171419), and funded by Science and Technology Project of Hebei Education Department (No. ZD2022037) and the Natural Science Foundation of Hebei Province (A2022201002).

REFERENCES

[1] D. Azé, A unified theory for metric regularity of set-valued mappings, J. Convex Anal., 13 (2006), 225-252.
[2] J.F. Bonnans, A. Shapiro, Perturbation Analysis of Optimization Problems, Springer, New York, 2000.
[3] A.D. Ioffe, Metric regularity and subdifferential calculus, Russian Math. Survey. 55 (2000), 103-162.
[4] D. Klatte, B. Kummer, Nonsmooth Equations in Optimization, Regularity, Calculus, Methods and Applications, Nonconvex Optimization and its Application 60, Kluwer Academic Publishers, Dordrecht, 2002.
[5] W. Li, Abadie's constraint qualification, metric regularity, and error bounds for differentiable convex inequalities, SIAM J. Optim. 7 (1997), 966-978.
[6] B.S. Mordukhovich, Complete characterization of openness, metric regularity, and Lipschitzian properties of set-valued mappings, Trans. Amer. Math. Soc. 340 (1993), 1-35.
[7] B.S. Mordukhovich, Variational Analysis and Generalized Differentiation I/II, Springer-verlag, Berlin Heidelberg, 2006.
[8] A.L. Dontchev, R.T. Rockafellar, Regularity and conditioning of solution mappings in variational analysis, Set-Valued Anal. 12 (2004), 79-109.
[9] L. Huang, Q. He, Z. Wei, On metric subregularity for convex constraint systems by primal equivalent conditions, Optim. Lett. 11 (2017), 1713–1728.
[10] A.D. Ioffe, V. Outrata, On metric and calmness qualification conditions in subdifferential calculus, Set-Valued Anal. 16 (2008), 199-227.
[11] J.-P. Penot, Error bounds, calmness and their applications in nonsmooth analysis. In: Nonlinear analysis and optimization II. Optimization, Contemporary Mathematics, vol. 514, pp. 225-247. American Mathematical Society, Providence, 2010.
[12] Z. Wei, J.-C. Yao, X.Y. Zheng, Strong Abadie CQ, ACQ, calmness and linear regularity, Math. Program. 145 (2014), 97-131.
[13] C. Zălinescu, Weak sharp minima, well-behaving functions and global error bounds for convex inequalities in Banach spaces, Proc. 12th Baikal Internat. Conf. on Optimization Methods and Their Applications, pp. 272-284, Irkutsk, 2001.
[14] B. Zhang, W. Ouyang, Coincidence points for set-valued mappings with directional regularity, Fixed Point Theory 22 (2021), 391-406.
[15] B. Zhang, X.Y. Zheng, Well-posedness and generalized metric subregularity with respect to an admissible function, Sci. China Math. 62 (2019), 809-822.

[16] B. Zhang, J. Zhu, Pseudo metric subregularity and its stability in Asplund spaces, Positivity, 25 (2021), 469-494.
[17] X.Y. Zheng, K. F. Ng, Metric regularity and constraint qualifications for convex inequalities on Banach spaces, SIAM J. Optim. 14 (2004), 757-772.
[18] R. Henrion, R. Jourani, J. Outrata, On the calmness of a class of multifunctions, SIAM J. Optim. 13 (2002), 603-618.
[19] R. Henrion, J. Outrata, A subdifferential condition for calmness of multifunctions. J. Math. Anal. Appl. 258 (2001), 110–130.
[20] R. Henrion, J. Outrata, Calmness of constraint systems with applications, Math. Program. 104 (2005), 437-464.
[21] X.Y. Zheng, K.F. Ng, Metric subregularity and constraint qualifications for Convex Generalized equations in Banach spaces, SIAM J. Optim. 18 (2007), 437-460.
[22] X.Y. Zheng, K.F. Ng, Metric subregularity and calmness for nonconvex generalized equatioins in Banach spaces, SIAM. J. Optim., 20 (2010), 2119-2136.
[23] H. Gfrerer, First order and second order characterizations of metric subregularity and calmness of constrant set mapping, SIAM J. Optim. 21 (2011), 1439-1474.
[24] L. Huang, Q. He, Z. Wei, BCQ and strong BCQ for nonconvex generalized equations with applications to metric subregularity, Set-Valued Var. Anal. 22 (2014), 747-762.
[25] A.Y. Kruger, Error bounds and metric subregularity, Opitimization 64 (2015), 49-79.
[26] A. Y. Kruger, Error bounds and Hölder metric subregularity, Set-Valued Var. Anal. 23 (2015), 705-736.
[27] G. Li, B.S. Mordukhovich, Hölder metric subregularity with applications to proximal point methods, SIAM J. Optim. 22 (2012), 1655-1684.
[28] B.S. Mordukhovich, W. Ouyang, Higher-order metric subregularity and its applications, J. Global Optim. 63 (2015), 777-795.
[29] H.V. Ngai, P.N. Tinh, Metric subregularity of set-valued mappings: first and second order infinitesimal characterizations, Math. Oper. Res. 40 (2015), 703-724.
[30] Z. Wei, C. Tammer, J.-C. Yao, Characterizations for strong Abadie constraint qualification and applications to calmness, J. Optim. Theory Appl. 189 (2021), 1-18.
[31] Z. Wei, J.-C. Yao, Abadie constraint qualifications for convex constraint systems and applications to calmness property, J. Optim. Theory Appl. 174 (2017), 388-407.
[32] J. P. Aubin, H. Frankowska, Set-valued Analysis. Birkhäuser, Boston, 1990.
[33] Z. Shen, J.-C. Yao, X.Y. Zheng, Calmness and the Abadie CQ for multifunctions and linear regularity for a collection of closed sets, SIAM J. Optim. 29 (2019), 2291-2319.