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ON SPLIT EQUALITY GENERALIZED EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS OF BREGMAN RELATIVELY NONEXPANSIVE SEMIGROUPS

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Abstract. In this paper, we introduce a new split inverse problem called the split equality generalized equilibrium problem which is more general than the split feasibility problem, the split equilibrium problem, and the split equality equilibrium problem. We develop an iterative algorithm for approximating a common solution of this problem and the split equality fixed point problem for Bregman relatively nonexpansive semigroups in p-uniformly convex and uniformly smooth Banach spaces. Using our iterative algorithm, we prove a strong convergence theorem and investigate a split equality convex optimization problem as an application. Finally, we present some numerical experiments to demonstrate the applicability of our proposed method.
Keywords. Bregman relatively nonexpansive semigroup; Generalized equilibrium problem; p-uniformly convex and uniformly smooth Banach spaces; Split feasibility problem.

1. Introduction

Let C be a nonempty, closed, and convex subset of a Banach space E, and let $T : C \to C$ be a nonlinear mapping. The Fixed Point Problem (FPP) is formulated as finding a point $x^* \in C$ such that $Tx^* = x^*$. The point x^* is called a fixed point of T. We denote by $Fix(T)$, the set of all the fixed points of T, i.e., $Fix(T) = \{x^* \in C : Tx^* = x^*\}$. Several problems in engineering and sciences can be formulated as the FPP of nonlinear mappings.

Recently, the problem of finding a common solution of the FPP and some optimization problem (OP) has attracted great research attention. The motivation for studying such a common solution problem lies in its potential applications to mathematical models with more than constraints. This arises in various practical problems, such as network resource allocation, signal processing, and image recovery. An instance is in network bandwidth allocation problem for two services in a heterogeneous wireless access networks in which the bandwidth of the services are mathematically related (see, e.g., [1, 2, 3]).

Let C be a nonempty, closed, and convex subset of a real Banach space E, and let $F : C \times C \to \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of real numbers. The Equilibrium Problem (EP) is to find

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$z^* \in C$ such that

$$F(z^*,x) \geq 0, \ \forall\, x \in C. \tag{1.1}$$

We denote by $EP(F)$, the set of solutions of (1.1). It is known that many problems in economics can be reduce to problem (1.1). Since the introduction of EP (1.1) by Blum and Oettli [4], many authors have used various iterative algorithms, such as Halpern, viscosity, hybrid, cyclic, shrinking, and so on to approximate solutions of EP (1.1) in Hilbert and Banach space; see, e.g., [5, 6, 7, 8, 9, 10, 11, 12, 13] and the references therein.

An important generalization of the EP is the Generalized Equilibrium Problem (GEP) (see [14, 15]) defined as follows: Find $z^* \in C$ such that

$$F(z^*,x) + \phi(z^*,x) - \phi(z^*,z^*) \geq 0, \ \forall\, x \in C; \tag{1.2}$$

where C is a nonempty, closed, and convex subset of a real Banach space E and $F : C \times C \to \mathbb{R}$ and $\phi : C \times C \to \mathbb{R}$ are bifunctions. We denote by $GEP(F,\phi)$ the set of solutions of (1.2).

Remark 1.1. If $\phi = 0$, then the GEP (1.2) reduces to EP (1.1).

In order to model the inverse problems which arises from phase retrievals and medical image reconstruction (see [16]), Censor and Elfving [17] introduced the Split Feasibility Problem (SFP) in 1994, which is to find

$$u^* \in C \text{ such that } Au^* \in Q; \tag{1.3}$$

where C and Q are nonempty, closed, and convex subsets of real Banach spaces E_1 and E_2, respectively, and $A : E_1 \to E_2$ is a bounded linear operator.

Following the idea of the SFP (1.3), many other optimization problems, such as the Split Equilibrium Problem (SEP), the Split Variational Inclusion Problem (SVIP), the Split Variational Inequality Problem (SVP), the Split Minimization Problem (SMP), and so on, have been introduced; see, e.g., [18, 19, 20, 21] and the references therein.

In 2013, Kazmi and Rizvi [18] introduced the so-called SEP in Hilbert spaces, which is to:

$$\text{find } u^* \in C \text{ such that } F(u^*,x) \geq 0, \ \forall\, x \in C; \tag{1.4}$$

and

$$v^* = Au^* \in Q \text{ solves } G(v^*,y) \geq 0, \ \forall\, y \in Q; \tag{1.5}$$

where C and Q are nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively, $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ are bifunctions with a bounded linear operator $A : H_1 \to H_2$.

Moudafi [19] proposed a new SFP: Split Equality Problem (SEP). Let H_1, H_2, and H_3 be real Hilbert spaces. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators, and let $C \subset H_1$ and $Q \subset H_2$ be two nonempty, closed, and convex sets. The SEP is formulated as follows:

$$\text{find } x \in C, \ y \in Q \text{ such that } Ax = By. \tag{1.6}$$

If $C := Fix(S)$ and $Q := Fix(T)$ in (1.6), where $S : H_1 \to H_1$ and $T : H_2 \to H_2$ are two nonlinear mappings, then the SEP becomes the Split Equality Fixed Point Problem (SEFPP).

Following the idea of SFP (1.6), Ma et al. [22] introduced the Split Equality Equilibrium Problem (SEEP) in Banach spaces. Let E_1, E_2, and E_3 be three Banach spaces, and let C and

Q be nonempty, closed, and convex subsets of E_1 and E_2, respectively. Let $F : E_1 \times E_1 \to \mathbb{R}$ and $G : E_2 \times E_2 \to \mathbb{R}$ be two bifunctions and $A : E_1 \to E_3$, $B : E_2 \to E_3$ be two bounded linear operators. The SEEP is to find $u^* \in C$ and $v^* \in Q$ such that

$$F(u^*,u) \geq 0, \quad G(v^*,v) \geq 0 \; \forall \; u \in C, \; v \in Q \text{ and } Au^* = Bv^*. \tag{1.7}$$

We denote by SEEP(F,G), the set of solutions of SEEP (1.7). Furthermore, Ma et al. [22] proved the following weak and strong convergence theorem for finding a common element in the set of solutions of the SEFPP with nonexpansive mappings and the set of solutions of the SEEP in three Banach spaces as follows.

Theorem 1.1. *Let E_1, E_2 be real uniformly convex and 2-uniformly smooth Banach spaces satisfying Opial's condition with the smoothness constant k satisfying $0 < k \leq \frac{1}{\sqrt{2}}$, and let E_3 be a smooth, reflexive and strictly convex Banach space. Let $F_1 : E_1 \times E_1 \to \mathbb{R}$ and $F_2 : E_2 \times E_2 \to \mathbb{R}$ be bifunctions satisfying Assumption 2.1. Let $T : E_1 \to E_1$ and $S : E_2 \to E_2$ be two nonexpansive mappings with $Fix(T) \neq \emptyset$ and $Fix(S) \neq \emptyset$, respectively. Assume $A : E_1 \to E_3$ and $B : E_2 \to E_3$ are two bounded linear operators with adjoints A^* and B^*, respectively. Let the sequence $\{(x_n, y_n)\}$ in $E_1 \times E_2$ be generated for arbitrary $(x_1, y_1) \in E_1 \times E_2$ by*

$$\begin{cases} F_1(u_n,u) + \frac{1}{r}\langle u - u_n, J_1u_n - J_1x_n\rangle \geq 0, \; \forall \; u \in E_1; \\ F_2(v_n,v) + \frac{1}{r}\langle v - v_n, J_2v_n - J_2y_n\rangle \geq 0; \; \forall \; v \in E_2; \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)T(u_n - \psi J_1^{-1}A^*J_3(Au_n - Bv_n)); \\ y_{n+1} = \alpha_n y_n + (1-\alpha_n)S(v_n + \psi J_2^{-1}B^*J_3(Au_n - Bv_n)), \end{cases}$$

where $r \in (0,\infty)$, $(||A||^2 + ||B||^2)^{-1} < \psi < 2(||A||^2 + ||B||^2)^{-1}$ and $\{\alpha_n\}$ is a sequence in $[a,b]$ for some $a,b \in (0,1)$. If $\Gamma := SEFPP(T,S) \cap SEEP(F_1,F_2) \neq \emptyset$, then
(i) $(x_n,y_n) \rightharpoonup (p,q) \in \Gamma$.
(ii) Furthermore, if S and T are semi-compact, then $(x_n,y_n) \to (p,q) \in \Gamma$.

Recently, Cholamjiak and Sunthrayuth [23] proposed an Halpern-type iterative scheme for finding a solution of the SFP and the fixed point problem of Bregman relatively nonexpansive semigroups in the framework of p-uniformly convex and uniformly smooth Banach spaces. They proved the following strong convergence result.

Theorem 1.2. *Let E_1 and E_2 be two real p-uniformly convex and uniformly smooth Banach spaces, let C and Q be a nonempty, closed and convex subsets of E_1 and E_2, respectively. Let $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be adjoint of A. Let $V = \{T(t)\}_{t\geq 0}$ be u.a.r. Bregman relatively nonexpansive semigroup and uniformly Lipschitzian mapping of C into E_1 with a bounded measurable function $L(t) : (0,\infty) \to [0,\infty)$ such that $Fix(V) := \bigcap_{h\geq 0} Fix(T(h)) \neq \emptyset$. Suppose that $Fix(V) = \hat{Fix}(V)$ and $Fix(V) \bigcap SFP \neq \emptyset$. For given $u \in E_1$, let $\{u_n\}$ be a sequence generated by $u_1 \in C$ and*

$$\begin{cases} x_n = \Pi_C J^q_{E_1^*}(J^q_{E_1}(u_n) - \lambda_n A^* J^p_{E_2}(I - P_Q)Au_n); \\ u_{n+1} = \Pi_C J^q_{E_1^*}[\alpha_n J^p_{E_1}(u) + (1-\alpha_n)(\beta_n J^p_{E_1}(x_n) + (1-\beta_n)J^p_{E_1}T(t_n)x_n)], \; \forall \; n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$, $\{t_n\}$ is a real positive divergent sequence and $\{\lambda_n\}$ is a real positive sequence which satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < a \leq \beta_n \leq b < 1$;
(iii) $0 < c \leq \lambda_n \leq d < \left(\frac{q}{k_q||A||^q}\right)^{\frac{1}{q-1}}$.
Then, the sequence $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to an element* $x^* = \Pi_{Fix(V)\cap SFP}u$.

In 2018, Kazmi et al. [24] introduced an hybrid iterative method for finding a common solution of the GEP and the fixed point problem of Bregman relatively nonexpansive mappings in reflexive Banach spaces. They proved the following theorem.

Theorem 1.3. *Let C be a nonempty, closed, and convex subset of a reflexive Banach space E with dual E^* such that $C \subset int(domf)$. Let $f : E \to (-\infty, +\infty]$ be a coercive Legendre function which is bounded, uniformly Frechet differentiable, and totally convex on bounded subsets of E. Let $G : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 2.1 and $\phi : C \times C \to \mathbb{R}$ satisfy Assumption 2.2. Let $T : C \to C$ be a Bregman relatively nonexpansive mapping. Assume $\Omega = GEP(G,\phi) \cap Fix(T) \neq \emptyset$. Let $\{x_n\}$ and $\{z_n\}$ be the sequences generated by the iterative schemes:*

$$\begin{cases} x_0, z_0 \in C; \\ u_n = \bigtriangledown f^*(\alpha_n \bigtriangledown f(z_n) + (1-\alpha_n) \bigtriangledown f(Tx_n)); \\ z_{n+1} = res^f_{G,\phi} u_n; \\ C_n = \{z \in C : \Delta_f(z, z_{n+1}) \leq \Delta_f(z, z_n) + (1-\alpha_n)\Delta_f(z, x_n)\}; \\ Q_n = \{z \in C : \langle \bigtriangledown f(x_0) - \bigtriangledown f(x_n), z - x_n \rangle \leq 0\}; \\ x_{n+1} = proj^f_{C_n \bigcap Q_n} x_0, \ \forall\, n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\lim_{n\to\infty} \alpha_n = 0$. Then, $\{x_n\}$ converges strongly to $proj^f_{\Omega} x_0$, where $proj^f_{\Omega} x_0$ is the Bregman projection of C onto Ω.

Motivated by the ongoing research in this direction and the idea of GEP (1.2) and SEP (1.6), we introduce the following Split Equality Generalized Equilibrium Problem (SEGEP).

Definition 1.1. Let E_1, E_2, and E_3 be three Banach spaces. Let C and Q be nonempty, closed, and convex subsets of E_1 and E_2 respectively. Let $F : C \times C \to \mathbb{R}, G : Q \times Q \to \mathbb{R}$ and $\phi : C \times C \to \mathbb{R}, \psi : Q \times Q \to \mathbb{R}$ be bifunctions. Then the SEGEP is to find $x^* \in C$ and $y^* \in Q$ such that

$$F(x^*, x) + \phi(x^*, x) - \phi(x^*, x^*) \geq 0, \ \forall\, x \in C; \tag{1.8}$$

and

$$G(y^*, y) + \psi(y^*, y) - \psi(y^*, y^*) \geq 0, \ \forall\, y \in Q; \text{ such that } Ax^* = By^*. \tag{1.9}$$

We denote by SEGEP(F, G, ϕ, ψ), the set of solutions of SEGEP (1.8)-(1.9). Furthermore, we introduce an iterative algorithm to approximate the common solution of the SEGEP and the split equality fixed point problem of Bregman relatively nonexpansive semigroups in the framework of p-uniformly convex and uniformly smooth Banach spaces. We obtain a strong convergence result for the proposed algorithm and apply our result to split equality convex minimization problems. Finally, we present some numerical experiments to illustrate the applicability of our proposed method. The result obtained in this article generalizes the results of [22], [23], [24], and other related results in the literature.

2. PRELIMINARIES

We state some known and useful results which are be needed in the proof of our main theorem. In the sequel, we denote strong and weak convergence by "$\to$" and "$\rightharpoonup$", respectively. Let E be a real Banach space with norm $||.||$, and let E^* be the dual space of E. Let $K(E) := \{x \in E : ||x|| = 1\}$ denote the unit sphere of E. The modulus of convexity is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\big\{1 - \frac{||x+y||}{2} : x, y \in K(E),\ ||x-y|| \geq \varepsilon\big\}.$$

The space E is said to be uniformly convex if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. Let $p > 1$, then E is said to be p-uniformly convex (or to have a modulus of convexity of power type p) if there exists $c_p > 0$ such that $\delta_E(\varepsilon) \geq c_p \varepsilon^p$ for all $\varepsilon \in (0,2]$. Note that every p-uniformly convex space is uniformly convex. The modulus of smoothness of E is the function $\rho_E : \mathbb{R}^+ := [0,\infty) \to \mathbb{R}^+$ defined by

$$\rho_E(\tau) = \sup\big\{\frac{||x+\tau y + ||x-\tau y||}{2} - 1 : x, y \in K(E)\big\}.$$

The space E is said to be uniformly smooth if $\frac{\rho_E(\tau)}{\tau} \to 0$ as $\tau \to 0$. Let $q > 1$, then a Banach space E is said to be q-uniformly smooth if there exists $\kappa_q > 0$ such that $\rho_E(\tau) \leq \kappa_q \tau^q$ for all $\tau > 0$. It is known that E is p-uniformly convex if and only if E^* is q-uniformly smooth. Moreover, a Banach space E is p-uniformly convex if and only if E^* is q-uniformly smooth, where p and q satisfy $\frac{1}{p} + \frac{1}{q} = 1$, (see [25, 26] for details and other geometry properties on Banach spaces). Let $p > 1$ be a real number, the generalized duality mapping $J_E^p : E \to 2^{E^*}$ is defined by

$$J_E^p(x) = \{\bar{x} \in E^* : \langle x, \bar{x}\rangle = ||x||^p, ||\bar{x}|| = ||x||^{p-1}\},$$

where $\langle .,. \rangle$ denotes the duality pairing between elements of E and E^*. In particular, $J_E^p = J_E^2$ is called the normalized duality mapping. If E is p-uniformly convex and uniformly smooth, then E^* is q-uniformly smooth and uniformly convex. In this case, the generalized duality mapping J_E^p is one-to-one, single-valued, and satisfies $J_E^p = (J_{E^*}^q)^{-1}$, where $J_{E^*}^q$ is the generalized duality mapping of E^*. Furthermore, if E is uniformly smooth, then the duality mapping J_E^p is norm-to-norm uniformly continuous on bounded subsets of E (see [27] for more details). Let $f : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous, and convex function, then the Frenchel conjugate of f denoted as $f^* : E^* \to (-\infty, +\infty]$ is define as

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\},\ x^* \in E^*.$$

Let the domain of f be denoted as $(domf) = \{x \in E : f(x) < +\infty\}$. For any $x \in int(domf)$ and $y \in E$, we define the right-hand derivative of f at x in the direction y by

$$f^0(x,y) = \lim_{t \to 0^+} \frac{f(x+ty) - f(x)}{t}.$$

The function f is said to be G$\hat{a}$teaux differentiable at x if $\lim_{t\to 0^+} \frac{f(x+ty)-f(x)}{t}$ exists for any y. In this case, $f^0(x,y)$ coincides with $\triangledown f(x)$ (the value of the gradient $\triangledown f$ of f at x). The function f is said to be G$\hat{a}$teaux differentiable if it is G$\hat{a}$teaux differentiable for any $x \in int(domf)$. The function f is said to be Frechet differentiable at x if its limit is attained uniformly in $||y|| = 1$. In conclusion, f is said to be uniformly Frechet differentiable on a subset C of E if the above limit is attained uniformly for $x \in C$ and $||y|| = 1$. A function f is said to be Legendre if it satisfies the following conditions:

(1) The interior of the domain of f, $int(domf)$ is nonempty, f is Gâteaux differentiable on $int(domf)$ and $dom \bigtriangledown f = int(domf)$.

(2) The interior of the domain of f^*, $int(domf^*)$ is nonempty, f^* is Gâteaux differentiable on $int(domf^*)$ and $dom \bigtriangledown f^* = int(domf)$.

Definition 2.1. Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function $\Delta_f : E \times E \to [0, +\infty)$ defined by $\Delta_f(x,y) := f(x) - f(y) - \langle \bigtriangledown f(y), x - y \rangle$ is called the Bregman distance with respect of f.

It is known that Bregman distance Δ_f does not satisfy the properties of a metric because Δ_f fails to satisfy the symmetric and triangular inequality properties. Moreover, it is known that the duality mapping J_E^p is the sub-differential of the functional $f_p(.) = \frac{1}{p}||.||^p$ for $p > 1$ (see [28]). Then the Bregman distance Δ_p is defined with respect to f_p as follows:

$$\begin{aligned} \Delta_p(x,y) &= \frac{1}{p}\|x\|^p - \frac{1}{p}\|y\|^p - \langle J_E^p y, x - y \rangle \\ &= \frac{1}{p}\|x\|^p - \langle J_E^p y, x \rangle + \frac{1}{q}\|y\|^p \\ &= \frac{1}{q}\|y\|^p - \frac{1}{q}\|x\|^p - \langle J_E^p y - J_E^p x, x \rangle. \end{aligned} \tag{2.1}$$

Let $T : C \to int(domf)$ be a mapping.

(i) A point $p \in C$ is called an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} ||Tx_n - x_n|| = 0$. We denote by $\hat{Fix}(T)$ the set of asymptotic fixed points of T.

(ii) T is said to be Bregman quasi-nonexpansive if

$$Fix(T) \neq \emptyset \text{ and } \Delta_f(p, Tx) \leq \Delta_f(p, x),\ \forall\, x \in C,\ p \in Fix(T);$$

(iii) T is said to be Bregman relatively nonexpansive if

$$\hat{Fix}(T) = Fix(T) \neq \emptyset \text{ and } \Delta_f(p, Tx) \leq \Delta_f(p, x),\ \forall\, x \in C,\ p \in Fix(T).$$

Let E be a real Banach space. A one parameter family $V = \{S(t) : t \geq 0\}$ of mappings from E into E is said to be a nonexpansive semigroup if it satisfies the following conditions:

(A1) $S(0)x = x$ for all $x \in E$;

(A2) $S(t+u) = S(t)S(u)$ for all $t, u \geq 0$;

(A3) for each $x \in E$, the mapping $t \longmapsto S(t)x$ is continuous;

(A4) for each $t \geq 0$, $S(t)$ is nonexpansive, i.e., $||S(t)x - S(t)y|| \leq ||x - y||$, $\forall x, y \in E$. We denote by $Fix(V)$, the set of all fixed points of V, i.e., $Fix(V) = \{x \in C : S(t)x = x, t \geq 0\} = \bigcap_{t \geq 0} Fix(S(t))$.

Recall that a one-parameter family $V = \{S(t)\}_{t \geq 0} : C \to E$ is said to be Bregman relatively nonexpansive semigroup if it satisfies conditions (A1)-(A3) and the following conditions:

(a) $Fix(V)$ is nonempty;

(b) $Fix(V) = \hat{Fix}(V)$;

(c) $\Delta_p(S(t)x, z) \leq \Delta_p(x, z)$, $\forall\, x \in C$, $z \in Fix(V)$ and $t \geq 0$.

Definition 2.2. [23] A continuous operator semigroup $V=\{S(t)\}_{t\geq 0}:C\to E$ is said to be uniformly asymptotically regular (in short, u.a.r.) if, for all $u\geq 0$ and any bounded subset B of C, $\lim_{t\to\infty}\sup_{x\in B}||J_E^p(S(t)x)-J_E^p\big(S(u)S(t)x\big)||=0$.

A Bregman relatively nonexpansive semigroup $V=\{S(t)\}_{t\geq 0}:C\to E$ is said to be a uniformly Lipschitzian mapping if there exists a bounded measurable function $L(t):(0,\infty)\to[0,\infty)$ such that $||S(t)x-S(t)y||\leq L(t)||x-y||$, $\forall x,y\in C$. Recall that the metric projection P_C from E onto C satisfies the following property: $||x-P_Cx||\leq\inf_{y\in C}||x-y||$, $\forall x\in E$. It is known that P_Cx is the unique minimizer of the norm distance. Moreover, P_Cx is characterized by the following properties: $\langle J_E^p(x-P_Cx),y-P_Cx\rangle\leq 0$, $\forall y\in C$. The Bregman projection from E onto C denoted by Π_C also satisfies the property $\Delta_p(x,\Pi_C(x))=\inf_{y\in C}\Delta_p(x,y)$, $\forall x\in E$. Also, if C is a nonempty, closed, and convex subset of a p-uniformly convex and uniformly smooth Banach space E and $x\in E$. Then the following assertions holds: see [23]

(i) $z=\Pi_Cx$ if and only if

$$\langle J_E^p(x)-J_E^p(z),\ y-z\rangle\leq 0,\ \forall\ y\in C;$$

(ii)

$$\Delta_p(\Pi_Cx,y)+\Delta_p(x,\Pi_Cx)\leq\Delta_p(x,y),\ \forall\ y\in C. \tag{2.2}$$

Lemma 2.1. [28] *Let E be a Banach space and $x,y\in E$. If E is q-uniformly smooth, then there exists $C_q>0$ such that $||x-y||^q\leq||x||^q-q\langle J_q^E(x),y\rangle+C_q||y||^q$.*

Lemma 2.2. [29] *Let E be a real p-uniformly convex and uniformly smooth Banach space. Let $z,x_k\in E$ $(k=1,2,...,N)$ and $\alpha_k\in(0,1)$ with $\sum_{k=1}^N\alpha_k=1$. Then,*

$$\Delta_p(J_q^{E^*}(\sum_{k=1}^N\alpha_kJ_p^E(x_k)),z)\leq\sum_{k=1}^N\alpha_k\Delta_p(x_k,z)-\alpha_i\alpha_jg_r^*(||J_p^E(x_i)-J_p^E(x_j)||),$$

for all $i,j\in 1,2,...,N$ and $g_r^*:\mathbb{R}^+\to\mathbb{R}^+$ being a strictly increasing function such that $g_r^*(0)=0$.

Lemma 2.3. [30] *Let E be a real p-uniformly convex and uniformly smooth Banach space. Let $V_p:E^*\times E\to[0,+\infty)$ be defined by*

$$V_p(x^*,x)=\frac{1}{q}||x^*||^q-\langle x^*,x\rangle+\frac{1}{p}||x||^p,\ \forall\ x\in E,x^*\in E.$$

Then the following assertions hold:

(i) V_p is nonnegative and convex in the first variable.
(ii) $\Delta_p(J_q^{E^*}(x^*),x)=V_p(x^*,x)$, $\forall x\in E,\ x^*\in E$.
(iii) $V_p(x^*,x)+\langle y^*,J_q^{E^*}(x^*)-x\rangle\leq V_p(x^*+y^*,x)$, $\forall x\in E,\ x^*,y^*\in E$.

Lemma 2.4. [23] *Let E be a real p-uniformly convex and uniformly smooth Banach space. Suppose that $\{x_n\}$ and $\{y_n\}$ are bounded sequences in E. Then the following assertions are equivalent:*

(i) $\lim_{n\to\infty}\Delta_p(x_n,y_n)=0$;
(ii) $\lim_{n\to\infty}||x_n-y_n||=0$.

To solve EP (1.1) and GEP (1.2), we need the following assumptions ([31]):

Assumption 2.1.
(i) $F(x,x) = 0, \forall\, x \in C$;
(ii) F is monotone, i.e. $F(x,y) + F(y,x) \leq 0, \forall\, x,y \in C$;
(iii) For each $x,y,z \in C$; $\limsup_{t\to 0} F(tz + (1-t)x, y) \leq F(x,y)$;
(iv) For each $x \in C, y \longmapsto F(x,y)$ is convex and lower semicontinuous.

Assumption 2.2.
(i) $\phi(x,x) - \phi(x,y) - \phi(y,x) - \phi(y,y) \geq 0, \forall x,y \in C$, that is, ϕ is skew-symmetric;
(ii) ϕ is convex in the second argument;
(iii) ϕ is continuous.

Lemma 2.5. [24] *Let $f : E \to (-\infty, +\infty]$ be a coercive and Gateaux differentiable function. If $G : C \times C \to \mathbb{R}$ is a bifunction satisfying Assumption 2.1 and $\phi : C \times C \to \mathbb{R}$ satisfying Assumption 2.2, then $dom(res^f_{G,\phi}) = E$.*

Lemma 2.6. [24] *Let C be a nonempty, closed, and convex subset of a real reflexive Banach space E. Let $G : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 2.1 and $\phi : C \times C \to \mathbb{R}$ satisfy Assumption 2.2. Let $f : E \to (-\infty, +\infty]$ be a coercive Legendre function, and let $res^f_{G,\phi} : E \to 2^C$ be the resolvent associated with G and ϕ defined as follows:*

$$res^f_{G,\phi}(x) = \{z \in C : G(z,y) + \langle \bigtriangledown f(z) - \bigtriangledown f(x), y - z\rangle + \phi(z,y) - \phi(z,z) \geq 0,\ \forall\, y \in C\},\ \forall\, x \in E.$$

Then
(a) $res^f_{G,\phi}$ is single-valued Bregman firmly nonexpansive type mapping;
(b) $F(res^f_{G,\phi}) = Sol(GEP)$ is closed and convex;
(c) $\Delta_f(q, res^f_{G,\phi}x) + \Delta_f(res^f_{G,\phi}x, x) \leq \Delta_f(q,x),\ \forall\, q \in F(res^f_{G,\phi})$ and $x \in E$;
(d) $res^f_{G,\phi}$ is Bregman quasi-nonexpansive.

Lemma 2.7. [32] *Let $f : E \to (-\infty, +\infty]$ be uniformly Frechet differentiable and bounded on bounded subsets of E. Then f is uniformly continuous on bounded subsets of E and $\bigtriangledown f$ is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Lemma 2.8. [33] *Assume that $\{a_n\}$ is a real number sequence such that there exists a real subsequence $\{n_i\}$ of $\{n\}$ with $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing real sequence $\{m_k\} \subset \mathbb{N}$ with $m_k \to \infty$ and the following conditions are satisfied for all (sufficiently large) numbers $k \in \mathbb{N}$, $a_{m_k} \leq a_{m_k+1}$ and $a_k \leq a_{m_k+1}$. In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.*

Lemma 2.9. [34] *Assume $\{a_n\}$ is a nonnegative real sequence with $a_{n+1} \leq (1-\sigma_n)a_n + \sigma_n\delta_n$, where $\{\sigma_n\}$ is a reak sequence in $(0,1)$ and $\{\delta_n\}$ is a real sequence with (i) $\sum_{n=1}^{\infty}\sigma_n = \infty$; (ii) $\limsup_{n\to\infty}\delta_n \leq 0$ or $\sum_{n=1}^{\infty}|\sigma_n\delta_n| < \infty$. Then $\lim_{n\to\infty} a_n = 0$.*

3. Main Results

Lemma 3.1. *Let E_1, E_2, and E_3 be three p-uniformly convex and uniformly smooth Banach spaces. Let C and Q be nonempty, closed, and convex subsets of E_1 and E_2 with duals E_1^* and E_2^*, respectively. Let $f : E_1 \to (-\infty, +\infty]$ and $g : E_2 \to (-\infty, +\infty]$ be coercive Legendre* *functions which are bounded, uniformly Frechet differentiable, and totally convex on bounded subsets of E_1 and E_2, respectively. Let $F : C \times C \to \mathbb{R}$, $G : Q \times Q \to \mathbb{R}$ and $\phi : C \times C \to \mathbb{R}$, $\psi : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumptions 2.1 and 2.2, respectively. Let $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be bounded linear operators, and let $A^* : E_3^* \to E_1^*$, $B : E_3^* \to E_2^*$ be adjoint operators of A and B, respectively. Let $U = \{T(t)\}_{t\geq 0}$ and $V = \{S(t)\}_{t\geq 0}$ be an u.a.r Bregman relatively nonexpansive semigroup and uniformly Lipschitzian mappings of C and Q into E_1 and E_2, respectively with bounded measurable functions $L(t) : (0,\infty) \to [0,\infty)$ and $D(t) : (0,\infty) \to [0,\infty)$ such that $Fix(U) := \bigcap_{h\geq 0} Fix(T(h)) \neq \emptyset$ and $Fix(V) := \bigcap_{k\geq 0} Fix(S(k)) \neq \emptyset$. Supposethat $Fix(U) = \hat{Fix}(U)$ and $Fix(V) = \hat{Fix}(V)$, and assume that $\Gamma = \{(\bar{x},\bar{y}) : \bar{x} \in Fix(U)\bigcap SEGEP(F,\phi),\ \bar{y} \in Fix(V)\bigcap SEGEP(G,\psi), A\bar{x} = B\bar{y}\} \neq \emptyset$. For a fixed $u \in E_1$ and $v \in E_2$, let $\{(x_n,y_n)\}$ be a sequence generated iteratively by*

$$\begin{cases} u_n = res^f_{F,\phi} x_n; \\ v_n = res^g_{G,\psi} y_n; \\ w_n = \Pi_C J^q_{E_1^*}[J^p_{E_1} u_n - \gamma_n A^* J^p_{E_3}(Au_n - Bv_n)]; \\ z_n = \Pi_Q J^q_{E_2^*}[J^p_{E_2} v_n + \gamma_n B^* J^p_{E_3}(Au_n - Bv_n)]; \\ x_{n+1} = J^q_{E_1^*}[\alpha_n J^p_{E_1}(u) + (1-\alpha_n)(\beta_n J^p_{E_1}(w_n) + (1-\beta_n) J^p_{E_1} T(t_n) w_n)]; \\ y_{n+1} = J^q_{E_2^*}[\alpha_n J^p_{E_2}(v) + (1-\alpha_n)(\beta_n J^p_{E_2}(z_n) + (1-\beta_n) J^q_{E_2} S(s_n) z_n)];\ \forall\, n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$, $\{s_n\}$ and $\{t_n\}$ are real positive sequences, $0 < \gamma \leq \gamma_n \leq \rho \leq \left(\frac{q}{C_q||A||^q}\right)^{\frac{1}{q-1}}$, and $0 < \gamma \leq \gamma_n < \rho \leq \left(\frac{q}{D_q||B||^q}\right)^{\frac{1}{q-1}}$. Then $\{(x_n,y_n)\}$ is bounded.

Proof. Let $(\bar{x},\bar{y}) \in \Gamma$,

$$\xi_n := J^q_{E_1^*}[J^p_{E_1} u_n - \gamma_n A^* J^p_{E_3}(Au_n - Bv_n)],$$

and

$$\omega_n := J^q_{E_2^*}[J^p_{E_2} v_n + \gamma_n B^* J^p_{E_3}(Au_n - Bv_n), \bar{x}].$$

Then from (3.1) and Lemma 2.1, we have that

$$\begin{aligned} &\Delta_p(w_n,\bar{x}) \leq \Delta_p(\xi_n,\bar{x}) \\ &= \frac{1}{q}||J^p_{E_1} u_n - \gamma_n A^* J^p_{E_3}(Au_n - Bv_n)||^q - \langle J^p_{E_1} u_n, \bar{x}\rangle + \gamma_n \langle J^p_{E_3}(Au_n - Bv_n), A\bar{x}\rangle + \frac{1}{p}||\bar{x}||^p \\ &\leq \frac{1}{q}||J^p_{E_1} u_n||^q - \gamma_n \langle Au_n, J^p_{E_3}(Au_n - Bv_n)\rangle + \frac{C_q(\gamma_n||A||)^q}{q}||J^p_{E_3}(Au_n - Bv_n)||^q \\ &- \langle J^p_{E_1} u_n, \bar{x}\rangle + \gamma_n \langle J^p_{E_3}(Au_n - Bv_n), A\bar{x}\rangle + \frac{1}{p}||\bar{x}||^p \\ &= \frac{1}{q}||u_n||^p - \langle J^p_{E_1} u_n, \bar{x}\rangle + \frac{1}{p}||\bar{x}||^p + \gamma_n \langle J^p_{E_3}(Au_n - Bv_n), A\bar{x} - Au_n\rangle \\ &+ \frac{C_q(\gamma_n||A||)^q}{q}||J^p_{E_3}(Au_n - Bv_n)||^q \\ &= \Delta_p(u_n,\bar{x}) + \gamma_n \langle J^p_{E_3}(Au_n - Bv_n), A\bar{x} - Au_n\rangle + \frac{C_q(\gamma_n||A||)^q}{q}||(Au_n - Bv_n)||^p. \end{aligned} \tag{3.2}$$

Following (3.2), we have that

$$\begin{aligned}\Delta_p(z_n,\bar{y}) &\leq \Delta_p(\omega_n,\bar{y})\\ &\leq \Delta_p(v_n,\bar{y}) - \gamma_n\langle J^p_{E_3}(Au_n - Bv_n), B\bar{y} - Bv_n\rangle + \frac{D_q(\gamma_n||B||)^q}{q}||(Au_n - Bv_n)||^p. \quad (3.3)\end{aligned}$$

Adding (3.2) and (3.3) and using the fact that $A\bar{x} = B\bar{y}$, we obtain that

$$\begin{aligned}\Delta_p(w_n,\bar{x}) + \Delta_p(z_n,\bar{y}) &\leq \Delta_p(\xi_n,\bar{x}) + \Delta_p(\omega_n,\bar{y})\\ &\leq \Delta_p(u_n,\bar{x}) + \Delta_p(v_n,\bar{y}) - \gamma_n\langle J^p_{E_3}(Au_n - Bv_n), Au_n - Bv_n\rangle\\ &\quad + \frac{C_q(\gamma_n||A||)^q}{q}||(Au_n - Bv_n)||^p + \frac{D_q(\gamma_n||B||)^q}{q}||(Au_n - Bv_n)||^p\\ &= \Delta_p(u_n,\bar{x}) + \Delta_p(v_n,\bar{y})\\ &\quad - \left[\gamma_n - \left(\frac{C_q(\gamma_n||A||)^q}{q} + \frac{D_q(\gamma_n||B||)^q}{q}\right)\right]||Au_n - Bv_n||^p. \quad (3.4)\end{aligned}$$

Using (3.1), we have that

$$\Delta_p(u_n,\bar{x}) + \Delta_p(v_n,\bar{y}) \leq \Delta_p(x_n,\bar{x}) + \Delta_p(y_n,\bar{y}). \quad (3.5)$$

Now, let $a_n = J^q_{E_1^*}(\beta_n J^p_{E_1}(w_n) + (1-\beta_n)J^p_{E_1}T(t_n)w_n)$ and $b_n = J^q_{E_2^*}(\beta_n J^p_{E_2}(z_n) + (1-\beta_n)J^q_{E_2}S(s_n)z_n)$. From Lemma 2.2, we have that

$$\begin{aligned}\Delta_p(a_n,\bar{x}) &\leq \beta_n\Delta_p(w_n,\bar{x}) + (1-\beta_n)\Delta_p(T(t_n)w_n,\bar{x})\\ &\quad - \beta_n(1-\beta_n)g_r^*(||J^p_{E_1}(w_n) - J^p_{E_1}T(t_n)w_n||)\\ &\leq \Delta_p(w_n,\bar{x}) - \beta_n(1-\beta_n)g_r^*(||J^p_{E_1}(w_n) - J^p_{E_1}T(t_n)w_n||)\\ &\leq \Delta_p(w_n,\bar{x}). \quad (3.6)\end{aligned}$$

Hence $\Delta_p(b_n,\bar{y}) \leq \Delta_p(z_n,\bar{y})$ and

$$\Delta_p(a_n,\bar{x}) + \Delta_p(b_n,\bar{y}) \leq \Delta_p(w_n,\bar{x}) + \Delta_p(z_n,\bar{y}). \quad (3.7)$$

From (3.1) and (3.6), we have that

$$\begin{aligned}\Delta_p(x_{n+1},\bar{x}) &\leq \alpha_n\Delta_p(u,\bar{x}) + (1-\alpha_n)\Delta_p(a_n,\bar{x})\\ &\leq \alpha_n\Delta_p(u,\bar{x}) + (1-\alpha_n)\Delta_p(w_n,\bar{x}). \quad (3.8)\end{aligned}$$

Similarly, we have that $\Delta_p(y_{n+1},\bar{y}) \leq \alpha_n\Delta_p(v,\bar{y}) + (1-\alpha_n)\Delta_p(z_n,\bar{y})$, which together with (3.5) and (3.8) obtains that

$$\begin{aligned}\Delta_p(x_{n+1},\bar{x}) + \Delta_p(y_{n+1},\bar{y}) &\leq \alpha_n[\Delta_p(u,\bar{x}) + \Delta_p(v,\bar{y})] + (1-\alpha_n)[\Delta_p(x_n,\bar{x}) + \Delta_p(y_n,\bar{y})]\\ &\leq \max\{\Delta_p(u,\bar{x}) + \Delta_p(v,\bar{y}), \Delta_p(x_n,\bar{x}) + \Delta_p(y_n,\bar{y})\}\\ &\ \vdots\\ &\leq \max\{\Delta_p(u,\bar{x}) + \Delta_p(v,\bar{y}), \Delta_p(x_1,\bar{x}) + \Delta_p(y_1,\bar{y})\}.\end{aligned}$$

Therefore, we conclude that $\{\Delta_p(x_n,\bar{x}) + \Delta_p(y_n,\bar{y})\}$ is bounded. Consequently, $\{\Delta_p(u_n,\bar{x}) + \Delta_p(v_n,\bar{y})\}$ and $\{\Delta_p(w_n,\bar{x}) + \Delta_p(z_n,\bar{y})\}$ are bounded. □

Theorem 3.1. *Let E_1, E_2, and E_3 be three p-uniformly convex and uniformly smooth Banach spaces. Let C and Q be nonempty, closed, and convex subsets of E_1 and E_2 with duals E_1^* and E_2^*, respectively. Let $f : E_1 \to (-\infty, +\infty]$ and $g : E_2 \to (-\infty, +\infty]$ be coercive Legendre functions which are bounded, uniformly Frechet differentiable, and totally convex on bounded subsets of E_1 and E_2, respectively. Let $F : C \times C \to \mathbb{R}$, $G : Q \times Q \to \mathbb{R}$ and $\phi : C \times C \to \mathbb{R}$, $\psi : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumptions 2.1 and 2.2, respectively. Let $A : E_1 \to E_3$, $B : E_2 \to E_3$ be bounded linear operators, and let $A^* : E_3^* \to E_1^*$, $B : E_3^* \to E_2^*$ be adjoint operators of A and B, respectively. Let $U = \{T(t)\}_{t\geq 0}$ and $V = \{S(t)\}_{t\geq 0}$ be an u.a.r Bregman relatively nonexpansive semigroup and uniformly Lipschitzian mappings of C and Q into E_1 and E_2 respectively with bounded measurable functions $L(t) : (0,\infty) \to [0,\infty)$ and $D(t) : (0,\infty) \to [0,\infty)$ such that $Fix(U) := \bigcap_{h\geq 0} Fix(T(h)) \neq \emptyset$ and $Fix(V) := \bigcap_{k\geq 0} Fix(S(k)) \neq \emptyset$. Suppose $Fix(U) = \hat{Fix}(U)$ and $Fix(V) = \hat{Fix}(V)$, and assume that $\Gamma = \{(\bar{x}, \bar{y}) : \bar{x} \in Fix(U) \bigcap SEGEP(F, \phi),\ \bar{y} \in Fix(V) \bigcap SEGEP(G, \psi), A\bar{x} = B\bar{y}\} \neq \emptyset$. For a fixed $u \in E_1$ and $v \in E_2$, let $\{(x_n, y_n)\}$ be the sequence generated iteratively by (3.1), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ satisfying the following conditions:*

(i) $\lim_{n\to\infty} s_n = +\infty, \lim_{n\to\infty} t_n = +\infty, \lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = +\infty, 0 < a \leq \beta_n \leq b < 1$;

(ii) $0 < \gamma \leq \gamma_n \leq \rho \leq \left(\frac{q}{C_q||A||^q}\right)^{\frac{1}{q-1}},\ 0 < \gamma \leq \gamma_n < \rho \leq \left(\frac{q}{D_q||B||^q}\right)^{\frac{1}{q-1}}$.

Then $\{(x_n, y_n)\}$ converges strongly to $(x^*, y^*) \in \Gamma$.

Proof. Let $(\bar{x}, \bar{y}) \in \Gamma$. Then, from (3.1) and Lemma 2.3, (3.6), we obtain that

$$\begin{aligned}
\Delta_p(x_{n+1}, \bar{x}) &= V_p(\alpha_n J_{E_1}^p(u) + (1-\alpha_n) J_{E_1}^p(a_n), \bar{x}) \\
&\leq V_p(\alpha_n J_{E_1}^p(u) + (1-\alpha_n) J_{E_1}^p(a_n) - \alpha_n (J_{E_1}^p(u) - J_{E_1}^p(\bar{x}), \bar{x})) \\
&\quad - \langle -\alpha_n (J_{E_1}^p(u) - J_{E_1}^p(\bar{x})), J_{E_1^*}^q[\alpha_n J_{E_1}^p(u) + (1-\alpha_n) J_{E_1}^p(a_n)] - \bar{x}\rangle \\
&\leq \alpha_n V_p(J_{E_1}^p(\bar{x}), \bar{x}) + (1-\alpha_n) V_p(J_{E_1}^p(a_n), \bar{x}) + \alpha_n \langle J_{E_1}^p(u) - J_{E_1}^p(\bar{x}), x_{n+1} - \bar{x}\rangle \\
&= \alpha_n \Delta_p(\bar{x}, \bar{x}) + (1-\alpha_n)\Delta_p(a_n, \bar{x}) + \alpha_n \langle J_{E_1}^p(u) - J_{E_1}^p(\bar{x}), x_{n+1} - \bar{x}\rangle \\
&\leq (1-\alpha_n)\Delta_p(w_n, \bar{x}) - \beta_n(1-\beta_n) g_r^*(||J_{E_1}^p(w_n) - J_{E_1}^p(T(t_n)w_n||)) \\
&\quad + \alpha_n \langle J_{E_1}^p(u) - J_{E_1}^p(\bar{x}), x_{n+1} - \bar{x}\rangle. \qquad (3.9)
\end{aligned}$$

Similarly, we have that

$$\begin{aligned}
\Delta_p(y_{n+1}, \bar{y}) &\leq (1-\alpha_n)\Delta_p(z_n, \bar{y}) - \beta_n(1-\beta_n) g_r^*(||J_{E_2}^p(z_n) - J_{E_2}^p(S(s_n)z_n||)) \\
&\quad + \alpha_n \langle J_{E_2}^p(v) - J_{E_2}^p(\bar{y}), y_{n+1} - \bar{y}\rangle. \qquad (3.10)
\end{aligned}$$

On adding (3.9) and (3.10), and substituting (3.4) and (3.5), we have that

$$\begin{aligned}
&\Delta_p(x_{n+1}, \bar{x}) + \Delta_p(y_{n+1}, \bar{y}) \\
&\leq (1-\alpha_n)[\Delta_p(x_n, \bar{x}) + \Delta_p(y_n, \bar{y})] \\
&\quad - \beta_n(1-\beta_n) g_r^*[||J_{E_1}^p(w_n) - J_{E_1}^p(T(t_n)w_n||) + ||J_{E_2}^p(z_n) - J_{E_2}^p(S(s_n)z - n||)] \\
&\quad - [\gamma_n - (\frac{Cq(\gamma_n||A||)^q}{q}) + \frac{D_q(\gamma_n||B||)^q}{q}]||Au_n - Bv_n||^p \\
&\quad + \alpha_n[\langle J_{E_1}^p(u) - J_{E_1}^p(\bar{x}), x_{n+1} - \bar{x}\rangle + \langle J_{E_2}^p(v) - J_{E_2}^p(\bar{y}), y_{n+1} - \bar{y}\rangle]. \qquad (3.11)
\end{aligned}$$

We now divide our proof in two cases.

CASE A: Suppose that $\{\Delta_p(x_n,\bar{x})+\Delta_p(y_n,\bar{y})\}$ is monotone non-increasing. Then $\{\Delta_p(x_n,\bar{x})+\Delta_p(y_n,\bar{y})\}$ is convergent. Hence,

$$\lim_{n\to\infty}\{(\Delta_p(x_n,\bar{x}))+\Delta_p(y_n,\bar{y})-(\Delta_p(x_{n+1},\bar{x})+\Delta_p(y_{n+1},\bar{y}))\}=0.$$

From (3.11), condition (i) and (ii) of (3.1), we have that

$$\begin{aligned}&\beta_n(1-\beta_n)g_r^*[||J_{E_1}^p(w_n)-J_{E_1}^p(T(t_n)w_n)||+||J_{E_2}^p(z_n)-J_{E_2}^p(S(s_n)z_n)||]\\&\leq(1-\alpha_n)[\Delta_p(x_n,\bar{x})+\Delta_p(y_n,\bar{y})]-[\Delta_p(x_{n+1},\bar{x})+\Delta_p(y_{n+1},\bar{y})]\\&\quad+\alpha_n[\langle J_{E_1}^p(u)-J_{E_1}^p(\bar{x}),x_{n+1}-\bar{x}\rangle+\langle J_{E_2}^p(v)-J_{E_2}^p(\bar{y}),y_{n+1}-\bar{y}\rangle],\end{aligned}$$

which implies by the property of g_r^* that

$$\lim_{n\to\infty}[||J_{E_1}^p(w_n)-J_{E_1}^p(T(t_n)w_n)||+||J_{E_2}^p(z_n)-J_{E_2}^p(S(s_n)z_n)||]=0.$$

Consequently, we have

$$\lim_{n\to\infty}||J_{E_1}^p(w_n)-J_{E_1}^p(T(t_n)w_n)||=\lim_{n\to\infty}||J_{E_2}^p(z_n)-J_{E_2}^p(S(s_n)z_n)||=0. \tag{3.12}$$

Since $J_{E_1^*}^q$ and $J_{E_2^*}^q$ are norm-to-norm uniformly continuous on bounded subsets E_1^* and E_2^* respectively, we have that

$$\lim_{n\to\infty}||w_n-T(t_n)w_n||=\lim_{n\to\infty}||z_n-S(s_n)z_n||=0. \tag{3.13}$$

From Lemma (2.4), we also have that

$$\lim_{n\to\infty}\Delta_p(T(t_n)w_n,w_n)=\lim_{n\to\infty}\Delta_p(S(s_n)z_n,z_n)=0. \tag{3.14}$$

Since $\{T(t)\}_{t\geq0}$ and $\{S(t)\}_{t\geq0}$ are uniformly Lipschitzian with bounded measurable functions $L(t)$ and $D(t)$ respectively. Then, we have from (3.13) that

$$\begin{aligned}||T(t)T(t_n)w_n-T(t)w_n||&\leq L(t)||T(t_n)w_n-w_n||\\&\leq\sup_{t\geq0}\{L(t)\}||T(t_n)w_n-w_n||\to0,\ n\to\infty.\end{aligned} \tag{3.15}$$

Similarly, we obtain

$$\begin{aligned}||S(t)S(s_n)z_n-S(t)z_n||&\leq D(t)||S(s_n)z_n-z_n||\\&\leq\sup_{t\geq0}\{D(t)\}||S(s_n)z_n-z_n||\to0,\ n\to\infty.\end{aligned} \tag{3.16}$$

From (3.15) and (3.16) and the fact that $J_{E_1}^p$ and $J_{E_2}^p$ are uniformly continuous on bounded subsets of E_1 and E_2 respectively, we have

$$\lim_{n\to\infty}||J_{E_1}^p(T(t)T(t_n)w_n)-J_{E_1}^p(T(t)w_n)||=0=\lim_{n\to\infty}||J_{E_2}^p(S(t)S(s_n)z_n)-J_{E_2}^p(S(t)z_n)||. \tag{3.17}$$

For each $t\geq0$, we have that

$$\begin{aligned}||J_{E_1}^p(w_n)-J_{E_1}^p(T(t)w_n)||&\leq||J_{E_1}^p(w_n)-J_{E_1}^p(T(t_n)w_n)||+||J_{E_1}^p(T(t_n)w_n)-J_{E_1}^p(T(t)T(t_n)w_n)||\\&\quad+||J_{E_1}^p(T(t)T(t_n)w_n)-J_{E_1}^p(T(t)w_n)||\\&\leq||J_{E_1}^p(w_n)-J_{E_1}^p(T(t_n)w_n)||+||J_{E_1}^p(T(t)T(t_n)w_n)-J_{E_1}^p(T(t)w_n)||\\&\quad+\sup_{w\in\{w_n\}}||J_{E_1}^p(T(t_n)w)-J_{E_1}^p(T(t)T(t_n)w||.\end{aligned} \tag{3.18}$$

Similarly, we have that

$$||J^p_{E_2}(z_n) - J^p_{E_2}(S(t)z_n)|| \leq ||J^p_{E_2}(z_n) - J^p_{E_2}(S(s_n)z_n)|| + ||J^p_{E_2}(S(t)S(s_n)z_n) - J^p_{E_2}(S(t)z_n)|| \\ + \sup_{z\in\{z_n\}} ||J^p_{E_2}(S(s_n)z) - J^p_{E_1}(S(t)S(s_n)z||. \tag{3.19}$$

From (3.18), (3.19), (3.12), and (3.17), we have

$$\lim_{n\to\infty} ||J^p_{E_1}(w_n) - J^p_{E_1}T(t)w_n)|| = 0 = \lim_{n\to\infty} ||J^p_{E_2}(z_n) - J^p_{E_2}(S(t)z_n)||. \tag{3.20}$$

Since $J^p_{E_1^*}$ and $J^p_{E_2^*}$ are norm-to-norm uniformly continuous on bounded subsets of E_1^* and E_2^*. Then, we have from (3.20) that

$$\lim_{n\to\infty} ||w_n - T(t)w_n|| = 0 = \lim_{n\to\infty} ||z_n - S(t)z_n|| = 0. \tag{3.21}$$

Now, we have from (3.11) and condition (i) of (3.1) that

$$\lim_{n\to\infty} \left[\gamma_n - \left(\frac{Cq(\gamma_n||A||)^q}{q} + \frac{D_q(\gamma_n||B||)^q}{q}\right)\right] ||Au_n - Bv_n||^p = 0.$$

Since

$$0 < \gamma\left[1 - \left(\frac{C_q(\gamma_n||A||)^q}{q} + \frac{D_q(\gamma_n||B||)^q}{q}\right)\right] \leq \left[\gamma_n - \left(\frac{Cq(\gamma_n||A||)^q}{q} + \frac{D_q(\gamma_n||B||)^q}{q}\right)\right],$$

we have $\lim_{n\to\infty} ||Au_n - Bv_n|| = 0$. From the definition of ξ_n and ω_n, we obtain

$$||J^p_{E_1}\xi_n - J^p_{E_1}u_n|| \leq \gamma_n ||A^*|| ||J^p_{E_3}(Au_n - Bv_n)|| \\ \leq \left(\frac{q}{C_q||A||^q}\right)^{\frac{1}{q-1}} ||A|| ||Au_n - Bv_n||^{p-1} \to 0,\ n \to \infty. \tag{3.22}$$

Similarly, we obtain

$$||J^p_{E_2}\omega_n - J^p_{E_2}v_n|| \leq \left(\frac{q}{D_q||B||^q}\right)^{\frac{1}{q-1}} ||B|| ||Au_n - Bv_n||^{p-1} \to 0,\ n \to \infty. \tag{3.23}$$

Since $J^p_{E_1^*}$ and $J^p_{E_2^*}$ are norm-to-norm uniformly continuous on bounded subsets of E_1^* and E_2^* respectively, we have from (3.22) and (3.23) that

$$\lim_{n\to\infty} ||\xi_n - u_n|| = 0 = \lim_{n\to\infty} ||\omega_n - v_n||. \tag{3.24}$$

Furthermore, from (2.2) we have that

$$\Delta_p(\xi_n, w_n) \leq \Delta_p(\xi_n, \bar{x}) - \Delta_p(\Pi_C \xi_n, \bar{x}) \\ = \Delta_p(\xi_n, \bar{x}) - \Delta_p(w_n, \bar{x}). \tag{3.25}$$

Similarly, we obtain that

$$\Delta_p(\omega_n, z_n) \leq \Delta_p(\omega_n, \bar{y}) - \Delta_p(z_n, \bar{y}). \tag{3.26}$$

On adding (3.25) and (3.26) and substituting (2.1), (3.4), (3.5), and (3.8), we have that

$$\begin{aligned}\Delta_p(\xi_n,w_n)+\Delta_p(\omega_n,z_n) &\le \Delta_p(\xi_n,\bar{x})-\Delta_p(w_n,\bar{x})+[\Delta_p(\omega_n,\bar{y})-\Delta_p(z_n,\bar{y})]\\ &\le \Delta_p(u_n,\bar{x})+\Delta_p(v_n,\bar{y})-[\Delta_p(w_n,\bar{x})+\Delta_p(z_n,\bar{y})]\\ &\le [\Delta_p(x_n,\bar{x})+\Delta_p(y_n,\bar{y})]-[\Delta_p(w_n,\bar{x})+\Delta_p(z_n,\bar{y})]\\ &\le [\Delta_p(x_n,\bar{x})+\Delta_p(y_n,\bar{y})]-[\Delta_p(x_{n+1},\bar{x})+\Delta_p(y_{n+1},\bar{y})]\\ &\quad+\alpha_n[\Delta_p(u,\bar{x})+\Delta_p(v,\bar{y})]+(1-\alpha_n)[\Delta_p(w_n,\bar{x})+\Delta_p(z_n,\bar{y})]\\ &\quad-[\Delta_p(w_n,\bar{x})+\Delta_p(z_n,\bar{y})].\end{aligned}$$

From condition (i) of (3.1), we have that $\lim_{n\to\infty}[\Delta_p(\xi_n,w_n)+\Delta_p(\omega_n,z_n)]=0$, which implies that $\lim_{n\to\infty}\Delta_p(\xi_n,w_n)=0=\lim_{n\to\infty}\Delta_p(\omega_n,z_n)$. Hence,

$$\lim_{n\to\infty}||\xi_n-w_n||=0=\lim_{n\to\infty}||\omega_n-z_n||. \tag{3.27}$$

Hence, we have from (3.24) and (3.27) that

$$\lim_{n\to\infty}||w_n-u_n||=0=\lim_{n\to\infty}||z_n-v_n||. \tag{3.28}$$

From (3.4) and (3.8), we have that

$$\begin{aligned}\Delta_p(x_{n+1},\bar{x})+\Delta_p(y_{n+1},\bar{y}) &\le \alpha_n[\Delta_p(u,\bar{x})+\Delta_p(v,\bar{y})]+(1-\alpha_n)[\Delta_p(w_n,\bar{x})+\Delta_p(z_n,\bar{y})]\\ &\le \alpha_n[\Delta_p(u,\bar{x})+\Delta_p(v,\bar{y})]+(1-\alpha_n)[\Delta_p(u_n,\bar{x})+\Delta_p(v_n,\bar{y})].\end{aligned}$$

This implies that

$$-[\Delta_p(u_n,\bar{x})+\Delta_p(v_n,\bar{y})]\le \alpha_n[\Delta_p(u,\bar{x})+\Delta_p(v,\bar{y})]-[\Delta_p(x_{n+1},\bar{x})+\Delta_p(y_{n+1},\bar{y})]. \tag{3.29}$$

Now, using Lemma 2.6 (d), we have that

$$\Delta_p(u_n,x_n)\le \Delta_p(x_n,\bar{x})-\Delta_p(u_n,\bar{x}), \tag{3.30}$$

and

$$\Delta_p(v_n,y_n)\le \Delta_p(y_n,\bar{y})-\Delta_p(v_n,\bar{y}). \tag{3.31}$$

Adding (3.30) and (3.31) and substituting (3.29), we have that

$$\begin{aligned}\Delta_p(u_n,x_n)+\Delta_p(v_n,y_n) &\le \Delta_p(x_n,\bar{x})+\Delta_p(y_n,\bar{y})-[\Delta_p(u_n,\bar{x})+\Delta_p(v_n,\bar{y})]\\ &\le \Delta_p(x_n,\bar{x})+\Delta_p(y_n,\bar{y})+\alpha_n[\Delta_p(u,\bar{x})+\Delta_p(v,\bar{y})]\\ &\quad-[\Delta_p(x_{n+1},\bar{x})+\Delta_p(y_{n+1},\bar{y})].\end{aligned} \tag{3.32}$$

Using condition (i) of (3.1), we have that $\lim_{n\to\infty}\Delta_p(u_n,x_n)=0=\Delta_p(v_n,y_n)$. This also implies that

$$\lim_{n\to\infty}||u_n-x_n||=0=\lim_{n\to\infty}||v_n-y_n||=0. \tag{3.33}$$

From (3.28) and (3.33), we have that

$$||w_n-x_n||\le ||w_n-u_n||+||u_n-x_n||\to 0 \text{ as } n\to\infty. \tag{3.34}$$

Similarly, we have from (3.28) and (3.33) that

$$||z_n-y_n||\le ||z_n-v_n||+||v_n-y_n||\to 0, \text{ as } n\to\infty. \tag{3.35}$$

From (3.1), we have that

$$\Delta_p(a_n, w_n) \leq (1-\beta_n)\Delta_p(T(t_n)w_n, w_n). \tag{3.36}$$

It follows from (3.14) that $\lim_{n\to\infty}\Delta_p(a_n, w_n) = 0$, which implies that

$$\lim_{n\to\infty} ||a_n - w_n|| = 0. \tag{3.37}$$

Using the same approach as in (3.36) and (3.37), we have from (3.1) and (3.14) that

$$\lim_{n\to\infty} ||b_n - z_n|| = 0. \tag{3.38}$$

In view of (3.1), we have that

$$\begin{aligned}\Delta_p(x_{n+1}, a_n) &= \Delta_p(J^q_{E_1^*}[\alpha_n J^p_{E_1}(u) + (1-\alpha_n)J^p_{E_1}a_n], a_n)\\ &\leq \alpha_n\Delta_p(u, a_n) + (1-\alpha_n)\Delta_p(a_n, a_n).\end{aligned} \tag{3.39}$$

From condition (i) of (3.1), we have that $\lim_{n\to\infty}\Delta_p(x_{n+1}, a_n) = 0$, which implies that

$$\lim_{n\to\infty} ||x_{n+1} - a_n|| = 0. \tag{3.40}$$

Following the same approach as in (3.39) and (3.40), we have that

$$\lim_{n\to\infty} ||y_{n+1} - b_n|| = 0. \tag{3.41}$$

By (3.34) and (3.37), we obtain that

$$\lim_{n\to\infty} ||a_n - x_n|| = 0. \tag{3.42}$$

Also, from (3.35) and (3.38), we have that

$$\lim_{n\to\infty} ||b_n - z_n|| = 0. \tag{3.43}$$

Using (3.40) and (3.42), we obtain that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{3.44}$$

We also obtain from (3.41) and (3.43) that $\lim_{n\to\infty} ||y_{n+1} - y_n|| = 0$. Since $\{x_n\}$ is bounded in E_1 and E_1 is reflexive, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to x^*. By (3.21) and (3.34), we have that $x^* \in Fix(U) = \hat{Fix}(U)$. Also, since $\{y_n\}$ is bounded in E_2 and E_2 is reflexive, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ which converges weakly to y^*. By using similar argument as in above, we obtain that $y^* \in Fix(V) = \hat{Fix}(V)$.

Next, we show that $x^* \in \Omega$ which is the solution of Generalized Equilibrium Problem (GEP). From (3.33), there exist subsequences $\{u_{n_k}\}$ and $\{v_{n_k}\}$ such that $\{u_{n_k}\}$ and $\{v_{n_k}\}$ converges weakly to (x^*, y^*) as $k \to \infty$. From (3.1), since $u_n = res^f_{F,\phi}x_n$,, we have that

$$F(u_{n_k}, r) + \langle \bigtriangledown f(u_{n_k}) - \bigtriangledown f(x_{n_k}), r - u_{n_k}\rangle + \phi(r, u_{n_k}) - \phi(u_{n_k}, u_{n_k}) \geq 0,\ \forall\, r \in C.$$

Using Assumption 2.1, we have that

$$\langle \bigtriangledown f(u_{n_k}) - \bigtriangledown f(x_{n_k}), r - u_{n_k}\rangle \geq F(r, u_{n_k}) - \phi(r, u_{n_k}) + \phi(u_{n_k}, u_{n_k}), \forall\, r \in C.$$

Since F is lower semicontinuous in the second argument, ϕ is continuous, and f is uniformly Frechet differentiable, we obtain from Lemma 2.7 and inequality (3.33) that

$$0 \geq F(r, x^*) - \phi(r, x^*) + \phi(x^*, x^*),\ \forall\, r \in C.$$

Set $r_t = ty + (1-t)x^*$, $\forall\, t \in (0,1]$ and $r \in C$. Then, $r_t \in C$ and $F(r_t, x^*) - \phi(r_t, x^*) + \phi(x^*, x^*) \leq 0$. Now,

$$\begin{aligned} 0 = F(r_t, r_t) &\leq tF(r_t, y) + (1-t)F(r_t, x^*) \\ &\leq tF(r_t, y) + (1-t)t[\phi(y, x^*) - \phi(x^*, x^*)]. \end{aligned}$$

Since $t > 0$, we have $F(x^*, y) + \phi(y, x^*) - \phi(x^*, x^*) \geq 0$, $\forall y \in C$. Therefore, $x^* \in \Omega$. Following the same step as above, we have that $y^* \in \Omega$. We now show that $Ax^* = By^*$. Since $A : E_1 \to E_3$ and $B : E_2 \to E_3$ are bounded linear operators and $\{x_n\}$ and $\{y_n\}$ converges weakly to x^* and y^* respectively, we have $h \in E_3^*$,

$$\begin{aligned} h(Ax_n) &= (h \circ A)(x_n) \to (h \circ A)(x^*) = h(Ax^*) \\ h(By_n) &= (h \circ B)(y_n) = (h \circ B)(y^*) = h(By^*). \end{aligned}$$

This convergence implies that $Ax_n - By_n \to Ax^* - By^*$. Also, by weakly semi-continuity of the norm, it follows that $||Ax^* - By^*|| \leq \liminf_{n\to\infty} ||Ax_n - By_n|| = 0$. That is, $Ax^* = By^*$. Therefore $(x^*, y^*) \in \Gamma$. Now, we show that $\{(x_n, y_n)\}$ converges strongly to (x^*, y^*). From (3.11), we have that

$$\begin{aligned} &\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*) \\ &\leq (1-\alpha_n)[\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)] \\ &\quad + \alpha_n[\langle J_{E_1}^p(u) - J_{E_1}^p(x^*), x_{n+1} - x^*\rangle + \langle J_{E_2}^p(v) - J_{E_2}^p(y^*), y_{n+1} - y^*\rangle]. \end{aligned} \tag{3.45}$$

Since $\{x_n\}$ is bounded, we choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, such that $x_{n_k} \rightharpoonup x^*$, and

$$\begin{aligned} \limsup_{n\to\infty}\langle J_{E_1}^p(u) - J_{E_1}^p(x^*),\ x_n - x^*\rangle &= \limsup_{k\to\infty}\langle J_{E_1}^p(u) - J_{E_1}^p(x^*),\ x_{n_k} - x^*\rangle \\ &= \langle J_{E_1}^p(u) - J_{E_1}^p(x^*),\ \omega - x^*\rangle. \end{aligned}$$

Hence, we have from (3.44) that

$$\begin{aligned} \limsup_{n\to\infty}\langle J_{E_1}^p(u) - J_{E_1}^p(x^*),\ x_{n+1} - x_n\rangle &= \limsup_{n\to\infty}\langle J_{E_1}^p(u) - J_{E_1}^p(x^*),\ x_{n+1} - x_n\rangle \\ &\quad + \limsup_{n\to\infty}\langle J_{E_1}^p(u) - J_{E_1}^p(x^*),\ x_n - x^*\rangle = 0. \end{aligned} \tag{3.46}$$

Similarly, we obtain that

$$\limsup_{n\to\infty}\langle J_{E_2}^p(v) - J_{E_2}^p(y^*),\ y_{n+1} - y^*\rangle = 0. \tag{3.47}$$

Therefore, applying Lemma 2.9 in (3.45), we conclude that $\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*) \to 0$, $n \to \infty$. Therefore, $(x_n, y_n) \to (x^*, y^*)$.

CASE B: Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\Delta_p(x_{n_i}, x^*) + \Delta_p(y_{n_i}, y^*) \leq \Delta_p(x_{n_i+1}, x^*) + \Delta_p(y_{n_i+1}),\ \forall\, i \in \mathbb{N}.$$

By Lemma 2.8, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$. For all $k \in \mathbb{N}$, we have $\Delta_p(x_{m_k}, x^*) + \Delta_p(y_{m_k}, y^*) \leq \Delta_p(x_{m_k+1}, x^*) + \Delta_p(y_{m_k+1}, y^*)$, and

$$\Delta_p(x_k, x^*) + \Delta_p(y_k, y^*) \leq \Delta_p(x_{m_k+1}, x^*) + \Delta_p(y_{m_k+1}, y^*). \tag{3.48}$$

Then, by the same arguments as in (3.7) and (3.11), we have that $\lim_{k\to\infty} ||T(t_{n_k})w_{n_k} - w_{n_k}|| = 0$, and $\lim_{k\to\infty} ||S(s_{n_k})z_{n_k} - z_{n_k}|| = 0$. Also, from (3.32), we have that $\lim_{k\to\infty} ||u_{n_k} - x_{n_k}|| = 0$, and

$\lim_{k\to\infty}||v_{n_k}-y_{n_k}||=0$. From (3.11), we have that

$$\begin{aligned}&\Delta_p(x_{m_k+1},x^*)+\Delta_p(y_{m_k+1},y^*)\\&\le(1-\alpha_{m_k})[\Delta_p(x_{m_k},x^*)+\Delta_p(y_{m_k},y^*)]\\&\quad+\alpha_{m_k}[\langle J_{E_1}^p(u)-J_{E_1}^p(x^*),x_{m_k+1}-x^*\rangle+\langle J_{E_2}^p(v)-J_{E_2}^p(y^*),y_{m_k+1}-y^*\rangle],\end{aligned}$$

which implies that

$$\begin{aligned}&\alpha_{m_k}[\Delta_p(x_{m_k},x^*)+\Delta_p(y_{m_k},y^*)]\\&\le[\Delta_p(x_{m_k},x^*)+\Delta_p(y_{m_k},y^*)]-[\Delta_p(x_{m_k+1},x^*)+\Delta_p(y_{m_k+1},y^*)]\\&\quad+\alpha_{m_k}[\langle J_{E_1}^p(u)-J_{E_1}^p(x^*),x_{m_k+1}-x^*\rangle+\langle J_{E_2}^p(v)-J_{E_2}^p(y^*),y_{m_k+1}-y^*\rangle].\end{aligned}$$

That is,

$$[\Delta_p(x_{m_k},x^*)+\Delta_p(y_{m_k},y^*)]\le[\langle J_{E_1}^p(u)-J_{E_1}^p(x^*),x_{m_k+1}-x^*\rangle+\langle J_{E_2}^p(v)-J_{E_2}^p(y^*),y_{m_k+1}-y^*\rangle],$$

which implies from (3.46) and (3.47) that $\lim_{k\to\infty}[\Delta_p(x_{m_k},x^*)+\Delta_p(y_{m_k},y^*)]=0$, which together with (3.48) yields that $\Delta_p(x_k,x^*)+\Delta_p(y_k,y^*)\le\Delta_p(x_{m_k+1},x^*)+\Delta_p(y_{m_k+1},\ y^*)\to 0$ as $k\to\infty$. This implies that $\{(x_k,y_k)\}$ converges strongly to (x^*,y^*). Thus $\{(x_n,y_n\}$ converges strongly to $(x^*,y^*)\in\Gamma$. □

Remark 3.1. The iterative scheme considered in this article has an advantage over the one considered in [24] in the sense that we do not use any projection of a point on the intersection of closed and convex sets which creates some difficulties in a practical calculation of the iterative sequence. The Halpern iteration considered in this article provides more flexibility in defining the algorithm parameters which are important for the numerical implementation perspective.

Remark 3.2. (i) The problem considered in this article generalizes the one considered in [23]. (ii) Our result also generalizes the result of [22] as we were able to remove the compactness condition imposed on their mappings. In addition, the problem considered in this article generalizes the one considered in [22]. Ma et al. [22] considered uniformly convex and 2-uniformly smooth Banach space whereas we considered p-uniformly convex and uniformly smooth Banach space. Finally, the map considered in this article generalizes the one considered in [22] and other related results.

In the result stated below, we consider the split equality fixed point problem of relatively nonexpansive semigroup in Banach spaces.

Corollary 3.1. *Let $E_1,\ E_2$, and E_3 be three p-uniformly convex and uniformly smooth Banach spaces. Let C and Q be nonempty, closed, and convex subsets of E_1 and E_2 with duals E_1^* and E_2^*, respectively. Let $A:E_1\to E_3,\ B:E_2\to E_3$ be bounded linear operators and $A^*:E_3^*\to E_1^*,\ B:E_3^*\to E_2^*$ be adjoint operators of A and B respectively. Let $U=\{T(t)\}_{t\ge0}$ and $V=\{S(t)\}_{t\ge0}$ be an u.a.r Bregman relatively nonexpansive semigroup and uniformly Lipschitzian mappings of C and Q into E_1 and E_2 with bounded measurable function $L(t):(0,\infty)\to[0,\infty)$ and $D(t):(0,\infty)\to[0,\infty)$ such that $Fix(U):=\bigcap_{h\ge0}Fix(T(h))\ne\emptyset$ and $Fix(V):=\bigcap_{k\ge0}Fix(S(k))\ne\emptyset$. Suppose that $Fix(U)=\hat{Fix}(U)$ and $Fix(V)=\hat{Fix}(V)$, and assume that $\Gamma=\{(\bar{x},\bar{y}):\bar{x}\in Fix(U),\ \bar{y}\in Fix(V),A\bar{x}=B\bar{y}\}\ne\emptyset$. For a fixed $u\in E_1$ and $v\in E_2$,* *let $\{(x_n, y_n)\}$ be a sequence generated iteratively by*

$$\begin{cases} w_n = \Pi_C J^q_{E_1^*}[J^p_{E_1} u_n - \gamma_n A^* J^p_{E_3}(Au_n - Bv_n)]; \\ z_n = \Pi_Q J^q_{E_2^*}[J^p_{E_2} v_n + \gamma_n B^* J^p_{E_3}(Au_n - Bv_n)]; \\ x_{n+1} = J^q_{E_1^*}[\alpha_n J^p_{E_1}(u) + (1-\alpha_n)(\beta_n J^p_{E_1}(w_n) + (1-\beta_n) J^p_{E_2} T(t_n) w_n)]; \\ y_{n+1} = J^q_{E_1^*}[\alpha_n J^p_{E_2}(v) + (1-\alpha_n)(\beta_n J^p_{E_2}(z_n) + (1-\beta_n) J^p_{E_2} S(s_n) z_n)];\ \forall\, n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ satisfying the following conditions:
(i) $\lim_{n\to\infty} s_n = +\infty, \lim_{n\to\infty} t_n = +\infty, \lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = +\infty, 0 < a \leq \beta_n \leq b < 1$;
(ii) $0 < \gamma \leq \gamma_n \leq \rho \leq \left(\frac{q}{C_q ||A||^q}\right)^{\frac{1}{q-1}},\ 0 < \gamma \leq \gamma_n < \rho \leq \left(\frac{q}{D_q ||B||^q}\right)^{\frac{1}{q-1}}$.
Then $\{(x_n, y_n)\}$ converges strongly to $(x^*, y^*) \in \Gamma$.

We also consider the split equality equilibrium problem and fixed point problem of Bregman strongly nonexpansive mappings in Banach spaces as follows.

Corollary 3.2. *Let E_1, E_2, and E_3 be three p-uniformly convex and uniformly smooth Banach spaces. Let C and Q be nonempty, closed, and convex subsets of E_1 and E_2 with duals E_1^* and E_2^* respectively. Let $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumption 2.1. Let $A : E_1 \to E_3$, $B : E_2 \to E_3$ be bounded linear operators and $A^* : E_3^* \to E_1^*$, $B : E_3^* \to E_2^*$ be adjoint operators of A and B respectively. Let $T : E_1 \to E_2$ and $S : E_2 \to E_3$ be right Bregman strongly nonexpansive mappings such that $Fix(T) = \hat{Fix}(T)$ and $Fix(S) = \hat{Fix}(S)$. Assume that $\Gamma = \{(\bar{x}, \bar{y}) : \bar{x} \in Fix(U) \cap SEEP(F),\ \bar{y} \in Fix(V) \cap SEEP(G), A\bar{x} = B\bar{y}\} \neq \emptyset$. For a fixed $u \in E_1$ and $v \in E_2$, let $\{(x_n, y_n)\}$ be a sequence generated iteratively by*

$$\begin{cases} u_n = res_F x_n; \\ v_n = res_G y_n; \\ w_n = \Pi_C J^q_{E_1^*}[J^p_{E_1} u_n - \gamma_n A^* J^p_{E_3}(Au_n - Bv_n)]; \\ z_n = \Pi_Q J^q_{E_2^*}[J^p_{E_2} v_n + \gamma_n B^* J^p_{E_3}(Au_n - Bv_n)]; \\ x_{n+1} = J^q_{E_1^*}[\alpha_n J^p_{E_1}(u) + (1-\alpha_n)(\beta_n J^p_{E_1}(w_n) + (1-\beta_n) J^p_{E_1} T w_n)]; \\ y_{n+1} = J^q_{E_2^*}[\alpha_n J^p_{E_2}(v) + (1-\alpha_n)(\beta_n J^p_{E_2}(z_n) + (1-\beta_n) J^p_{E_2} S z_n)];\ \forall\, n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ satisfying the following conditions:
(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = 0, 0 < a \leq \beta_n \leq b < 1$;
(ii) $0 < \gamma \leq \gamma_n \leq \rho \leq \left(\frac{q}{C_q ||A||^q}\right)^{\frac{1}{q-1}},\ 0 < \gamma \leq \gamma_n < \rho \leq \left(\frac{q}{D_q ||B||^q}\right)^{\frac{1}{q-1}}$.
Then $\{(x_n, y_n)\}$ converges strongly to $(x^*, y^*) \in \Gamma$.

As an application, we now study the following Split Equality Convex Optimization Problem (SECOP): Find

$$x^* \in \arg\min_{x \in C} h(x),\ y^* \in \arg\min_{y \in Q} a(y) : Ax^* = By^*,$$

where C and Q are nonempty, closed, and convex subsets of real Banach spaces, and $h : C \to \mathbb{R}$ and $a : Q \to \mathbb{R}$ are convex and a lower semicontinuous functional. Let $F : C \times C$ be defined by

$F(x^*,x) := h(x) - h(x^*)$ and $G(y^*,y) := a(y) - a(y^*)$. Let us now consider the SECOP: Find $x^* \in C$ and $y^* \in Q$ such that

$$F(x^*,x) \geq 0 \text{ and } G(y^*,y) \geq 0,$$

for all $x \in C$ and $y \in Q$. It is obvious that F and G satisfy Assumptions **1.3**. We denote by $\Omega := \{(\bar{x},\bar{x}) : \bar{x} \in Fix(U) \cap SECOP(F,\phi),\ \bar{y} \in Fix(V) \cap SECOP(G,\psi), A\bar{x} = B\bar{y}\} \neq \emptyset$.

Theorem 3.2. *Let E_1, E_2, and E_3 be three p-uniformly convex and uniformly smooth Banach spaces. Let C and Q be nonempty, closed, and convex subsets of E_1 and E_2 with duals E_1^* and E_2^* respectively. Let $f : E_1 \to (-\infty,+\infty]$ and $g : E_2 \to (-\infty,+\infty]$ be coercive Legendre functions which are bounded, uniformly Frechet differentiable and totally convex on bounded subsets of E_1 and E_2 respectively. Let $h : C \to \mathbb{R}$, $a : Q \to \mathbb{R}$ and $\phi : C \times C \to \mathbb{R}$, $\psi : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumptions 2.1 and 2.2, respectively. Let $A : E_1 \to E_3$, $B : E_2 \to E_3$ be bounded linear operators and $A^* : E_3^* \to E_1^*$, $B : E_3^* \to E_2^*$ be adjoint operators of A and B respectively. Let $U = \{T(t)\}_{t\geq 0}$ and $V = \{S(t)\}_{t\geq 0}$ be u.a.r Bregman relatively nonexpansive semigroup and uniformly Lipschitzian mapping of C and Q into E_1 and E_2 respectively with bounded measurable function $L(t) : (0,\infty) \to [0,\infty)$ and $D(t) : (0,\infty) \to [0,\infty)$ such that $Fix(U) := \bigcap_{h\geq 0} Fix(T(h)) \neq \emptyset$ and $Fix(V) := \bigcap_{k\geq 0} Fix(S(k)) \neq \emptyset$. Suppose $Fix(U) = \hat{Fix}(U)$ and $Fix(V) = \hat{Fix}(V)$, and assume that $\Omega \neq \emptyset$. For a fixed $u \in E_1$ and $v \in E_2$, let $\{(x_n,y_n)\}$ be a sequence generated iteratively by*

$$\begin{cases} u_n = res^f_{F,\phi} x_n; \\ v_n = res^g_{G,\psi} y_n; \\ w_n = \Pi_C J^q_{E_1^*}[J^p_{E_1} u_n - \gamma_n A^* J^p_{E_3}(Au_n - Bv_n)]; \\ z_n = \Pi_Q J^q_{E_2^*}[J^p_{E_2} v_n + \gamma_n B^* J^p_{E_3}(Au_n - Bv_n)]; \\ x_{n+1} = J^q_{E_1^*}[\alpha_n J^p_{E_1}(u) + (1-\alpha_n)(\beta_n J^p_{E_1}(w_n) + (1-\beta_n) J^p_{E_1} T(t_n) w_n)]; \\ y_{n+1} = J^q_{E_2^*}[\alpha_n J^p_{E_2}(v) + (1-\alpha_n)(\beta_n J^p_{E_2}(z_n) + (1-\beta_n) J^q_{E_2} S(s_n) z_n)];\ \forall\, n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} s_n = +\infty, \lim_{n\to\infty} t_n = +\infty, \lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = +\infty, 0 < a \leq \beta_n \leq b < 1$*;*

(ii) $0 < \gamma \leq \gamma_n \leq \rho \leq \left(\frac{q}{C_q||A||^q}\right)^{\frac{1}{q-1}},\ 0 < \gamma \leq \gamma_n < \rho \leq \left(\frac{q}{D_q||B||^q}\right)^{\frac{1}{q-1}}$.

Then $\{(x_n,y_n)\}$ converges strongly to $(x,y^*) \in \Omega$.

4. Numerical Examples

In this section, we present some numerical examples to illustrate and support the convergence of our proposed method (Theorem 3.2). All numerical computations were carried out using Matlab version R2021(b). In the numerical computations, we choose $\alpha_n = \frac{2}{3n+1}, \beta_n = \frac{2n}{4n+1}, s_n = n+1$, and $t_n = 2n$.

Example 4.1. Let $E_1 = E_2 = E_3 = \mathbb{R}$ and $C = Q = [0,10]$. Let the mappings $A, B : \mathbb{R} \to \mathbb{R}$ be defined by $Ax = \frac{2x}{5}$ and $Bx = \frac{3x}{7}$. Then, A and B are bounded linear operators. Also, we define $S(t)(x) = e^{-t}x$ and $T(t)(x) = e^{-2t}x$ for all $t \geq 0$, $x \in \mathbb{R}$. Let the bifunctions $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ be defined by $F(x,y) = y^2 + 6xy - 7x^2$ and $G(x,y) = 2y^2 + 6xy - 8x^2$,

$\forall(x,y) \in \mathbb{R}\times\mathbb{R}$. Also, we define $\phi : C\times C \to \mathbb{R}$ and $\psi : Q\times Q \to \mathbb{R}$ by $\phi(x,y) = y^2 - 1$ and $\psi(x,y) = y^2 - 2$, $\forall(x,y) \in \mathbb{R}\times\mathbb{R}$. Next, we find $u \in C$ such that, for all $z \in C$,

$$\begin{aligned} 0 &\le F(u,z) + \phi(u,z) - \phi(u,u) + \langle z-u, u-x\rangle \\ &= z^2 + 6uz - 7u^2 + z^2 - u^2 + \langle z-u, u-x\rangle \\ &\Leftrightarrow \\ 0 &\le 2z^2 + 3uz - 5u^2 + 3u(z-u) + (z-u)(u-x) \\ &= 2z^2 + 3uz - 5u^2 + 3u(z-u) + uz - xz - u^2 + ux \\ &= 2z^2 + (7u-x)z + (-9u^2 + ux). \end{aligned}$$

Suppose $h(z) = 2z^2 + (7u-x)z + (-9u^2+ux)$. Then, $h(z)$ is a quadratic function of z with coefficients $a = 2$, $b = (7u-x)$, and $c = (-9u^2+ux)$. We determine the discriminant $\triangle$ of $h(z)$ as $\triangle = (7u-x)^2 - 4(2)(-9u^2+ux) = (11u-x)^2$. According to Lemma 2.6, $res^f_{F,\phi}$ is single-valued. Therefore, it follows that $h(z)$ has at most one solution in $\mathbb{R}$. Thus $u = \frac{x}{11}$. This implies that $res^f_{F,\phi}(x) = \frac{x}{11}$. Following similar procedure, we have that $res^f_{G,\psi}(y) = \frac{y-1}{11}$. We choose different initial values as follows:
Case I: $x_1 = -1.0, y_1 = 2.0$;
Case II: $x_1 = 3.5, y_1 = 8.4$;
Case III: $x_1 = -7.9, y_1 = -5.1$;
Case IV: $x_1 = 4.0, y_1 = 9.0$

The stopping criterion used for this example is $|x_{n+1} - x_n| < 10^{-9}$. We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Figures 1, 2, 3, and 4.

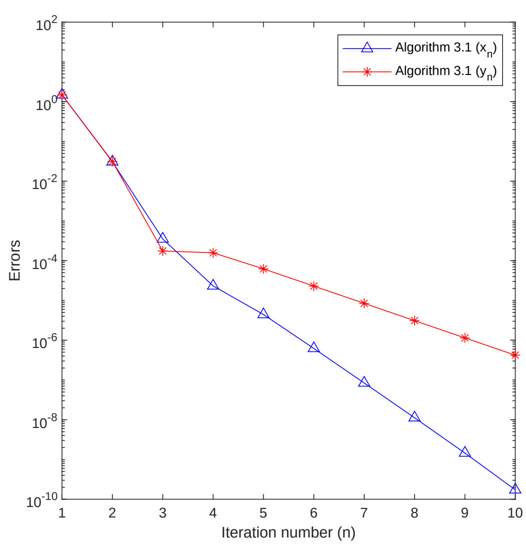

FIGURE 1. Example 4.1: Case I

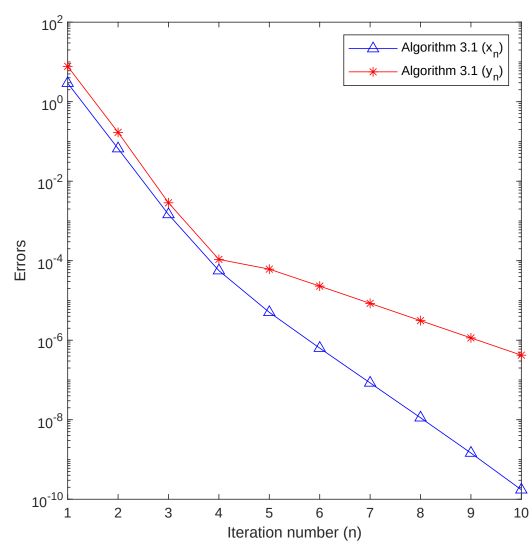

FIGURE 2. Example 4.1: Case II

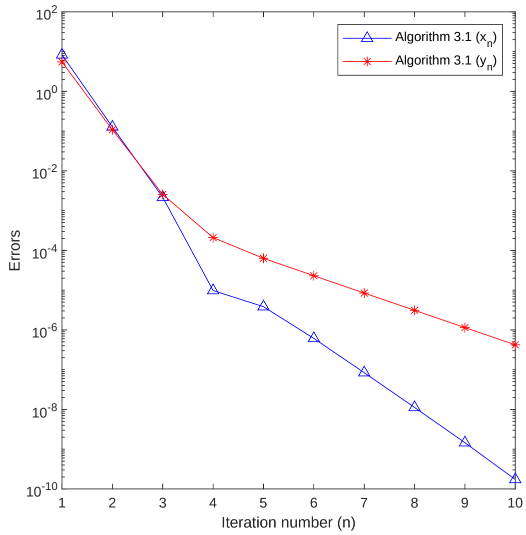

FIGURE 3. Example 4.1: Case III

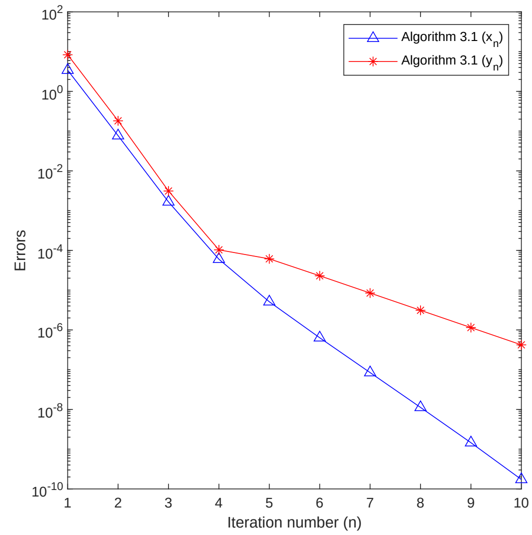

FIGURE 4. Example 4.1: Case IV

Example 4.2. Let $E_1 = E_2 = E_3 = L_2([0,1])$ be endowed with inner product

$$\langle x,y\rangle = \int_0^1 x(t)y(t)dt, \quad \forall x,y \in L_2([0,1])$$

and norm

$$||x|| := \Big(\int_0^1 |x(t)|^2\Big)^{\frac{1}{2}} \quad \forall x,y \in L_2([0,1]).$$

We define $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ by $F(x,y) = \langle L_1 x, y-x\rangle$ and $G(x,y) = \langle L_2 x, y-x\rangle$, where $L_1 x(t) = \frac{5x(t)}{6}$ and $L_2 x(t) = \frac{7x(t)}{10}$. Also, we define $\phi(x,y) = \psi(x,y) = y(t) - 1\ \forall y \in L_2([0,1])$. Moreover, let $A,B : L_2([0,1]) \to L_2([0,1])$ be defined by $Ax(t) = \frac{2x(t)}{5}$ and $Bx(t) = \frac{x(t)}{2}$. Then A and B are bounded linear operators. Also, we define $S(t)(x) = e^{-3t}x$ and $T(t)(x) = e^{-5t}x$ for all $t \geq 0$, $x \in L_2([0,1])$. Next, we find $x \in E_1$ such that, for all $u \in E_1$,

$$\begin{aligned} & f_1(x,u) + \phi(x,u) - \phi(x,x) + \langle u-x, x-z\rangle \geq 0 \\ \Longleftrightarrow & \frac{5x}{6}(u-x) + (u-x)(x-z) \geq 0 \\ \Longleftrightarrow & (u-x)[11x + 6 - 6z] \geq 0. \end{aligned}$$

According to Lemma 2.6, $res^f_{F,\phi}$ is single-valued. Hence, $x = \frac{6z-6}{11}$. This implies that $res^f_{F,\phi}(z) = \frac{6z-6}{11}$. Following a similar procedure as above, we obtain $res^f_{G,\psi}(w) = \frac{10w-10}{17}$. We choose different initial values as follows:

Case I: $x_1 = 2t^3 + 4t - 1, y_1 = 4t + 5$;
Case II: $x_1 = 3t^2 + 2, y_1 = 2\sin t$;
Case III: $x_1 = 2t\cos t, y_1 = \frac{3t^2}{5}$;
Case IV: $x_1 = 2t^3 - 3, y_1 = \exp(3t)$.

The stopping criterion used for this example is $\|x_{n+1} - x_n\| < 10^{-9}$. We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in the following four figures.

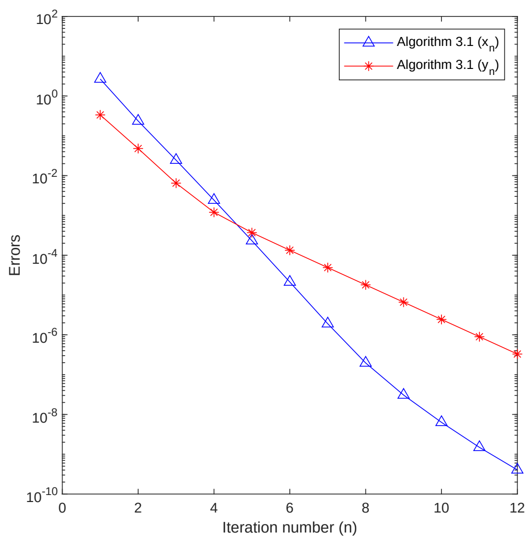

FIGURE 5. Example 4.2: Case I

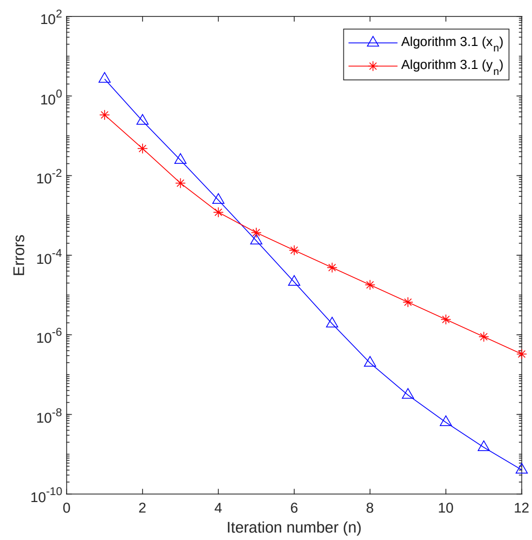

FIGURE 6. Example 4.2: Case II

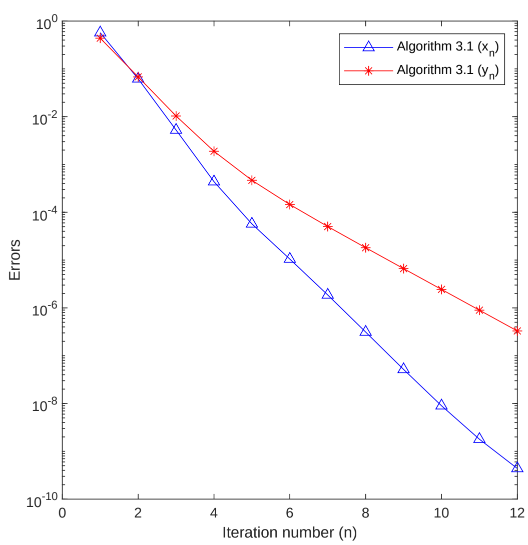

FIGURE 7. Example 4.2: Case III

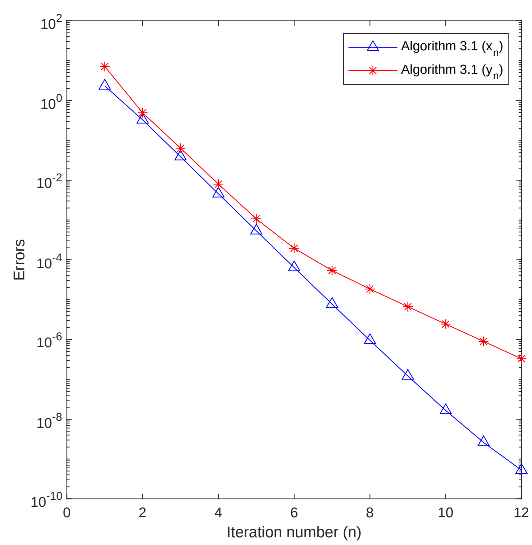

FIGURE 8. Example 4.2: Case IV

5. Conclusion

In this paper, we introduced and studied a new split inverse problem called split equality generalized equilibrium problem. We proposed a new iterative method for approximating a common solution of this problem and the split equality fixed point problem with Bregman relatively nonexpansive semigroups in Banach spaces. Moreover, we proved a strong convergence result for the proposed algorithm. Finally, we applied our result to a split equality convex optimization problem and presented some numerical examples to illustrate and support the convergence of our proposed method.

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